A Majorize-Minimize line search algorithm for barrier function optimization

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EUSIPCO, Glasgow, August 27th 2009

Barrier functions

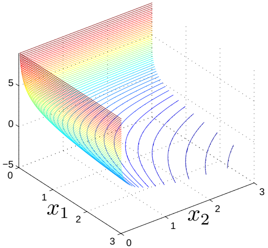
Definition

B is said *barrier function* associated to the constraint $\boldsymbol{x} \in \mathcal{C}$ if $B(\boldsymbol{x})$ is unbounded at the boundary of $\mathcal{C}$

E.g.: Logarithmic barrier function for positivity contraints

$$B(\boldsymbol{x}) = -\sum_n \log(x_n)$$

$$\mathcal{C} = \{\boldsymbol{x} > 0\}$$



$$B(x) = -\log(x_1) - \log(x_2)$$

$\Rightarrow$ If a criterion contains a barrier function, its minimizers belong to $\mathcal{C}$.

Criteria involving barrier functions

① *Interior point methods:*

$$\min_{c_i(\boldsymbol{x})>0} P(\boldsymbol{x}) \Leftrightarrow \min P(\boldsymbol{x}) - \mu \sum_i \log(c_i(\boldsymbol{x})), \{\mu\} \to 0$$

② *Emission tomography:*

$$\hat{\boldsymbol{x}} = \arg\min \sum_i [\boldsymbol{H}\boldsymbol{x}]_i - y_i \log[\boldsymbol{H}\boldsymbol{x}]_i + \lambda R(\boldsymbol{x}) \Rightarrow \boldsymbol{H}\hat{\boldsymbol{x}} > 0$$

③ *Maximum entropy:*

$$\hat{\boldsymbol{x}} = \arg\min \|\boldsymbol{H}\boldsymbol{x} - \boldsymbol{y}\|^2 + \lambda \sum_n x_n \log x_n \Rightarrow \hat{\boldsymbol{x}} > 0$$

General formulation

Objective function

$$\text{minimize } \left(F(\boldsymbol{x}) = P(\boldsymbol{x}) + \mu B(\boldsymbol{x})\right), \quad \mu > 0 \tag{1}$$

- $B(\boldsymbol{x}) = \sum_{i=1}^{I} b_i(\boldsymbol{a}_i^T \boldsymbol{x} + \rho_i)$: barrier function,
  e.g., $b_i(u) = -\log u$ or $u \log u$

Algorithmic scheme

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \alpha_k \boldsymbol{d}_k, \quad \text{for} \quad k = 1, \dots, K$$

- $\boldsymbol{d}_k$ *descent direction* i.e., $\boldsymbol{d}_k^T \nabla F(\boldsymbol{x}_k) < 0$: (P)CG, L-BFGS, ...
- $\alpha_k > 0$ *stepsize* given by a **1D nonquadratic MM algorithm**

Outline

Line search strategies

Goal: Given $\boldsymbol{x}_k$ and $\boldsymbol{d}_k$, find α_k that ensures the convergence of the algorithm

- Classical strategies: Dichotomy, backtracking, cubic interpolation, quadratic approximation...
- Iterative minimization of $F(\boldsymbol{x}_k + \alpha \boldsymbol{d}_k)$
- Identifying α_k that fulfills some convergence conditions

e.g.: Wolfe conditions

Problem statement

$$f(\alpha) = F(\boldsymbol{x}_k + \alpha \boldsymbol{d}_k) = P(\boldsymbol{x}_k + \alpha \boldsymbol{d}_k) + \mu \underbrace{B(\boldsymbol{x}_k + \alpha \boldsymbol{d}_k)}_{\text{barrier term } b(\alpha)}$$

b undefined for $\alpha > \bar{\alpha}$ if there exists i such that $c_i(\boldsymbol{x}_k + \bar{\alpha}\boldsymbol{d}_k) = 0$.

