On the Blind Equalization of Continuous Phase Modulated Signals using the Constant Modulus Criterion

Pascal Bianchi, Philippe Loubaton

Abstract

The use of the Constant Modulus (CM) criterion to achieve blind equalization of Continuous Phase Modulated signals is investigated in this paper. Solutions to the CM criterion are characterized. It is shown that blind equalization using the CM criterion may be unsuccessful in the sense that the output equalizer does not necessarily coincides with a delayed and rotated version of the input signal. An indeterminacy indeed remains and this residual indeterminacy is characterized. As an application of our results, a blind equalization criterion which allows to eliminate the latter indeterminacy is proposed. Simulations sustain theoretical claims and show that the proposed criterion leads to attractive performance.

P. Bianchi is with the École Supérieure d’Électricité Département Télécommunications, Plateau du Moulon, 91192 Gif-sur-Yvette, France (e-mail: pascal.bianchi@supelec.fr)

P. Loubaton is with the Université de Marne-la-Vallée, Équipe Signal et Communications, UMR 8049 IGM-LabInfo, Institut Gaspard Monge/Électronique, 5 boulevard Descartes, 77454 Marne-la-Vallée, France (e-mail: loubaton@univ-mlv.fr).

Permission to publish this abstract separately is granted
I. Introduction

Continuous phase modulation is a widespread scheme thanks to its attractive spectral efficiency and its constant modulus property. In particular, it is used in the European second generation mobile system GSM, in the professional mobile communications system Tetrapol, as well as in a number of military systems.

Here, we address the issue of blind equalization of Continuous Phase Modulated (CPM) signals. The present paper is essentially motivated by applications to passive listening. We assume that an unknown CPM signal is intercepted. The received signal is corrupted by a frequency selective channel and by a possible additive white Gaussian noise. Applications to passive listening usually require blind compensation for the distortion induced by the channel, blind estimation of unknown technical parameters (symbol rate, modulation index, etc.), and detection of data symbols.

A number of works has been devoted to the blind estimation of the technical parameters of CPM signals. We refer to [1], [2] and references therein for further information on this topic. Comparatively, the issue of blind compensation for the channel has been investigated in fewer works. Apart from the case of CPM signals with modulation index $\frac{1}{2}$ for which particular methods can be used [3], [4], [5], most of the existing approaches consist in the blind identification of channel’s coefficients [6] [7] [8]. These methods are based on the observation that received samples can be written as an Hidden Markov Model: the Expectation-Maximization (EM) algorithm [9] is thus used to identify channel’s response. Nevertheless, blind channel identification methods based on the EM algorithm may be difficult for the following reasons. Firstly, it has been shown in [10] that the computational cost of each iteration of the EM algorithm is proportional to $N^2$, where $N$ is the number of observed samples. Secondly, each iteration of the EM algorithm leads to the maximization of a multidimensional non convex function. Therefore, the EM algorithm must be properly initialized. Finally, blind channel identification based on the EM algorithm requires the prior knowledge of the symbol period and the modulation index of the transmitted CPM signal, whereas these parameters may be unknown in a passive listening context.

In this paper, we consider the use of a blind equalizer as an alternative to the above blind identification methods. The Constant Modulus Algorithm (CMA) proposed by Godard [11] is one of the most popular algorithm used to compensate for distortions induced by the
channel. The Constant Modulus (CM) criterion basically consists in selecting the equalizer which minimizes the variations of the modulus of the output signal. The behavior of the CMA has been thoroughly studied in the literature [11], [12], [13], [14], [15], [16], [17], [18]. When the transmitted signal is a linear modulation of an independent and identically distributed (i.i.d.) symbol sequence it is well-known that, in the noiseless case and under certain technical assumptions, the minimization of the CM criterion allows to recover a delayed and rotated version of the transmitted symbol sequence at the output equalizer. Hence, the CMA successfully cancels the Inter-Symbol Interference (ISI) for (most) classical linear modulations. However, such a powerful result does not hold for CPM signals. This is due to the fact that a CPM signal cannot be written as a linear modulation of an i.i.d. sequence. Nevertheless, we note that in [17], the result stating that the CM criterion allows to recover input symbols (up to a delay and phase shift) is generalized to the case of constant modulus but non necessarily i.i.d. sources. Unfortunately, CPM sources do not verify some properties implicitly assumed by [17], so that the conclusion of [17] again does not apply to CPM signals. To our knowledge, no result in the literature indicates whether the CMA ensures successful equalization of CPM signals or not. Here, our aim is therefore to characterize the solutions to the CM criterion when a Continuous Phase Modulation is used by the transmitter.

The paper is organized as follows. Section II introduces the signal model. Section III describes the theoretical framework of the blind equalization. Indeed, the aim of the equalization step in case of CPM signals is slightly different from the case of classical linear modulations. In Section IV, solutions to the CM criterion are characterized. It is shown that the output equalizer does not necessarily coincide with a delayed and rotated version of the input CPM signal. Section V provides an example of possible application of our results: a novel equalization criterion is introduced and shown to be free from undesired global minima. Section VI sustains theoretical claims and shows the attractive performance of the proposed criterion.

II. Signal Model

The complex envelope $s_a(t)$ of a CPM signal can be expressed as follows:

$$s_a(t) = \exp i \psi_a(t) ,$$  

(1)
where the phase $\psi_a(t)$ is given by:

$$\psi_a(t) = \pi h \int_{-\infty}^{t} \sum_{j \in \mathbb{Z}} a_j g_a(u - jT_s) du$$

$$= \pi h \sum_{j \in \mathbb{Z}} a_j \varphi_a(t - jT_s).$$  \(2\)

Here, $(a_n)_{n \in \mathbb{Z}}$ represents the symbol sequence. It is assumed that for each $n$, $a_n$ is equally likely $\pm 1$ and that the sequence is i.i.d. $T_s$ is the symbol period. Function $g_a(t)$ is classically called the shaping filter, is positive on the interval $[0, LT_s]$, and is zero outside $[0, LT_s]$, where $L$ is a positive integer. Function $g_a(t)$ is normalized in such a way that $\int_{0}^{LT_s} g_a(t) dt = 1$. Therefore, function $\phi_a(t)$ defined by $\phi_a(t) = 0$ if $t < 0$ and $\phi_a(t) = \int_{0}^{t} g_a(s) ds$ satisfies $\phi_a(t) = 1$ for $t \geq LT_s$. Parameter $h$ is called the modulation index and typically lies in the interval $[0, 1]$. In the sequel, we focus on full response CPM signals, i.e. the length $L$ of the shaping filter is equal to one. This assumption allows to simplify the following proofs as well as the presentation of the results. Although the generalization of our results to the case of partial response signaling (i.e. $L > 1$) is possible, it requires much longer proofs and is not addressed here due to the lack of space. We refer to [19] for further details. When $L = 1$, it is quite useful to mention that, due to the previous definition of $\phi_a(t)$, $s_a(t)$ can be written as follows for each $nT_s \leq t < (n + 1)T_s$,

$$s_a(t) = \exp i \pi h \left( \sum_{k=-\infty}^{n-1} a_k + a_n \phi_a(t - nT) \right).$$  \(3\)

Therefore, the phase variation induced by one symbol is equal to $\pi h$.

We denote by $y_a(t)$ the complex envelope of the received signal. Signal $y_a(t)$ can be interpreted as the output of an analog filter of impulse response $h_a(t)$ excited by $s_a(t)$ and corrupted by an additive white Gaussian noise $w_a(t)$ of spectral density $N_0$ in a limited bandwidth. For each $t$,

$$y_a(t) = h_a(t) \ast s_a(t) + w_a(t),$$

where $\ast$ stands for the convolution. The frequency response of the channel is furthermore assumed to be non zero in the bandwidth of the input signal $s_a(t)$. A digital equalizer with transfer function $g(z) = \sum_k g_k z^{-k}$ is used on a sampled version of the received signal. We focus on the case where the sampling rate coincides with the symbol rate $1/T_s$. Our results

\(^1\)Note that signal $s_a(t)$ has actually an infinite bandwidth. However, in practice, the spectral density of $s_a(t)$ is clearly negligible at high enough frequencies.
can be generalized to the case where the sampling rate is different from $1/T_s$. However, this case is not addressed in this paper due to the lack of space: we refer to [20], [19] for further details. We denote by $y(n) = y_a(nT_s)$ the received samples and by $z(n)$ the output of the equalizer defined by

$$z(n) = \sum_{k \in \mathbb{Z}} g_k y(n - k).$$ \hspace{1cm} (4)

Coefficients $(g_k)_{k \in \mathbb{Z}}$ are assumed to be selected so as to minimize the Constant Modulus criterion introduced in the next section.

