ON THE BLIND EQUALIZATION OF CONTINUOUS PHASE MODULATED SIGNALS USING A CONSTANT MODULUS CRITERION

P. Bianchi, Ph. Loubaton, F. Sirven[*]

Equipe Signal pour les Communications, UMLV and ENST/ URA 820, [*] Thales Communication
5, boulevard Descartes 77454 Marne-La-Vallée Cedex 2, France.
email : bianchi,loubaton@univ-mlv.fr, Francois.Sirven@fr.thalesgroup.com

ABSTRACT

The use of a constant modulus (CM) criterion to achieve blind equalization of Continuous Phase Modulated signals is investigated in this paper. A characterization of the solutions is provided. It is shown that the constant modulus criterion may be, theoretically, unsuccessful because of the existence of undesired solutions.

1. INTRODUCTION

Continuous phase modulation is a widespread scheme thanks to its attractive spectral efficiency and its constant modulus property. In particular, it is used in the European second generation mobile system GSM, in the professional mobile communications system Tetrapol, as well as in a number of military systems.

This paper is devoted to the blind equalization problem of continuous phase modulations. In contrast with the linear modulation case, this problem seems to have been addressed only by a few authors. The most popular approach (see e.g. [5], [6], [7]) consists in estimating channel coefficients in the maximum likelihood sense, so as to be able to recover the symbol sequence by some appropriate algorithm (Viterbi algorithm, Bahl algorithm,....). To this end, the so-called Expectation Maximization (EM) algorithm has been proposed to avoid calculating the likelihood for every possible symbol sequence. Note that this type of methods requires a relevant initial estimate of channel coefficients and the *a priori* knowledge of certain modulation parameters such as symbol rate, modulation index, and shaping filter. Since this kind of information is sometimes not available (e.g. in passive listening), it is relevant to study other approaches.

As CPM signals have constant modulus, using the constant modulus (CM) algorithm introduced by Godard seems to be quite natural. This approach consists in looking for an equalizer which produces a constant modulus output when driven by the received signal. However, the theoretical behavior of the CM algorithm in case of CPM signals seems not to have been addressed thoroughly in previous works.

Recall that the theoretical behavior of Godard's algorithm has been studied extensively by several authors in the linear modulation case (see e.g. [3]). In particular, it has been shown that if the symbol sequence is circular and independent identically distributed (i.i.d.), then the CM equalizer output coincides with the symbol sequence. In this paper, we characterize the set of equalizers producing a constant modulus output signal, and show that the CM criterion is, theoretically, not successful.

2. SIGNAL MODEL

The complex envelope $x_a(t)$ of a CPM signal can be written as $x_a(t) = \exp i\psi_a(t)$, where the phase $\psi_a(t)$ of $x_a(t)$ is given by

$$\psi_a(t) = \pi h \left(\sum_{n\in\mathbb{Z}} a_n \phi_a(t - nT_s) \right). \quad (1)$$

$(a_n)_{n\in\mathbb{Z}}$ is the symbol sequence. It is assumed that $a_n = \pm 1$ for each n, and that the sequence is centered and independent identically distributed. T_s is the symbol period. Function $\phi_a(t)$ is generally an increasing function such that $\phi_a(t) = 0$ for each $t \leq 0$, $0 \leq \phi_a(t) \leq 1$ for $0 \leq t \leq LT_s$ and $\phi_a(t) = 1$ for each $t \geq LT_s$, where L is an integer. Parameter h is called the modulation index, and is characterized by the fact that the phase variation induced by a symbol is equal to $\pm\pi h$. We assume that $0 < h < 1$. In this paper, we assume that $L = 1$, i.e. the transmitted signal is a full response CPM signal. The case of partial response CPM signals (i.e. $L > 1$) will be addressed in a forthcoming work. Using the previous properties of ϕ_a, we easily get a more convenient expression of phase $\psi_a(t)$ for each t such as $nT_s \leq t < (n+1)T_s$,

$$\psi_a(t) = \pi h \left(\sum_{k=-\infty}^{n-1} a_k \right) + \pi h a_n \phi_a(t - nT_s). \quad (2)$$

