Exact and high order discretization schemes for Wishart processes and an SDE on Correlation matrices

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What is a Wishart process?

Wishart processes have initially been introduced and studied by Bru in her PhD thesis on Escherichia Coli (1987), and have recently been extended by Cuchiero, Filipovic, Mayerhofer and Teichmann (2009). A Wishart process \((X_t)_{t \geq 0}\) of dimension \(d\) is defined on nonnegative symmetric matrices \(S^+_d(\mathbb{R})\) and solves the following SDE:

\[
\begin{align*}
    dX_t &= (\alpha a^T a + bX_t + X_t b^T)dt + \sqrt{X_t}dW_ta + a^T dW_t^T \sqrt{X_t}, \ t \geq 0, \\
    X_0 &= x \in S^+_d(\mathbb{R}).
\end{align*}
\]

Here, \(\alpha \in \mathbb{R}, a, b \in \mathcal{M}_d(\mathbb{R})\) and \(\sqrt{X_t}\) is the square root of the nonnegative matrix \(X_t\): if \(X_t = O_t \text{diag}(\Lambda_1^t, \ldots, \Lambda_d^t)O_t^{-1}\),

\[
\sqrt{X_t} := O_t \text{diag}(\sqrt{\Lambda_1^t}, \ldots, \sqrt{\Lambda_d^t})O_t^{-1}.
\]

\(W_t\) denotes a \(d \times d\) matrix whose components are independent standard Brownian motions.

**\(d = 1\) CIR diffusion:** \(dX_t = (\alpha a^2 + 2bX_t)dt + 2a\sqrt{X_t}dW_t, t \geq 0\).
When is it well-defined?

We have the following results (Bru, Cuchiero et al., Mayerhofer et al.):

- When $\alpha \geq d + 1$, the SDE has a unique strong solution on the positive symmetric matrices $S^{+,*}_d(\mathbb{R})$.
- When $d - 1 \leq \alpha < d + 1$, the SDE has a unique weak solution on $S^+_d(\mathbb{R})$.

When $d = 1$, $\alpha \geq 2$ ensures that the CIR never reaches 0. However, we know in that case that there is a strong solution for any $\alpha \geq 0$. 
An explicit characteristic function

Let $X^x_t \sim WIS_d(x, \alpha, b, a; t)$ follow a Wishart distribution. Its Fourier transform is known explicitly:

$$\forall \nu \in S_d(\mathbb{R}), \mathbb{E}[\exp(i\text{Tr}(\nu X^x_t))] = \frac{\exp(\text{Tr}[i\nu(I_d - 2iq_tv)^{-1}m_t x m^T_t])}{\det(I_d - 2iq_tv)^{\alpha/2}}, \quad (2)$$

where $q_t = \int_0^t \exp(sb)a^T a \exp(sb^T)ds$, $m_t = \exp(tb)$.

In particular, if $\tilde{X}^x_t \sim WIS_d(x, \alpha, 0, I^n_d; t)$, where $(I^n_d)_{i,j} = 1_{i=j\leq n}$,

$$\forall \nu \in S_d(\mathbb{R}), \mathbb{E}[\exp(i\text{Tr}(\nu \tilde{X}^x_t))] = \frac{\exp(\text{Tr}[i\nu(I_d - 2itI^n_d v)^{-1}x])}{\det(I_d - 2itI^n_d v)^{\alpha/2}}.$$
Affine diffusions on nonnegative symmetric matrices

Cuchiero et al. consider the following dynamics for General Affine processes (we exclude here jumps):

\[ dX_t = (\bar{\alpha} + B(X_t))dt + \sqrt{X_t}dW_t a + a^T dW_t^T \sqrt{X_t}, \quad X_0 = x \in S_d(\mathbb{R}). \quad (3) \]

\( \bar{\alpha} \in S_d(\mathbb{R}), \ a \in M_d(\mathbb{R}) \) and \( B : S_d(\mathbb{R}) \to S_d(\mathbb{R}) \) is a linear mapping such that \( \text{Tr}(B(x)z) \geq 0 \) if \( \text{Tr}(xz) = 0 \) for \( x, z \in S_d^+(\mathbb{R}) \).

The Wishart SDE (1) is the particular case where

\[ \bar{\alpha} = \alpha a^T a, \ B(x) = bx + xb^T. \]

- If \( \bar{\alpha} - (d + 1)a^T a \in S_d^+(\mathbb{R}), (3) \) has a unique strong solution.
- If \( \bar{\alpha} - (d - 1)a^T a \in S_d^+(\mathbb{R}), (3) \) has a unique weak solution.

The characteristic function of \( X_t \) can be obtained by solving ODEs.
An application of Wishart processes in finance

Gourieroux and Sufana (2004) consider the following dynamics for \( d \) assets:

\[
\begin{bmatrix}
\log(S^1_t) \\
\vdots \\
\log(S^d_t)
\end{bmatrix} = 
\begin{bmatrix}
(X_t)_{1,1}/2 \\
\vdots \\
(X_t)_{d,d}/2
\end{bmatrix} dt + \sqrt{X_t}dZ_t,
\]

(4)

where \( X_t \) solves (1) and \((Z_t, t \geq 0)\) is a \( d \)-dimensional Brownian motion independent of \((W_t, t \geq 0)\), and \( D_i \in \mathcal{M}_d(\mathbb{R}) \).

- \( d = 1 \): Heston model without correlation.
- The characteristic function of \((\log(S^1_t), \ldots, \log(S^d_t))^T\) can be calculated by solving ODEs.
- Dependence between \( W \) and \( Z \) has been considered in Da Fonseca and al. (2008) to keep the Affine structure.
- \( X_t \) is the instantaneous covariance matrix, i.e.
  \[ \langle d \log(S^i_t), d \log(S^j_t) \rangle = (X_t)_{i,j} \ dt. \]
Existing results on the simulation of Wishart processes

**Exact simulation.** To the best of our knowledge, exact simulation algorithms for Wishart distribution only exist in the literature for integer degrees $\alpha \in \mathbb{N}, \alpha \geq d - 1$ (Odell and Feiveson (1966) and Gleser (1975)).

