EFFICIENCY OF LINE SEARCH STRATEGIES IN INTERIOR POINT METHODS FOR LINEARLY CONSTRAINED SIGNAL RESTORATION

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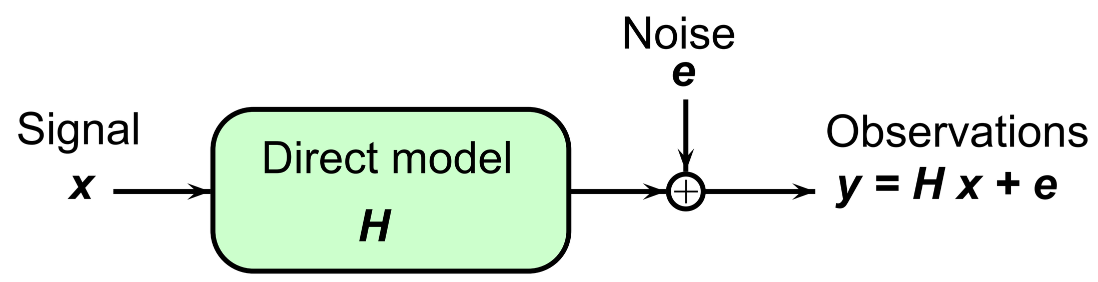
LINEARLY CONSTRAINED SIGNAL RESTORATION



AIM Given noisy measurements $\boldsymbol{y}$, reconstruction of $\boldsymbol{x}$ that fulfills some linear constraints $C(\boldsymbol{x}) \geqslant \boldsymbol{0}$

$$\min_{\boldsymbol{x}\in\mathbb{R}^N} F(\boldsymbol{x}) \quad \text{s.t} \quad C(\boldsymbol{x}) = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{\rho} \geqslant \boldsymbol{0} \quad \text{with} \quad \boldsymbol{C} \in \mathbb{R}^{M\times N} \quad \text{and} \quad \boldsymbol{\rho} \in \mathbb{R}^M \qquad (1)$$

INTERIOR POINT FRAMEWORK

Replace problem (1) by a sequence of subproblems $\mathcal{P}_\mu$ with barrier parameter values $\mu \searrow 0$.

Primal method: $\mathcal{P}_\mu^{(1)}$

⇝ Find $\boldsymbol{x}$ that minimizes

$$F_\mu(\boldsymbol{x}) = F(\boldsymbol{x}) + \mu B(\boldsymbol{x})$$

with the barrier function

$$B(\boldsymbol{x}) = -\sum_{i=1}^{M} \log\left(C_i(\boldsymbol{x})\right).$$

Primal-dual method: $\mathcal{P}_\mu^{(2)}$

⇝ Find $(\boldsymbol{x}, \boldsymbol{\lambda})$ such that

$$\begin{cases} \nabla F(\boldsymbol{x}) - \boldsymbol{C}^T\boldsymbol{\lambda} = \boldsymbol{0} \\ \lambda_i C_i(\boldsymbol{x}) = \mu, \quad i = 1, \dots, M \end{cases}$$

under the constraint $(C(\boldsymbol{x}), \boldsymbol{\lambda}) \geqslant \boldsymbol{0}$.

LINE SEARCH FOR BARRIER FUNCTION MINIMIZATION

PRACTICAL IMPLEMENTATION

Problem $\mathcal{P}_\mu^{(1)}$ (or $\mathcal{P}_\mu^{(2)}$) is solved iteratively using Newton's algorithm

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \alpha_k \boldsymbol{d}_k$$

where α_k ensures a sufficient decrease of a merit function

$$f_\mu(\alpha) = F_\mu(\boldsymbol{x}_k + \alpha \boldsymbol{d}_k).$$

For example, α_k fulfills the strong Wolfe conditions:

$$\begin{cases} f_\mu(\alpha_k) \leqslant f_\mu(0) + c_1\alpha_k \dot{f}_\mu(0) \\ |\dot{f}_\mu(\alpha_k)| \leqslant c_2 |\dot{f}_\mu(0)| \end{cases}, \quad \text{with} \quad (c_1, c_2) \in (0,1).$$

DIFFICULTY F_μ is a **barrier function** for the constrained domain.

➪ $f_\mu(\alpha)$ **unbounded** for $\alpha \notin (\alpha_-, \alpha_+)$ where

$$\alpha_- = \max_{i|[\boldsymbol{C}\boldsymbol{d}_k]_i > 0} -\frac{[\boldsymbol{C}\boldsymbol{x}_k + \boldsymbol{\rho}]_i}{[\boldsymbol{C}\boldsymbol{d}_k]_i}, \qquad \alpha_+ = \min_{i|[\boldsymbol{C}\boldsymbol{d}_k]_i < 0} -\frac{[\boldsymbol{C}\boldsymbol{x}_k + \boldsymbol{\rho}]_i}{[\boldsymbol{C}\boldsymbol{d}_k]_i}.$$

➪ **Special-purpose** line search procedures have to be designed.

