Exact and high order discretization schemes for Wishart processes and an SDE on Correlation matrices

Aurélien Alfonsi

CERMICS Mathfi Project, Ecole des Ponts - Paris-Est University
Joint work with Abdelkoddousse Ahdida

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Structure of the talk

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What is a Wishart process ?

Wishart processes have initially been introduced and studied by Bru in her phD thesis on Escherichia Coli (1987), and have recently been extended by Cuchiero, Filipovic, Mayerhofer and Teichmann (2009). A Wishart process $(X_t)_{t\geq 0}$ of dimension d is defined on nonnegative symmetric matrices $\mathcal{S}_d^+(\mathbb{R})$ and solves the following SDE :

$$\begin{aligned} dX_t &= (\alpha a^T a + bX_t + X_t b^T)dt + \sqrt{X_t}dW_t a + a^T dW_t^T \sqrt{X_t},\ t \geq 0, \quad (1) \\ X_0 &= x \in \mathcal{S}_d^+(\mathbb{R}). \end{aligned}$$

Here, $\alpha \in \mathbb{R}$, $a, b \in \mathcal{M}_d(\mathbb{R})$ and $\sqrt{X_t}$ is the square root of the nonnegative matrix X_t : if $X_t = O_t diag(\Lambda_t^1, \dots, \Lambda_t^d) O_t^{-1}$, $\sqrt{X_t} := O_t diag(\sqrt{\Lambda_t^1}, \dots, \sqrt{\Lambda_t^d}) O_t^{-1}$. W_t denotes a $d \times d$ matrix whose components are independent standard Brownian motions.
$d = 1$ CIR diffusion : $dX_t = (\alpha a^2 + 2bX_t)dt + 2a\sqrt{X_t}dW_t, t \geq 0.$

When is it well-defined ?

We have the following results (Bru, Cuchiero et al., Mayerhofer et al.) :

- When $\alpha \geq d+1$, the SDE has a unique strong solution on the positive symmetric matrices $\mathcal{S}_d^{+,*}(\mathbb{R})$.
- When $d-1 \leq \alpha < d+1$, the SDE has a unique weak solution on $\mathcal{S}_d^{+}(\mathbb{R})$.

When $d=1$, $\alpha \geq 2$ ensures that the CIR never reaches 0. However, we know in that case that there is a strong solution for any $\alpha \geq 0$.

An explicit characteristic function

Let $X_t^x \sim WIS_d(x, \alpha, b, a; t)$ follow a Wishart distribution. Its Fourier transform is known explicitly :

$$\forall v \in \mathcal{S}_d(\mathbb{R}), \mathbb{E}[\exp(i\mathrm{Tr}(vX_t^x))] = \frac{\exp(\mathrm{Tr}[iv(I_d - 2iq_t v)^{-1} m_t x m_t^T])}{\det(I_d - 2iq_t v)^{\alpha/2}}, \quad (2)$$

where $q_t = \int_0^t \exp(sb) a^T a \exp(sb^T) ds, m_t = \exp(tb)$.
In particular, if $\tilde{X}_t^x \sim WIS_d(x, \alpha, 0, I_d^n; t)$, where $(I_d^n)_{i,j} = \mathbf{1}_{i=j\leq n}$,

$$\forall v \in \mathcal{S}_d(\mathbb{R}), \mathbb{E}[\exp(i\mathrm{Tr}(v\tilde{X}_t^x))] = \frac{\exp(\mathrm{Tr}[iv(I_d - 2itI_d^n v)^{-1} x])}{\det(I_d - 2itI_d^n v)^{\alpha/2}}.$$

Affine diffusions on nonnegative symmetric matrices

Cuchiero et al. consider the following dynamics for General Affine processes (we exclude here jumps) :

$$dX_t = (\bar{\alpha} + B(X_t))dt + \sqrt{X_t}dW_t a + a^T dW_t^T \sqrt{X_t}, X_0 = x \in \mathcal{S}_d(\mathbb{R}). \quad (3)$$

