AN EM APPROACH FOR POISSON-GAUSSIAN NOISE MODELING

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CONTENTS

The importance of the Poisson-Gaussian noise

Where does it appear?

- CCD camera images [Healey *et al.* 1994]
- Medical images [Nichols *et al.* 2002]
- Biological images (fluorescence microscopy) [Pawley 1994]
- Astronomical images [Benvenuto *et al.* 2008]

What are the signal processing challenges?

- noise parameters identification
- signal recovery

When is it especially difficult?

- low level signal (The Anscombe transform introduces a significant bias)
- non-zero background (The model parameters cannot be estimated from first and second order statistics)

General model

Observations

$\forall s \in \{1, \ldots, S\}$
$\forall t \in \{1, \ldots, T\}$

$$R_{s,t} = \alpha Q_{s,t} + N_{s,t}$$

Assumptions:

1. $(Q_{s,t})_{\substack{1 \le s \le S \\ 1 \le t \le T}}$, $(N_{s,t})_{\substack{1 \le s \le S \\ 1 \le t \le T}}$ - independent sequences
2. $Q_{s,t}$ and $N_{s,t}$ mutually independent

GENERAL MODEL

Poisson noise

Observations

$Q_{s,t} \sim \mathcal{P}(u_s)$

$(u_s)_{1 \leq s \leq S} \geq 0$ - "clean" image

$\alpha \in]0, +\infty[$ - scaling param.

$\forall s \in \{1, \ldots, S\}$

$\forall t \in \{1, \ldots, T\}$

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General model

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$\alpha \in \left]0, +\infty\right[$ - scaling param.

$$R_{s,t} = \alpha\, Q_{s,t} + N_{s,t}$$

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Gaussian noise

$N_{s,t} \sim \mathcal{N}(c, \sigma^2)$
$c \in \mathbb{R}$ - mean
σ^2 - variance
$\sigma^2 \geq 0$

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Problem - to identify:

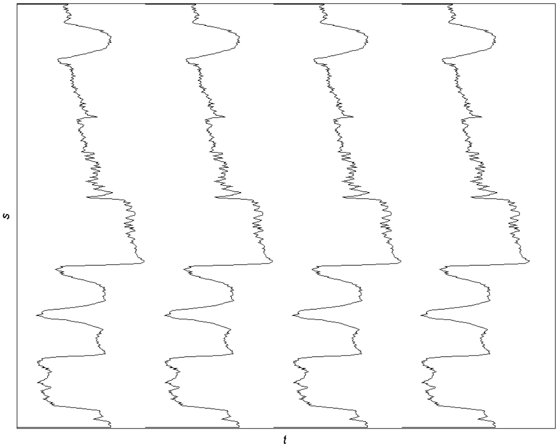
- α
- c
- σ^2

u_s - unknown

Gaussian noise

$N_{s,t} \sim \mathcal{N}(c, \sigma^2)$
$c \in \mathbb{R}$ - mean
σ^2 - variance
$\sigma^2 \ge 0$