- Line search constrained to $\alpha \in [0, \bar{\alpha})$
- Vertical asymptote at $\bar{\alpha}$
- Classical methods not suited

The Majorize-Minimize (MM) principle *[Hunter04]*

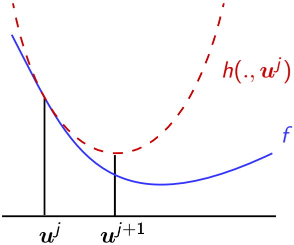
Goal: find $\boldsymbol{u}$ that minimizes f over E

For all $\boldsymbol{v} \in E$, let $h(., \boldsymbol{v})$ a *tangent majorant* for f at $\boldsymbol{v}$ i.e.,

$$h(\boldsymbol{u}, \boldsymbol{v}) \geqslant f(\boldsymbol{u}), \quad \forall \boldsymbol{u} \in E,$$
$$h(\boldsymbol{v}, \boldsymbol{v}) = f(\boldsymbol{v})$$

MM algorithm:

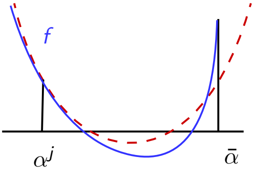
$$\boldsymbol{u}^{j+1} = \arg\min_{\boldsymbol{u} \in E} h(\boldsymbol{u}, \boldsymbol{u}^j)$$



Proposed 1D nonquadratic MM stepsize strategy

Curvature of the barrier term unbounded at $\bar{\alpha}$

$\Rightarrow$ No quadratic majorizing approximation for f.



Finding majorizing approximations of f of the form

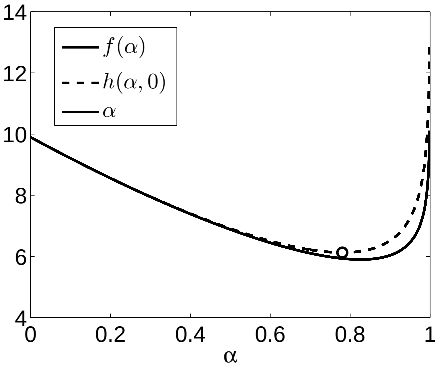
$$h(\alpha, \alpha^j) = p_0 + p_1\alpha + p_2\alpha^2 - p_3 \log(\bar{\alpha} - \alpha)$$

- Construction of h fairly easy for $-log$ and "entropic" barriers
- $\arg\min_\alpha h(\alpha, \alpha^j)$ is a root of degree 2 polynomial

Toy Example

$$f(\alpha) = (\alpha - 5)^2 - \sum_{i=1}^{10} \log(i - \alpha)$$

$\bar{\alpha} = 1$
$m^0 = 2$, $\gamma^0 = 1.55$
$\alpha^1 = 0.7805$
comparing with $\alpha^* = 0.8258$



Convergence analysis results *[Chouzenoux09]*

Goal: Discussing the convergence of the iterative descent algorithm

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \alpha_k \boldsymbol{d}_k, \quad k = 1, \ldots, K$$

Stepsize properties

After any number of MM subiterates,

- Armijo and Zoutendijk conditions hold
- The stepsize bounded away from 0

$\Rightarrow$ Convergence of several optimization algorithms:

Steepest descent, CG, truncated Newton, (L)BFGS ...

Monodimensional Nuclear Magnetic Resonance

NMR model

$$y(\tau) = \int_0^{\infty} x(T) \exp^{-\frac{\tau}{T}} \, dT$$

T: relaxation time
τ: echo time
$y(\tau)$: measured echo
$x(T)$: spectrum to estimate
After discretization,

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{\epsilon}, \qquad \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{y} \in \mathbb{R}^m$$

Goal: Estimate $\boldsymbol{x}$ from $\boldsymbol{y}$ subject to $\boldsymbol{x} \geqslant 0$

Maximum entropy method

$$\min_{\boldsymbol{x} \geqslant 0} F(\boldsymbol{x}) = P(\boldsymbol{x}) + \mu B(\boldsymbol{x})$$