**Approximation schemes.**

- the Euler-Maruyama scheme is not well-defined exactly as for the CIR process.
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A first simple remark

We consider two SDEs $X_{t}^{1, x}$ and $X_{t}^{2, x}$ associated respectively to the operators $L_1$ and $L_2$ and defined on the same domain $\mathbb{D}$. We assume that:

$$L_1 L_2 = L_2 L_1.$$ 

Then,

$$\mathbb{E}[f(X_{t}^{1, x}, X_{t}^{2, x})] = \mathbb{E}[\mathbb{E}[f(X_{t}^{1, x}, X_{t}^{2, x})|X_{t}^{2, x}]] = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbb{E}[L_1^k f(X_{t}^{2, x})] = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^{k+l}}{k!l!} L_2^l L_1^k f(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (L_1 + L_2)^k f(x).$$ 

Thus, if we have exact schemes for $L_1$ and $L_2$, then we have an exact scheme for $L_1 + L_2$ simply by taking the composition of both schemes.
The extended Cholesky decomposition

Let \( q \in S_d^+ (\mathbb{R}) \) be a matrix with rank \( r \). Then there is a permutation matrix \( p \), an invertible lower triangular matrix \( c_r \in \mathcal{G}_r (\mathbb{R}) \) and \( k_r \in \mathcal{M}_{d-r \times r} (\mathbb{R}) \) such that :

\[
pqp^T = cc^T, \quad c = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix}.
\]

The triplet \( (c_r, k_r, p) \) is called an extended Cholesky decomposition of \( q \). Besides, \( \tilde{c} = \begin{pmatrix} c_r & 0 \\ k_r & I_{d-r} \end{pmatrix} \in \mathcal{G}_d (\mathbb{R}) \), and we have :

\[
q = (\tilde{c}^T p)^T I_d \tilde{c}^T p,
\]

where \( (I_d^r)_{i,j} = 1_{i=j\leq r} \).
Reduction to the case $b = 0, a = I_d^n$

We use the characteristic function (2). Let $n = \text{Rk}(q_t)$. There is $	heta_t \in G_d(\mathbb{R})$ such that $q_t/t = \theta_t I_d^n \theta_t^T$.

$$\begin{align*}
\det(I_d - 2iq_t \nu) &= \det(\theta_t(\theta_t^{-1} - 2itI_d^n \theta_t^T \nu)) = \det(I_d - 2itI_d^n \theta_t^T \nu \theta_t), \\
\text{Tr}[i\nu(I_d - 2iq_t \nu)^{-1} m_t x m_t^T] &= \text{Tr}[i(\theta_t^{-1})^T \theta_t^T \nu(\theta_t^{-1} - 2i\theta_t I_d^n \theta_t^T \nu \theta_t) - 1 m_t x m_t^T ] \\
&= \text{Tr}[i\theta_t^T \nu \theta_t(I_d - 2ii_t \theta_t^T \nu \theta_t)^{-1} \theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T].
\end{align*}$$

$$\implies \mathbb{E}[\exp(i\text{Tr}(\nu X_t))] = \mathbb{E}[\exp(i\text{Tr}(\theta_t^T \nu \theta_t \tilde{X}_t \theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T))] = \mathbb{E}[\exp(i\text{Tr}(\nu \theta_t \tilde{X}_t \theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T \theta_t^T))],$$

i.e.

$$WIS_d(x, \alpha, b, a; t) = \theta_t WIS_d(\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T, \alpha, 0, I_d^n; t) \theta_t^T. \quad (5)$$

It is therefore sufficient to simulate exactly a Wishart process with $a = I_d^n$ and $b = 0$. 

A remarkable splitting when $a = I^n_d$ and $b = 0$

The infinitesimal generator of a Wishart process with $b = 0$ and $a = I^n_d$ is:

$$L = \text{Tr}(\alpha I^n_d D) + 2\text{Tr}(xDI^n_d D), \text{ with } D_{i,j} = \partial_{i,j}$$

for $f : \mathcal{M}_d(\mathbb{R}) \to \mathbb{R}$ s.t. $f(x) = f(x^T)$ for $1 \leq i, j \leq d$.

$I^n_d = \sum_{i=1}^n e^i_d$ with $(e^i_d)^{k,l} = 1_{k=l=i}$. We set $L_i = \text{Tr}(\alpha e^i_d D) + 2\text{Tr}(xDe^i_d D)$.

**Proposition 1**

$$L = L_1 + \cdots + L_n, \text{ with } L_i L_j = L_j L_i, \text{ and where}$$

- $L_i$ is the same operator as $L_1$ by permuting $i$th and first coordinates.
- $L_1$ is the operator of a Wishart process with $b = 0$ and $a = I^n_d$, which is thus well defined on $S^+_d(\mathbb{R})$.

**Consequence** : It is sufficient to sample an exact scheme for $L_1$ to get an exact scheme for $L$. This can be done!
Exact scheme for $L_1$ when $d = 2$ ($\alpha \geq d - 1 = 1$)

For $f : S_d(\mathbb{R}) \rightarrow \mathbb{R}, L_1f(x) =$

\[\alpha \partial_{\{1,1\}} f(x) + 2x_{1,1} \partial_{\{1,1\}}^2 f(x) + 2x_{1,2} \partial_{\{1,1\}} \partial_{\{1,2\}} f(x) + \frac{x_{2,2}}{2} \partial_{\{1,2\}}^2 f(x).\]

It is associated to the following SDE when $(X_0)_{2,2} > 0$

\[
\begin{align*}
    d(X_t)_{1,1} &= \alpha dt + 2\sqrt{(X_t)_{1,1}} dB_t^1 + 2\frac{(X_t)_{1,2}}{\sqrt{(X_t)_{2,2}}} dB_t^2 \\
    d(X_t)_{1,2} &= \sqrt{(X_t)_{2,2}} dB_t^2, \quad (X_t)_{2,1} = (X_t)_{1,2}, \\
    d(X_t)_{2,2} &= 0
\end{align*}
\]

and if $(X_0)_{2,2} = 0$:

\[
    d(X_t)_{1,1} = \alpha dt + 2\sqrt{(X_t)_{1,1}} dB_t^1, \quad d(X_t)_{1,2} = d(X_t)_{2,2} = 0.
\]

In the second case: CIR that can be simulated exactly (e.g. Glasserman) In the first case, we set $U_t = (X_t)_{1,1} - ((X_t)_{1,2})^2 / (X_t)_{2,2}$:

\[
dU_t = (\alpha - 1) dt + \sqrt{U_t} dB_t^1 : \text{CIR indep. of } (X_t)_{1,2} \sim \mathcal{N}((X_0)_{1,2}, (X_0)_{2,2} t).
\]
Exact scheme for $L_1$ when $d \geq 3$ ($\alpha \geq d - 1$) I

Up to a permutation, $(x)_{2 \leq i,j \leq d} = \left( \begin{array}{cc} c_r & 0 \\ k_r & 0 \end{array} \right) \left( \begin{array}{cc} c_r^T & k_r^T \\ 0 & 0 \end{array} \right) =: cc^T$.

