1

$$x(t) = \sum_n a_n\, g(t-nT)$$

$$E\left(x(t+\tau)\, x^*(t)\right) = E\left[\underbrace{\left(\sum_n a_n\, g(t+\tau-nT)\right)\left(\sum_m a_m^*\, g(t-mT)\right)}\right]$$

$$= E\left[\sum_{n,m} a_n\, a_m^*\, g(t+\tau-nT)\, g(t-mT)\right]$$

$$= \sum_{n,m} E\left(a_n\, a_m^*\right) g(t+\tau-nT)\, g(t-mT)$$

$$E\left(a_n\, a_m^*\right) = \left|\begin{array}{l} \text{si } n=m,\ E|a_n|^2 = 1 \\ \text{si } n \neq m\ \ E(a_n a_m^*) = E(a_n)\, E(a_m^*) = 0 \end{array}\right.$$

$$E\left(x(t+\tau)\, x^*(t)\right) = \sum_n E|a_n|^2\, g(t+\tau-nT)\, g(t-nT)$$

$$= \sum_n g(t+\tau-nT)\, g(t-nT)$$

2

τ fixé $\quad t \longrightarrow E(x(t+\tau)\,x^*(t)) = \sum_{m\in\mathbb{Z}} g(t-mT+\tau)\,g(t-mT)$

$$E(x(t+T+\tau)\,x^*(t+T)) = \sum_{m\in\mathbb{Z}} g(t+T-mT+\tau)\,g(t+T-mT)$$

$$= \sum_{m\in\mathbb{Z}} g(t-(m-1)T+\tau)\,g(t-(m-1)T)$$

$m = m-1 \qquad = \sum_{m\in\mathbb{Z}} g(t-mT+\tau)\,g(t-mT)$

$t \to g(t+\tau)\,g(t)$

3

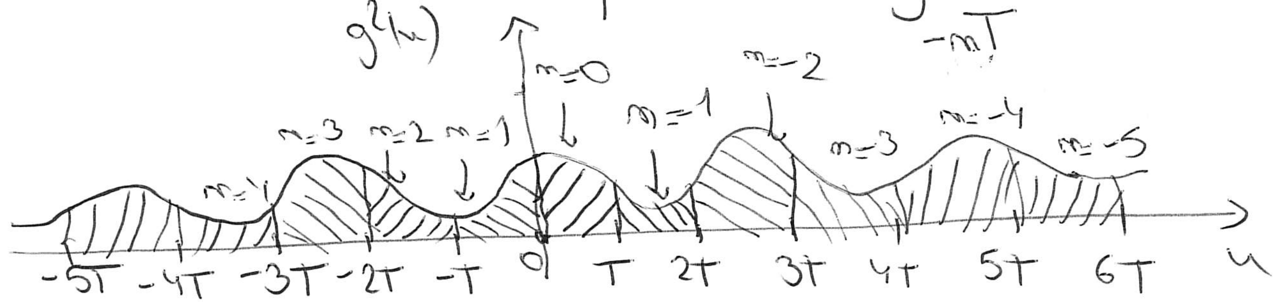
$$E\left(x(t+\tau)\,x^*(t)\right) = \sum_m g(t+\tau-mT)\,g(t-mT)$$

$$R(\tau) = \frac{1}{T}\int_0^T E\left(x(t+\tau)x^*(t)\right)dt \qquad R(0) = \frac{1}{T}\int_{-\infty}^{+\infty}\left(g(t)\right)^2 dt$$

$$R(0) = \frac{1}{T}\int_0^T E\left(|x(t)|^2\right)dt = \frac{1}{T}\int_0^T\left(\sum_m \left(g(t-mT)\right)^2\right)dt$$

$$= \frac{1}{T}\sum_m \underbrace{\int_0^T \left(g(t-mT)\right)^2 dt}_{u=t-mT,\quad u\in[-mT,\,-(m-1)T]}$$

$$= \frac{1}{T}\sum_m \int_{-mT}^{-(m-1)T} g^2(u)\,du = \frac{1}{T}\int_{-\infty}^{+\infty} g^2(u)\,du$$