Example

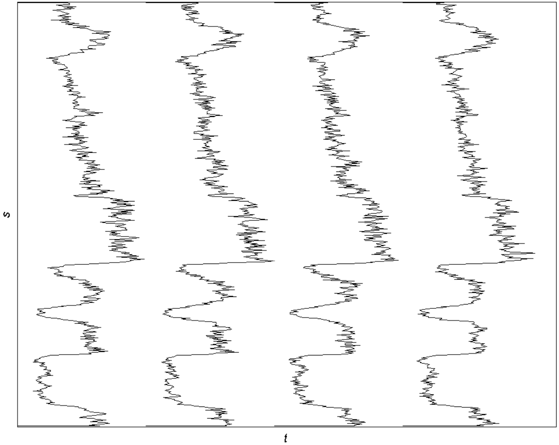


$U_{s,t}$

$T = 4$,
$S = 512$

Original signal

EXAMPLE

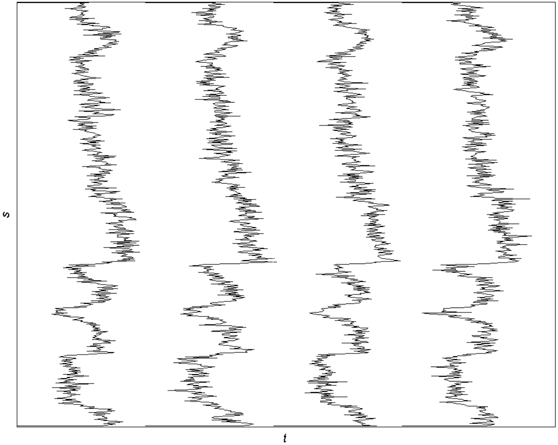


$Q_{s,t}$

$T = 4$,
$S = 512$

Signal corrupted by Poisson noise

EXAMPLE

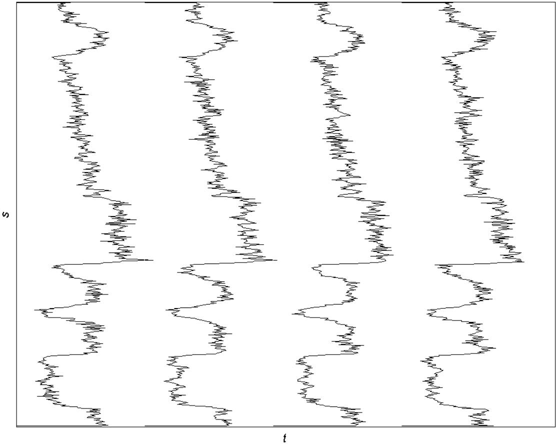


$U_{s,t} + N_{s,t}$

$T = 4$,
$S = 512$

Signal corrupted by Gaussian noise

Example



$\alpha Q_{s,t} + N_{s,t}$

$T = 4,$
$S = 512$

Signal corrupted by Poisson and Gaussian noises

MOMENT BASED APPROACH

$$\kappa_n[R_{s,t}] = \alpha^n \kappa_n[Q_{s,t}] + \kappa_n[N_{s,t}]$$

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Cumulant of order n

Moment based approach

$$\kappa_n[R_{s,t}] \;=\; \alpha^n \kappa_n[Q_{s,t}] \;+\; \kappa_n[N_{s,t}]$$

Then:

- mean value

$$\kappa_1[R_{s,t}] = \mathsf{E}[R_{s,t}] = \alpha u_s + c$$

- variance

$$\kappa_2[R_{s,t}] = \mathsf{Var}[R_{s,t}] = \alpha^2 u_s + \sigma^2$$

- higher-order cumulants

$$\kappa_n[R_{s,t}] = \alpha^n u_s, \qquad n \geq 3$$

PROPOSED CUMULANT BASED METHOD

From:

$$\begin{cases} \mathsf{E}[R_{s,t}] = \alpha u_s + c \\ \mathsf{Var}[R_{s,t}] = \alpha^2 u_s + \sigma^2 \end{cases}$$

After some algebraic manipulations, we obtain:

$$\widehat{\alpha} = \frac{S \sum_{s=1}^{S} \widehat{\mathsf{E}}[r_{s,t}] \widehat{\mathsf{Var}}[r_{s,t}] - \sum_{s=1}^{S} \widehat{\mathsf{E}}[r_{s,t}] \sum_{s=1}^{S} \widehat{\mathsf{Var}}[r_{s,t}]}{S \sum_{s=1}^{S} (\widehat{\mathsf{E}}[r_{s,t}])^2 - \left(\sum_{s=1}^{S} \widehat{\mathsf{E}}[r_{s,t}] \right)^2}$$

- The estimation of $\widehat{\alpha}$ is precise as only cumulants of order 1 and 2 are used

PROPOSED CUMULANT BASED METHOD

Estimate of σ^2:

$$\widehat{\sigma}^2 = \sum_{s \in \mathbb{I}} \widehat{\mathsf{Var}}[r_{s,t}] - \widehat{\alpha}^{-1} \widehat{\kappa}_3[r_{s,t}]$$

Estimation error increases:

- when T is small
- when u_s is large

PROPOSED CUMULANT BASED METHOD

Estimate of σ^2:

$$\widehat{\sigma}^2 = \frac{\sum_{s \in \mathbb{I}} \widehat{\mathsf{Var}}[r_{s,t}]^{-6} \left(\widehat{\mathsf{Var}}[r_{s,t}] - \widehat{\alpha}^{-1} \widehat{\kappa}_3 [r_{s,t}] \right)}{\sum_{s \in \mathbb{I}} \widehat{\mathsf{Var}}[r_{s,t}]^{-6}}$$

Weighted estimator

PROPOSED CUMULANT BASED METHOD

Estimate of σ^2:

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Weighted estimator

Set $\mathbb{I}$ may be defined as:

$$\mathbb{I} = \left\{s \in \{1,\dots,S\} \mid \widehat{\mathsf{Var}}[r_{s,t}] - \widehat{\alpha}^{-1}\widehat{\kappa}_3[r_{s,t}] \geq 0\right\}$$

- Estimate $\widehat{\sigma}^2$ is precise, because weighting the observations effectively discards the unreliable ones.