III. Blind Equalization using the CM criterion

Before presenting our analysis of the CM criterion, it is worth understanding the specificity of blind equalization in case of CPM signals. Indeed, the objective that underlies blind equalization in case of CPM signals is somewhat different to the case of classical linear modulations. From now on, we focus on the noiseless case $w_a(t) = 0$.

A. Objective of the equalization step

In the classical case where the transmitted signal is a linear modulation of data symbols, a digital equalizer excited by received samples at symbol rate $T_s$ can directly provide an estimate of data symbols. This is due to the fact that received samples at symbol rate can be written as the output of a digital filter excited by the symbol sequence. Thus, it is clear that the inversion of the latter digital filter is sufficient to recover the symbol sequence without ISI.

However, the case of CPM signals is more complicated. The transmitted CPM signal is indeed a non linear function of data symbols. As a consequence, the equalization step does not allow to directly recover data symbols. At best, we may expect from the equalization step that the output equalizer coincide with the signal that would have been observed in the absence of multipath. As a consequence, we shall use the term successful equalization to refer to the case where for each $n$,

$$z(n) = \lambda s_a(nT_s - \tau)$$ \hspace{1cm} (5)

where $\tau$ is a delay and $\lambda$ is a complex gain. After the equalization step, the recovery of data symbols requires the use of a CPM detector [21], [22], [23], possibly coupled with a synchronization procedure.
B. Constant Modulus Criterion

As mentioned previously, equalizer $g(z)$ is selected so that the so-called CM criterion is minimum. Following [11] [13], we define the CM criterion by

$$J_{T_s}(g) = E \left( (|z(n)|^2 - 1)^2 \right)$$

(6)

where $E(.)$ denotes the expectation. Subscript “$T_s$” recalls that definition (6) rests on the fact that the sampling rate is equal to $T_s$. It can be easily verified that $(z(n))_{n \in \mathbb{Z}}$ is a stationary sequence, so that (6) does not depend on $n$. Clearly, the minimization of (6) allows to select the equalizer which minimizes the variations of the modulus of the output signal.

Remark 1: Function $J_{T_s}(g)$ is greater than or equal to zero. Moreover, in case of successful equalization, sequence $z(n)$ has the form $z(n) = \lambda s_a(nT_s - \tau)$ and has therefore a constant modulus. If $|\lambda| = 1$, then definition (6) immediately lead to $J_{T_s}(g) = 0$. Hence, a successful equalization ensures that the CM criterion is minimum. The main goal of the present paper is to investigate whether the converse is true, i.e. does the minimization of the CM criterion ensure successful equalization?

IV. Solutions to the CM criterion

In this paragraph, we aim to characterize the solutions to equation $J_{T_s}(g) = 0$. More precisely, we provide the general expression of the output equalizer $z(n)$ when $J_{T_s}(g) = 0$. To begin, it is worth noting that equation $E \left( (|z(n)|^2 - 1)^2 \right) = 0$ holds if and only if

$$\forall n, \ |z(n)| = 1 \text{ a.s.}$$

Notation a.s. stands for almost surely. In other words, the characterization of the minima of the CM criterion is equivalent to the characterization of the constant modulus sequences $z(n)$ which verify (4). As mentioned in Section II, we focus on the case of full response CPM signals, i.e. $L = 1$. Laurent [24] has shown that full response CPM signals can be written has a linear modulation of a certain random sequence $x(n)$ which is a function of data symbols $a_n$. For the sake of completeness, we briefly recall the proof of this result.
A. Laurent’s decomposition of CPM signals

Assume that $L = 1$. It can be easily shown that for non integer values of the modulation index $h$, for each $n T_s \leq t < (n + 1) T_s$,

$$\exp(i \pi h a_n \phi_a(t - n T_s)) = e^{i \pi h a_n} \frac{\sin(\pi h \phi_a(t - n T_s))}{S} + \frac{\sin(\pi h - \pi h \phi_a(t - n T_s))}{S}$$

(7)

where $S = \sin(\pi h)$. As a consequence, expression (3) can be written as

$$s_a(t) = x(n) \frac{\sin(\pi h \phi_a(t - n T_s))}{S} + x(n - 1) \frac{\sin(\pi h - \pi h \phi_a(t - n T_s))}{S}$$

(8)

where sequence $x(n)$ is defined as

$$x(n) = \exp \left( i \pi h \sum_{k=-\infty}^{n} a_k \right).$$

(9)

Sequence $x(n)$ is classically called the pseudo-symbols sequence. It depends on data symbols $a_n$. From (8), it is straightforward to show that CPM signal $s_a(t)$ can be written as a linear modulation of pseudo-symbols sequence $x(n)$:

$$s_a(t) = \sum_{n \in \mathbb{Z}} x(n) c(t - n T_s),$$

(10)

where $c(t)$ is the Laurent’s shaping pulse defined from $\phi_a(t)$ by

$$c(t) = \begin{cases} 
\frac{\sin \pi h \phi_a(t)}{S} & \text{if } 0 \leq t \leq T_s \\
\frac{\sin(\pi h - \pi h \phi_a(t - T_s))}{S} & \text{if } T_s \leq t \leq 2 T_s \\
0 & \text{otherwise.}
\end{cases}$$

(11)

We stress the fact that shaping pulse $c(t)$ is non zero only in the interval $[0, 2T_s]$. It is also interesting to note that for each $n$, $s_a(n T_s) = x(n - 1)$. This means that pseudo-symbols can also be interpreted as samples of the transmitted signal at symbol rate.

In the noiseless case i.e. $y_a(t) = h_a(t) * s_a(t)$, equation (10) implies that the received signal can itself be written as a linear modulation of pseudo-symbols sequence $x(n)$. More precisely, received samples $y(n) = y_a(n T_s)$ can be written as follows for each $n$:

$$y(n) = \sum_{k \in \mathbb{Z}} h_k x(n - k),$$

November 28, 2005 SUBMITTED VERSION
where \( h_k = \int_R h_a(t)c(kT_s - t)dt \). As a consequence, the output \( z(n) \) of the equalizer can be written as the output of a digital filter with transfer function \( f(z) = \sum_k f_k z^{-k} \) excited by sequence \( x(n) \):

\[
z(n) = \sum_{k \in \mathbb{Z}} f_k x(n-k),
\]

where for each \( k \), \( f_k = \sum_l g_l h_{k-l} \). Filter \( f(z) \) represents the combined effect of the shaping pulse, the propagation channel and the equalizer on the transmitted pseudo-symbols sequence \( x(n) \). Hence, the characterization of the solutions to the Constant Modulus criterion is equivalent to the characterization of the set of filters \( f(z) = \sum_k f_k z^{-k} \) such that the following constant modulus condition holds almost surely:

\[
\forall n \in \mathbb{Z}, \left| \sum_{k \in \mathbb{Z}} f_k x(n-k) \right| = 1.
\]

The set of solutions to this problem depends on whether modulation index \( h \) is equal to or different from \( 1/2 \).

B. Case \( h \neq \frac{1}{2} \)

The following result is proved in the appendix.

**Theorem 1:** Assume that \( h \) is not an integer multiple of \( \frac{1}{2} \). Condition (13) holds almost surely if and only if \( f(z) \) has the form

\[
f(z) = (e^{i\varphi} z^{-K}) \tilde{f}(z),
\]

where \( K \) is an integer, \( \varphi \in (-\pi, \pi] \) and where coefficients \( \{ \tilde{f}_k \}_{k \in \mathbb{Z}} \) of filter \( \tilde{f}(z) = \sum_{k \in \mathbb{Z}} \tilde{f}_k z^{-k} \) are defined by

\[
\begin{align*}
\tilde{f}_0 &= \frac{\sin \theta}{\sin \pi h} \\
\tilde{f}_1 &= \frac{\sin(\pi h - \theta)}{\sin \pi h} \\
\tilde{f}_2 &= 0
\end{align*}
\]

and \( \tilde{f}_k = 0 \) for each \( k \) different from 0, 1, 2. Here, \( \theta \) is any parameter of interval \([0, \pi)\).

We respectively designate by **Type I filters** and **Type II filters** the two families of filters defined by (15). Theorem 1 states that the constant modulus condition holds at the output of the equalizer if and only if filter \( f(z) \) coincides either with a Type I or a Type II filter,
up to a certain delay and phase shift. In particular, at most three coefficients of filter $f(z)$ may be non zero. We now make the following comments.