$\bar{\alpha} \in \mathcal{S}_d(\mathbb{R})$, $a \in \mathcal{M}_d(\mathbb{R})$ and $B : \mathcal{S}_d(\mathbb{R}) \to \mathcal{S}_d(\mathbb{R})$ is a linear mapping such that $\mathrm{Tr}(B(x)z) \geq 0$ if $\mathrm{Tr}(xz) = 0$ for $x, z \in \mathcal{S}_d^+(\mathbb{R})$.
The Wishart SDE (1) is the particular case where

$$\bar{\alpha} = \alpha a^T a,\ B(x) = bx + xb^T.$$

- If $\bar{\alpha} - (d+1)a^T a \in \mathcal{S}_d^+(\mathbb{R})$, (3) has a unique strong solution.
- If $\bar{\alpha} - (d-1)a^T a \in \mathcal{S}_d^+(\mathbb{R})$, (3) has a unique weak solution.

The characteristic function of X_t can be obtained by solving ODEs.

An application of Wishart processes in finance

Gourieroux and Sufana (2004) consider the following dynamics for d assets :

$$d \begin{bmatrix} \log(S_t^1) \\ \vdots \\ \log(S_t^d) \end{bmatrix} = \left(r - \begin{bmatrix} (X_t)_{1,1}/2 \\ \vdots \\ (X_t)_{d,d}/2 \end{bmatrix} \right) dt + \sqrt{X_t} dZ_t, \quad (4)$$

where X_t solves (1) and $(Z_t, t \geq 0)$ is a d-dimensional Brownian motion independent of $(W_t, t \geq 0)$, and $D_i \in \mathcal{M}_d(\mathbb{R})$.

- $d = 1$: Heston model without correlation.
- The characteristic function of $(\log(S_t^1), \ldots, \log(S_t^d))^T$ can be calculated by solving ODEs.
- Dependence between W and Z has been considered in Da Fonseca and al. (2008) to keep the Affine structure.
- X_t is the instantaneous covariance matrix, i.e. $\langle d\log(S_t^i), d\log(S_t^j) \rangle = (X_t)_{i,j}$ dt.

Existing results on the simulation of Wishart processes

Exact simulation. To the best of our knowledge, exact simulation algorithms for Wishart distribution only exist in the literature for integer degrees $\alpha \in \mathbb{N}, \alpha \geq d-1$ (Odell and Feiveson (1966) and Gleser (1975)).

Approximation schemes.

- the Euler-Maruyama scheme is not well-defined exactly as for the CIR process.
- Very few and recent literature. Gauthier and Possamaï (2009) and Benabid, Bensusan and El Karoui (2010).

A first simple remark

We consider two SDEs $X_t^{1,x}$ and $X_t^{2,x}$ associated respectively to the operators L_1 and L_2 and defined on the same domain $\mathbb{D}$. We assume that :

$$L_1 L_2 = L_2 L_1.$$

Then, $\mathbb{E}[f(X_t^{1,X_t^{2,x}})] = \mathbb{E}[\mathbb{E}[f(X_t^{1,X_t^{2,x}})|X_t^{2,x}]] = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbb{E}[L_1^k f(X_t^{2,x})] = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{t^{k+l}}{k!l!} L_2^l L_1^k f(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!}(L_1+L_2)^k f(x)$.
Thus, if we have exact schemes for L_1 and L_2, then we have an exact scheme for L_1+L_2 simply by taking the composition of both schemes.

The extended Cholesky decomposition

Let $q \in \mathcal{S}_d^+(\mathbb{R})$ be a matrix with rank r. Then there is a permutation matrix p, an invertible lower triangular matrix $c_r \in \mathcal{G}_r(\mathbb{R})$ and $k_r \in \mathcal{M}_{d-r\times r}(\mathbb{R})$ such that :

$$pqp^T = cc^T,\ c = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix}.$$

The triplet (c_r, k_r, p) is called an extended Cholesky decomposition of q. Besides, $\tilde{c} = \begin{pmatrix} c_r & 0 \\ k_r & I_{d-r} \end{pmatrix} \in \mathcal{G}_d(\mathbb{R})$, and we have :

$$q = (\tilde{c}^T p)^T I_d^r \tilde{c}^T p,$$

where $(I_d^r)_{i,j} = \mathbf{1}_{i=j\leq r}$.