A fundamental result due to Laurent (the so-called Laurent's representation [1]) states that if $L = 1$, signal $x_a(t)$ can be written as a linear modulation. In order to introduce this useful result, we denote from now on by $(x_n)_{n\in\mathbb{Z}}$ the sequence defined by :

$$x_n = \exp i\pi h \left(\sum_{k=-\infty}^{n-1} a_k \right). \qquad (3)$$

It is easily seen that $(x_n)_{n\in\mathbb{Z}}$ is a centered stationary sequence which satisfies $x_{n+1} = \exp i\pi h a_n x_n$. Then, it can be shown that

$$x_a(t) = \sum_{n=0}^{\infty} x_{n+1} c_0(t - nT_s), \qquad (4)$$

where $c_0(t)$ is a pulse which is non zero only for $0 < t < 2T_s$ and which is defined as follows.

$$\begin{array}{ll} c_0(t) = \frac{\sin \pi h \phi_a(t)}{\sin \pi h} & \text{for } 0 \le t \le T_s \\ c_0(t) = \frac{\sin \pi h(1-\phi_a(t-T_s))}{\sin \pi h} & \text{for } T_s \le t \le 2T_s. \end{array} \qquad (5)$$

Equation (4) states that CPM signal $x_a(t)$ can be interpreted as the linear modulation of sequence (x_n) using shaping filter $c_0(t)$. However, (x_n) is not an i.i.d. sequence. Thus, classical results on the behavior of the CM algorithm in the linear modulation context do not hold yet.

3. PROBLEM FORMULATION

Fig 1 represents the basic equalization model. We assume that the continuous-time received signal $y_a(t)$ is a filtered version of the transmitted CPM signal $x_a(t) = \exp i\psi_a(t)$, where $\psi_a(t)$ is defined by (1). $H(f)$ denotes the (analog) frequency response of the filter which is assumed to be non zero in the bandwidth of the input signal $x_a(t)$ [1]. $y_a(t)$ is passed through an analog equalization filter $G(f)$. Output signal $z_a(t)$ is sampled at rate $1/T_e$. Sequence $z_n = z_a(nT_e)$ denotes the output samples. In the classical linear modulation context, the CM algorithm should be used at symbol rate, i.e. $T_e = T_s$. However in our context, since transmitted signal $x_a(t)$ has constant modulus, it may also be relevant to use the CM algorithm with other sampling rates.

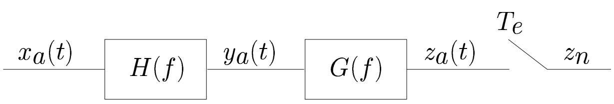

Fig. 1. Equalization model

[1]Note that signal $x_a(t)$ has actually an infinite bandwidth. However, in practice, the spectral density of $x_a(t)$ is clearly negligible at high enough frequencies.

In practice, the equalizer is implemented by a digital filter. We thus note that our structure is actually different from the practical one. However, the analog context is easier to present, while covering the digital one if the rate at which operates the digital equalizer is large enough. In the sequel, we study the equalizers for which

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} E(|z_n|^2 - 1)^2, \qquad (6)$$

is identically zero.

For this, we use Laurent's representation (4) and rather characterize the combined transfer function $F(f) = C_0(f)G(f)H(f)$ associated to the constant modulus equalizers $G(f)$. Here, $C_0(f)$ represents the Fourier transform of function $c_0(t)$. If we denote by $f(t)$ the inverse Fourier transform of $F(f)$, continuous-time signal $z_a(t)$ can be written as

$$z_a(t) = \sum_{n=0}^{\infty} x_{n+1} f(t - nT_s). \qquad (7)$$

Note that in order that the right-hand side of (7) converges in the mean square sense, function $f(t)$ must be square integrable. Because it is moreover natural to restrict $f(t)$ to be continuous [2], we characterize in the sequel the set of all functions $f(t)$ for which (6) is zero.