**Comparison to the classical case of a linear modulation of an i.i.d. sequence**

Solutions to the constant modulus condition are well-known in the classical case where the transmitted signal is a linear modulation of an i.i.d. symbol sequence, say $(\pi(n))_{n\in\mathbb{Z}}$ [11]. If $T_e = T_s$, samples at the output of the equalizer can be written as $\pi(n) = \sum_k f_k \pi(n-k)$, where $f(z) = \sum_k f_k z^{-k}$ represents the combined effect of the channel and the equalizer. A very popular result due to Godard states that if $x(n)$ is an i.i.d. sequence of zero mean, circular at the second order ($E(|x(n)+k)x(n)|^2) = 0$ for each $k$) and with strictly negative fourth-order cumulant ($E(|x(n)|^4) - 2E(|x(n)|^2)^2 < 0$), then criterion $E((|\pi(n)|^2-1)^2)$ is minimum if and only if $f(z)$ is a trivial filter of the form $f(z) = \lambda z^{-K}$, where $\lambda$ is a complex coefficient and where $K$ is an integer. In other words, the minimization of the CM criterion ensures that the output equalizer $\pi(n)$ coincides with the transmitted sequence $\pi(n)$ up to a delay and a complex gain. In the case where a full response CPM signal is transmitted, the output equalizer has the form $z(n) = \sum_k f_k x(n-k)$, where $x(n)$ is the pseudo-symbol sequence defined by (9). The output equalizer thus has the same form than in case of a classical linear modulation. Despite this similarity, the previous well-known result which holds for classical linear modulations does not hold for CPM signal. Indeed, in case of CPM signals, filter $f(z)$ may have two or three non zero coefficients. In particular, $f(z)$ is not necessarily a trivial filter. This difference is due to the fact that the input pseudo-symbol sequence $x(n)$ is not an i.i.d. sequence ($x(n)$ verifies indeed the relation $x(n) = \exp(i\pi h \omega_n) x(n-1)$). This is why classical result of [11] is not applicable in our context.

**Analysis of solutions**

Theorem 1 states in particular that a filter $f(z)$ minimizing the CM criterion is not necessarily a trivial filter. However, this does not mean that the equalization failed. We shall see in the sequel that certain non trivial solutions given by (15) are actually interpolation filters. For these solutions, the output equalizer verifies the relation $z(n) = s_a(nT_s - \tau)e^{i\phi}$ for each $n$ and for some delay $\tau$. This shows that even non trivial solution filters $f(z)$ may correspond to a successful equalization as defined by (5). Therefore it is important to determine what solutions among those given by (15) correspond to interpolation filters.
• Type I filters.

Assume that \( f(z) \) is a Type I filter. For the sake of simplicity, we assume without restriction that delay \( K \) and phase shift \( \varphi \) are both zero. Then, for each integer \( n \),

\[
z(n) = \frac{\sin \theta}{S} x(n) + \frac{\sin(\pi h - \theta)}{S} x(n - 1),
\]

(16)

where \( \theta \in [0, \pi) \). Using the fact that \( x(n) = \exp(i\pi ha_n) x(n - 1) \) and using a relation similar to (7), it is straightforward to show that

\[
z(n) = \exp \left( \frac{i\pi h}{\delta} \sum_{j=-\infty}^{n-1} a_j + i\theta a_n \right).
\]

(17)

We now wonder which condition should be verified by \( \theta \) so that the above signal can be interpreted as a sampled version of transmitted signal \( s_a(t) \).

If \( 0 \leq \theta \leq \pi h \). As \( \phi_a(t) \) is an increasing continuous function equal to zero at point \( t = 0 \) and equal to one at point \( t = T_s \), it is a bijection of \([0, T_s]\) in \([0, 1]\). Therefore, it exists a certain real \( \delta \in [0, T_s] \) such that \( \theta = \pi h \phi_a(\delta) \). In this case, we obtain immediately:

\[
z(n) = s_a(nT_s + \delta)
\]

(18)

Thus, the output equalizer can be written as a sampled version of the transmitted signal. Filter \( f(z) \) is an interpolation filter.

If \( \pi h < \theta < \pi \). Then no real \( \delta \) is likely to verify equality \( \theta = \pi h \phi_a(\delta) \). Filter \( f(z) \) is not an interpolation filter.

• Type II filters.

After some algebra (cf. Appendix), it can be shown that if \( f(z) \) is a Type II filter, then for each \( n \),

\[
z(n) = a_{n-1} \exp \left( i\pi h \sum_{j=-\infty}^{n-1} a_j + i\theta a_{n-1} a_n \right).
\]

(19)

It is thus clear that \( z(n) \) does not correspond to any sampled version of the transmitted signal. Type II filters are not interpolation filters.

**Consequences of Theorem 1 for data recovery**

The above observations show that the equalization step based on the CMA may be unsuccessful. Indeed, residual indeterminacies remain after the equalization step. However, Theorem 1 provides a parametrization of residual filter \( f(z) \). Hence, identification of \( f(z) \)
is equivalent to the identification of a limited number of parameters (namely \( \theta, \varphi \) and \( \epsilon \), where \( \epsilon \) is a Boolean parameter equal to one if \( f(z) \) is a Type I filter and to zero if \( f(z) \) is a Type II filter). In order to estimate both the unknown parameters \( \theta, \varphi, \epsilon \), and the symbol sequence \( (a_n)_{n \in \mathbb{Z}} \), the most natural approach seems to be the use the maximum likelihood criterion. However, a direct maximization of the likelihood is impractical in case of CPM signals. Different methods have been proposed in the literature to achieve joint parameter identification and symbol detection. Some approximations have been formulated by [2] so as to reduce the computational cost of the rigorous likelihood maximization. For example, the so-called Optimum-Detection Estimation-Directed (OD-ED) approach is certainly the most popular of these approximations, see e.g. [22]. In the present case, this approach basically consists in using a classical CPM detector ([22], [23]) at every point \((\theta, \varphi, \epsilon)\) of a discrete grid. We refer to [22] and [2] for further details. Other efficient methods, such as Per-Survivor Processing, have also been proposed for such a task.

C. Case \( h = \frac{1}{2} \)

The case where modulation index \( h \) is equal to \( \frac{1}{2} \) (e.g. Minimum Shift Keying (MSK) modulations) requires a different approach. In this case, pseudo-symbol sequence defined by (9) simply verifies \( x(n) = ia_n x(n-1) \) for each \( n \). Furthermore, \( x(0) \) belongs to the alphabet \( \{1, i, -1, -i\} \). As a consequence, pseudo-symbols \( x(n) \) are alternatively real and imaginary. We thus put for each \( n \)

\[
u(n) = \frac{x(n)}{i^n}.
\]

Hence, \((u(n))_{n \in \mathbb{Z}}\) is a stationary zero mean i.i.d. sequence which verifies equality \( u(n) = a_n u(n-1) \) for each \( n \). Furthermore, this sequence is either real or imaginary, depending on the value of \( x(0) \). We assume without restriction that \( u(n) \) is a real sequence.

Output signal \( z(n) \) can be written as

\[ z(n) = i^n \sum_{k \in \mathbb{Z}} \hat{f}_k u(n - k), \]

where coefficients \( \hat{f}_k \) of filter \( \hat{f}(z) = \sum_k \hat{f}_k z^{-k} \) are defined by \( \hat{f}_k = i^{-k} f_k \). The characterization of the set of filters \( f(z) \) which verify property (13) is thus equivalent to the characterization of the set of filters \( \hat{f}(z) \) which are such that sequence \( \sum_k \hat{f}_k u(n - k) \) has constant modulus for (almost) each binary random sequence \((u(n))_{n \in \mathbb{Z}}\). The latter set of filters has been characterized by [12]. Using this result, one can easily show the following theorem:
Theorem 2: Assume that \( h = \frac{1}{2} \). Property (13) holds if and only if \( f(z) \) verifies
\[
f(z) = (e^{i\varphi}z^{-K})\tilde{f}(z),
\]
(22)
where \( K \) is an integer, \( \varphi \in (-\pi, \pi] \) and where coefficients \( \tilde{f}_k \) of filter \( \tilde{f}(z) = \sum_{k \in \mathbb{Z}} \tilde{f}_k z^{-k} \) are defined by
\[
\begin{align*}
\tilde{f}_0 &= i(-P+1) \sin \theta \\
\tilde{f}_P &= \cos \theta
\end{align*}
\]
(23)
and \( \tilde{f}_k = 0 \) for each \( k \) different from 0, \( P \).