- Fit to data: Least square $P(\boldsymbol{x}) = \|\boldsymbol{H}\boldsymbol{x} - \boldsymbol{y}\|^2$
- Regularization: Entropy measure $B(\boldsymbol{x}) = \sum_n x_n \ln x_n$

Optimization strategy

- NLCG algorithm with specific SVD preconditionner
- Comparison of a classical Wolfe line search *[Moré94]* with the proposed MM line search.

	c_1	c_2	K	$T(s)$
MT	10^{-3}	0.1	25	1.08
	10^{-3}	0.5	28	1.27
	10^{-3}	0.9	34	1.28
	10^{-3}	0.99	49	1.69
MM	J		K	$T(s)$
	1		24	0.86
	2		26	1.17
	5		28	1.73
	10		27	2.44

K: Iterates number
T: Time before convergence
(c_1, c_2): Wolfe parameters
J: Number of MM subiterates

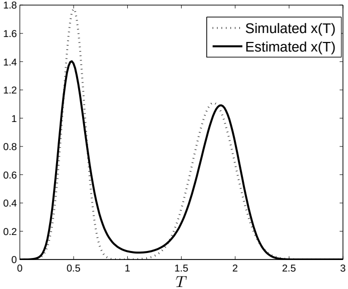


Figure: NMR reconstruction of synthetic data with $SNR = 40dB$

Interior point for quadratic programming

Convex quadratic programming problem

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} F_0(\boldsymbol{x}) = \rho_0 + a_0^T\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}_0\boldsymbol{x} \tag{2}$$

$$\text{s.t. : } c_i(\boldsymbol{x}) = -\frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}_i\boldsymbol{x} + a_i^T\boldsymbol{x} + \rho_i \geqslant 0, \quad 1 \leqslant i \leqslant m$$

Augmented criterion

$$F_\mu(\boldsymbol{x}) = F_0(\boldsymbol{x}) - \mu\sum_{i=1}^{m}\log(c_i(\boldsymbol{x}))$$

Interior point: Solve $\arg\min F_\mu$ for a series $\{\mu\} \rightarrow 0$.

Interior Point algorithm *[Boyd04]*

1) Set $\mu = 1$, tolerances ϵ, ξ and select a feasible point $\boldsymbol{x}$
2) **WHILE** $\frac{1}{2}(\boldsymbol{d}^T \nabla F_\mu)^2 > \xi$,

    Compute Newton direction $\boldsymbol{d}$ of F_μ
    Compute step size α
    Update $\boldsymbol{x} \leftarrow \boldsymbol{x} + \alpha \boldsymbol{d}$

3) **IF** $\mu < \epsilon$, **RETURN**
   ELSE Set $\mu \leftarrow \theta\mu$ and **GO TO** step 2.

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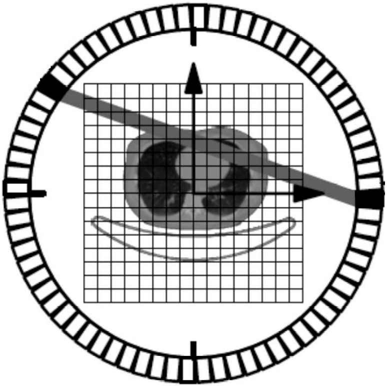
Results

50 random problems of size $n = 400$, $m = 200$

Backtracking	273.24 iter	5636.87 s
Damping *[Nesterov94]*	135.30 iter	465.07 s
Proposed MM linesearch	64.40 iter	225.03 s

Emission tomography reconstruction



See article

Conclusion and future work

Main result

MM linesearch strategy for criteria involving barrier functions

- Simple stepsize scheme
- Strong convergence results
- Efficient in practice

Constrained optimization

Possible adaptation to

- Primal interior point *[Johnson2000]*
- L-BFGS-B *[Byrd1995]*
- Multiplicative algorithm *[Lanteri2001]*
- Kullback proximal algorithm *[Teboulle1997,Chrétien2000]*

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