The meaning of (6) depends on the choice of the sampling period T_e. In order to precise this point, we recall that signal $z_a(t)$ is periodically correlated with period T_s. In particular, function $t \to E(|z_a(t)|^2 - 1)^2$ is periodic of period T_s. Expanding this function in Fourier series, it is easy to check that if $\frac{T_e}{T_s}$ is irrational, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} E(|z_n|^2 - 1)^2 = \frac{1}{T_s} \int_0^{T_s} E(|z_a(t)|^2 - 1)^2 \, dt.$$

Therefore, if $\frac{T_e}{T_s}$ is irrational, (6) is zero if and only if $|z_a(t)| = 1$ for each t a.s.[3] On the other hand, if $\frac{T_e}{T_s} = p/q$ where p and q are coprime integers, sequence $n \to E(|z_n|^2 - 1)^2$ is periodic of period q. Therefore, the average (6) coincides with the finite sum $\frac{1}{q}\sum_{n=0}^{q-1} E(|z_n|^2 - 1)^2$, and is zero if and only if $|z_a(\frac{nT_s}{q})| = 1$ for each $n \in \mathbb{Z}$ a.s. Therefore, using an appropriate sampling rate T_e allows either to force the modulus of $|z_a(t)|$ to 1 for each t, either to force the modulus 1 condition at every integer multiples of $\frac{T_s}{q}$.

[2]If e.g. the transfer function $G(f)$ of the equalizer is assumed to be square integrable, then $f(t)$ is easily seen to be continuous as soon as $H(f)$ is bounded.

[3]notation a.s. stands for almost surely

We finally remark that condition $|z_a(t)| = 1$ for each t a.s. is a priori relevant. Indeed, it seems intuitively reasonable to conjecture that condition $|z_a(t)| = 1$ for each t a.s. may force the equalizer frequency response $G(f)$ to be equal to $\frac{1}{H(f)}$ up to a scaling factor or a delay. If this holds, equalizer output $z_a(t)$ would coincide with a scaled and delayed version of transmitted signal $x_a(t)$, from which symbol sequence (a_n) could be extracted by a standard CPM demodulation algorithm. Similarly, using the CM algorithm at symbol rate $T_e = T_s$ seems also to be reasonable. Condition $|z_a(nT_s)| = 1$ for each n a.s. would be appropriate if it forced equalizer output samples $z_a(nT_s)$ to coincide with the modulus one sequence $(x_n)_{n\in\mathbb{Z}}$ (see equation (7)). In this case, transmitted symbols could also be detected by using a simple demodulation algorithm.

However, we show in the next section that the behavior of the CM algorithm is in both cases more complicated.

4. EXPRESSION OF CONSTANT MODULUS SOLUTIONS

4.1. The case $T_e = T_s$.

We first study the behavior of the equalizer structure when the output equalizer sampling rate is equal to symbol rate $\frac{1}{T_s}$. To this end, it is sufficient to look for the square summable sequences $(f_k)_{k\in\mathbb{Z}}$ for which the stationary sequence z_n defined by

$$z_n = \sum_{k\in\mathbb{Z}} f_k x_{n-k}$$

satisfies $|z_n| = 1$ for each n a.s. This problem has been investigated thoroughly in the case where sequence x_n is a circular i.i.d. sequence (see e.g. [3]). In this context, the modulus one constraint forces the output equalizer to coincide with a delayed and rotated version of $(x_n)_{n\in\mathbb{Z}}$.

Unfortunately, the present sequence $(x_n)_{n\in\mathbb{Z}}$ is not i.i.d., and the results are quite different.

4.1.1. *The case $h \neq \frac{1}{2}$.*

We first consider the case $h \neq \frac{1}{2}$.

Theorem 1 *Assume that the index h is not equal to $\frac{1}{2}$. Then, $|\sum_{k\in\mathbb{Z}} f_k x_{n-k}| = 1$ for each n a.s. if and only if sequence $(f_k)_{k\in\mathbb{Z}}$ is a delayed and rotated version of one of the following sequences :*

$$\left\{\begin{array}{l} f_0 = \frac{\sin\theta}{\sin\pi h} \\ f_1 = \frac{\sin(\pi h-\theta)}{\sin\pi h} \\ f_2 = 0, \end{array}\right. \quad or \quad \left\{\begin{array}{l} f_0 = \frac{\sin\theta}{\sin\pi h} \\ f_1 = \frac{e^{-i\theta}}{i\tan\pi h} \\ f_2 = \frac{i\cos\theta}{\sin\pi h}, \end{array}\right.$$

and $f_k = 0$ for each k different from 0, 1, 2. Here, θ is any parameter of $]-\pi,\pi]$.