Proposed cumulant based method

Estimate of c:

$$\widehat{c} = \frac{1}{S}\sum_{s=1}^{S}\left(\widehat{\mathsf{E}}[r_{s,t}] - \widehat{\alpha}^{-1}\widehat{\mathsf{Var}}[r_{s,t}]\right) + \frac{\widehat{\sigma}^2}{\widehat{\alpha}}$$

Summary:

- All parameters estimated using cumulants of order at most 3
- α estimator is a function of 1^{st} and 2^{st} order moments
- Precise estimator of σ^2
- c derived from α and σ^2

RESULTS

S	T	$\hat{\sigma}^2$ bias	$\hat{\sigma}^2$ std	$\hat{c}$ bias	$\hat{c}$ std	$\hat{\alpha}$ bias	$\hat{\alpha}$ std	$\overline{\text{err}}$
1024	50	-14.83	20.29	-1.74	5.07	-0.21	0.14	0.22
	100	-8.58	13.86	-0.54	4.04	-0.09	0.11	0.16
	200	-11.69	10.13	1.18	2.7	-0.04	0.04	0.14
	500	-1.93	6.24	-0.26	1.61	-0.02	0.04	0.06
	1000	-1.16	4.55	-0.24	1.1	-0.01	0.03	0.04

σ^2	$\hat{\sigma}^2$ bias	$\hat{\sigma}^2$ std	$\hat{c}$ bias	$\hat{c}$ std	$\hat{\alpha}$ bias	$\hat{\alpha}$ std	$\overline{\text{err}}$
0.25	-0.14	0.03	-0.11	5.15	-0.19	0.13	0.18
25	-3.59	3.89	- 0.18	4.19	-0.19	1.14	0.19
400	-11.45	84.32	-0.84	10.53	-0.22	0.15	0.34

c	$\hat{\sigma}^2$ bias	$\hat{\sigma}^2$ std	$\hat{c}$ bias	$\hat{c}$ std	$\hat{\alpha}$ bias	$\hat{\alpha}$ std	$\overline{\text{err}}$
5	-11.70	19.43	-1.20	5.07	-0.20	0.13	0.33
15	-10.16	26.15	-0.72	5.23	-0.18	0.13	0.17
100	-11.47	24.10	0.21	5.19	-0.17	0.13	0.09

α	$\hat{\sigma}^2$ bias	$\hat{\sigma}^2$ std	$\hat{c}$ bias	$\hat{c}$ std	$\hat{\alpha}$ bias	$\hat{\alpha}$ std	$\overline{\text{err}}$
1	12.99	6.17	13.73	6.29	-0.02	0.02	0.64
5	-2.93	5.69	-0.25	2.83	-0.09	0.07	0.15
50	1.07	56.4	-3.55	25.7	-1.10	0.67	0.81

The reliability of cumulant based method:

- increases with T
- decreases as σ^2 and α increase
- decreases for small value of α and c

Results

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MAXIMUM LIKELIHOOD ESTIMATOR

$$(\widehat{u}, \widehat{\alpha}, \widehat{c}, \widehat{\sigma}) = \underset{(u,\alpha,c,\sigma)}{\operatorname{argmax}} \; f_R(r \mid u, \alpha, c, \sigma)$$

Joint density function for all observations r

MAXIMUM LIKELIHOOD ESTIMATOR

$$(\widehat{u}, \widehat{\alpha}, \widehat{c}, \widehat{\sigma}) = \operatorname*{argmax}_{(u,\alpha,c,\sigma)} f_R(r \mid u, \alpha, c, \sigma)$$

Joint density function for all observations r

$$f_R(r \mid u, \alpha, c, \sigma) = \sum_{q \in \mathbb{N}^{ST}} p_{R,Q}(r, q \mid u, \alpha, c, \sigma)$$

Mixed continuous-discrete probability distribution of (R, Q)

Maximum Likelihood Estimator

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$$p_{R,Q}(r, q \mid u, \alpha, c, \sigma) = f_{R|Q=q}(r \mid u, \alpha, c, \sigma)\mathsf{P}(Q = q \mid u)$$

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$$p_{R,Q}(r, q \mid u, \alpha, c, \sigma) = f_N(r - \alpha q \mid c, \sigma) \mathsf{P}(Q = q \mid u)$$

Joint density function for all observations r

Mixed continuous-discrete probability distribution of (R, Q)

Maximum Likelihood Estimator

Joint density function $f_R(r \mid u, \alpha, c, \sigma)$ for Poisson + Gaussian:

$$\frac{1}{(2\pi)^{ST/2}\sigma^{ST}} \prod_{s=1}^{S} \exp(-Tu_s) \prod_{t=1}^{T} \sum_{q_{s,t}=1}^{+\infty} \exp\left(-\frac{(r_{s,t} - \alpha q_{s,t} - c)^2}{2\sigma^2}\right) \frac{u_s^{q_{s,t}}}{q_{s,t}!}$$

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The likelihood takes an intricate form which renders the computation of the ML estimator too difficult.

