Comment relâcher la contrainte de trame ajustée dans le cadre des méthodes proximales en restauration d'images?

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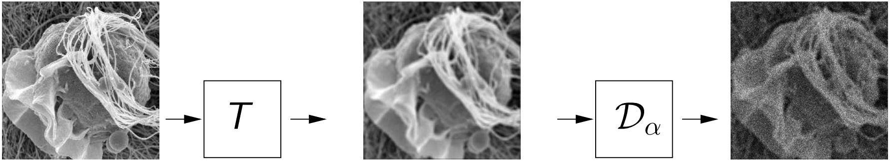
Degradation model: convolution and noise



Original image $\overline{y} \in \ell^2(\mathbb{Z})$

Blurred image $(T\overline{y})$

Degraded image $z \in \ell^2(\mathbb{Z})$ $\alpha = 0.1$

Degradation model

$$z = \mathcal{D}_\alpha(T\overline{y})$$

- $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$: Convolution operator
- $\mathcal{D}_\alpha$: Effect of noise (Gaussian noise, Poisson noise, Laplace noise, ...)
- α: noise parameter (variance for Gaussian noise, scaling parameter for Poisson noise, ...)

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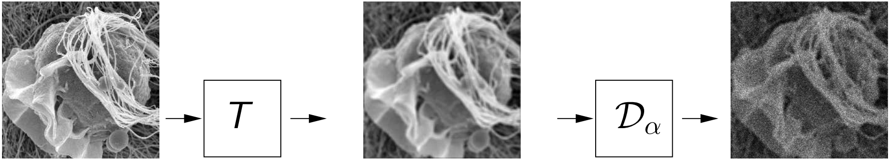

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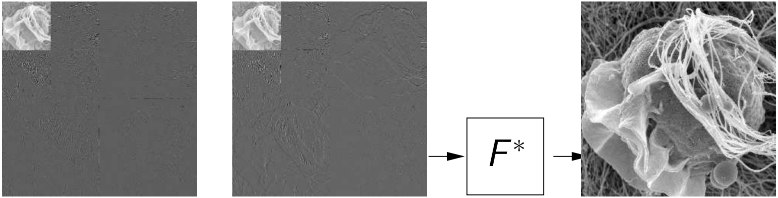
$\Rightarrow$ Our objective is to recover an image $\hat{y}$, the closest as possible to $\overline{y}$.

Existing wavelet-based variational methods

Gaussian noise + convolution

- Quadratic regularization techniques (Wiener filtering).
- Multiresolution analyses were used for denoising ($T = \mathrm{Id}$).
- Redundant **frame representations** were substituted for wavelet bases for denoising.

Frame representation



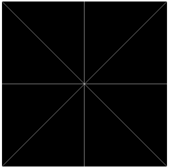
Frame coefficients (x) Image (y)

- $x \in \ell^2(\mathbb{Z})$: **Frame coefficients** of the image y.
- $F^* : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$: **Frame synthesis operator**

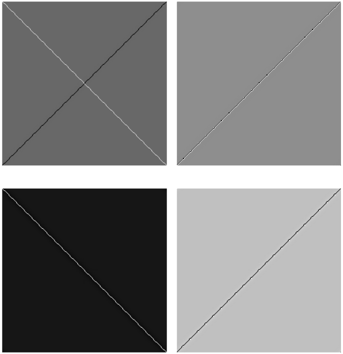
$$\boxed{y = F^* x}$$

- $F : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$: **Frame analysis operator**
- Frame is tight if $\exists \mu \in]0, +\infty[,\ F^* F = \mu \mathrm{Id}$.

Tight frame drawback: DTT example



Original



Diagonal decomposition coefficients
for the first resolution level
without prefiltering (top) and with prefiltering (bottom)
after a dual-tree decomposition
with Meyer filter $M = 2$.

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Laplace, Poisson, ... noise + convolution

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Laplace, Poisson, ... noise + convolution

- Douglas-Rachford (DR) algorithm [Combettes&Pesquet 2007]
- Parallel Proximal Algo. (PPXA) [Combettes&Pesquet 2008]
- ADMM [Afonso et al. 2009, Setzer et al. 2009]
- PPXA+: unifying framework for PPXA and ADMM [Pesquet 2010]

Existing wavelet-based variational methods

Gaussian noise + convolution

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Laplace, Poisson, ... noise + convolution

- Douglas-Rachford (DR) algorithm [Combettes&Pesquet 2007] TF
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- PPXA+: unifying framework for PPXA and ADMM [Pesquet 2010] TF