Reduction to the case $b = 0, a = I_d^n$

We use the characteristic function (2). Let $n = \mathrm{Rk}(q_t)$. There is $\theta_t \in \mathcal{G}_d(\mathbb{R})$ such that $q_t/t = \theta_t I_d^n \theta_t^T$.

$$\begin{aligned}
\det(I_d - 2iq_t v) &= \det(\theta_t(\theta_t^{-1} - 2itI_d^n\theta_t^T v)) = \det(I_d - 2itI_d^n\theta_t^T v\theta_t),\\
\mathrm{Tr}[iv(I_d - 2iq_t v)^{-1} m_t x m_t^T] &= \mathrm{Tr}[i(\theta_t^{-1})^T\theta_t^T v(\theta_t\theta_t^{-1} - 2i\theta_t I_d^n\theta_t^T v\theta_t\theta_t^{-1})^{-1} m_t x m_t^T]\\
&= \mathrm{Tr}[i\theta_t^T v\theta_t(I_d - 2iI_d^n\theta_t^T v\theta_t)^{-1}\theta_t^{-1} m_t x m_t^T(\theta_t^{-1})^T].
\end{aligned}$$

$\implies \mathbb{E}[\exp(i\mathrm{Tr}(vX_t^x))] = \mathbb{E}[\exp(i\mathrm{Tr}(\theta_t^T v\theta_t \tilde{X}_t^{\theta_t^{-1}m_t x m_t^T(\theta_t^{-1})^T}))] =$ $\mathbb{E}[\exp(i\mathrm{Tr}(v\theta_t \tilde{X}_t^{\theta_t^{-1}m_t x m_t^T(\theta_t^{-1})^T}\theta_t^T))]$, i.e.

$$WIS_d(x, \alpha, b, a; t) \underset{Law}{=} \theta_t WIS_d(\theta_t^{-1} m_t x m_t^T(\theta_t^{-1})^T, \alpha, 0, I_d^n; t)\theta_t^T. \qquad (5)$$

It is therefore sufficient to simulate exactly a Wishart process with $a = I_d^n$ and $b = 0$.

A remarkable splitting when $a = I_d^n$ and $b = 0$

The infinitesimal generator of a Wishart process with $b = 0$ and $a = I_d^n$ is :

$$L = \mathrm{Tr}(\alpha I_d^n D) + 2\mathrm{Tr}(xDI_d^n D), \text{ with } D_{i,j} = \partial_{i,j}$$

for $f : \mathcal{M}_d(\mathbb{R}) \to \mathbb{R}$ s.t. $f(x) = f(x^T)$ for $1 \le i, j \le d$.
$I_d^n = \sum_{i=1}^n e_d^i$ with $(e_d^i)_{k,l} = \mathbf{1}_{k=l=i}$. We set $L_i = \mathrm{Tr}(\alpha e_d^i D) + 2\mathrm{Tr}(xDe_d^i D)$.

Proposition 1

$$L = L_1 + \cdots + L_n, \text{ with } L_iL_j = L_jL_i, \text{ and where}$$

- *L_i is the same operator as L_1 by permuting ith and first coordinates.*
- *L_1 is the operator of a Wishart process with $b = 0$ and $a = I_d^1$, which is thus well defined on $\mathcal{S}_d^+(\mathbb{R})$.*

Consequence : It is sufficient to sample an exact scheme for L_1 to get an exact scheme for L. This can be done !