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→ Expectation-Maximization algorithm

EXPECTATION-MAXIMIZATION ALGORITHM

Expectation step

$$J(\theta \mid \theta^{(n)}) = \mathsf{E}_{Q|R=r,\theta^{(n)}}[\ln p_{R,Q}(R, Q \mid \theta)]$$

Maximization step

$$(\forall n \in \mathbb{N}) \qquad \theta^{(n+1)} = \underset{\theta}{\operatorname{argmin}} - J(\theta \mid \theta^{(n)})$$

Importance of having a good enough initialization

Proposed cumulant-based method

Expectation-maximization algorithm

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Vector of parameters:

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Observed data

Vector of parameters:

- $\theta = (u, \alpha, c, \sigma^2)$

Maximization step

$$(\forall n \in \mathbb{N}) \qquad \theta^{(n+1)} = \underset{\theta}{\operatorname{argmin}} - J(\theta \mid \theta^{(n)})$$

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Observed data

Vector of parameters:

- $\theta = (u, \alpha, c, \sigma^2)$

Missing data

Maximization step

$$(\forall n \in \mathbb{N}) \qquad \theta^{(n+1)} = \underset{\theta}{\operatorname{argmin}} - J(\theta \mid \theta^{(n)})$$

Importance of having a good enough initialization

Proposed cumulant-based method

EM MAXIMIZATION STEP

$$(\forall s \in \{1, \ldots, S\}) \quad u_s^{(n+1)} = \frac{1}{T} \sum_{t=1}^{T} \mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}]$$

$$\begin{bmatrix} ST & \mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}] \\ \mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}] & \sum_{s,t} \mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}^2] \end{bmatrix} \begin{bmatrix} c^{(n+1)} \\ \alpha^{(n+1)} \end{bmatrix} = \begin{bmatrix} \sum_{s,t} r_{s,t} \\ \sum_{s,t} r_{s,t} \mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}] \end{bmatrix}$$

$$(\sigma^2)^{(n+1)} = \frac{1}{ST} \sum_{s=1}^{S} \sum_{t=1}^{T} r_{s,t} \left(r_{s,t} - \alpha^{(n+1)} \mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}] - c^{(n+1)} \right)$$

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Updated in EM Expectation step

$$(\sigma^2)^{(n+1)} = \frac{1}{ST} \sum_{s=1}^{S} \sum_{t=1}^{T} r_{s,t} \left(r_{s,t} - \alpha^{(n+1)} \mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}] - c^{(n+1)} \right)$$

EM Expectation step

$$\mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}] = \sum_{q_{s,t}=1}^{+\infty} q_{s,t} \mathsf{P}(Q_{s,t} = q_{s,t} \mid R = r, \theta^{(n)})$$

EM EXPECTATION STEP

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After some calculations, we have:

EM EXPECTATION STEP

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$$\mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}] = \frac{\zeta_{s,t}^{(n)}}{\eta_{s,t}^{(n)}}$$

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$$\zeta_{s,t}^{(n)} = \sum_{q_{s,t}=1}^{+\infty} \exp\left(-\frac{(r_{s,t} - \alpha^{(n)}q_{s,t} - c^{(n)})^2}{2(\sigma^2)^{(n)}}\right) \frac{(u_s^{(n)})^{q_{s,t}}}{(q_{s,t}-1)!}$$

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$$\eta_{s,t}^{(n)} = \sum_{q_{s,t}=0}^{+\infty} \exp\left(-\frac{(r_{s,t} - \alpha^{(n)} q_{s,t} - c^{(n)})^2}{2(\sigma^2)^{(n)}}\right) \frac{(u_s^{(n)})^{q_{s,t}}}{q_{s,t}!}$$