Exemples de densités spectrales

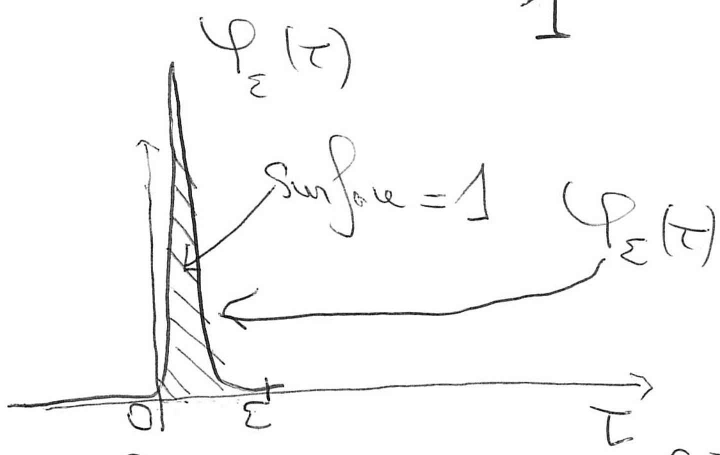
- Le bruit blanc $\quad E(n(t)) = 0, \quad E\left(n(t+\tau)\, n^*(t)\right) = N_0\, \delta(\tau) = R(\tau)$

$$S(f) = \int_{-\infty}^{+\infty} R(\tau)\, e^{-2i\pi f \tau}\, d\tau = N_0 \underbrace{\int_{-\infty}^{+\infty} \delta(\tau)\, e^{-2i\pi f\tau}\, d\tau}_{1} = N_0$$

$$\int_{-\infty}^{+\infty} f(\tau)\,\delta(\tau)\, d\tau = f(0)$$

$$\|$$

$$\int_{-\infty}^{+\infty} f(\tau)\,\delta(\tau)\, d\tau = f(0) \underbrace{\int_{-\infty}^{+\infty} \delta(\tau)\, d\tau}_{1} = f(0)$$



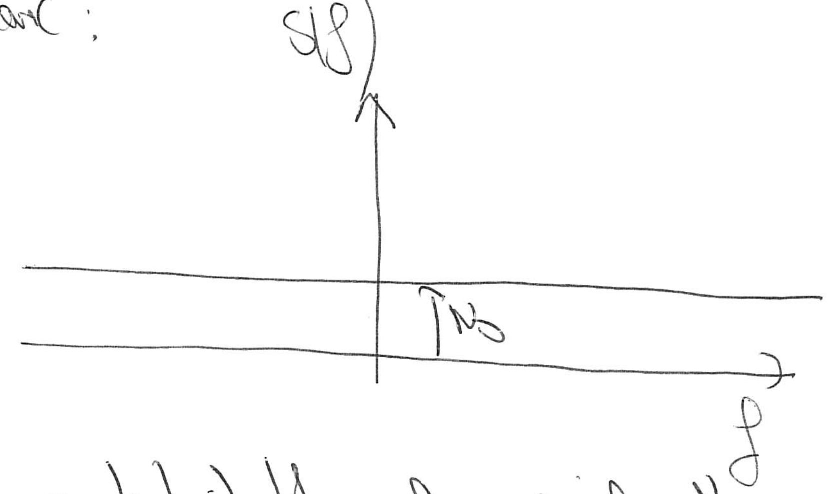
$\delta(\tau) : \lim\limits_{\varepsilon \to 0} \varphi_\varepsilon(\tau)$, avec $\displaystyle\int_{-\infty}^{+\infty} \varphi_\varepsilon(\tau)\, d\tau = 1$

$\displaystyle\int_{-\infty}^{+\infty} \delta(\tau)\, d\tau = 1.$ $\qquad f(\tau)\,\delta(\tau) = f(0)\,\delta(\tau)$

5

Densité spectrale du bruit blanc :