We can show that $L_1$ is the generator of the SDE:

\[
\begin{align*}
d(X_t^x)_{1,1} &= \alpha dt + 2 \sqrt{(X_t^x)_{1,1}} - \sum_{k=1}^{r} \left( \sum_{l=1}^{r} (c_{r}^{-1})_{k,l} (X_t^x)_{1,l+1} \right)^2 dZ_t^1 \\
&\quad + 2 \sum_{k=1}^{r} \sum_{l=1}^{r} (c_{r}^{-1})_{k,l} (X_t^x)_{1,l+1} dZ_t^{k+1} \\
d(X_t^x)_{1,i} &= \sum_{k=1}^{r} c_{i-1,k} dZ_t^{k+1} = d(X_t^x)_{i,1}, \ i = 2, \ldots, d \\
d(X_t^x)_{l,k} &= 0, \ \text{for} \ 2 \leq k, l \leq d.
\end{align*}
\]

The SDE associated to $L_1$ can be solved explicitly as for $d = 2$, and requires the sampling of 1 CIR distribution and $r - 1$ standard Gaussian variables that are independent:
Exact scheme for $L_1$ when $d \geq 3$ ($\alpha \geq d - 1$) II

\[
X^x_t = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_r & 0 \\
0 & k_r & I_{d-r-1}
\end{pmatrix} \times \left( (U^u_t)_{1,1} + \sum_{k=1}^r ((U^u_t)_{1,k+1})^2 \right) \begin{pmatrix}
((U^u_t)_{1,1})_{1 \leq l \leq r}^T \\
((U^u_t)_{1,l+1})_{1 \leq l \leq r} \\
0
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & c_r^T & k_r^T \\
0 & 0 & I_{d-r-1}
\end{pmatrix},
\]

where

\[
\begin{align*}
    d(U^u_t)_{1,1} &= (\alpha - r)dt + 2\sqrt{(U^u_t)_{1,1}}dZ^1_t, \\
    u_{1,1} &= x_{1,1} - \sum_{k=1}^r (u_{1,k+1})^2 \geq 0, \\
    (u^u_t)_{1,l+1})_{1 \leq l \leq r} &= (dZ^1_t)_{1 \leq l \leq r}, \\
    (u_{1,l+1})_{1 \leq l \leq r} &= c_r^{-1}(x_{1,l+1})_{1 \leq l \leq r}.
\end{align*}
\]
Remarks on the exact scheme

- when the initial value $x \in S_d^+ \times (\mathbb{R})$, we only make usual Cholesky decompositions.
- For $x = 0$, $b = 0$ and $t = 1$, our exact scheme gives back the Bartlett’s decomposition (1933):

\[
\begin{pmatrix}
(L_{i,j})_{1 \leq i, j \leq n} & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
(L_{i,j}^T)_{1 \leq i, j \leq n} & 0 \\
0 & 0
\end{pmatrix}
\sim WIS_d(0, \alpha, 0, I_d^n; 1),
\]

where $L_{i,j} \sim \mathcal{N}(0, 1)$, $i > j$ and $(L_{i,i})^2 \sim \chi^2(\alpha - i + 1)$ are independent and $L_{i,j} = 0$ for $i < j$.

- The exact scheme has a complexity of $O(d^4)$ operations ($n \leq d$ Cholesky), $O(d^2)$ Gaussian samples, $O(d)$ CIR samples. [A totally different exact scheme in $O(d^3)$ is possible for $\alpha \geq 2d - 1$]
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Assumptions

\[ t \geq 0, \quad X^x_t = x + \int_0^t b(X^x_s)ds + \int_0^t \sigma(X^x_s)dB_s. \]

Assumption : domain \( \mathbb{D} \subset \mathbb{R}^\zeta, \forall x \in \mathbb{D}, \mathbb{P}(\forall t \geq 0, X^x_t \in \mathbb{D}) = 1; \) \( b_i(x), (\sigma(x)\sigma^T(x))_{i,j} \in \mathcal{C}^\infty_{\text{pol}}(\mathbb{D}). \)

\[ \mathcal{C}^\infty_{\text{pol}}(\mathbb{D}) = \{ f \in \mathcal{C}^\infty(\mathbb{D}, \mathbb{R}), \forall \gamma \in \mathbb{N}^\zeta, \exists C_{\gamma} > 0, e_{\gamma} \in \mathbb{N}^*, \forall x \in \mathbb{D}, \left| \partial_\gamma f(x) \right| \leq C_{\gamma}(1 + \|x\|^{e_{\gamma}}) \} \]

Associated operator :
\[ Lf(x) = \sum_{i=1}^\zeta b_i(x) \partial_i f(x) + \frac{1}{2} \sum_{i,j,k=1}^\zeta \sigma_{i,k}(x)\sigma_{j,k}(x) \partial_i \partial_j f(x). \]

Rem : \( f \in \mathcal{C}^\infty_{\text{pol}}(\mathbb{D}) \implies Lf \in \mathcal{C}^\infty_{\text{pol}}(\mathbb{D}). \)
Notations for discretization schemes

**Definition 2**

A family of transition probabilities \((\hat{p}_x(t)(dz), t > 0, x \in \mathbb{D})\) on \(\mathbb{D}\) is s.t. \(\hat{p}_x(t)\) is a probability law on \(\mathbb{D}\) for \(t > 0\) and \(x \in \mathbb{D}\). We note \(\hat{X}_t^x\) a r.v. with law \(\hat{p}_x(t)(dz)\).

**Associated discretization scheme** : \((\hat{X}_{t_i}^n, 0 \leq i \leq n)\) sequence of \(\mathbb{D}\)-valued r.v. s.t. \(\hat{X}_{t_i}^n\) is sampled according to \(\hat{p}_{\hat{X}_{t_i}^n}(T/n)(dz)\).

Example (Euler) : \(\hat{X}_t^x = x + b(x)t + \sigma(x)W_t\), \(\hat{p}_x(t)\) : law density of \(\hat{X}_t^x\).
Talay-Tubaro Theorem (1990)

If

\[ f : \mathbb{D} \rightarrow \mathbb{R} \ 	ext{s. t.} \ u(t, x) = \mathbb{E}[f(X^x_T - t)] \] is defined on \([0, T] \times \mathbb{D}\), solves for \( t \in [0, T] \), \( x \in \mathbb{D} \), \( \partial_t u(t, x) = -Lu(t, x) \), and has “good bounds” on all its derivatives \( \partial_t^l \partial_\gamma u \), i.e.

\[ \forall l \in \mathbb{N}, \gamma \in \mathbb{N}^\varepsilon, \exists C_{l,\gamma}, e_{l,\gamma} > 0, \forall x \in \mathbb{D}, t \in [0, T], |\partial_t^l \partial_\gamma u(t, x)| \leq C_{l,\gamma}(1 + \|x\|^{e_{l,\gamma}}). \]

the scheme is a potential weak \( \nu \)th-order discr. scheme for \( L : \)

\[ \mathbb{E}[f(\hat{X}^x_t)] \mid_{t \to 0^+} = f(x) + \sum_{k=1}^{\nu} \frac{1}{k!} t^k L^k f(x) + \text{Remainder in } t^{\nu+1} \]

and \((\hat{X}^n_t, i = 0, \ldots, n)\) has uniformly bounded moments.

then, \(|\mathbb{E}[f(\hat{X}^n_t)] - \mathbb{E}[f(X^x_T)]| \leq K/n^{\nu}. \)
Composition of discretization schemes I

\( \hat{p}_x^1(t)(dz), \hat{p}_x^2(t)(dz) : \) potential \( \nu \)th-order schemes for \( L_1, L_2 \).

