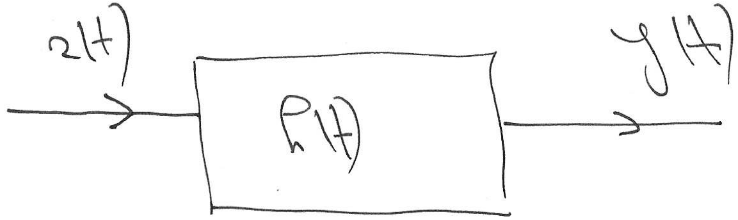
Exercice 2, TD1

(1)

$h(t) = \frac{1}{T}$ si $t \in [0,T]$, $h(t)$ sinon.

<u>Question préliminaire</u>



Expression de $y(t)$?

$$y(t) = (h * x)(t) = \int_{-\infty}^{+\infty} h(s)\, x(t-s)\, ds$$

$$= \int_0^T h(s)\, x(t-s)\, ds = \frac{1}{T}\int_0^T x(t-s)\, ds$$

$t - s = u \quad du = -ds$

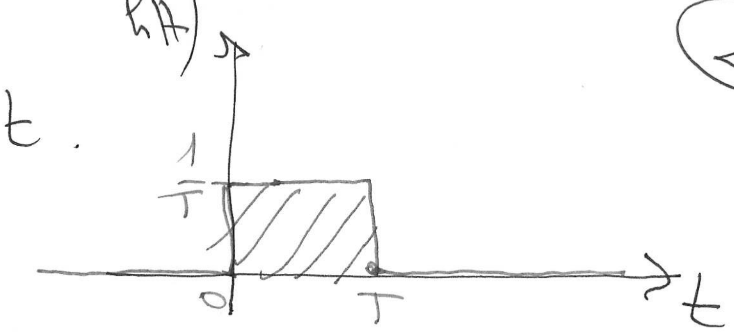
Si s varie entre 0 et T, u varie entre t et $t-T$

$$\frac{1}{T}\int_0^T x(t-s)\, ds = \frac{1}{T}\int_t^{t-T} x(u)(-du) = -\frac{1}{T}\int_t^{t-T} x(u)\, du = \frac{1}{T}\int_{t-T}^{t} x(u)\, du$$

: justifie le terme de filtre moyenneur.

(2)

$$\int_0^{\infty} |h(t)|\,dt = \int_0^{\infty} h(t)\,dt \quad \text{car } h(t) \geq 0 \quad \forall t.$$



$$= \int_0^{T} h(t)\,dt = \int_0^{T} \frac{1}{T}\,dt = \frac{1}{T}\int_0^{T} dt = 1 < +\infty \Rightarrow \text{le filtre est stable}$$

$$y(t) = \frac{1}{T}\int_{t-T}^{t} x(u)\,du \ , \quad |y(t)| = \frac{1}{T}\left|\int_{t-T}^{t} x(u)\,du\right| \leq \frac{1}{T}\int_{t-T}^{t} |x(u)|\,du.$$

Si il existe une constante $m > 0$ telle que $|x(u)| \leq m \ \forall u$,

$$\frac{1}{T}\int_{t-T}^{t} |x(u)|\,du \leq \frac{1}{T}\int_{t-T}^{t} (m)\,du = m \underbrace{\frac{1}{T}\int_{t-T}^{t} dt}_{1} = m$$

③

$$\hat{h}(\nu) = \int h(t)\, e^{-2i\pi\nu t}\, dt = e^{-i\pi\nu T} \frac{\sin \pi\nu T}{\pi\nu T}\ ?$$

$$= \int_0^T \frac{1}{T} e^{-2i\pi\nu t}\, dt = \frac{1}{T}\int_0^T e^{-2i\pi\nu t}\, dt \quad \text{si } \nu \neq 0, \int_0^T e^{-2i\pi\nu t}\, dt = -\frac{1}{2i\pi\nu}\left[e^{-2i\pi\nu t}\right]_0^T$$