Considered problems

Synthesis formulation (SF):

$$\hat{y} = F^*\hat{x} \qquad \text{such that} \qquad \hat{x} \in \underset{x\in\ell^2(\mathbb{Z})}{\operatorname{Argmin}} \sum_{i=1}^{n} f_i(L_i F^* x) + \sum_{j=1}^{m} g_j(x)$$

Analysis formulation (AF):

$$\hat{y} \in \underset{y\in\ell^2(\mathbb{Z})}{\operatorname{Argmin}} \sum_{i=1}^{n} f_i(L_i y) + \sum_{j=1}^{m} g_j(F y)$$

- For every $i \in \{1, \dots, n\}$, $f_i \colon \ell^2(\mathbb{Z}) \mapsto \,]-\infty, +\infty]$ is a convex, l.s.c and proper function and $L_i \colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is a convolutive operator.
- For every $j \in \{1, \dots, m\}$, $g_j \colon \ell^2(\mathbb{Z}) \mapsto \,]-\infty, +\infty]$ is a convex, l.s.c and proper function.

Proximity operator

Definition [Moreau (1965)]

- **Definition**: For every u in a Hilbert space $\mathcal{H}$, the function $v \mapsto \varphi(v) + \|u - v\|^2/2$, where φ is a convex, l.s.c and proper function, achieves its minimum at a unique point denoted by $\mathrm{prox}_\varphi u$, i.e.,

$$(\forall u \in \mathcal{H}), \qquad \boxed{\mathrm{prox}_\varphi u = \arg\min_{v \in \mathcal{H}} \varphi(v) + \frac{1}{2}\|u - v\|^2}$$

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Examples:

- If $\varphi = \iota_C \Rightarrow \mathrm{prox}_{\iota_C} = P_C$ where ι_C denotes the indicator function over a closed convex set C and P_C denotes the projection onto C
- If $\varphi = \chi|\,.\,|$ with $\chi > 0 \Rightarrow \mathrm{prox}_{\chi|.|}$ is a soft-thresholding with threshold value χ
- If $\varphi = \omega|\,.\,|^p + \chi|\,.\,|$ with $\omega > 0 \Rightarrow$ Closed forms for several $p \geq 1$

PPXA+: $\underset{x\in\ell^2(\mathbb{Z})}{\text{minimize}} \sum_{i=1}^n f_i(L_i F^* x) + \sum_{j=1}^m g_j(x)$

Initialization

$$\left\lfloor \begin{array}{l} (\eta_i)_{1\le i\le n} \in \;]0,+\infty[^n, (\kappa_j)_{1\le j\le m} \in \;]0,+\infty[^m \\ (v_{i,0})_{1\le i\le n} \in (\ell^2(\mathbb{Z}))^n, (w_{j,0})_{1\le j\le m} \in (\ell^2(\mathbb{Z}))^m \\ x_0 = \arg\min_{u\in\ell^2(\mathbb{Z})} \sum_{i=1}^n \eta_i \|L_i F^* u - v_{i,0}\|^2 + \sum_{j=1}^m \kappa_j \|u - w_{j,0}\|^2 \end{array}\right.$$

For $\ell = 0, 1, \ldots$

$$\left\lfloor \begin{array}{l} \text{For } i = 1,\ldots,n \\ \lfloor \quad p_{i,\ell} = \mathrm{prox}_{f_i/\eta_i} v_{i,\ell} \\ \text{For } j = 1,\ldots,m \\ \lfloor \quad r_{j,\ell} = \mathrm{prox}_{g_j/\kappa_j} w_{j,\ell} \\ \lambda_\ell \in]0,2[\\ c_\ell = \arg\min_{u\in\ell^2(\mathbb{Z})} \sum_{i=1}^n \eta_i \|L_i F^* u - p_{i,\ell}\|^2 + \sum_{j=1}^m \kappa_j \|u - r_{j,\ell}\|^2 \\ \text{For } i = 1,\ldots,n \\ \lfloor \quad v_{i,\ell+1} = v_{i,\ell} + \lambda_\ell\big(L_i F^*(2c_\ell - x_\ell) - p_{i,\ell}\big) \\ \text{For } j = 1,\ldots,m \\ \lfloor \quad w_{j,\ell+1} = w_{j,\ell} + \lambda_\ell\big(2c_\ell - x_\ell - r_{j,\ell}\big) \\ x_{\ell+1} = x_\ell + \lambda_\ell(c_\ell - x_\ell) \end{array}\right.$$

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For $i = 1, \ldots, n$

$v_{i,\ell+1} = v_{i,\ell} + \lambda_\ell\big(L_i(2c_\ell - y_\ell) - p_{i,\ell}\big)$