\( \hat{p}^2(\lambda_2 t) \circ \hat{p}_x^1(\lambda_1 t)(dz) = \int \hat{p}_y^1(\lambda_2 t)(dz)\hat{p}_x^1(\lambda_1 t)(dy) : \) scheme that amounts to first use the scheme 1 with a time step \( \lambda_1 t \) and then the scheme 2 with a time step \( \lambda_2 t \). \( \hat{X}_{x,\lambda_2 t,\lambda_1 t}^{2\circ 1} \) a r.v. with this law.

**Proposition 3**

\[
\mathbb{E}[f(\hat{X}_{x,\lambda_2 t,\lambda_1 t}^{2\circ 1})] = \lim_{t \to 0^+} \sum_{l_1+l_2 \leq \nu} \frac{\lambda_1^{l_1} \lambda_2^{l_2}}{l_1! l_2!} t^{l_1+l_2} L_1^{l_1} L_2^{l_2} f(x) + \text{Remainder}
\]

\[
(= [I + \lambda_1 t L_1 f + \ldots + \frac{(\lambda_1 t)^\nu}{\nu!} L_1^\nu f][I + \lambda_2 t L_2 f + \ldots + \frac{(\lambda_2 t)^\nu}{\nu!} L_2^\nu f] + \text{Rem})
\]

Csq: a scheme acts on \( f \) “as” the operator \( I + t L f + \ldots + \frac{t^\nu}{\nu!} L^\nu f + \text{Rem} \).

Composition of schemes = Composition of operators.
Composition of discretization schemes II

**Corollary 4**

If \( \hat{p}^1_x, \hat{p}^2_x \) are potential \( \nu \)th-order schemes for \( L_1, L_2 \) and \( L_1L_2 = L_2L_1 \),
\( \hat{p}^1(t) \circ \hat{p}^2_x(t) \) is a potential \( \nu \)th-order schemes for \( L_1 + L_2 \).

**Corollary 5**

\( \hat{p}^1_x, \hat{p}^2_x \) : potential 2\(^{nd} \) order schemes for \( L_1, L_2 \). Then,

\[
\hat{p}^2(t/2) \circ \hat{p}^1(t) \circ \hat{p}^2_x(t/2) \quad \text{(Strang 1968)} \quad (8)
\]

\[
\frac{1}{2} (\hat{p}^2(t) \circ \hat{p}^1_x(t) + \hat{p}^1(t) \circ \hat{p}^2_x(t)) \quad \text{(9)}
\]

are potential second order schemes for \( L_1 + L_2 \).

Proof for (9):

\[
(I + tL_1 + t^2/2L^2_1 + ...)(I + tL_2 + t^2/2L^2_2 + ...) =
I + t(L_1 + L_2) + t^2/2(L^2_1 + L^2_2 + 2L_1L_2) + ...
\]

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High order schemes for the operator $L_1$

**Result:** If we replace in the exact scheme for $L_1$ (7),

- $(U_t^u)_{1,1}$ by $(\hat{U}_t^u)_{1,1}$ sampled with a potential $\nu$th-order scheme for the CIR: $dU_t = (\alpha - r)dt + 2\sqrt{U_t}dZ_t$,
- $(U_t^u)_{1,l+1}$ by $\sqrt{t}\hat{G}^{l+1}$ where $\hat{G}^{l+1}$ is a bounded variable s.t. $\forall k \leq 2\nu + 1$, $\mathbb{E}[(\hat{G}^{l+1})^k] = \mathbb{E}[G^k]$, where $G \sim \mathcal{N}(0, 1)$,

we can show that we get a potential $\nu$th order scheme for $L_1$.

Second and third order schemes for the CIR can be found in A. 2008. Here are some matching-moment variables for $\mathcal{N}(0, 1)$ for $\nu = 2, 3$:

$$
P(\hat{G}^i = \sqrt{3}) = P(\hat{G}^i = -\sqrt{3}) = \frac{1}{6} \quad \text{and} \quad P(\hat{G}^i = 0) = \frac{2}{3}
$$

(resp. $\mathbb{P}\left(\hat{G}^i = \varepsilon \sqrt{3 + \sqrt{6}}\right) = \frac{\sqrt{6} - 2}{4\sqrt{6}}$, $\mathbb{P}\left(\hat{G}^i = \varepsilon \sqrt{3 - \sqrt{6}}\right) = \frac{1}{2} - \frac{\sqrt{6} - 2}{4\sqrt{6}}$, $\varepsilon \in \{-1, 1\}$).
A third order scheme for Wishart processes

We use once again the splitting given by Proposition 1.

- We have a third order scheme for $L_1$.
- By a permutation of the first and $i^{th}$ coordinate, we get also a third order scheme for $L_i$.
- By Corollary 4, we get a third order scheme $\hat{X}_t^\chi$ for a Wishart process with $a = I^n_d$ and $b = 0$.
- Last, we can show from (5) (under some assumptions) that $\theta_t \hat{X}_t \theta^{-1}_t m_t x_m^T (\theta^{-1}_t)^T \theta^T_t$ is a third order scheme.
Second order scheme for general Affine processes I

A First remark:
Let $dX_t = (\bar{\alpha} + B(X_t))dt + \sqrt{X_t}dW_t a + a^T dW_t^T \sqrt{X_t}$, $X_0 = x \in S_d(\mathbb{R})$.
There is $u \in G_d(\mathbb{R})$ such that

$$ (X_t)_{t \geq 0} = (u^T \tilde{X}_t u)_{t \geq 0}, $$

where

- $d\tilde{X}_t = (\tilde{\delta} + B_u(\tilde{X}_t))dt + \sqrt{\tilde{X}_t}dW_t I_d^n + I_d^n dW_t^T \sqrt{\tilde{X}_t}$, $\tilde{X}_0 = (u^{-1})^T xu^{-1}$,
- $B_u(x) = (u^{-1})^T B(u^T xu) u^{-1}$,
- $\tilde{\delta}$ is a diagonal matrix such that $\tilde{\delta} - (d - 1) I_d^n \in S_d^+(\mathbb{R})$

$\implies$ It is sufficient to get a scheme for $\tilde{X}_t$ (i.e. when $a = I_d^n$ and $\bar{\alpha}$ is a diagonal matrix).
Second order scheme for general Affine processes II