We respectively denote by type 1 and type 2 the previous families of sequences. A sketch of proof is provided in the appendix. This result shows that the constant modulus criterion does not allow to reconstruct directly sequence $(x_n)_{n\in\mathbb{Z}}$, but a particular filtered version of it.

This result is actually not surprising. We first consider type 1 filters. For each θ, it is clear that $|\cos\theta x_n + ia_n\sin\theta x_n| = 1$ for each n. But, ia_nx_n can also be written as $ia_nx_n = \frac{1}{\sin\pi h}(x_{n+1} - \cos\pi h x_n)$, so that $e^{i\theta a_n}x_n = \cos\theta x_n + ia_n\sin\theta x_n$ coincides with the output of the first family filter $f(z) = \frac{\sin\theta}{\sin\pi h} + z^{-1}\frac{\sin(\pi h-\theta)}{\sin\pi h}$ driven by x_n. We note that if θ coincides with the phase $\pi h\phi_a(\tau)$ for some $\tau\in]0,T_s[$, then one can easily check using (4) and (5) that $e^{i\theta a_n}x_n$ coincides with $x_a(nT+\tau)$. In other words, the first family of filters contains two taps FIR interpolating filters.

In order to illustrate the properties of the second family filter, we consider $\cos\theta a_{n-1}x_n + ia_n\sin\theta x_n$, which is still of modulus one for each n. Using that $-ia_{n-1}x_n = \frac{1}{\sin\pi h}(x_{n-1} - \cos\pi h x_n)$, we get immediately that $\cos\theta a_{n-1}x_n + ia_n\sin\theta x_n$ is the output of the second filter with parameter θ driven by sequence x. In contrast with the first family, the second family does not seem to contain interpolating filters.

4.1.2. *The case $h = \frac{1}{2}$.*

In this case, another approach is required to solve the constant modulus condition. This particular type of modulation is called Minimum Shift Keying (MSK). We get from (3) that $x_{n+1} = ia_nx_n$, thus sequence (x_n) is alternately real and imaginary. We put $x_n = i^nb_n$, where b_n is recursively defined by $b_{n+1} = a_nb_n$. (b_n) is a real i.i.d. sequence. Output samples z_n can be written as $z_n = i^n(\sum_{k\in\mathbb{Z}}\tilde{f}_kb_{n-k})$ where $\tilde{f}_k = f_ki^{-k}$. Therefore, $|z_n| = 1$ if and only if $\sum_{k\in\mathbb{Z}}\tilde{f}_kb_{n-k}$ is modulus 1. As (b_n) is a real i.i.d. sequence, the set of all filters $\tilde{f}(z)$ satisfying this CM condition has been recently characterized by Houcke et al. ([8]). Using this result, we get immediately that $|z_n| = 1$ if and only if (f_n) is a delayed and rotated version of the following sequence :

$$\left\{\begin{array}{l} f_0 = \sin\theta \\ f_N = i^{-(N-1)}\cos\theta, \end{array}\right.$$

where $N\in\mathbb{N}$, and $f_n = 0$ for each n different from 0, N.

4.2. The case $T_e \neq T_s$.

We now address the case $T_e \neq T_s$, and just focus on T_e/T_s irrational. The case T_e/T_s rational can be similarly treated. We thus have to characterize the square integrable continuous functions for which $z_a(t)$ given by (7) satisfies $|z_a(t)| = 1$ for each t. We just address the case $h \neq \frac{1}{2}$ in the following.