Exact scheme for L_1 when $d = 2$ ($\alpha \geq d-1=1$)

For $f : \mathcal{S}_d(\mathbb{R}) \to \mathbb{R}$, $L_1 f(x) = \alpha \partial_{\{1,1\}} f(x) + 2x_{1,1}\partial^2_{\{1,1\}} f(x) + 2x_{1,2}\partial_{\{1,1\}}\partial_{\{1,2\}} f(x) + \frac{x_{2,2}}{2}\partial^2_{\{1,2\}} f(x)$. It is associated to the following SDE when $(X_0)_{2,2} > 0$

$$\begin{cases} d(X_t)_{1,1} = \alpha dt + 2\sqrt{(X_t)_{1,1} - \frac{(X_t)^2_{1,2}}{(X_t)_{2,2}}} dB^1_t + 2\frac{(X_t)_{1,2}}{\sqrt{(X_t)_{2,2}}} dB^2_t \\ d(X_t)_{1,2} = \sqrt{(X_t)_{2,2}} dB^2_t, \ (X_t)_{2,1} = (X_t)_{1,2}, \\ d(X_t)_{2,2} = 0 \end{cases}$$

and if $(X_0)_{2,2} = 0$:

$$d(X_t)_{1,1} = \alpha dt + 2\sqrt{(X_t)_{1,1}} dB^1_t, \ d(X_t)_{1,2} = d(X_t)_{2,2} = 0.$$

In the second case : CIR that can be simulated exactly (e.g. Glasserman) In the first case, we set $U_t = (X_t)_{1,1} - ((X_t)_{1,2})^2/(X_t)_{2,2}$:

$dU_t = (\alpha - 1)dt + \sqrt{U_t} dB^1_t$: CIR indep. of $(X_t)_{1,2} \sim \mathcal{N}((X_0)_{1,2}, (X_0)_{2,2}t)$.

Exact scheme for L_1 when $d \geq 3$ $(\alpha \geq d-1)$ I

Up to a permutation, $(x)_{2\leq i,j\leq d} = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix} \begin{pmatrix} c_r^T & k_r^T \\ 0 & 0 \end{pmatrix} =: cc^T$.
We can show that L_1 is the generator of the SDE :

$$\begin{aligned}
d(X_t^x)_{1,1} &= \alpha dt + 2\sqrt{(X_t^x)_{1,1} - \sum_{k=1}^r \left(\sum_{l=1}^r (c_r^{-1})_{k,l}(X_t^x)_{1,l+1}\right)^2} dZ_t^1 \\
&\quad +2\sum_{k=1}^r \sum_{l=1}^r (c_r^{-1})_{k,l}(X_t^x)_{1,l+1} dZ_t^{k+1} \\
d(X_t^x)_{1,i} &= \sum_{k=1}^r c_{i-1,k} dZ_t^{k+1} = d(X_t^x)_{i,1},\ i = 2,\dots,d \\
d(X_t^x)_{l,k} &= 0, \quad \text{for } 2 \leq k,l \leq d.
\end{aligned} \tag{6}$$

The SDE associated to L_1 can be solved explicitly as for $d=2$, and requires the sampling of 1 CIR distribution and $r-1$ standard Gaussian variables that are independent :

Exact scheme for L_1 when $d \geq 3$ ($\alpha \geq d-1$) II

$$X_t^x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix} \times$$

$$\begin{pmatrix} (U_t^u)_{1,1} + \sum_{k=1}^{r} ((U_t^u)_{1,k+1})^2 & ((U_t^u)_{1,l+1})^T_{1\leq l \leq r} & 0 \\ ((U_t^u)_{1,l+1})_{1\leq l\leq r} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r^T & k_r^T \\ 0 & 0 & I_{d-r-1} \end{pmatrix},$$

where

$$\begin{aligned} d(U_t^u)_{1,1} &= (\alpha - r)dt + 2\sqrt{(U_t^u)_{1,1}}dZ_t^1, \\ u_{1,1} &= x_{1,1} - \textstyle\sum_{k=1}^r (u_{1,k+1})^2 \geq 0, \\ d((U_t^u)_{1,l+1})_{1\leq l\leq r} &= (dZ_t^{l+1})_{1\leq l\leq r}, \\ (u_{1,l+1})_{1\leq l\leq r} &= c_r^{-1}(x_{1,l+1})_{1\leq l\leq r}. \end{aligned} \tag{7}$$