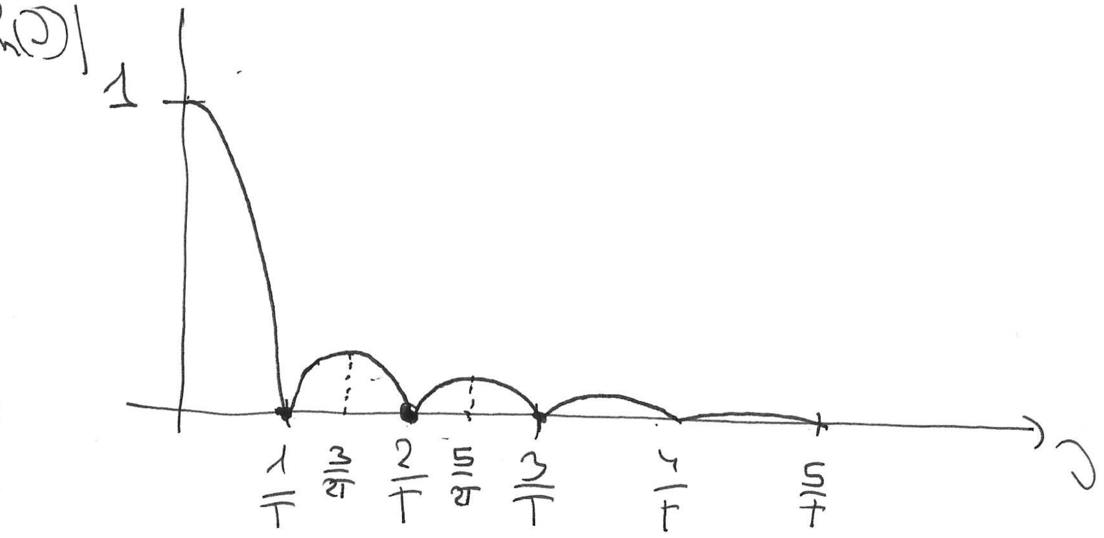
$$= -\frac{1}{2i\pi\nu}\left(e^{-2i\pi\nu T} - 1\right)$$

$$\hat{h}(\nu) = -\frac{1}{2i\pi\nu T}\left(e^{-2i\pi\nu T} - 1\right) = \frac{1}{2i\pi\nu T}\left(1 - e^{-2i\pi\nu T}\right) = \frac{1}{2i\pi\nu T} e^{-i\pi\nu T} \cdot e^{i\pi\nu T}\left(1 - e^{-2i\pi\nu T}\right)$$

$$= \frac{1}{2i\pi\nu T} e^{-i\pi\nu T} \underbrace{\left(e^{i\pi\nu T} - e^{-i\pi\nu T}\right)}_{2i \sin \pi\nu T}$$

$$= e^{-i\pi\nu T} \frac{\sin \pi\nu T}{\pi\nu T}$$

$$\left|\operatorname{sinc} \pi\nu T\right| = |\hat{h}(\nu)| = \left|\frac{\sin \pi\nu T}{\pi\nu T}\right|$$ Par définition, on pose $\operatorname{sinc} x = \frac{\sin x}{x}$



$$\hat{h}(\nu) = \left| \frac{\sin \pi \nu T}{\pi \nu T} \right|$$

$\hat{h}(\nu) = 0$ si $\sin \pi \nu T = 0$

$\Leftrightarrow \pi \nu T = k\pi, \; k \neq 0$

$\hat{h}(\nu) = 0 \Leftrightarrow \nu = \frac{k}{T}, \; k > 0$

si $\nu \in [0, \frac{1}{T}]$, $\pi \nu T \in [0, \pi]$, $\begin{cases} \sin \pi \nu T \text{ croît de } 0 \text{ à } 1 \text{ si } \nu \in [0, \frac{1}{2T}] \\ \sin \pi \nu T \text{ décroît de } 1 \text{ à } 0 \text{ si } \nu \in [\frac{1}{2T}, \frac{1}{T}] \end{cases}$

si $\nu \in [\frac{1}{2T}, \frac{1}{T}]$, $\hat{h}(\nu)$ décroît

si $\nu \in [0, \frac{1}{2T}]$, $\hat{h}(\nu)$ est le produit d'une fonction croissante et d'une fonction décroissante mais $\hat{h}(\nu)$ reste décroissante

On applique à l'entrée $x(t) = A\cos(2\pi\nu_0 t + \varphi)$ pour tout t.