Here, \( P \) is a strictly positive integer and \( \theta \) is a real number in the interval \([0, \pi)\).

Now consider a filter \( f(z) \) which belongs to the above set of solutions. Assume for the sake of simplicity that both delay \( K \) and phase shift \( \varphi \) are zero. For each integer \( n \),
\[
z(n) = i^{-P+1} \sin \theta \ x(n) + \cos \theta \ x(n-P).
\]
(24)
Using definition (9) of pseudo-symbols, \( x(n) \) can be written as the following function of \( x(n-P) \):
\[
x(n) = i^P \left( \prod_{j=0}^{P-1} a_{n-j} \right) x(n-P).
\]
Plugging the above equality into (24), one obtain
\[
z(n) = \left( \cos \theta + i \left( \prod_{j=0}^{P-1} a_{n-j} \right) \sin \theta \right) x(n-P).
\]
As symbols are binary, product \( \prod_{j=0}^{P-1} a_{n-j} \) is equal to \( \pm 1 \). Finally, output equalizer \( z(n) \) verifies the following equality:
\[
z(n) = \exp \left( i\theta \prod_{j=0}^{P-1} a_{n-j} \right) x(n-P)
\]
(25)
\[
= \exp \left( \frac{i\pi}{2} \sum_{j=-\infty}^{n-P} a_j + i\theta \prod_{j=0}^{P-1} a_{n-j} \right).
\]
(26)
In particular, it is clear that sequence \( z(n) \) has a constant modulus. More precisely, for a given \( n \), \( z(n) \) belongs to the alphabet \( \{e^{i\theta}, ie^{i\theta}, -e^{i\theta}, -ie^{i\theta}\} \).

As before, it is useful to emphasize the solutions \( f(z) \) which correspond interpolation filters, i.e. which lead to successful equalization.
• If $P = 1$, equation (26) becomes

$$z(n) = \exp \left( i\pi \frac{n-1}{2} \sum_{j=-\infty}^{n-1} a_j + i\theta a_n \right).$$

- If $0 \leq \theta \leq \frac{\pi}{2}$. Then it exists a real $\delta \in [0, T_s]$ such that $\theta = \frac{\pi}{2} \phi_a(\delta)$. In this case, the output equalizer verifies $z(n) = s_a(n T_s + \delta)$ for each $n$. The solution is an interpolation filter.
- If $\frac{\pi}{2} < \theta < \pi$. In this case, $f(z)$ is an undesired solution.

• If $P > 1$,
  - If parameter $\theta$ belongs to the set $\{0, \frac{\pi}{2}\}$, then $f(z)$ has a single non zero coefficient and is thus a trivial filter.
  - For any other value of $\theta$, $f(z)$ is an undesired solution.

V. AN APPLICATION: CONSTRUCTION OF A NOVEL EQUALIZATION CRITERION

In the previous Section, solutions to the CM criterion have been characterized. In particular, it has been shown that the classical CMA does not ensure successful equalization. For instance when $h \neq \frac{1}{2}$, Theorem 1 states that the CM criterion is minimum if the equalized sequence $z(n)$ coincides with the output of a Type I or a Type II filter excited by pseudo-symbols sequence $x(n)$. Consequently, classical CPM detectors cannot be directly used on the received sequence. The use of more involved detectors is necessary. As discussed in Section IV and confirmed in Section VI, identification of the above Type I or Type II indeterminacy is required before data recovery. Therefore, it is of practical interest to propose novel criteria which ensure successful equalization of CPM signals. This allows to avoid the latter residual filter identification step and to directly use a classical CPM detector on the equalizer’s output. In this section, we provide an example of the way our results may be used to that end. We propose an equalization criterion $J_{T_s}(G)$ such that, if $J_{T_s}(G)$ is minimum, then the output equalizer coincides with a delayed and rotated version of sequence $(x(n))_{n \in \mathbb{Z}}$. Such a property is of practical interest for the following reasons. Recalling that $x(n) = s_a((n-1) T_s)$, the minimization of the proposed criterion $J_{T_s}(G)$ thus allows to recover a sampled version of the transmitted signal $s_a(t)$. Successful equalization is therefore guaranteed. Furthermore, as the output equalizer coincides with samples of the transmitted signal at instants $(n T_s)_{n \in \mathbb{Z}}$, no timing recovery is needed. The proposed criterion allows to both equalize and synchronize the received signal.
From now on, due to the lack of space, we focus on the case \( h \neq \frac{1}{2} \). We motivate the introduction of the proposed criterion by giving some insights. Our aim is to find a condition on output sequence \( z(n) \) which ensures that \( z(n) \) coincides with a delayed and rotated version of sequence \( x(n) \), i.e. \( z(n) = e^{i\varphi} x(n - K) \) for some \( \varphi \) and some integer delay \( K \). We first make the following observation.

**Claim 1:** Consider any real number \( \psi \) in the interval \([0, \pi)\). If the desired condition \( z(n) = e^{i\varphi} x(n - K) \) holds for some \( \varphi \) and some \( K \), then both following conditions (27a) and (27b) are verified:

\[
\forall n \in \mathbb{Z}, \quad |z(n)| = 1 \tag{27a}
\]

\[
\forall n \in \mathbb{Z}, \quad \left| \frac{\sin \psi}{S} z(n) + \frac{\sin(\pi h - \psi)}{S} z(n - 1) \right| = 1 \tag{27b}
\]

**Proof:** Assume that \( z(n) = e^{i\varphi} x(n - K) \). Due to (9), \( x(n) \) has a constant modulus equal to one. Therefore, (27a) is trivially verified. Equation (27b) is a direct consequence of Theorem 1. Indeed, if \( z(n) = e^{i\varphi} x(n - K) \), then sequence \( \frac{\sin \psi}{S} z(n) + \frac{\sin(\pi h - \psi)}{S} z(n - 1) \) coincides with sequence

\[
e^{i\varphi} \left( \frac{\sin \psi}{S} x(n - K) + \frac{\sin(\pi h - \psi)}{S} x(n - K - 1) \right), \tag{28}
\]

The above sequence can be written as the output of filter \( e^{i\varphi} z^{-K} \left( \frac{\sin \psi}{S} + \frac{\sin(\pi h - \psi)}{S} \right) z^{-1} \) excited by pseudo-symbols sequence \( x(n) \). As the latter filter is clearly a Type I filter, Theorem 1 implies that (28) is a modulus one sequence. Condition (27b) follows.

Therefore, conditions (27a) and (27b) are necessary conditions so that desired result \( z(n) = e^{i\varphi} x(n - K) \) holds. The equalization criterion proposed in the sequel is based on the stronger observation that (27a) and (27b) are not only necessary but also sufficient conditions. In other words, if one ensures that both conditions (27a) and (27b) are verified, then \( z(n) = e^{i\varphi} x(n - K) \) for some \( \varphi \) and some integer \( K \). Then, recalling the fact that \( s_a(nT_s) = x(n - 1) \) due to (8), output \( z(n) \) coincides with a delayed and rotated version of the samples \( s_a(nT_s) \) of the transmitted signal. In the sequel, we simply put \( \psi = \frac{\pi h}{2} \) for the sake of illustration. In this case, sequence \( \frac{\sin \psi}{S} z(n) + \frac{\sin(\pi h - \psi)}{S} z(n - 1) \) simply coincides with sequence \( \rho (z(n) + z(n - 1)) \), where \( \rho = \sin(\frac{\pi h}{2})/S \).