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PPXA+ convergence

The convergence of the sequence $(x_\ell)_{\ell\in\mathbb{N}}$ (resp. $(y_\ell)_{\ell\in\mathbb{N}}$) generated by algorithm PPXA+ for SF (resp. algorithm PPXA+ for AF) is established under the following assumptions:

1. $\left(\bigcap_{i=1}^{n} \operatorname{ri}\operatorname{dom} f_i \circ L_i F^*\right) \cap \left(\bigcap_{j=1}^{m} \operatorname{ri}\operatorname{dom} g_j\right) \neq \varnothing$
   (resp. $\left(\bigcap_{i=1}^{n} \operatorname{ri}\operatorname{dom} f_i \circ L_i\right) \cap \left(\bigcap_{j=1}^{m} \operatorname{ri}\operatorname{dom} g_j \circ F\right) \neq \varnothing$).
2. There exists $\underline{\lambda} \in]0,2[$ such that $(\forall \ell \in \mathbb{N})$, $\underline{\lambda} \leq \lambda_{\ell+1} \leq \lambda_\ell$.

Difficulties to use PPXA+ and frame representations

- Analysis formulation

PPXA+ iteration:

$$c_\ell = \Big(\sum_{i=1}^{n} \eta_i L_i^* L_i + \sum_{j=1}^{m} \kappa_j F^* F\Big)^{-1} \Big(\sum_{i=1}^{n} \eta_i L_i^* p_{i,\ell} + \sum_{j=1}^{m} \kappa_j F^* r_{j,\ell}\Big)$$

- Synthesis formulation

PPXA+ iteration:

$$c_\ell = \Big(\sum_{i=1}^{n} \eta_i F L_i^* L_i F^* + \sum_{j=1}^{m} \kappa_j \mathrm{I}\Big)^{-1} \Big(\sum_{i=1}^{n} \eta_i F L_i^* p_{i,\ell} + \sum_{j=1}^{m} \kappa_j r_{j,\ell}\Big)$$

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Difficulty:

$$\left(\sum_{i=1}^n \eta_i L_i^* L_i + \kappa F^* F \right)^{-1} \qquad \text{by setting} \qquad \kappa = \sum_{j=1}^m \kappa_j$$

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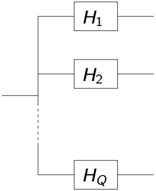
Difficulty:

$$\kappa \left(\sum_{i=1}^n \eta_i F L_i^* L_i F^* + \kappa \mathrm{Id} \right)^{-1} = \mathrm{Id} - F \left(\sum_{i=1}^n \eta_i L_i^* L_i \right) \left(\kappa \mathrm{I} + F^* F \left(\sum_{i=1}^n \eta_i L_i^* L_i \right) \right)^{-1} F^*$$

$\Rightarrow$ How can we compute efficiently $(\quad \cdot \quad + F^* F \cdot)^{-1}$?

Q-channel undecimated filter bank



$$\Longrightarrow \qquad F = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_Q \end{bmatrix}$$

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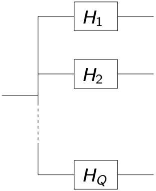


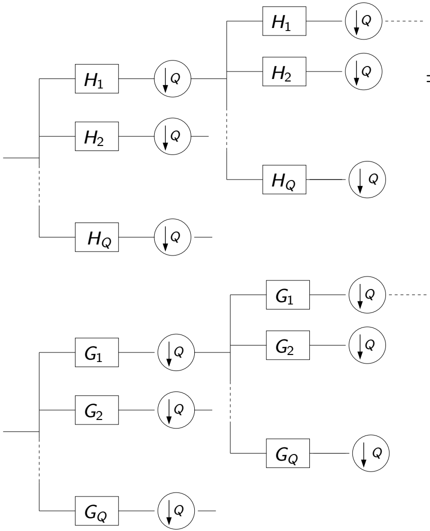
$$\Longrightarrow \qquad F = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_Q \end{bmatrix}$$

- V_q: **filtering operator** such that

$$(\forall \omega \in [-\pi, \pi]^2) \qquad \sum_{q=1}^{Q} |\hat{v}_q(\omega)|^2 \geq \mu,$$

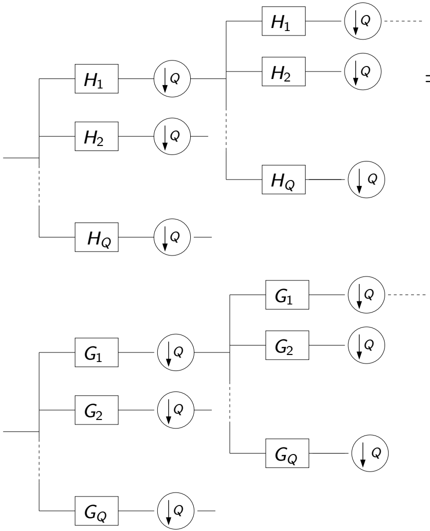
and $\hat{v}_q(\omega)$: frequency response of the filter associated with the convolutive operator V_q.