Let $\delta_{\min} = \min_{i=1,\ldots,n} \delta_{i,i} \geq d - 1$. We split the generator of $\tilde{X}_t$:

$$
L = \text{Tr}([\bar{\delta} + B(x)]D^S) + 2\text{Tr}(xD^S I_d^n D^S)
= \underbrace{\text{Tr}([\bar{\delta} - \delta_{\min} I_d^n + B_u(x)]D^S)}_{L_{ODE}} + \underbrace{\delta_{\min} \text{Tr}(D^S) + 2\text{Tr}(xD^S I_d^n D^S)}_{L_{WIS_d}(x, \delta_{\min}, 0, I_d^n)},
$$

where $L_{ODE}$ is associated to the affine ODE $x'(t) = \bar{\delta} - \delta_{\min} I_d^n + B_u(x(t))$ that can be solved explicitly and is such that $x(t) \in S_d^+(\mathbb{R})$ for $t \geq 0$. By Corollary 5, we get a second order scheme for $\tilde{X}_t$ and thus for $X_t$. 
A faster second order scheme when $\bar{\alpha} - dI_d^n \in S_d^+(\mathbb{R})$

All the previous schemes rely on the splitting given by Proposition 1 and require thus $O(d^4)$ operations.

**Remark**: We can check that if $c^Tc = x$, $(c + W_tI_d^n)^T(c + W_tI_d^n)$ is a Wishart process with $\alpha = d$, $a = I_d^n$, $b = 0$ starting from $x$. Also, $(c + \sqrt{t}\hat{G}I_d^n)^T(c + \sqrt{t}\hat{G}I_d^n)$ is a potential second order scheme for $WIS_d(x, d, 0, I_d^n)$ where $\hat{G}$ is a matrix with independent elements sampled according to (10).

**Consequence**: By using the splitting:

$$L = \underbrace{\text{Tr}([\bar{\delta} - dI_d^n + B_u(x)]D^S)}_{L_{ODE}} + \underbrace{d\text{Tr}(D^S) + 2\text{Tr}(xD^SI_d^nD^S)}_{L_{WIS_d}(x, d, 0, I_d^n)},$$

we get a by Corollary 5 a second order scheme for $\tilde{X}_t$ in $O(d^3)$ operations.
1. Wishart and Affine processes on nonnegative symmetric matrices
2. Exact simulation of Wishart processes
3. Discretization schemes obtained by composition
4. High order discretization schemes for Wishart and Affine processes

5. Numerical results
6. A mean-reverting SDE on correlation matrices
7. Construction of MRC processes
A modified Euler scheme

The Euler scheme for the Affine diffusion (3) is:

\[ \hat{X}_{t_{i+1}} = \hat{X}_{t_i} + (\bar{\alpha} + B(\hat{X}_{t_i}))(t_{i+1} - t_i) + \sqrt{\hat{X}_{t_i}}(W_{t_{i+1}} - W_{t_i})a + a^T(W_{t_{i+1}} - W_{t_i})^T \sqrt{\hat{X}_{t_i}}. \]

It is not well-defined since \( \hat{X}_{t_{i+1}} \) may not be nonnegative. Corrected Euler scheme:

\[
\hat{X}_{t_{i+1}} = \hat{X}_{t_i} + (\bar{\alpha} + B(\hat{X}_{t_i}))(t_{i+1} - t_i) \\
+ \sqrt{(\hat{X}_{t_i})^+}(W_{t_{i+1}} - W_{t_i})a + a^T(W_{t_{i+1}} - W_{t_i})^T \sqrt{(\hat{X}_{t_i})^+},
\]

where \( \sqrt{(x^+)} := odiag(\sqrt{\lambda_1^+}, \ldots, \sqrt{\lambda_d^+})o^{-1} \) for \( x \in S_d(\mathbb{R}) \) and \( x = odiag(\lambda_1, \ldots, \lambda_d^+)o^{-1} \).
## A time comparison (10^6 samples, N time-steps)

<table>
<thead>
<tr>
<th>Schemes</th>
<th>N = 10</th>
<th></th>
<th>Time</th>
<th>N = 30</th>
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<th>Time</th>
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<td></td>
<td>R. value</td>
<td>Im. value</td>
<td></td>
<td>R. value</td>
<td>Im. value</td>
<td></td>
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<tr>
<td>Exact (1 step)</td>
<td>−0.526852</td>
<td>−0.227962</td>
<td>12</td>
<td>−0.526486</td>
<td>−0.229078</td>
<td>125</td>
</tr>
<tr>
<td>2\textsuperscript{nd} order bis</td>
<td>−0.526229</td>
<td>−0.228663</td>
<td>41</td>
<td>−0.526489</td>
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<tr>
<td>2\textsuperscript{nd} order</td>
<td>−0.526577</td>
<td>−0.228923</td>
<td>76</td>
<td>−0.526574</td>
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<td>3\textsuperscript{rd} order</td>
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<td>Exact (N steps)</td>
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<td>−0.526891</td>
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<td>Corrected Euler</td>
<td>−0.525627*</td>
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<td>225</td>
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\(\alpha = 3.5, d = 3, \Delta_R = 10^{-3}, \Delta_{Im} = 10^{-3}\), exact value R. = −0.527090 and Im. = −0.228251

<table>
<thead>
<tr>
<th>Schemes</th>
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<td>Exact (1 step)</td>
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<td>Corrected Euler</td>
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<td>−0.589735*</td>
<td>−0.042002*</td>
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\(\alpha = 2.2, d = 3, \Delta_R = 0.9 \times 10^{-3}, \Delta_{Im} = 1.3 \times 10^{-3}\), exact value R. = −0.591411 and Im. = −0.036346

<table>
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<td>Im. value</td>
<td></td>
<td>R. value</td>
<td>Im. value</td>
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<td>−0.056882*</td>
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</table>

\(\alpha = 10.5, d = 10, \Delta_R = 1.4 \times 10^{-3}, \Delta_{Im} = 1.3 \times 10^{-3}\), exact value R. = 0.063960 and Im. = −0.063544

<table>
<thead>
<tr>
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<th></th>
<th>N = 30</th>
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<td>Im. value</td>
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<td>Im. value</td>
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<td>Exact (1 step)</td>
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<td>−0.030118*</td>
<td>−0.088988*</td>
<td>7144</td>
</tr>
</tbody>
</table>

\(\alpha = 9.2, d = 10, \Delta_R = 1.4 \times 10^{-3}, \Delta_{Im} = 1.4 \times 10^{-3}\), exact value R. = −0.036064 and Im. = −0.093275
Weak convergence I