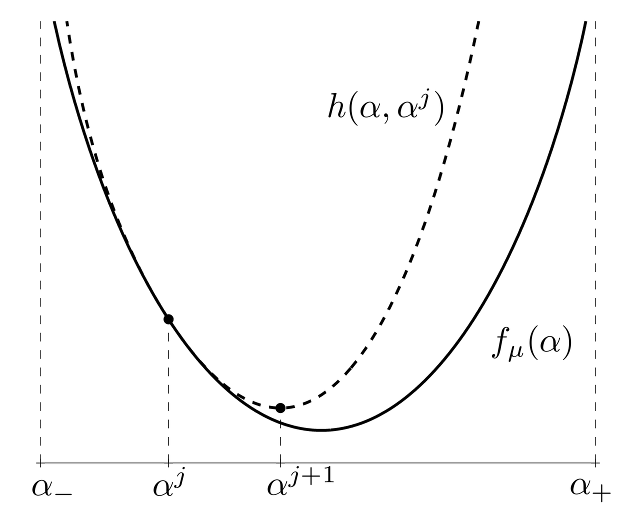
DAMPED BACKTRACKING LINE SEARCH [1, 2]

- Initialization with $\alpha^0 = \theta\alpha_+$, θ close to one
- $\alpha^{j+1} = \tau\alpha^j$, $\tau \in (0,1)$, until α^j satisfies the first Wolfe condition.

INTERPOLATION-BASED LINE SEARCH [3]

- Initialization with $\alpha^0 = \theta\alpha_+$, θ close to one
- Trial steps $\{\alpha^j\}$ defined from an interpolation procedure, until α^j satisfies the two Wolfe conditions.
- Specific interpolating functions $f_0 + f_1\alpha + f_2\alpha^2 - \mu\log(f_3 - \alpha)$.

PROPOSED MAJORIZE-MINIMIZE LINE SEARCH [4]



$h(., \alpha^j)$ **tangent majorant** for f_μ at α^j i.e.,

$$\begin{cases} h(\alpha, \alpha^j) \geqslant f_\mu(\alpha) \\ h(\alpha^j, \alpha^j) = f_\mu(\alpha^j) \end{cases}$$

Majorize-Minimize recurrence

$\alpha^{j+1} = \arg\min_\alpha h(\alpha, \alpha^j)$, $j \leqslant J$

with $h(\alpha, \alpha^j) =$

$p_0 + p_1\alpha + p_2\alpha^2 - p_3\log(\alpha - \alpha_\pm)$.

APPLICATION TO SPARSE SIGNAL RECONSTRUCTION

AIM Recovering a sparse spike train sequence $\boldsymbol{x}^o \in \mathbb{R}^N$ from $\boldsymbol{y} = \boldsymbol{h} \star \boldsymbol{x}^o + \boldsymbol{e} \in \mathbb{R}^P$, where $\boldsymbol{h}$ is a filter of length L and $\boldsymbol{e}$ is a white centered Gaussian additive noise.

BASIS PURSUIT RECONSTRUCTION

$$\min_{\boldsymbol{x}\in\mathbb{R}^N} \|\boldsymbol{y} - \boldsymbol{h} \star \boldsymbol{x}\|_2^2 + \eta\,\|\boldsymbol{x}\|_1,$$

reformulated as a quadratic programming problem

$$\min_{\boldsymbol{x},\boldsymbol{u}} F(\boldsymbol{x}, \boldsymbol{u}) = \|\boldsymbol{y} - \boldsymbol{h} \star \boldsymbol{x}\|_2^2 + \eta \sum_{i=1}^{N} u_i$$

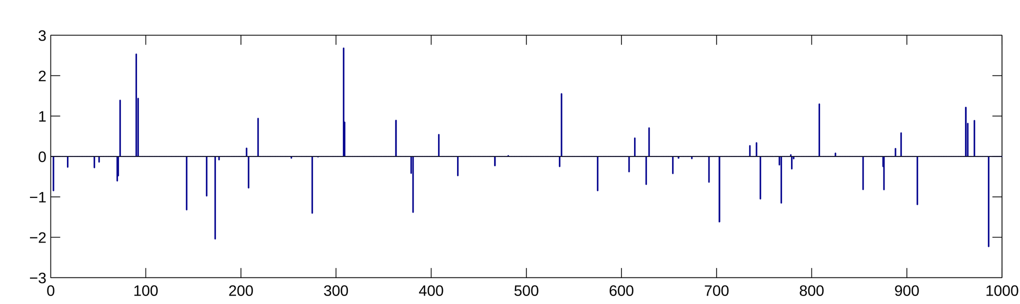
$$\text{subject to} \quad -u_i \leqslant x_i \leqslant u_i, \quad i = 1, \dots, N.$$