Proposition 1 *Assume that $\frac{T_e}{T_s}$ is irrational and that $h \neq \frac{1}{2}$. $|z_a(t)| = 1$ for each t a.s. if and only if for each $\tau \in [0, T_s[$, sequence $(f(\tau + kT_s))_{k\in\mathbb{Z}}$ coincides with a delayed or rotated version of one of the two sequences given in theorem 1.*

We first give some simple examples illustrating that condition $|z_a(t)| = 1$ for each t a.s. may hold even if $z_a(t)$ does not coincide with $x_a(t)$. The first one corresponds to the interpolators. Given a time delay $\delta \in [0, T_s]$, we put $f(t) = c_0(t - \delta)$, so that $z_a(t) = x_a(t - \delta)$. Proposition 1 clearly holds for $f(t)$: using definition (5) of c_0, one can easily check that for each $\tau \in [0, T_s[$, $(f(\tau + kT_s))_{k\in\mathbb{Z}}$ is a delayed version of a type 1 sequence, where parameter θ coincides either with $\pi h \phi_a(\tau - \delta)$ if $\tau \geq \delta$, or with $\pi h \phi_a(\tau - \delta + T_s)$ if $\tau < \delta$.

In order to exhibit a second example, we consider a function $\tilde{\phi}_a(t)$ which satisfy the same requirements as $\phi_a(t)$, and define function $f(t) = \tilde{c}_0(t)$ from $\tilde{\phi}_a(t)$ through formula (5). Then, signal $z_a(t)$ is of course a CPM signal with the same index as $x_a(t)$ and with the same symbol sequence (a_n), but which have a different phase. In other words, if the equalizer $G(f)$ is chosen in such a way that $C_0(f)H(f)G(f) = \tilde{C}_0(f)$ for each f, then output signal $z_a(t)$ still coincide with a CPM signal, but whose shaping filter $\tilde{\phi}_a(t)$ is different from those of the transmitted signal (i.e. $\phi_a(t)$). Note that for each t, sequence $(\tilde{c}_0(t+kT_s))_{k\in\mathbb{Z}}$ still coincide with a type 1 sequence.

In the previous examples, sequence $(f(t + kT_s))_{k\in\mathbb{Z}}$ is for each t a type 1 sequence, and the function $f(t)$ has a support of width $2T_s$. It is however possible to construct more complicated examples such that $f(t)$ has a larger support, and such that $(f(t+kT_s))_{k\in\mathbb{Z}}$ can be, depending on t, a type 1 or a type 2 sequence.

We put $h = 0.7$ and we denote by $t \rightarrow \theta(t)$ a piecewise linear function defined on interval $[0, T_s]$ whose variations are resumed on figure (2).

t	0		$\frac{T_s}{3}$		$\frac{T_s}{2}$		$\frac{2T_s}{3}$		T_s
$\theta(t)$	0	$\nearrow$	$\frac{\pi}{2}$	$\nearrow$	π	$\searrow$	$\frac{\pi}{2}$	$\nearrow$	πh

Fig. 2. Variations of function $\theta(t)$ on interval $[0, T_s]$

We now consider function $f(t)$ defined from $\theta(t)$ by :

$$f(t) = \begin{cases} \frac{\sin\theta(t)}{\sin\pi h} & \text{for } t \in [0, T_s], \\ \frac{\sin(\pi h - \theta(t-T_s))}{\sin\pi h} & \text{for } t \in [T_s, T_s + \frac{T_s}{3}] \cup [T_s + \frac{2T_s}{3}, 2T_s], \\ \frac{e^{-i\theta(t-T_s)}}{i\tan\pi h} & \text{for } t \in [T_s + \frac{T_s}{3}, T_s + \frac{2T_s}{3}], \\ \frac{i\cos\theta(t-2T_s)}{\sin\pi h} & \text{for } t \in [2T_s + \frac{T_s}{3}, 2T_s + \frac{2T_s}{3}], \\ 0 & \text{otherwise.} \end{cases}$$

Figure 3 represents the modulus of filter $f(t)$. Signal $z_a(t)$

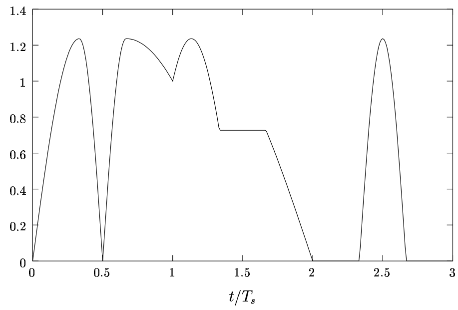

Fig. 3. Modulus of filter $f(t)$

is clearly not a CPM signal because the support of $f(t)$ is larger than $2T_s$.