Q-band DTT [Kingsbury 2001, Chaux et al. 2006]



$$\Longrightarrow \quad F = R \begin{bmatrix} U_1 & 0 & \dots & 0 \\ 0 & U_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & U_Q \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_Q \end{bmatrix}$$

Q-band DTT [Kingsbury 2001, Chaux et al. 2006]



$$\Longrightarrow \quad F = R \begin{bmatrix} U_1 & 0 & \dots & 0 \\ 0 & U_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & U_Q \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_Q \end{bmatrix}$$

- V_q: related to discretization
- U_q: ortho. matrices (Q-band orthon. wav. decomp. in parallel)
- R: orthogonal combination of the subbands applied to ensure directionality (2D)

A specific class of frame representation: DTT

$$F = U \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_Q \end{bmatrix} \qquad \Rightarrow F^* F = \mu_U \sum_{q=1}^{Q} V_q^* V_q.$$

- U: tight frame analysis matrix with constant $\mu_U \in \,]0, +\infty[$.
- V_q: **prefiltering operator** such that

$$(\forall \omega \in [-\pi, \pi]^2) \qquad \mu_U \sum_{q=1}^{Q} |\hat{v}_q(\omega)|^2 \geq \mu,$$

and $\hat{v}_q(\omega)$: frequency response of the filter associated with the convolutive operator V_q.

A specific class of frame representation: DTT

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- Analysis formulation:

$$\Big(\sum_{i=1}^{n} \eta_i L_i^* L_i + \kappa F^* F\Big)^{-1} = \Big(\sum_{i=1}^{n} \eta_i L_i^* L_i + \kappa \mu_U \sum_{q=1}^{Q} V_q^* V_q\Big)^{-1}$$

- Synthesis formulation:

$$\mathrm{Id} - F\big(\sum_{i=1}^{n} \eta_i L_i^* L_i\big)\Big(\kappa \mathrm{I} + F^* F\big(\sum_{i=1}^{n} \eta_i L_i^* L_i\big)\Big)^{-1} F^*$$

$$= \mathrm{Id} - F\big(\sum_{i=1}^{n} \eta_i L_i^* L_i\big)\Big(\kappa \mathrm{I} + \mu_U \sum_{q=1}^{Q} V_q^* V_q\big(\sum_{i=1}^{n} \eta_i L_i^* L_i\big)\Big)^{-1} F^*$$

Results: Cropped versions of Barbara image. Images are restored using SF and complex DTT

Original

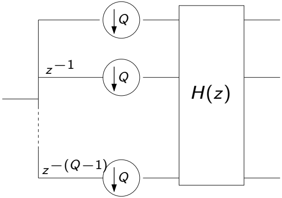
Degraded
SNR = 11.4 dB , SSIM = 0.53

Results: Cropped versions of Barbara image. Images are restored using SF and complex DTT

Tight-frame Complex DTT
SNR = 13.3 dB , SSIM = 0.69

Complex DTT
SNR = 14.2 dB , SSIM = 0.73

Filter bank – polyphase implementation



$$\Longrightarrow \quad F = V\,\Pi_Q \quad \text{with} \quad V = \begin{bmatrix} V_{1,1} & \dots & V_{1,Q} \\ \vdots & & \vdots \\ V_{P,1} & \dots & V_{P,Q} \end{bmatrix}$$

- $Q \in \mathbb{N}^*$: decimation factor.
- $P \in \mathbb{N}^*$: number of channels.
- $P/Q \geq 1$: redundancy introduced by such a filter bank structure.
- Π_Q: **polyphase decomposition** from $\ell^2(\mathbb{Z})$ to $\left(\ell^2(\mathbb{Z})\right)^Q$, $\Pi_Q y = (y^{(j)})_{1\leq j\leq Q}$ where $y^{(j)} = \big(y(Qn-j+1)\big)_{n\in\mathbb{Z}}$ is the j-th polyphase component of order Q of the signal y.
- $V_{i,j}: \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$: **SISO** (Single-Input Single-Ouput) stable filter.