**Theorem 3:** Assume that \( h \) is not an integer multiple of \( \frac{1}{2} \). Define

\[
\mathcal{J}_{T_s}(g) = E \left( (|z(n)|^2 - 1)^2 \right) + E \left( (\rho^2 |z(n) + z(n - 1)|^2 - 1)^2 \right). \tag{29}
\]
Then,
\[ J_{T_s}(g) \geq 0. \]
\[ J_{T_s}(g) = 0 \text{ if and only if } z(n) \text{ is a delayed and rotated version of sequence } s_n(nT_s). \]

**Proof:** Assume that \( J_{T_s}(G) = 0. \) In particular, this implies that \( E((|z(n)|^2 - 1)^2) = 0. \) Due to Theorem 1, the output equalizer \( z(n) \) can be written as the output of digital filter \( f(z) \) excited by \( x(n) \), where \( f(z) \) is either a Type I or a Type II filter with parameter \( \theta \) up to a delay and a phase shift. For the sake of simplicity, we assume without restriction that the latter delay and phase shift are both zero. On the otherhand, \( E((| \rho(z(n) + z(n - 1)|^2 - 1)^2) = 0. \) Due to Theorem 1 again, this implies that \( \rho(1 + z^{-1})f(z) \) is either a Type I or a Type II filter with parameter \( \hat{\theta} \). First consider the case where both \( f(z) \) and \( \rho(1 + z^{-1})f(z) \) are Type I filters. In this case, for some delay \( K \) and some phase shift \( \varphi \),

\[
\rho(1 + z^{-1}) \left( \frac{\sin \theta}{S} + \frac{\sin(\pi h - \theta)}{S} z^{-1} \right) = e^{i\varphi} z^{-K} \left( \frac{\sin \hat{\theta}}{S} + \frac{\sin(\pi h - \hat{\theta})}{S} z^{-1} \right).
\]

Identifying coefficients of the lefthand and the righthand polynomials of the above equation, we obtain that parameter \( \theta \) is equal to zero or \( \pi h \). Thus \( f(z) \) has only one non zero coefficient, so that the output equalizer coincides with a delayed and rotated version of the pseudo-symbols sequence. The three remaining possible cases (\( f(z) \) and \( \rho(1 + z^{-1})f(z) \) may also be Type II filters) can be treated the same way.

In this case, output sequence \( z(n) \) coincides with the samples of the transmitted signal at symbol rate: the proposed criterion \( J_{T_s}(G) \) ensures successful equalization. Furthermore, the minimization of \( J_{T_s}(G) \) also ensures timing recovery. Indeed, the delay \( \tau \) in equation (5) is an integer multiple of the symbol period \( T_s \). No additional timing synchronization step is required. Classical CPM detectors can directly be used on the equalized sequence.

**VI. Simulations**

Firstly, we illustrate the effect of both the classical CM criterion and the criterion proposed in Section V. For the sake of illustration, we provide the results of the equalization step considering two different examples of channel realizations. Secondly, we characterize the performance of the different criteria based on a large number of channel realizations.

Due to the lack of space, we focus on the case where \( h \neq \frac{1}{2} \). Parameters used in the simulations are the following. Modulation index \( h \) is equal to \( h = 0.75 \). The shaping pulse used by the transmitter is a raised cosine filter, i.e. \( g_a(t) = 1 - \cos(2\pi \frac{t}{T_s}). \) The received signal
is corrupted by multipath and sampled at rate $T_s$. In order to compare empirical results to theoretical results of Section IV, we first study the noiseless case. A 40-tap equalizer is used at the receiver side. Coefficients of the equalizer are calculated using a gradient search algorithm which minimizes the following empirical CM cost function:

$$J^{(N)}_T(g) = \frac{1}{N} \sum_{n=0}^{N-1} (|z(n)|^2 - 1)^2,$$

where $z(n)$ is the output equalizer and where sample size $N$ is set to $N = 2000$. 500 iterations of the gradient search algorithm have been used. At each iteration, the optimal step of the gradient descent is calculated. Figure 1 has been obtained using the following three-path channel. Delays are $[0.55T_s, 3.30T_s, 3.59T_s]$ and associated complex magnitudes are $[0.053 + 0.83i, 0.63 - 1.60i, 0.07 - 0.58i]$. In the sequel, we refer to this channel as channel 1. Figure 1a represents the constellation of pseudo-symbols $x(n)$ in the complex plane. Pseudo-symbols all have modulus one. Figure 1b represents the constellation of received samples. This constellation does no longer belong to the unit circle. Figure 1c represents the constellation corresponding to the output equalizer after the use of the CMA. As expected, points of the latter constellation belong to the unit circle. However, comparing Figures 1a and 1c, one can easily see that output equalizer $z(n)$ is far from coinciding with transmitted pseudo-symbol sequence $x(n)$. Transfer function $f(z) = \sum_k f_k z^{-k}$ relating $z(n)$ to $x(n)$ is illustrated by Figure 1d. Two consecutive coefficients of $f(z)$ are significant. More precisely, $f(z)$ can be approximated by a Type I filter with parameter $\theta = 1.76$. Figures 1e and 1f represent the results of the equalization step for channel 1 when the proposed criterion of Section V is used. In practice, coefficients of the equalizer are selected so that the following empirical cost function

$$\tilde{J}^{(N)}_T(g) = \frac{1}{N} \sum_{n=0}^{N-1} (|z(n)|^2 - 1)^2 + \left(\rho^2 |z(n) + z(n-1)|^2 - 1 \right)^2,$$

is minimum. As for the classical CM criterion, 500 iterations of a gradient search algorithm with optimal step are used to minimize the above function. Figure 1e represents the constellation observed at the output of the equalizer. This constellation approximately coincides with a rotated version of the transmitted one. Furthermore, Figure 1f shows that the residual filter $f(z)$ corresponding to the combined effect of the channel and the equalizer has only one non zero coefficient. As expected, $f(z)$ is a trivial filter. The proposed criterion allows to eliminate the inevitable indeterminacy associated
to the CM criterion. Figure 2 illustrates the results of the equalization step for a different channel. Delays respectively associated to each path of the channel are the following: 

$$[0.66T_s, 0.80T_s, 2.03T_s, 2.16T_s, 3.36T_s, 3.71T_s]$$,

while the corresponding complex magnitudes are 

$$[0.01 - 0.44i, -0.24 - 0.49i, -0.1i, 0.30 - 1.97i, 1.55 - 0.39i, 0.52 - 0.79i]$$. We refer to the latter channel as channel 2. Again, the output equalizer does not coincide with the input sequence. In this case, residual filter \(f(z)\) contains three significant coefficients. More precisely, \(f(z)\) can be approximated by a Type II filter with parameter \(\theta = 0.71\). Figures 2e and 2f represent the results of the equalization step for channel 2 when the proposed criterion \(J^{(N)}_{T_s}(g)\) is used. As previously, the residual filter \(f(z)\) has only one non zero coefficient. Therefore, output \(z(n)\) coincides with a delayed and rotated version of the pseudo-symbols sequence.

We now study the performance of both the classical CM criterion and the proposed criterion in the noisy case. In order to evaluate the performance, we consider the Bit Error Rate (BER). The BER is calculated as follows. Firstly, a blind equalizer is used on the received signal. Secondly, a symbol detector is used to estimate data \(a_n\). The BER is evaluated at the output of the latter detector. Of course, the BER depends on both the equalization criterion and on the symbol detection strategy used by the receiver. In the sequel, we compare the four following receiving scenarios.

**Scenario #1: CMA and OD-ED.** A classical CMA minimizing (30) is used on the received signal. After the equalization step, data recovery is achieved following the OD-ED approach depicted in Section IV.

We briefly recall the main idea of this approach. After the equalization step, the residual indeterminacy can be characterized by the set of parameters \((\varphi, \theta, \epsilon)\), where \(\varphi\) is the phase shift in (14), \(\theta\) is the parameter defined in (15) and \(\epsilon\) is the Boolean parameter equal to one if \(f(z)\) is a Type I filter and to zero if \(f(z)\) is a Type II filter. For each possible value of the set of indeterminate parameters \((\varphi, \theta, \epsilon)\) on a discrete grid, optimal detection of data symbols is achieved using a classical coherent detector, i.e. a Viterbi-like algorithm [21], [22], [23]. Finally, among the above set of estimated data sequences, the definitive estimate of the data sequence is defined as the sequence for which the output metric of the Viterbi-like algorithm is maximum. In our simulations, the latter metric is maximized w.r.t. all values of \(\theta\) on the grid \(\{0, \frac{\pi}{8}, \ldots, \frac{7\pi}{8}\}\), all \(\varphi\) on the grid \(\{0, \frac{\pi}{4}, \ldots, \frac{7\pi}{4}\}\) and all \(\epsilon \in \{0, 1\}\). When a CMA is used, the above method seems to be one of the most natural approaches to detect data symbols.
However, it requires the use of one detection step for each value of unknown parameters. Therefore, in practice, the receiver described by Scenario #1 is difficult to implement in a number of situations. It is thus interesting to consider the following simple approach.

**Scenario #2: CMA and direct detection.** A classical CMA minimizing (30) is used on the received signal. After the equalization step, classical coherent CPM detection [21], [22], [23] is used directly on the output equalizer without any parameter identification step.