Calculer $y(t)$.

$$x(t) \longrightarrow \boxed{\hat{h}(\nu)} \longrightarrow y(t) = A\,|\hat{h}(\nu_0)|\cos\left(2\pi\nu_0 t + \varphi + \mathrm{Ang}\,\hat{h}(\nu_0)\right)$$

$$= A\left|\frac{\sin \pi\nu_0 T}{\pi\nu_0 T}\right| \cos\left(2\pi\nu_0 t + \varphi + \underbrace{\mathrm{Ang}\,\hat{h}(\nu_0)}\right)$$

$\varphi(\nu_0) = \mathrm{Ang}\,\hat{h}(\nu_0)$?

$$\hat{h}(\nu_0) = e^{-i\pi\nu_0 T}\,\frac{\sin \pi\nu_0 T}{\pi\nu_0 T}$$

$$= |\hat{h}(\nu_0)|\,e^{i\varphi(\nu_0)}$$

$\varphi(\nu_0)$?

si $\sin \pi\nu_0 T \geq 0$, $|\hat{h}(\nu_0)| = \dfrac{\sin \pi\nu_0 T}{\pi\nu_0 T}$, $\mathrm{Ang}\,\hat{h}(\nu_0) = -\pi\nu_0 T$

si $\sin \pi\nu_0 T < 0$, $|\hat{h}(\nu_0)| = -\dfrac{\sin \pi\nu_0 T}{\pi\nu_0 T} = \left|\dfrac{\sin \pi\nu_0 T}{\pi\nu_0 T}\right|$

$\sin \pi\nu_0 T < 0$, $\quad \hat{h}(\nu_0) = e^{-i\pi\nu_0 T} \times -\dfrac{\sin \pi\nu_0 T}{\pi\nu_0 T} \times -1$

$$\hat{h}(\nu_0) = |\hat{h}(\nu_0)| \cdot -e^{-i\pi\nu_0 T}$$

$$-e^{-i\pi\nu_0 T} = e^{i\pi} e^{-i\pi\nu_0 T} = e^{i(\pi - \pi\nu_0 T)}$$

$$\hat{h}(\nu_0) = |\hat{h}(\nu_0)| \, e^{i(\pi - \pi\nu_0 T)}$$

$\operatorname{Arg} \hat{h}(\nu_0) = \pi - \pi\nu_0 T$

$$y(t) = A \left|\frac{\sin \pi\nu_0 T}{\pi\nu_0 T}\right| \cos(2\pi f_0 t + \varphi - \pi\nu_0 T) \quad \text{si} \quad \sin \pi\nu_0 T \geq 0$$

$$= A \left|\frac{\sin \pi\nu_0 T}{\pi\nu_0 T}\right| \cos(2\pi f_0 t + \varphi + \pi - \pi\nu_0 T) \quad \text{si} \quad \sin \pi\nu_0 T < 0$$

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$$x(t) = e^{-t} \quad \text{si } t \geq 0$$
$$= 0 \quad \text{si } t < 0$$

$$y(t) = (h * x)(t) \overset{?}{=} \frac{1}{T}\int_{t-T}^{t} x(u)\,du$$

si $t < 0$, $[t-T, t] \subset \mathbb{R}^-$, $x(u) = 0$ si $t-T \leq u \leq t$, $y(t) = 0$

si $t \geq 0$

$\cancel{\&}$ $t-T \leq 0$

$\Rightarrow t \in [0, T]$

$$y(t) = \frac{1}{T}\int_{t-T}^{t} x(u)\,du = \underbrace{\frac{1}{T}\int_{t-T}^{0} x(u)\,du}_{0} + \underbrace{\frac{1}{T}\int_{0}^{t} x(u)\,du}_{}$$

$$\frac{1}{T}\int_0^t e^{-u}\,du = -\frac{1}{T}\left(e^{-u}\right)_0^t$$
$$= -\frac{1}{T}\left(e^{-t} - 1\right) = \frac{1 - e^{-t}}{T}$$

Si $t > T$: $y(t) = \frac{1}{T}\int_{t-T}^{t} z(u)\,du = \frac{1}{T}\int_{t-T}^{t} e^{-u}\,du$

$$= -\frac{1}{T}\Big(e^{-u}\Big)_{t-T}^{t} = -\frac{1}{T}\left(e^{-t} - e^{-(t-T)}\right) = \frac{1}{T}\left(e^{-(t-T)} - e^{-t}\right)$$

$$= \frac{e^{-t}}{T}\left(e^{T} - 1\right)$$

$y(t) = 0$ si $t < 0$, $y(t) = \frac{1 - e^{-t}}{T}$ si $t \in [0, T]$, $y(t) = \frac{e^{-t}}{T}\left(e^{T} - 1\right)$

$t = T$ $\frac{e^{-T}}{T}\left(e^{T} - 1\right) = \frac{1}{T}\left(1 - e^{-T}\right)$