Filter bank – polyphase implementation

$$F = V\,\Pi_Q \quad \text{with} \quad \begin{bmatrix} V_{1,1} & \dots & V_{1,Q} \\ \vdots & & \vdots \\ V_{P,1} & \dots & V_{P,Q} \end{bmatrix} \qquad \Rightarrow \qquad F^* F = \Pi_Q^* V^* V \Pi_Q$$

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- Analysis formulation:

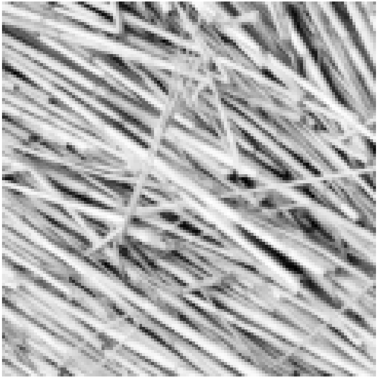
$$\Big(\sum_{i=1}^{n} \eta_i L_i^* L_i + \kappa F^* F\Big)^{-1} = \Pi_Q^*\Big(\sum_{i=1}^{n} \eta_i W_i^* W_i + \kappa V^* V\Big)^{-1}\Pi_Q$$

- Synthesis formulation:

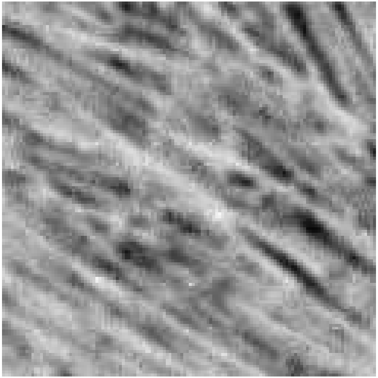
$$\mathrm{Id} - F\Big(\sum_{i=1}^{n} \eta_i L_i^* L_i\Big)\Big(\kappa \mathrm{I} + F^* F\Big(\sum_{i=1}^{n} \eta_i L_i^* L_i\Big)\Big)^{-1} F^*$$

$$= \mathrm{Id} - F\Pi_Q^*\Big(\sum_{i=1}^{n} \eta_i W_i^* W_i\Big)\Big(\kappa \mathrm{I} + V^* V\Big(\sum_{i=1}^{n} \eta_i W_i^* W_i\Big)\Big)^{-1}\Pi_Q F^*$$

Results: Image restored using AF with DTT and GenLOT

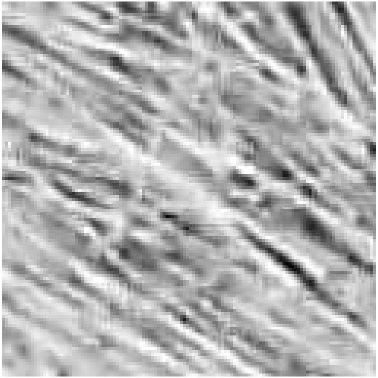


Original

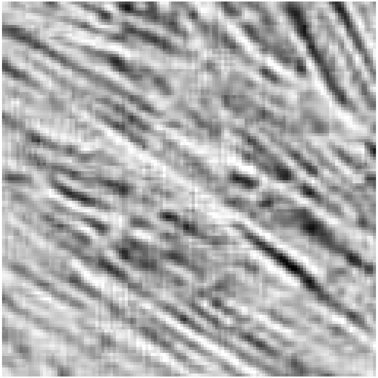


Degraded
SNR = 14.8 dB , SSIM = 0.42

Results: Image restored using AF with DTT and GenLOT



DTT
SNR = 16.7 dB , SSIM = 0.63



GenLOT
SNR = 17.1 dB , SSIM = 0.67

Generalization

Proposition

The operator

$$F = \Pi_R^* U V \Pi_Q$$

is a frame operator with frame constants

$$\underline{\mu} = \mu_U \inf_{\nu \in [-1/2, 1/2]} \sigma_{\min}(\nu) \qquad \text{and} \qquad \overline{\mu} = \mu_U \sup_{\nu \in [-1/2, 1/2]} \sigma_{\max}(\nu)$$

where, for every ν, $\sigma_{\min}(\nu)$ and $\sigma_{\max}(\nu)$ are the minimum and maximum eigenvalues of $\widehat{\mathbf{v}}(\nu)^{\mathrm{H}} \widehat{\mathbf{v}}(\nu)$. In addition, we have:

$$F^* F = \mu_U \Pi_Q^* V^* V \Pi_Q.$$

Conclusion

- Experimental results motivate the use of non tight frame representations.
- Allows us to deal with a large class of restoration problems with the considered class of non tight frame representations.
- Examples obtained with DTT and GenLOT but other forms of frame representations could be written as $F = \Pi_R^* U V \Pi_D$.