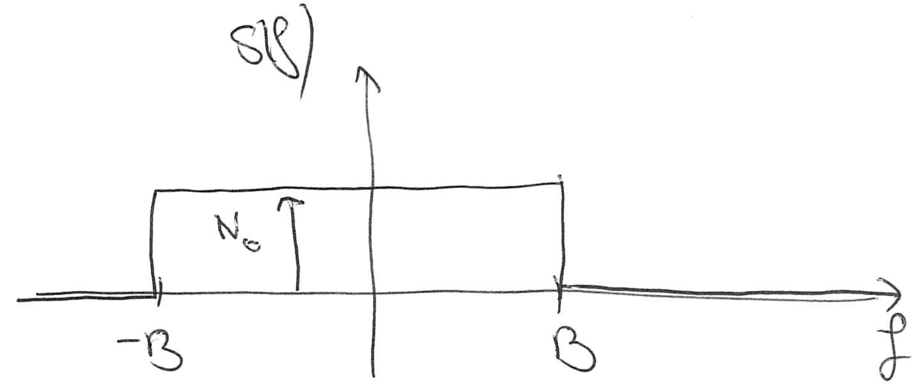
$$S(f) = N_0$$



$$R(0) = \int_{-\infty}^{+\infty} S(f)\,df = +\infty \Rightarrow$$ le bruit blanc est un signal de puissance moyenne infinie.

- Le bruit blanc passe-bas

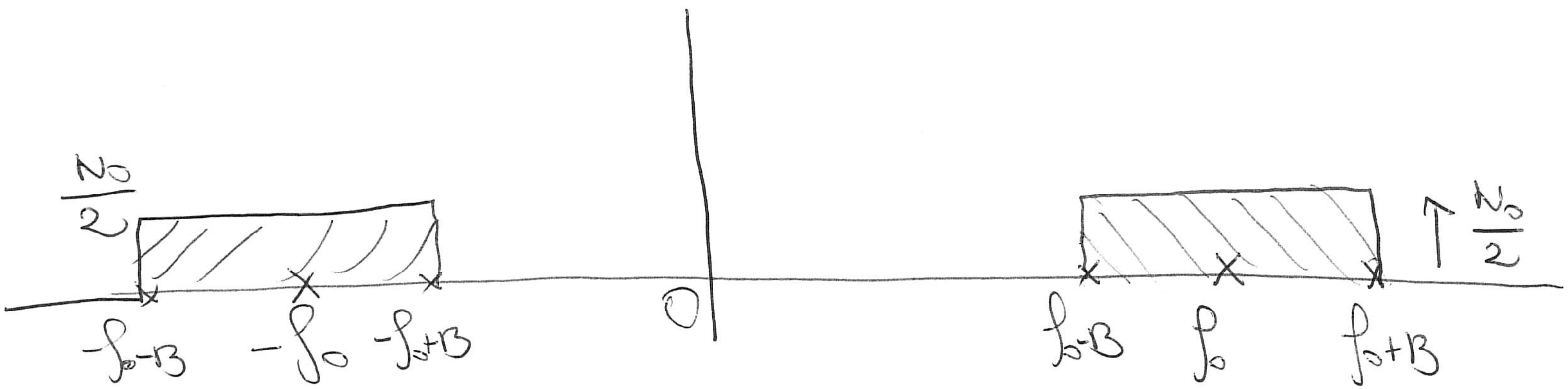
$$S(f) = N_0 \quad \text{si } f \in [-B, B]$$
$$ = 0 \quad \text{si } f \notin [-B, B]$$



$$R(0) = \int_{-\infty}^{+\infty} S(f)\,df = N_0 \int_{-B}^{B} df = 2 N_0 B$$

6

- Le bruit blanc passe-bande



$$R(0) = \int_{-\infty}^{+\infty} S(f)\,df = 2B\frac{N_0}{2} + 2B \times \frac{N_0}{2} = 2BN_0$$