Here, data symbols are thus detected ignoring the possible residual indeterminacy. In other words, data recovery is achieved as if the output equalizer were equal to a delayed and rotated version of the pseudo-symbols sequence. We remark that the detector of [21], [22], [23] used in our simulations is coherent in the sense that it assumes the knowledge of the phase shift $\varphi$ of the constellation at the input of the detector. For example, when $h = \frac{3}{4}$, our coherent detector assumes that the initial phase state belongs to the constellation $\{\exp(ik\frac{\pi}{4} + i\varphi), k = 0 \ldots 7\}$ (see [23] for details). Thus, strictly speaking, Scenario #2 still requires the estimation of one parameter (i.e. phase shift $\varphi$) together with data recovery. As for Scenario #1, we consider that possible values of $\varphi$ belong to the grid $\{0, \frac{\pi}{4}, \ldots, \frac{7\pi}{4}\}$. In this particular case, however, the set of possible initial phase states $\{\exp(ik\frac{\pi}{4} + i\varphi), k = 0 \ldots 7\}$ is independent from the value of $\varphi$ on the latter grid. Consequently, in our context, the coherent detector is used only one time on the output equalizer and directly provides the final data sequence estimate. On the other hand, it is worth remarking that in case of Scenario #1, a more involved coherent detector has to be used 16 times on the output equalizer, for different values of $\theta$, $\epsilon$.

We also study the performance associated to the criterion proposed in Section V. We recall that this criterion theoretically ensures inter-pseudo-symbols interference cancellation. As a consequence, no parameter identification step is required.

**Scenario #3: Proposed criterion and direct detection.** A gradient search algorithm minimizing the proposed criterion (31) is used on received samples. After the equalization step, a classical coherent CPM detection is used to recover data symbols.

Finally, we compare the results obtained using the above scenarios to the ideal case where the channel is perfectly known. This shall bring to the fore the performance degradation due to the ignorance of channel coefficients. In the sequel, we refer to *Wiener filter* as the transverse equalizer minimizing the Mean Square Error (MSE) between output sequence $z(n)$ and pseudo-symbols sequence $x(n)$.

**Scenario #4: Wiener filter and direct detection.** Assume that channel coefficients are per-
fectly known. Then a classical CPM detector is used on the output of the Wiener equalizer to recover data symbols.

In the sequel, the received signal is corrupted by a multipath channel and by an additive Gaussian noise. For each realization of the transmitted signal, the channel is randomly chosen as follows. The number of paths is randomly and uniformly chosen between 3 and 7. Delays are randomly and uniformly chosen between 0 and $5T_s$. Complex magnitudes of the different paths are i.i.d. Gaussian variables. A 40-tap digital equalizer is used on received samples. 500 iterations of the gradient search algorithms are used to adapt equalizer’s coefficients. For each realization of the transmitted signal and the channel, the BER is evaluated. It is worth noting that, for a few channel realizations, catastrophic values of the BER are likely to occur due to severe frequency selectivity. As these few channel realizations may have an important impact on the average BER, it is worth studying the distribution of BERs rather than only providing the average BER.

Figure 3 represents the normalized histogram of the BERs for each of the previous four receiving scenarios and when the signal-to-noise ratio $E_b/N_0$ is equal to 15dB. Normalized histogram can be interpreted as the empirical probability distribution of the BER. About 2000 realizations of the transmitted signal and the channel have been used to obtain the histograms. The average BER is mentioned on each figure. In order to provide informative results, the average BER is calculated ignoring the few catastrophic realizations of the BER possibly due to severe channel realizations. In case of Scenario #1, Figure 3a shows that the CMA followed by an OD-ED detector leads to a satisfying performance. No catastrophic value of the BER have been observed: symbol recovery is thus generally successful. Comparing Figures 3a and 3d, we note that the values of the BER in case of Scenario #1 are significantly larger than in the case where a Wiener filter is used (Scenario #4). This is however not surprising as Scenario #4 assumes perfect knowledge of the channel. On the otherhand, Figure 3b shows that the CMA leads to a catastrophic BER when directly followed by a classical CPM detector. As a consequence, when the CMA is used for blind equalization, the residual Type I or Type II indeterminacy has to be identified to obtain satisfying values of the BER: necessarily, more involved detectors such as the OD-ED detector have to be used. Finally, Figure 3c represents the performance of the proposed blind equalization algorithm directly followed by a simple CPM detector (Scenario #3). Comparing Figures 3c and Figure 3a, we observe that the proposed method generally outperforms
classical CMA followed by an OD-ED detector. The proposed method is therefore not only less complex to implement but is also generally more attractive in terms of performance. It should however be mentioned that three catastrophic values of the BER (greater than 0.3) have been observed among the 2000 realizations of the BER. The proposed algorithm failed three times whereas the CMA followed by OD-ED did not. Finally, we observe that the performance of the proposed algorithm is quite comparable to the performance of the detector based on a Wiener equalizer (Figure 3d).

Figure 4 represents the empirical distributions of the BER for each of the four scenarios, when $E_b/N_0 = 25$dB. Similar conclusions can be drawn in this case. Performances of both the CMA followed by OD-ED and of the proposed criterion directly followed by a simple CPM detector are satisfying. One catastrophic value of the BER among 2000 have been observed in case of the proposed method, none in the case of the CMA followed by OD-ED. In the general case, however, the proposed receiver (Figure 4c) outperforms the CMA followed by OD-ED (Figure 4a).

VII. Conclusion

Solutions to the CM criterion when a binary full response CPM signal is used by the transmitter have been characterized. It has been shown that the equalization may be unsuccessful in the sense that the output equalizer does not necessarily coincide with a delayed and rotated version of the input signal. An indeterminacy may indeed remain and this residual indeterminacy has been characterized. Firstly, our results can be used to design relevant CPM detectors which take into account the possible presence of the latter indeterminacy. Secondly, our results can be used to design novel equalization cost functions with the aim of eliminating the residual indeterminacy. An example of such a cost function has been provided. Implementation of the corresponding blind receiver is less complex than in case of classical CMA. Furthermore, simulations have shown that it generally outperforms classical CMA.
APPENDIX

I. PROOF OF THEOREM 1

Assume that modulation index $h$ is not an integer multiple of $\frac{1}{2}$. Assume that $f(z) = \sum_k f_k z^{-k}$ is such that sequence $z(n) = \sum_k f_k x(n - k)$ has constant modulus:

$$\forall n, \quad |z(n)| = 1 \quad \text{a.s.} \quad (32)$$

The above condition implies the following lemma:

Lemma 1: Consider two integers $n_0$ et $n_1$ such that $n_0 \leq n_1$. Then for each binary sequence $\alpha = [\alpha_{n_0}, \alpha_{n_0+1}, \ldots, \alpha_{n_1}]^T$,

$$E \left( |z(n)|^2 / [a_{n_0}, a_{n_0+1}, \ldots, a_{n_1}]^T = \alpha \right) = 1. \quad (33)$$

Proof: The probability measure associated to the set of sequences $(a_n)_{n \in \mathbb{Z}}$ verifying $a_n = \alpha_n$ for each $n \in \{n_0, \ldots, n_1\}$ is strictly positive. Therefore, claim (32) still holds conditionally on $[a_{n_0}, a_{n_0+1}, \ldots, a_{n_1}]^T = \alpha$. This implies (33). $\blacksquare$

We now use the following approach. For a given sequence $\alpha$, relation (33) provides an equation which involves unknown coefficients $(f_k)_{k \in \mathbb{Z}}$. Using (33) for well-chosen sequences $\alpha$, one can show that coefficients $(f_k)_{k \in \mathbb{Z}}$ are zero except for (at most) three consecutive integers $k$.