**Figure:** $d = 3$, $10^7$ MC samples, $T = 10$. $\mathfrak{R}(\mathbb{E}[\exp(-\text{Tr}(iv\hat{X}_N^N))])$ in fct of $T/N$. Left: $v = 0.05I_d$, $x = 0.4I_d$, $\alpha = 4.5$, $a = I_d$ and $b = 0$. Exact value: $0.054277$. Right: $v = 0.2I_d + 0.04q$, $x = 0.4I_d + 0.2q$, $\alpha = 2.22$, $a = I_d$ and $b = -0.5I_d$. Exact value: $0.239836$. $q_{ij} = 1_{i \neq j}$. 
Weak convergence II

**Figure:** $d = 10, 10^7$ MC samples, $T = 10$. Left: $\mathcal{I}(\mathbb{E}[\exp(-\text{Tr}(iv\hat{X}_{10}^N))])$ with $v = 0.009I_d$ in fct of $T/N$. $x = 0.4I_d$, $\alpha = 12.5$, $b = 0$ and $a = I_d$. Ex. value: $-0.361586$. Right: $\mathcal{N}(\mathbb{E}[\exp(-\text{Tr}(iv\hat{X}_{10}^N))])$ with $v = 0.009I_d$ in fct of $T/N$. $x = 0.4I_d$, $\alpha = 9.2$, $b = -0.5I_d$ and $a = I_d$. Ex. value $0.572241$. 

Aurélien Alfonsi (CERMICS, Projet Mathfi) UPEMLV, 15 Mars 2012
Convergence on pathwise expectations

**Figure:** $d = 3, 10^7$ MC samples, $T = 1$, $x = 0.4d + 0.2q$ with $q_{i,j} = 1_{i \neq j}$, $\alpha = 2.2$, $b = 0$ and $a = I_d$. Left, $\mathbb{E}[\max_{0 \leq k \leq N} \text{Tr}(\hat{X}_k^N)]$, right: $\mathbb{E}[\max_{0 \leq k \leq N} \text{Tr}(\hat{X}_k^N)] - \mathbb{E}[\max_{0 \leq k \leq N} \text{Tr}(X_k^N)]$ in fct of $T/N$. 
A scheme for the Gourieroux-Sufana model

In (4), the joint operator of \((S_t, X_t)\) is

\[
L = L^S + L^X, \quad \text{where } L^S = \sum_{i=1}^{d} rs_i \partial s_i + \frac{1}{2} \sum_{i,j=1}^{d} s_i s_j x_{i,j} \partial s_i \partial s_j,
\]

and \(L^X\) is the generator of a Wishart process. We can solve explicitly the SDE associated to \(L^S\): \(S_t^l = S_0^l \exp[(r - x_{l,l}/2)t + (\sqrt{x}Z_t)_l]\). By using a second order scheme for \(L^X\), we get a second order scheme for \(L\) by Corollary 5.
Put option in the Gourieroux-Sufana model

**Figure:** $\mathbb{E}[e^{-rT}(K - \max(\hat{S}_N^{1,N}, \hat{S}_N^{2,N}))^+]$ in fct of $T/N$. $d = 2, T = 1, K = 120, S_0 = S_0 = 100$, and $r = 0.02$. $x = 0.04I_d + 0.02q$ with $q_{i,j} = 1_{i \neq j}$, $a = 0.2I_d$, $b = 0.5I_d$ and $\alpha = 4.5$ (left), $\alpha = 1.05$ (right). $10^6$ Monte-Carlo samples.
Summary

We have obtained using splitting methods:

- an exact simulation algorithm for Wishart processes,
- second and third order schemes for Wishart processes,
- second order scheme for affine processes on nonnegative matrices.

The discretization schemes are much more accurate and less time-consuming than the modified Euler scheme.

Which scheme to use? We recommend the exact scheme to calculate expectations that depends on one or few dates. For pathwise expectations, we recommend instead to use discretization schemes: the second order scheme “bis” if it is defined and the second/third order scheme otherwise.
1. Wishart and Affine processes on nonnegative symmetric matrices

2. Exact simulation of Wishart processes

3. Discretization schemes obtained by composition

4. High order discretization schemes for Wishart and Affine processes

5. Numerical results

6. **A mean-reverting SDE on correlation matrices**

7. Construction of MRC processes
Wright-Fisher (or Jacobi) processes

Up to our knowledge, there is no literature on particular diffusions defined on correlation matrices in dimension $d$. With 2 assets, $X_t$ is a correlation matrix in dimension 2 iff

$$X_t = \begin{bmatrix} 1 & \rho_t \\ \rho_t & 1 \end{bmatrix}, \text{ with } \rho_t \in [-1, 1].$$

A frequent choice (e.g. Jun Ma 2009) is to consider a Wright-Fisher (or Jacobi) process

$$d\rho_t = \kappa (\bar{\rho} - \rho_t)dt + \sigma \sqrt{1 - \rho_t^2}dB_t,$$

$\kappa > 0, \bar{\rho} \in [-1, 1], \sigma \geq 0.$

Properties: mean-reversion, explicit calculation of moments, ergodic law.

The processes that we present extend naturally in dimension $d$ these processes.
Correlation processes in higher dimension

Up to our knowledge, the existing processes in the literature rely on a parametrization of a subset of correlation matrices $\mathcal{C}_d(\mathbb{R})$. For example, Driessen and Maenhout (2006) consider:

$$(C_t)_{i,j} = 1_{i=j} + \rho_t 1_{i \neq j}, \quad 1 \leq i, j \leq d,$$

with

$$d\rho_t = \kappa (\bar{\rho} - \rho_t) dt + \sigma \sqrt{\rho_t (1 - \rho_t)} dB_t,$$

$\kappa > 0$, $\bar{\rho} \in [0, 1]$, $\sigma \geq 0$. Other choices can be found in the working paper of Christopher Kaya Boortz (2008).
Motivations to get an SDE on correlation matrices

- As far as Index options are concerned, one factor correlation may be sufficient since the Index has a “mean effect” on correlations.
- However, one may want to incorporate some views on the market (correlation between some companies or sectors...) in order to price more exotic options.
- A one factor correlation or parametrization of subsets of correlation matrices include (at least theoretically !) some easy detectable arbitrage.
- Up to our knowledge, there is no literature on particular diffusions defined on correlation matrices.
The MRC process

\[ X_t = x + \int_0^t (\kappa (c - X_s) + (c - X_s)\kappa) \, ds \]

\[ + \sum_{n=1}^d a_n \int_0^t \left( \sqrt{X_s - X_se_n^T X_s} dW_s e_n^T + e_n^T dW_s^T \sqrt{X_s - X_se_n^T X_s} \right), \]

- \((e_d^n)_{i,j} = 1_{i=j=n}, 1 \leq i, j \leq d,\)
- \(W\) is a \(d\)-square matrix made with independent Brownian motions,
- \(x, c \in \mathcal{C}_d(\mathbb{R}),\)
- \(\kappa = \text{diag}(\kappa_1, \ldots, \kappa_d)\) and \(a = \text{diag}(a_1, \ldots, a_d)\) are nonnegative diagonal matrices.