5. CONCLUSION.

Previous results assert that the use of the CM algorithm to equalize CPM signals might be unsuccessful because of the existence of undesired global minima of the cost function defined by (6). Nevertheless, this method provides most of the time a very satisfying equalization. In effect, simulations have shown that constant modulus criterion used on a CPM signal corrupted by multipath leads in general to low Bit Error Rates. These simulation results will be provided in an extended version of the paper.

6. REFERENCES

[1] P.A. Laurent, "Exact and approximate construction of digital phase modulations by superposition of amplitude modulated pulse", *IEEE Trans. on Communications*, vol. COM-34, no. 2, pp. 150-160, February 1986.

[2] D.N. Godard, "Self-recovering equalization and carrier tracking in two-dimensional data communication systems", *IEEE Trans. on Communications*, vol. COM-28, no. 11, pp. 1967-1875, November 1980.

[3] O. Shalvi, E.Weinstein, “New criteria for blind deconvolution of nonminimum phase systems (channels)”, *IEEE Trans. on; information theory*, col. 36, no.2, pp. 312-321, March 1990.

[4] J.R. Treichler, B.G. Agee, “A new approach to multipath correction of constant modulus signals”, *IEEE Trans. on acous. speech and signal process.*, vol. ASSP-31, no. 2, pp. 459-472, April 1983.

[5] G.K. Kaleh, R.Vallet, ”Joint parameter estimation and symbol detection for linear or nonlinear unknown channels”, *IEEE Trans. on Communications*, vol. 42, pp. 2406-2413, July 1994.

[6] H. Cirpan, M.K. Tsatsanis, ”Blind receiver for non linearly modulated signals in multipath”, in Proc. SPAWC, pp. 149-152, Paris, 1997.

[7] Hoang Nguyen, B.C. Levy, “Blind ML detection of CPM signals via the EMV algorithm”, in Proc. ICASSP, pp 2457-2460, Orlando, May 2002.

[8] S. Houcke, A. Chevreuil, Ph. Loubaton, “Estimation of the symbol period : the frequency offset case”, in Proc. EUSIPCO, Toulouse, September 2002.

A. PROOF OF THEOREM 1

We assume that $T_e = T_s$ and that (6) is equal to zero. Thus, that $|z_n| = 1$ for each $n \in \mathbb{Z}$ a.s. This implies that for each integers k_0, k_1 such as $k_0 \leq k_1$ and for each binary sequence $(\alpha_{k_0}, \dots, \alpha_{k_1})$,

$$E(|z_n|^2/(a_{n-k_0}, \dots, a_{n-k_1}) = (\alpha_{k_0}, \dots, \alpha_{k_1})) = 1 \quad (8)$$

Using (8) for certain well-chosen sequences $(\alpha_{k_0}, \dots, \alpha_{k_1})$, we now show that coefficients $(f_k)_{k\in\mathbb{Z}}$ are equal to zero except for at most three consecutive integers k.

We assume that integers k_0 and k_1 are such as $f_{k_0} f_{k_1} \neq 0$ and $k_1 \geq k_0 + 3$ and show that these assumptions lead to a contradiction. We put for each integer k: $f_k = \rho_k e^{i\psi_k}$, where $\rho_k \geq 0$. Using (7), we get that for each n, $|z_n|^2$ is equal to :

$$\sum_k |\rho_k|^2 + 2 \sum_{k'} \sum_{k>k'} \rho_{k'} \rho_k \cos(\pi h \sum_{j=k'}^{k-1} a_{n-j} + \psi_k - \psi_{k'}).$$