- $z(t) = \sum_m a_m\, g(t-mT)$ $\qquad G(f)$: transformée de Fourier de $g(t)$

$$S(f) = \frac{1}{T}\left|G(f)\right|^2$$

Indication : Calculer $R(\tau) = \frac{1}{T}\int_0^T E\left(z(t+\tau)\,z^*(t)\right)dt = \frac{1}{T}\int_0^T \left(\left(\sum_m g(t+\tau-mT)\right) g(t-mT)\right)dt$

(7)

Montrer que $R(\tau) = \frac{1}{T}\int_{-\infty}^{+\infty} g(t+\tau)\,g(t)\,dt$

On part de $R(\tau) = \frac{1}{T}\int_{0}^{T}\left(\sum_m g(t+\tau-mT)\,g(t-mT)\right)dt$

$$= \sum_m \underbrace{\frac{1}{T}\int_{0}^{T} g(t+\tau-mT)\,g(t-mT)\,dt}_{u = t-mT}$$

$$= \frac{1}{T}\sum_m \int_{-mT}^{-(m-1)T} g(u+\tau)\,g(u)\,du = \frac{1}{T}\int_{-\infty}^{+\infty} g(t+\tau)\,g(t)\,dt$$

La TF de $\tau \longrightarrow \frac{1}{T}\int_{-\infty}^{+\infty} g(t+\tau)\,g(t)\,dt$ est $\frac{1}{T}|G(f)|^2$?

8

$$(g*g)(x) = \int_{-\infty}^{+\infty} g(t)\, g(x-t)\, dt$$

$$\int_{-\infty}^{+\infty} g(t)\, g(t+\tau)\, dt$$

$$(g*\tilde{g})(x) = \int_{-\infty}^{+\infty} g(t)\, \tilde{g}(x-t)\, dt$$

$$\tilde{g}(t) = g(-t)$$

$$(g*\tilde{g})(x) = \int_{-\infty}^{+\infty} g(t)\, g(t-x)\, dt$$

$$\tilde{G}(f) = \int_{-\infty}^{+\infty} g(-t)\, e^{-2i\pi f t}\, dt = \int_{-\infty}^{+\infty} g(u)\, e^{2i\pi f u}\, du = (G(f))^*$$

$$(g*\tilde{g})(\tau) = \underbrace{\int_{-\infty}^{+\infty} g(t)\, g(t-\tau)\, dt}_{u = t-\tau,\ t = u+\tau} = \int_{-\infty}^{+\infty} g(u+\tau)\, g(u)\, du$$

$$R(\tau) = \frac{1}{T}\int_{-\infty}^{+\infty} g(t)\, g(t+\tau)\, dt = \frac{1}{T}(g*\tilde{g})(\tau) \Rightarrow S(f) = \frac{1}{T} G(f)\, \tilde{G}(f) = \frac{1}{T}|G(f)|^2$$

$$S(f) = \frac{1}{T} |G(f)|^2$$

$$R(0) = \int_{-\infty}^{+\infty} S(f)\,df = \frac{1}{T} \underbrace{\int_{-\infty}^{+\infty} |G(f)|^2\,df}$$

$$= \frac{1}{T} \int_{-\infty}^{+\infty} (g(t))^2\,dt$$

$$\int_{-\infty}^{+\infty} (g(t))^2\,dt = \int_{-\infty}^{+\infty} |G(f)|^2\,df$$

Identité de Parseval

10

$$z(t) = \sum_m a_m \, g(t - mT)$$

$$S(f) = \frac{1}{T} |G(f)|^2 .$$

$$S(f) = 0 \iff G(f) = 0$$

Bande passante de $z(t)$ $\triangleq$ le plus grand intervalle en dehors duquel $S(f) \neq 0$

= Bande passante de $g(t)$

Vision plus simple et intuitive, mais mathématiquement fausse.

$$\int z(t)\, e^{-2i\pi f t}\, dt = \int \left(\sum_m a_m \, g(t-mT) \right) e^{-2i\pi f t}\, dt = \sum_m a_m \int g(t-mT)\, e^{-2i\pi f t}\, dt$$

$$= \sum_m a_m \, G(f)\, e^{-2i\pi f m T} = G(f) \left(\sum_m a_m \, e^{-2i\pi f m T} \right)$$