**Intuitive parameters**: mean-reversion towards \(c\) with a speed and a noise respectively tuned by \(\kappa, a\).

**Notations**: \(\text{MRC}_d(x, \kappa, c, a)\) law of \((X_t)_{t \geq 0}\).
Infinite simulator generator

Quadratic variation:

\[
\langle d(X_t)_{i,j}, d(X_t)_{k,l} \rangle = \left[ a_i^2 (1_{i=k}(X_t)_{j,l} - (X_t)_{i,j}(X_t)_{i,l}) + 1_{i=l}(X_t)_{j,k} - (X_t)_{i,j}(X_t)_{i,k}) + a_j^2 (1_{j=k}(X_t)_{i,l} - (X_t)_{j,i}(X_t)_{j,l}) + 1_{j=l}(X_t)_{i,k} - (X_t)_{j,i}(X_t)_{j,k}) \right] dt.
\]

\[
L = \sum_{i=1}^{d} \left( \sum_{1 \leq j \leq d \atop j \neq i} k_i (c_{i,j} - x_{i,j}) \partial_{\{i,j\}} + \frac{1}{2} \sum_{1 \leq j,k \leq d \atop j \neq i, k \neq i} a_i^2 (x_{j,k} - x_{i,j}x_{i,k}) \partial_{\{i,j\}} \partial_{\{i,k\}} \right).
\]
Existence and uniqueness results

**Theorem 6**

Let $x \in \mathcal{C}^*_d(\mathbb{R})$. If $\kappa c + c\kappa - da^2 \in S^+_d(\mathbb{R})$, there is a unique strong solution of the SDE (10) that is such that $\forall t \geq 0, X_t \in \mathcal{C}^*_d(\mathbb{R})$.

**Theorem 7**

If $\kappa c + c\kappa - (d-2)a^2 \in S^+_d(\mathbb{R})$ or $d = 2$, there is a unique weak solution $(X_t, t \geq 0)$ to SDE (10) such that $\mathbb{P}(\forall t \geq 0, X_t \in \mathcal{C}_d(\mathbb{R})) = 1$. 
Some properties I

- Each cross correlation follows a 1D WF process on $[-1, 1]$: 
  \[ d(X_t)_{i,j} = (\kappa_i + \kappa_j)(c_{i,j} - (X_t)_{i,j})dt + \sqrt{a_i^2 + a_j^2} \sqrt{1 - (X_t)^2_{i,j}}d\beta_t^{i,j}. \]

- Any principal sub-matrix of $X_t$ follows a MRC process: Let 
  \( I = \{k_1 < \cdots < k_{d'}\} \subset \{1, \ldots, d\} \) and denote for \( x \in M_d(\mathbb{R}) \), 
  \( (x^I)_{i,j} = x_{k_i,k_j} \) for \( 1 \leq i, j \leq d' \). We have: 
  \[ (X^I_t)_{t \geq 0} \overset{law}{=} MRC_{d'}(x^I, \kappa^I, c^I, a^I). \]
Some properties II

- Explicit calculation of moments (\(\implies\) weak uniqueness). Let 
  \(m \in S_d(\mathbb{N})\) such that \(m_{i,i} = 0\) for \(1 \leq i \leq d\).
  \(x^m = \prod_{1 \leq i \leq j \leq d} x^{m_{i,j}}\) and \(|m| = \sum_{1 \leq i \leq j \leq d} m_{i,j}\). We have

\[
\mathbb{E}[X^m_t] = x^m \exp(-tK_m) + \exp(-tK_m) \int_0^t \exp(sK_m) \mathbb{E}[f_m(X_s)] ds,
\]

with \(K_m = \sum_{i=1}^d \sum_{j=1}^d \kappa_i m_{i,j} + \frac{1}{2} \sum_{i=1}^d a_i^2 \sum_{j,k=1}^d m_{i,j} m_{i,k}\) and

\[
f_m(x) = \sum_{i=1}^d \sum_{j=1}^d \kappa_i c_{i,j} x^{m - e\{i,j\}_d} + \frac{1}{2} \sum_{i=1}^d a_i^2 \sum_{j,k=1}^d m_{i,j} m_{i,k} x^{m - e\{i,j\}_d + e\{i,k\}_d}
\]

is a polynomial function of degree smaller than \(|m| - 1\).

- Ergodic law:

\[
\mathbb{E}[X^m_\infty] = x^m \text{ if } m \in S_d(\mathbb{N}) \text{ is such that } m_{i,j} > 0 \iff \kappa_i = \kappa_j = 0,
\]

\[
\mathbb{E}[X^m_\infty] = \mathbb{E}[f_m(X_\infty)]/K_m \text{ otherwise}.
\]
A Girsanov Theorem

\[
\frac{dQ}{dP} \bigg|_{\mathcal{F}_T} = \exp \left( \int_0^T \text{Tr}(H_s^T dW_s) - \frac{1}{2} \int_0^T \text{Tr}(H_s^T H_s) ds \right),
\]

Let \( x \in C^*_d(\mathbb{R}) \), \((X_t, t \geq 0) \sim \text{MRC}_d(x, \kappa^1, c^1, a)\) s.t.
\[
\kappa^1 c^1 + c^1 \kappa^1 - da^2 \in S^+_d(\mathbb{R}).
\]
Let \( c^2, \kappa^2 \) such that \( a_i = 0 \implies \kappa^2_i = 0 \) and \( \kappa^1 c^1 + c^1 \kappa^1 + \kappa^2 c^2 + c^2 \kappa^2 - da^2 \in S^+_d(\mathbb{R}) \). We set:

\[
\lambda = \text{diag}(\lambda_1, \ldots, \lambda_d) \quad \text{with} \quad \lambda_i = \begin{cases} 
\kappa^2_i / a_i & \text{if } a_i > 0 \\
0 & \text{otherwise} 
\end{cases}
\]

and \( H_t = (\sqrt{X_t})^{-1} c^2 \lambda \).