Consider a given integer $n$. Denote by $n_0$ et $n_1$ two integers such that $n_1 \geq n_0 + 2$. Laurent’s expression allows to write $z(n) = \sum_k f_{n-k} x(k)$, where pseudo-symbols $x(k)$ are defined by (9). This leads to:

$$|z(n)|^2 = \sum_{k=-\infty}^{\infty} |f_k|^2 |x(k)|^2 + 2\Re \left( \sum_{k=-\infty}^{\infty} \sum_{k'=-\infty}^{k-1} f_{n-k} f_{n-k'}^* x(k) x(k')^* \right),$$

where $\Re[.]$ stands for real part of. First note that $|x(k)|^2 = 1$ for each $k$. Furthermore, using definition (9), for $k' < k$, $x(k) x(k')^* = \exp \left[ i\pi h \sum_{j=k'+1}^{k} a_j \right]$. Now consider any binary sequence $\alpha = [\alpha_{n_0}, \alpha_{n_0+1}, \ldots, \alpha_{n_1}]^T$. Using the above equality, condition (33) becomes:

$$\sum_{k=-\infty}^{\infty} |f_k|^2 + 2\Re \left( \sum_{k=-\infty}^{\infty} \sum_{k'=-\infty}^{k-1} f_{n-k} f_{n-k'}^* \gamma_{kk'}(\alpha) \right) = 1, \quad (34)$$

where $\gamma_{kk'}(\alpha)$ is defined for each integers $k < k'$ by:

$$\gamma_{kk'}(\alpha) = E \left( \exp \left[ i\pi h \sum_{j=k'+1}^{k} a_j \right] / [a_{n_0}, \ldots, a_{n_1}]^T = \alpha \right). \quad (35)$$

November 28, 2005 SUBMITTED VERSION
It is worth noticing that $\gamma_{kk'}(\alpha)$ depends on $\alpha_{n_0}$ only if $k' \leq n_0 - 1$ and $k \geq n_0$. We now use equation (34) for different binary sequences. To begin, we consider a “variation” of the first component $\alpha_0$: if $\alpha$ is a given binary sequence, then (34) is valid for sequence $\alpha$ as well as for sequence $\alpha' = [-\alpha_{n_0}, \alpha_{n_0+1}, \ldots, \alpha_{n_1}]^T$. Substracting the two corresponding equations and using the fact that difference $\gamma$ and $\alpha$ and using the fact that difference $\gamma_{kk'}(\alpha) - \gamma_{kk'}(\alpha')$ is non zero only if $k' \leq n_0 - 1$ and $k \geq n_0$, we finally obtain

$$\mathcal{R} \left[ \sum_{k=n_0}^{\infty} \sum_{k'=-\infty}^{n_0-1} f_{n-k} f_{n-k'}^* (\gamma_{kk'}(\alpha) - \gamma_{kk'}(\alpha')) \right] = 0. \quad (36)$$

To continue, we now consider a “variation” of the last component $\alpha_{n_1}$ of $\alpha$ in equation (36). Equation (36) is valid for all sequences $\alpha$ and $\alpha'$ which only differ in their first component $\alpha_{n_0}$. Thus, this equation is still valid for sequences $\tilde{\alpha} = [\alpha_{n_0}, \ldots, \alpha_{n_1-1}, -\alpha_{n_1}]^T$ and $\tilde{\alpha}' = [-\alpha_{n_0}, \alpha_{n_0+1}, \ldots, \alpha_{n_1-1}, -\alpha_{n_1}]^T$. Substraction of corresponding equations yields:

$$\mathcal{R} \left[ \sum_{k=n_0}^{\infty} \sum_{k'=-\infty}^{n_0-1} f_{n-k} f_{n-k'}^* ((\gamma_{kk'}(\alpha) - \gamma_{kk'}(\alpha') - (\gamma_{kk'}(\tilde{\alpha}) - \gamma_{kk'}(\tilde{\alpha}'))) \right] = 0. \quad (37)$$

Using (35), we moreover remark that $\gamma_{kk'}(\alpha)$ depends on $\alpha_{n_1}$ only if $k \geq n_1$. Therefore, differences $(\gamma_{kk'}(\alpha) - \gamma_{kk'}(\tilde{\alpha}))$ and $(\gamma_{kk'}(\alpha') - \gamma_{kk'}(\tilde{\alpha}'))$ are non zero only if $k \geq n_1$. This implies that coefficient $(\gamma_{kk'}(\alpha) - \gamma_{kk'}(\alpha')) - (\gamma_{kk'}(\tilde{\alpha}) - \gamma_{kk'}(\tilde{\alpha}'))$ is non zero only if $k \geq n_1$. Thus, (37) can be simplified as follows:

$$\mathcal{R} \left[ \sum_{k=n_1}^{\infty} \sum_{k'=-\infty}^{n_0-1} f_{n-k} f_{n-k'}^* ((\gamma_{kk'}(\alpha) - \gamma_{kk'}(\alpha') - (\gamma_{kk'}(\tilde{\alpha}) - \gamma_{kk'}(\tilde{\alpha}'))) \right] = 0. \quad (38)$$

In order to write (38) as a function of parameters $\alpha_{n_0}, \ldots, \alpha_{n_1}$, we have to evaluate $\gamma_{kk'}(\alpha)$ for each $k$ and $k'$ such that $k \geq n_1$ and $k' \leq n_0 - 1$. Using (35), we obtain

$$\gamma_{kk'}(\alpha) = E \left( \exp \left[ i \pi h \sum_{j=k'+1}^{n_0} a_j \right] \right) \exp \left[ i \pi h \sum_{j=n_0}^{n_1} \alpha_j \right] E \left( \exp \left[ i \pi h \sum_{j=n_1+1}^{k} a_j \right] \right). \quad (39)$$

Now we can simplify (39) by noting that

$$E \left( \exp \left[ i \pi h \sum_{j=k'+1}^{n_0} a_j \right] \right) = \prod_{j=k'+1}^{n_0-1} E(e^{i \pi h a_j}) = (\cos \pi h)^{n_0-k'-1}.$$

Defining $C = \cos \pi h$, $\gamma_{kk'}(\alpha)$ can finally be written as

$$\gamma_{kk'}(\alpha) = C^{n_0-k'-1} C^{k-n_1} \exp \left( i \pi h \sum_{j=n_0}^{n_1} \alpha_j \right). \quad (40)$$
Therefore, coefficient \((\gamma_{kk'}(\alpha) - \gamma_{kk'}(\alpha')) - (\gamma_{kk'}(\tilde{\alpha}) - \gamma_{kk'}(\tilde{\alpha}'))\) is equal to
\[
C^{n_0-k'-1} C^{n_1} \exp \left( i\pi h \sum_{j=n_0+1}^{n_1-1} \alpha_j \right) (2i\alpha_{n_0} \sin \pi h)(2i\alpha_{n_1} \sin \pi h). \tag{41}
\]
Plugging (41) into (38) and noting that product \(C^{n_0-n_1}(2i\alpha_{n_0} \sin \pi h)(2i\alpha_{n_1} \sin \pi h)\) is non zero, we finally obtain
\[
\mathcal{R} \left[ \left( \sum_{k'=-\infty}^{n_0-1} f_{n-k'}C^{-k'} \right)^* \left( \sum_{k=n_1}^{\infty} f_{n-k}C^k \right) \exp \left( i\pi h \sum_{j=n_0+1}^{n_1-1} \alpha_j \right) \right] = 0. \tag{42}
\]
Then we use the following lemma.

**Lemma 2:** Denote by \(Z\) any complex number. If \(h\) is not an integer multiple of \(\frac{1}{2}\) and if for each \(\epsilon \in \{-1, 1\}, \mathcal{R} [Ze^{i\pi h\epsilon}] = 0\), then \(Z = 0\).

As \(n_1 \geq n_0 + 2\), the sum inside the argument of the exponential function in (42) contains at least one term. Thus, we may use the above lemma with the following value of \(Z\):
\[
Z = \left( \sum_{k'=-\infty}^{n_0-1} f_{n-k'}C^{-k'} \right)^* \left( \sum_{k=n_1}^{\infty} f_{n-k}C^k \right) \exp \left( i\pi h \sum_{j=n_0+1}^{n_1-2} \alpha_j \right).
\]
The above equation is equal to zero due to lemma 2. This is equivalent to
\[
\sum_{k=-\infty}^{n_0-1} f_{n-k}C^{-k} = 0 \quad \text{or} \quad \sum_{k=n_1}^{\infty} f_{n-k}C^k = 0. \tag{43}
\]
This property is true for any \(n_0\) and \(n_1\) such that \(n_1 \geq n_0 + 2\). We now study the two cases corresponding to equation (43).

**Case #1.** Assume that it exists a certain integer \(N_0 \in \mathbb{Z}\) such that \(\sum_{k=-\infty}^{N_0-1} f_{n-k}C^{-k} \neq 0\).