Remarks on the exact scheme

- when the initial value $x \in \mathcal{S}_d^{+,*}(\mathbb{R})$, we only make usual Cholesky decompositions.
- For $x = 0$, $b = 0$ and $t = 1$, our exact scheme gives back the Bartlett's decomposition (1933) :

$$\begin{pmatrix} (L_{i,j})_{1\le i,j\le n} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (L_{i,j}^T)_{1\le i,j\le n} & 0 \\ 0 & 0 \end{pmatrix} \sim WIS_d(0, \alpha, 0, I_d^n; 1),$$

  where $L_{i,j} \sim \mathcal{N}(0,1)$, $i > j$ and $(L_{i,i})^2 \sim \chi^2(\alpha - i + 1)$ are independent and $L_{i,j} = 0$ for $i < j$.
- The exact scheme has a complexity of $O(d^4)$ operations ($n \le d$ Cholesky), $O(d^2)$ Gaussian samples, $O(d)$ CIR samples. [A totally different exact scheme in $O(d^3)$ is possible for $\alpha \ge 2d - 1$]

Assumptions

$$t \geq 0,\ X_t^x = x + \int_0^t b(X_s^x)ds + \int_0^t \sigma(X_s^x)dB_s.$$

Assumption : domain $\mathbb{D} \subset \mathbb{R}^\zeta$, $\forall x \in \mathbb{D}, \mathbb{P}(\forall t \geq 0, X_t^x \in \mathbb{D}) = 1$;
$b_i(x), (\sigma(x)\sigma^T(x))_{i,j} \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$.

$$\begin{aligned}\mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D}) &= \{f \in \mathcal{C}^\infty(\mathbb{D}, \mathbb{R}), \forall \gamma \in \mathbb{N}^\zeta, \exists C_\gamma > 0, e_\gamma \in \mathbb{N}^*, \forall x \in \mathbb{D}, \\ &\quad |\partial_\gamma f(x)| \leq C_\gamma(1 + \|x\|^{e_\gamma})\}\end{aligned}$$

Associated operator :
$Lf(x) = \sum_{i=1}^\zeta b_i(x)\partial_i f(x) + \frac{1}{2}\sum_{i,j,k=1}^\zeta \sigma_{i,k}(x)\sigma_{j,k}(x)\partial_i\partial_j f(x)$.
Rem : $f \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D}) \implies Lf \in \mathcal{C}^\infty_{\mathbf{pol}}(\mathbb{D})$.

Notations for discretization schemes

Definition 2

A family of transition probabilities $(\hat{p}_x(t)(dz), t > 0, x \in \mathbb{D})$ on $\mathbb{D}$ is s.t. $\hat{p}_x(t)$ is a probability law on $\mathbb{D}$ for $t > 0$ and $x \in \mathbb{D}$. We note $\hat{X}^x_t$ a r.v. with law $\hat{p}_x(t)(dz)$.

Associated discretization scheme : $(\hat{X}^n_{t^n_i}, 0 \leq i \leq n)$ sequence of $\mathbb{D}$-valued r.v. s.t. : $\hat{X}^n_{t^n_{i+1}}$ is sampled according to $\hat{p}_{\hat{X}^n_{t^n_i}}(T/n)(dz)$.

Example (Euler) : $\hat{X}^x_t = x + b(x)t + \sigma(x)W_t$,
$\hat{p}_x(t)$: law density of $\hat{X}^x_t$.

Talay-Tubaro Theorem (1990)

If

1. $f : \mathbb{D} \to \mathbb{R}$ s. t. $u(t,x) = \mathbb{E}[f(X^x_{T-t})]$ is defined on $[0,T] \times \mathbb{D}$, solves for $t \in [0,T], x \in \mathbb{D}, \partial_t u(t,x) = -Lu(t,x)$, and has "good bounds" on all its derivatives $\partial_t^l \partial_\gamma u$, i.e.

   $$\forall l \in \mathbb{N}, \gamma \in \mathbb{N}^\zeta, \exists C_{l,\gamma}, e_{l,\gamma} > 0, \forall x \in \mathbb{D}, t \in [0,T],\ |\partial_t^l \partial_\gamma u(t,x)| \leq C_{l,\gamma}(1+\|x\|^{e_{l,\gamma}}).$$

2. the scheme is a **potential weak νth-order discr. scheme** for L :

   $$\mathbb{E}[f(\hat{X}^x_t)] \underset{t \to 0^+}{=} f(x) + \sum_{k=1}^{\nu} \frac{1}{k!} t^k L^k f(x) + \textit{Remainder in } t^{\nu+1}$$

   and $(\hat{X}^n_{t^n_i}, i = 0, \ldots, n)$ has uniformly bounded moments.

then, $|\mathbb{E}[f(\hat{X}^n_{t^n_n})] - \mathbb{E}[f(X^x_T)]| \leq K/n^\nu$.