Then, \((X_t, t \geq 0) \sim \text{MRC}_d(x, \kappa, c, a)\) under \( Q \), where \( \kappa = \kappa^1 + \kappa^2 \)

\[
c \in C_d(\mathbb{R}) \quad \text{s.t.} \quad c_{i,j} = \begin{cases} 
\frac{(\kappa^1_i + \kappa^1_j)c^1_{i,j} + (\kappa^2_i + \kappa^2_j)c^2_{i,j}}{\kappa_i + \kappa_j} & \text{if } \kappa_i + \kappa_j > 0 \\
0 & \text{if } \kappa_i + \kappa_j = 0.
\end{cases}
\]
1. Wishart and Affine processes on nonnegative symmetric matrices
2. Exact simulation of Wishart processes
3. Discretization schemes obtained by composition
4. High order discretization schemes for Wishart and Affine processes
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7. Construction of MRC processes
Connection with Wishart processes I

For $x \in S_d^+(\mathbb{R})$ such that $x_{i,i} > 0$ for all $1 \leq i \leq d$, we define $p(x) \in \mathcal{C}_d(\mathbb{R})$ by

$$(p(x))_{i,j} = \frac{x_{i,j}}{\sqrt{x_{i,i}x_{j,j}}}, \ 1 \leq i, j \leq d. \quad (11)$$

A natural idea to construct a process on $\mathcal{C}_d(\mathbb{R})$ is to consider $X_t = p(Y_t)$, where $(Y_t)_{t \geq 0}$ is a Wishart process $WIS_d(x, \alpha, b, a)$, i.e.

$$dY_t = (\alpha a^T a + bY_t + Y_t b^T)dt + \sqrt{Y_t}dW_t a + a^T dW_t^T \sqrt{Y_t}, \ t \geq 0, \ Y_0 = x \in S_d^+(\mathbb{R}).$$

**Problem**: this does not lead in general to an autonomous SDE... unless in special cases!
Connection with Wishart processes II

Result: Let $\alpha \geq \max(1, d - 2)$ and $y \in S^+_d(\mathbb{R})$ such that $y_{i,i} > 0$ for $1 \leq i \leq d$. Let $(Y^y_t)_{t \geq 0} \sim \text{WIS}_d(y, \alpha + 1, 0, e^1_d)$. Then, $(Y^y_t)_{i,i} = y_{i,i}$ for $2 \leq i \leq d$ and $(Y^y_t)_{1,1}$ follows a squared Bessel process of dimension $\alpha + 1$ and a.s. never vanishes. We set

$$X_t = p(Y^y_t), \quad \phi(t) = \int_0^t \frac{1}{(Y^y_s)_{1,1}} ds.$$ 

The function $\phi$ is a.s. one-to-one on $\mathbb{R}_+$ and defines a time-change such that:

$$(X_{\phi^{-1}(t)}, t \geq 0) \overset{\text{law}}{=} MRC_d(p(y), \frac{\alpha}{2} e^1_d, I_d, e^1_d).$$

Besides, the processes $(X_{\phi^{-1}(t)}, t \geq 0)$ and $((Y^y_t)_{1,1}, t \geq 0)$ are independent.
Connection with Wishart processes III

Sketch of the proof: We have $d(Y_t^{y})_{i,j} = 0$ for $2 \leq i, j \leq d$ and

$$d(Y_t^{y})_{1,1} = (\alpha + 1)dt + 2 \sum_{k=1}^{d} (\sqrt{Y_t^{y}})_{1,k} (dW_t)_{k,1}, \quad d(Y_t^{y})_{1,i} = \sum_{k=1}^{d} (\sqrt{Y_t^{y}})_{i,k} (dW_t)_{k,1}. $$

$$d\langle(Y_t^{y})_{1,1}\rangle = 4(Y_t^{y})_{1,1}dt$$ and $(Y_t^{y})_{1,1}$ is a square Bessel process that never vanishes. We have:

$$d(X_t)_{1,i} = -\frac{\alpha}{2} (X_t)_{1,i} \frac{dt}{(Y_t^{y})_{1,1}} + \sum_{k=1}^{d} \left( \frac{(\sqrt{Y_t^{y}})_{i,k}}{(\sqrt{Y_t^{y}})_{1,1}y_{i,i}} - (X_t)_{1,i} \frac{(\sqrt{Y_t^{y}})_{i,k}}{(Y_t^{y})_{1,1}} \right) (dW_t)_{k,1}$$

$$d\langle(X_t)_{1,i}, (X_t)_{1,j}\rangle = \frac{1}{(Y_t^{y})_{1,1}} [(X_t)_{i,j} - (X_t)_{1,i}(X_t)_{1,j}]dt$$ and $d\langle(X_t)_{1,i}, (Y_t^{y})_{1,1}\rangle = 0.$
Consequences

- There is a weak solution to $MRC_d(x, \frac{\alpha}{2} e_1^1, I_d, e_1^1)$.

- Up to a permutation of the first and $l$th coordinate, there is also a weak solution to $MRC_d(x, \frac{\alpha}{2} e_l^1, I_d, e_l^1)$, where $(e_d^l)_{i,j} = 1_{i=j=l}$.

We will denote by $L^{l,C}$ the infinitesimal generator of $MRC_d(x, \frac{d-2}{2} e_l^1, I_d, e_1^1)$.

**Remark:** the operators $L^{l,C}$ and $L^{k,C}$ commute for $1 \leq k, l \leq d$. 
Linear ODEs on correlation matrices

Let $b \in S_d(\mathbb{R})$, $\kappa \in M_d(\mathbb{R})$ and consider the following linear ODE

$$x'(t) = b - (\kappa x(t) + x(t)\kappa^T), \quad x(0) = x \in \mathcal{C}_d(\mathbb{R}),$$

What are necessary and sufficient conditions on $\kappa$ and $b$ such that

$$\forall x \in \mathcal{C}_d(\mathbb{R}), \forall t \geq 0, x(t) \in \mathcal{C}_d(\mathbb{R})?$$

**Necessary condition:**

$$\exists c \in \mathcal{C}_d(\mathbb{R}), \exists \kappa_1, \ldots, \kappa_d \in \mathbb{R},$$

$$\forall i \neq j, \kappa_i + \kappa_j \geq 0, \kappa = \text{diag}(\kappa_1, \ldots, \kappa_d) \text{ and } b = \kappa c + c \kappa.$$

**Sufficient condition:**

$$\kappa = \text{diag}(\kappa_1, \ldots, \kappa_d) \text{ and } b = \kappa c + c \kappa \in S_d^+(\mathbb{R})$$
Putting the pieces together

Let \( L^{i,C} \) the infinitesimal generator of \( \text{MRC}_d(p(y), \frac{d-2}{2}e^i_d, I_d, e^i_d) \). The SDE (10) is nothing but the one associated to generator:

\[
\sum_{i=1}^{d} a^2_i L^{i,C} + L^{ODE},
\]

where \( L^{ODE} \) is the operator associated to

\[
\xi'(t, x) = \kappa(c-x) + (c-x)\kappa - \frac{d-2}{2} [a^2(I_d-x) + (I_d-x)a^2], \quad \xi(0, x) = x \in \mathcal{C}_d(\mathbb{R}).
\]

This (linear) ODE can be solved explicitly and such that

\( \forall t \geq 0, x \in \mathcal{C}_d(\mathbb{R}), \xi(t, x) \in \mathcal{C}_d(\mathbb{R}) \) if \( \kappa c + c\kappa - (d-2)a^2 \in S^+_d(\mathbb{R}) \).