Then for each \(n_1 \geq N_0 + 2\), \(\sum_{k=n_1}^{\infty} f_{n-k}C^k = 0\). As this claim is also true replacing \(n_1\) by \(n_1 + 1\), we obtain
\[
f_{n-n_1} = \frac{1}{C^n_1} \left( \sum_{k'=n_1}^{\infty} f_{n-k'}C^k - \sum_{k'=n_1+1}^{\infty} f_{n-k'}C^k \right) = 0. \tag{44}
\]
Generally speaking, \(f_{n-k} = 0\) for each \(k \geq N_0 + 2\). One can thus legitimately define \(N_1 = \max \{k / f_{n-k} \neq 0\}\). This definition directly leads to
\[
\sum_{k=N_1}^{\infty} f_{n-k}C^k \neq 0.
\]
Therefore, for each integer \(n_0\) verifying \(N_1 \geq n_0 + 2\),
\[
\sum_{k=-\infty}^{n_0-1} f_{n-k}C^{-k} = 0.
\]
Using the same kind of arguments, we conclude that \( f_{n-k} = 0 \) for each \( k \leq N_1 - 3 \). Finally, only three consecutive coefficients \( f_{n-N_1-2}, f_{n-N_1-1}, f_{n-N_1} \) are likely to be non zero.

**Case #2.** On the opposite, assume that for each integer \( n_0 \in \mathbb{Z} \), \( \sum_{k=0}^{n_0-1} f_{n-k} c^{-k} = 0 \). Then, for each \( n_0 \):

\[
f_{n-n_0+1} = \frac{1}{C_{n_0+1}} \left( \sum_{k=-\infty}^{n_0-1} f_{n-k} c^{-k} - \sum_{k=-\infty}^{n_0-2} f_{n-k} c^{-k} \right) = 0.
\]  

(45)

\( f(z) \) thus coincides with the zero filter. This is impossible due to the modulus one condition. Case #2 does not happen.

At this point, we have shown that at most three coefficients of filter \( f(z) \) are likely to be non zero and that these coefficients are consecutive. For the sake of simplicity, we assume without restriction that coefficients \( f_0, f_1 \) and \( f_2 \) are possibly non zero and that \( f_0 \) is non zero. We now have to characterize these three coefficients. The modulus of \( f_0 x(n) + f_1 x(n-1) + f_2 x(n-2) \) is equal to one for each \( n \), almost surely. We recall that \( x(n-1) = e^{i\pi h_{n-1}} x(n-2) \), \( x(n) = e^{i\pi h_{n+n-1}} x(n-2) \), and that \( |x(n-2)| = 1 \). Therefore, for each \( n \) and for each binary sequence \( [\alpha_n, \alpha_{n-1}]^T \), \( f_0 e^{i\pi h_{n+n-1}} + f_1 e^{i\pi h_{n-1}+} f_2 \) has modulus one. We denote by \( \Theta(\alpha_{n-1}, \alpha_n) \) its argument. One can easily verify that any function \( \Theta(\alpha_{n-1}, \alpha_n) \) of two binary variables can be written as \( \Theta(\alpha_{n-1}, \alpha_n) = \Theta'(\alpha_{n-1}) + \Theta''(\alpha_{n-1}) \alpha_n \). Therefore, for each \( n \),

\[
f_0 e^{i\pi h_{n+n-1}} + f_1 e^{i\pi h_{n-1}} + f_2 = \exp \left( i\alpha_n \Theta''(\alpha_{n-1}) + \Theta'(\alpha_{n-1}) \right).
\]  

(46)

Using (46) respectively for sequences \( [1, \alpha_{n-1}]^T \) and \( [-1, \alpha_{n-1}]^T \), where \( \alpha_{n-1} \) is any parameter of \( \{-1, 1\} \), and substracting the two equations, we obtain

\[
2i f_0 \sin(\pi h) e^{i\pi h \alpha_{n-1}} = 2i \sin \Theta''(\alpha_{n-1}) e^{i\Theta'(\alpha_{n-1})}.
\]  

(47)

We first consider the modulus of each side of the above equation. The modulus is a constant w.r.t. \( \alpha_{n-1} \). It is straightforward to show that this claim holds only if function \( \Theta''(\alpha_{n-1}) \) has the following form

\[
\Theta''(\alpha_{n-1}) = \epsilon(\alpha_{n-1}) \theta + \frac{1 - \epsilon'(\alpha_{n-1})}{2} \pi
\]

modulo \( 2\pi \), where \( \theta \) is a constant in \( [0, \pi] \) and where \( \epsilon(.) \) and \( \epsilon'(.) \) are two binary functions of \( \{-1, 1\} \) in \( \{-1, 1\} \). We now identify the arguments of both sides of (47). Using the above expression of \( \Theta''(\alpha_{n-1}) \), it is straightforward to show that

\[
\Theta'(\alpha_{n-1}) = \pi h \alpha_{n-1} + \varphi + \frac{1 - \epsilon'(\alpha_{n-1})}{2} \pi
\]

November 28, 2005

SUBMITTED VERSION
modulo $2\pi$, where $\varphi$ is a constant of $[-\pi, \pi)$ and where $\epsilon(\alpha_{n-1}) = \epsilon(\alpha_{n-1})\epsilon'(\alpha_{n-1})$. Putting all pieces together, we obtain that for each $n$,

$$f_0 e^{i\pi h (\alpha_n + \alpha_{n-1})} + f_1 e^{i\pi h \alpha_{n-1}} + f_2 = \epsilon(\alpha_{n-1}) \exp(i\theta \epsilon(\alpha_{n-1})\alpha_n + i\pi h \alpha_{n-1} + i\varphi).$$  \hfill (48)

We now remark that any function $\epsilon : \{-1,1\} \rightarrow \{-1,1\}$ verifies the following property: it exists $\delta \in \{-1, +1\}$ such that

$$\forall \alpha \in \{-1, +1\}, \ \epsilon(\alpha) = \delta \quad \text{or} \quad \forall \alpha \in \{-1, +1\}, \ \epsilon(\alpha) = \delta\alpha.$$

We thus distinguish two cases depending on the expression of function $\epsilon$.

Case #1. Assume that $\epsilon(\alpha_{n-1}) = \delta$ is a constant. We put $\varphi' = \varphi$ if $\delta = 1$ and $\varphi' = \varphi + \pi$ if $\delta = -1$. We also put $\theta' = \delta \theta$, so that (48) can be rewritten

$$f_0 e^{i\pi h (\alpha_n + \alpha_{n-1})} + f_1 e^{i\pi h \alpha_{n-1}} + f_2 = \exp(i(\theta'\alpha_n + \pi h \alpha_{n-1} + \varphi')).$$  \hfill (49)

It is now straightforward to show that the above equality holds for each $\alpha_n, \alpha_{n-1}$ only if

$$f_0 = \frac{\sin \theta'}{\sin \pi h} e^{i\varphi'}, \ f_1 = \frac{\sin(\pi h - \theta')}{\sin \pi h} e^{i\varphi'}, \ f_2 = 0.$$  \hfill (50)

Reciprocally, it is straightforward to show that such a filter $f(z)$ excited by sequence $x(n)$ produces a constant modulus sequence.

Case #2. Now assume that $\epsilon(\alpha_{n-1}) = \delta \alpha_{n-1}$. We put $\varphi' = \varphi$ if $\delta = 1$ and $\varphi' = \varphi + \pi$ if $\delta = -1$. We also define $\theta' = \delta \theta$, so that (48) reduces to

$$f_0 e^{i\pi h (\alpha_n + \alpha_{n-1})} + f_1 e^{i\pi h \alpha_{n-1}} + f_2 = \alpha_{n-1} \exp(i(\theta'\alpha_n + \pi h \alpha_{n-1} + \varphi')).$$  \hfill (51)

It can be shown that the above equality holds for each $\alpha_n, \alpha_{n-1}$ only if

$$f_0 = \frac{\sin \theta'}{\sin \pi h} e^{i\varphi'}, \ f_1 = \frac{e^{-i\theta'}}{i \tan \pi h} e^{i\varphi'}, \ f_2 = i \cos \theta' \frac{e^{i\varphi'}}{\sin \pi h}.$$  \hfill (52)

Reciprocally, such a filter excited by sequence $x(n)$ produces a constant modulus sequence.

REFERENCES


Fig. 1. Results of equalization step for channel 1 - $h = 0.75$
Fig. 2. Results of equalization step for channel 2 - $h = 0.75$
(a) CMA and OD-ED

Average BER = 1.1 \times 10^{-2}

(b) CMA and direct detection

Average BER = 0.13

(c) Proposed criterion and direct detection

Average BER = 3.8 \times 10^{-3}

(d) Wiener filter and direct detection

Average BER = 8. \times 10^{-4}

Fig. 3. Empirical distributions of BERs - $E_b/N_0 = 15\text{dB}$
(a) CMA followed by OD-ED

(b) CMA followed by direct detection

(c) Proposed criterion followed by direct detection

(d) Wiener filter followed by direct detection

Fig. 4. Empirical distributions of BERs - $E_b/N_0 = 25$dB