▶ Table 1 summarizes the computational results for:

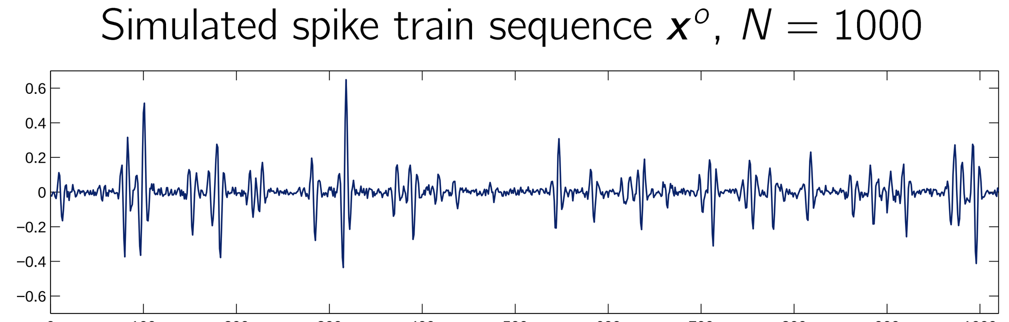
DB Damped backtracking line search for $\theta = 0.99$
INTERP Interpolation-based line search
MM Majorize-Minimize line search

			Primal [1]		Primal-Dual [2]	
DB	c_1		K	T	K	T
	0.5		502	1104	**9**	**5.5**
	0.2		169	381	**9**	**5.5**
	0.1		151	365	**9**	**5.5**
	0.01		144	358	**9**	**5.5**
INTERP	c_1	c_2	K	T	K	T
	10^{-1}	0.5	66	174	9	5.7
	10^{-1}	0.9	78	180	9	5.8
	10^{-1}	0.99	86	216	9	6
	10^{-2}	0.5	**67**	**175**	**9**	**5.5**
	10^{-2}	0.9	81	181	9	5.8
	10^{-2}	0.99	91	202	9	6
MM	J		K	T	K	T
	1		73	185	10	6.6
	2		**62**	**168**	**9**	**5.3**
	5		**62**	**173**	**9**	**5.5**
	10		**60**	**175**	9	5.9

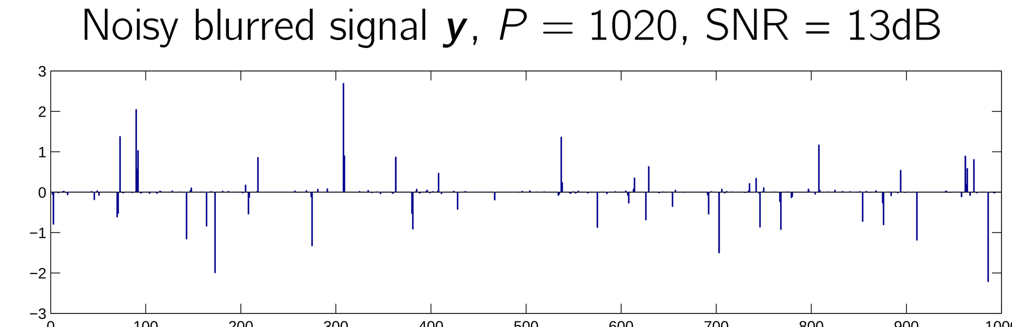
Table 1: *K denotes the sum of inner iterations and T the time before convergence (in s.), with tolerance parameter $\mu_{\min} = 10^{-8}$*



Simulated spike train sequence $\boldsymbol{x}^o$, $N = 1000$



Noisy blurred signal $\boldsymbol{y}$, $P = 1020$, SNR = 13dB



Reconstruction $\hat{\boldsymbol{x}}$ with similarity error 4%

CONCLUSION

▶ When dealing with the primal algorithm, the best stepsize strategy corresponds to a very rough minimization of the merit barrier function.
▶ The primal algorithm performs better, in term of convergence speed, when the stepsize is obtained with the proposed MM search.
▶ In contrast, the choice of the line search procedure has very little impact on the performances of the primal-dual algorithm.

In prospect:

➢ Non linearly constrained problems
➢ Line search for primal-dual algorithms
➢ First order algorithms for barrier minimization

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