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$$\mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}] = \frac{\zeta_{s,t}^{(n)}}{\eta_{s,t}^{(n)}}$$

$$\zeta_{s,t}^{(n)} = \sum_{q_{s,t}=1}^{+\infty} \exp\left(-\frac{(r_{s,t} - \alpha^{(n)} q_{s,t} - c^{(n)})^2}{2(\sigma^2)^{(n)}}\right) \frac{(u_s^{(n)})^{q_{s,t}}}{(q_{s,t}-1)!}$$

$$\eta_{s,t}^{(n)} = \sum_{q_{s,t}=0}^{+\infty} \exp\left(-\frac{(r_{s,t} - \alpha^{(n)} q_{s,t} - c^{(n)})^2}{2(\sigma^2)^{(n)}}\right) \frac{(u_s^{(n)})^{q_{s,t}}}{q_{s,t}!}$$

Problem: Infinite sums

EM Expectation step

Similarly:

$$\mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}^2] = \sum_{q_{s,t}=1}^{+\infty} (q_{s,t})^2 \mathsf{P}(Q_{s,t} = q_{s,t} \mid R = r, \theta^{(n)})$$

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Problem: Infinite sums

Solution: Adaptive truncation technique

ALGORITHM

Initialization:

$\theta^{(1)} \leftarrow \left(u^{(1)}, \alpha^{(1)}, c^{(1)}, (\sigma^2)^{(1)}\right)$ (Cumulant Method)

EM Algorithm:

for $n \leftarrow 1$ to N **do**

Expectation step

$$\mathsf{E}_{Q|R=r,\theta^{(n)}}[Q_{s,t}] \leftarrow \frac{\zeta_{s,t}^{(n)}}{\eta_{s,t}^{(n)}}$$

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Maximization step

$$\theta^{(n+1)} \leftarrow \left(u^{(n+1)}, \alpha^{(n+1)}, c^{(n+1)}, (\sigma^2)^{(n+1)}\right)$$

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end for

NUMERICAL RESULTS

Param.					Proposed EM algorithm						Error	
S	T	σ^2	c	α	$\hat{\sigma}^2$		$\hat{c}$		$\hat{\alpha}$		$\overline{\text{err}}_{\text{cum}}$	$\overline{\text{err}}_{\text{EM}}$
					bias	std	bias	std	bias	std		
1024	50	100	10	10	-1.86	14.02	-0.35	1.79	-0.21	0.05	0.22	0.11
1024	100	100	10	10	0.54	10.92	0.27	1.38	-0.08	0.02	0.16	0.08
1024	200	100	10	10	-0.91	7.89	0.09	0.88	-0.04	0.03	0.14	0.05
1024	50	100	5	10	1.30	13.16	0.08	1.61	-0.20	0.05	0.33	0.15
1024	50	25	10	1	-0.28	0.50	0.00	0.48	-0.02	0.01	0.64	0.03
1024	50	400	10	10	7.51	45.24	0.71	5.13	-0.22	0.07	0.34	0.21

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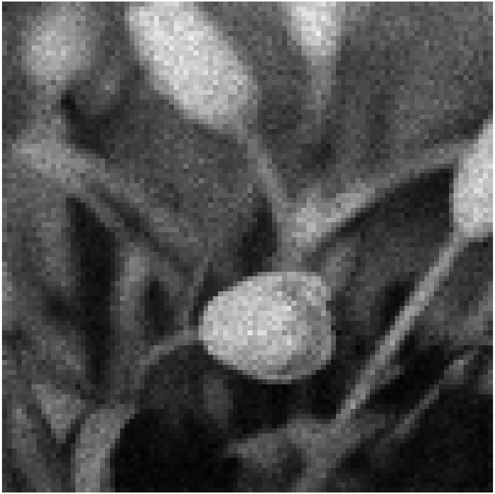
- Our EM algorithm performs well even if T is small
- Our EM does not appear to be sensitive to small values of α and c
- Our EM results improve when σ^2 is high.

VISUAL RESULTS

Original image. The range of values $(0, 255)$. The number of realizations $T = 40$

VISUAL RESULTS



Original image. The range of values $(0, 255)$. The number of realizations $T = 40$

Corrupted by noise with parameters $\alpha = 50$, $\sigma^2 = 416$ and $c = 10$.

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Reconstructed u_s values; SNR = 35.6 dB

Summary

- We have proposed a new EM-based approach dealing with Poisson plus Gaussian noise in case of non-zero background parameter estimation problems
- We have proposed a very accurate cumulant based initialization for this method
- The method is especially useful when a low level signal is expected
- These algorithms can estimate noise parameters for denoising or restoration procedures, in which they are often assumed to be known [Benvenuto 2008,Luisier 2011]
- We plan also to compare our method with a Bayesian approach