For a given binary sequence $\boldsymbol{\alpha} = (\alpha_{k_0}, \dots, \alpha_{k_1})$, an expression of $E(|z_n|^2/(a_{n-k_0} \dots a_{n-k_1}) = \boldsymbol{\alpha})$ can be obtained by calculating conditional expectations of every terms of the previous expression. We mention the case $k' \leq k_0 < k \leq k_1$, for which $E(\cos(\pi h \sum_{j=k'}^{k-1} a_{n-j} + \psi_k - \psi_{k'})/(a_{n-k_0} \dots a_{n-k_1}) = \boldsymbol{\alpha})$ is equal to $C^{k'-k_0} \cos\left(\pi h \sum_{j=k_0}^{k-1} \alpha_j + \psi_k - \psi_{k'}\right)$, where $C = cos \pi h$. Other cases can be similarly treated. We finally get a new expression of (8) which still holds when replacing $\alpha_{k_0}, \dots, \alpha_{k_1}$ by any other values. In particular, combining equations obtained by using sequences $(\alpha_{k_0}, \dots, \alpha_{k_1})$ and $(-\alpha_{k_0}, \alpha_{k_0+1}, \dots, \alpha_{k_1})$ yields after some algebra :

$$\sum_{k' \leq k_0} \left(\sum_{k=k_0+1}^{k_1+1} \rho_{k'} \rho_k C^{k_0-k'} \sin(\pi h \sum_{j=k_0+1}^{k-1} \alpha_j + \psi_k - \psi_{k'}) \right. \quad (9)$$

$$\left. + \sum_{k \geq k_0+2} \rho_{k'} \rho_k C^{k_0-k'+k-k_1-1} \sin(\pi h \sum_{j=k_0+1}^{k_1} \alpha_j + \psi_k - \psi_{k'}) \right) = 0.$$

Equation (9) still holds when replacing $(\alpha_{k_0}, \dots, \alpha_{k_1})$ by sequence $(\beta_{k_0}, \dots, \beta_{k_1})$ defined by $\beta_j = \alpha_j$ for $j = k_0, \dots, k_1 - 2$, $\beta_{k_1-1} = -\alpha_{k_1-1}$ and $\beta_{k_1} = -\alpha_{k_1}$. Moreover, we put $\alpha_{k_1-1} = 1$ and $\alpha_{k_1} = -1$ so that $\sum_{j=k_0+1}^{k_1} \alpha_j = \sum_{j=k_0+1}^{k_1} \beta_j$. As $\rho_{k_1} \neq 0$, combining equation (9) used with both sequences yields

$$\sum_{k' \leq k_0} \rho_{k'} C^{k_0-k'} \cos(\pi h \sum_{j=k_0+1}^{k_1-2} \alpha_j + \psi_{k_1} - \psi_{k'}) = 0, \quad (10)$$

for each sequence $(\alpha_{k_0+1}, \dots, \alpha_{k_1-2})$. It can be shown from (10) that k_0 is the smallest integer k such as $f_k \neq 0$. Indeed, using (10), we get that

$$\sum_{k \leq k_0} f_k C^{-k} = 0$$

. Now suppose that there exists an integer $k < k_0$ such as $f_k \neq 0$. We denote by k_2 the greatest such integer : $k_2 < k_0$. Clearly,

$$f_{k_0} C^{-k_0} = \sum_{k \leq k_0} f_k C^{-k} - \sum_{k \leq k_2} f_k C^{-k} = 0. \quad (11)$$

But (11) is in contradiction with assumption $f_{k_0} \neq 0$. Thus, such an integer k_2 does not exist. k_0 is therefore the smallest integer such as $f_{k_0} \neq 0$. This allows to reduce equation (10) to $\cos(\pi h \sum_{j=k_0+1}^{k_1-2} \alpha_j + \psi_{k_1} - \psi_{k_0}) = 0$. However, one can easily see that previous equality does not hold for every possible sequence $(\alpha_{k_0+1}, \dots, \alpha_{k_1-2})$ since $k_1 \geq k_0 + 3$ and $h \neq \frac{1}{2}$. This invalidate the assumption that $f_{k_0} f_{k_1} \neq 0$ for some integers such as $k_1 \geq k_0 + 3$. Thus, $(f_k)_{k\in\mathbb{Z}}$ is non zero for at most three consecutive integers, say $k = 0, 1, 2$. Using equation (8) with every possible values of symbol sequence $(\alpha_0, \alpha_1, \alpha_2)$ leads to a set of equations whose solutions correspond to theorem 1.