Composition of discretization schemes I

$\hat{p}^1_x(t)(dz), \hat{p}^2_x(t)(dz)$: potential νth-order schemes for L_1, L_2.
$\hat{p}^2(\lambda_2 t) \circ \hat{p}^1_x(\lambda_1 t)(dz) = \int_{\mathbb{D}} \hat{p}^1_y(\lambda_2 t)(dz) \hat{p}^1_x(\lambda_1 t)(dy)$: scheme that amounts to first use the scheme 1 with a time step $\lambda_1 t$ and then the scheme 2 with a time step $\lambda_2 t$. $\hat{X}^{2\circ 1,x}_{\lambda_2 t, \lambda_1 t}$ a r.v. with this law.

Proposition 3

$$\mathbb{E}[f(\hat{X}^{2\circ 1,x}_{\lambda_2 t, \lambda_1 t})] \underset{t\to 0^+}{=} \sum_{l_1+l_2\leq \nu} \frac{\lambda_1^{l_1}\lambda_2^{l_2}}{l_1! l_2!} t^{l_1+l_2} L_1^{l_1} L_2^{l_2} f(x) + Remainder$$

$(= [I + \lambda_1 t L_1 f + \dots + \frac{(\lambda_1 t)^\nu}{\nu!} L_1^\nu f][I + \lambda_2 t L_2 f + \dots + \frac{(\lambda_2 t)^\nu}{\nu!} L_2^\nu f] + Rem)$
Csq : a scheme acts on f "as" the operator $I + tLf + \dots + \frac{t^\nu}{\nu!} L^\nu f + Rem$.

Composition of schemes = Composition of operators.

Composition of discretization schemes II

Corollary 4

If $\hat{p}^1_x, \hat{p}^2_x$ are potential νth-order schemes for L_1, L_2 and $L_1 L_2 = L_2 L_1$, $\hat{p}^1(t) \circ \hat{p}^2_x(t)$ is a potential νth-order schemes for $L_1 + L_2$.

Corollary 5

$\hat{p}^1_x, \hat{p}^2_x$: potential 2nd order schemes for L_1, L_2. Then,

$$\hat{p}^2(t/2) \circ \hat{p}^1(t) \circ \hat{p}^2_x(t/2) \quad \textit{(Strang 1968)} \tag{8}$$

$$\frac{1}{2}\left(\hat{p}^2(t) \circ \hat{p}^1_x(t) + \hat{p}^1(t) \circ \hat{p}^2_x(t)\right) \tag{9}$$

are potential second order schemes for $L_1 + L_2$.

Proof for (9) : $(I + tL_1 + t^2/2L_1^2 + \dots)(I + tL_2 + t^2/2L_2^2 + \dots) =$
$I + t(L_1 + L_2) + t^2/2(L_1^2 + L_2^2 + 2L_1L_2) + \dots$

High order schemes for the operator L_1

Result : If we replace in the exact scheme for L_1 (7),

- $(U_t^u)_{1,1}$ by $(\hat{U}_t^u)_{1,1}$ sampled with a potential νth-order scheme for the CIR : $dU_t = (\alpha - r)dt + 2\sqrt{U_t}dZ_t$,
- $(U_t^u)_{1,l+1}$ by $\sqrt{t}\hat{G}^{l+1}$ where $\hat{G}^{l+1}$ is a bounded variable s.t. $\forall k \leq 2\nu + 1, \mathbb{E}[(\hat{G}^{l+1})^k] = \mathbb{E}[G^k]$, where $G \sim \mathcal{N}(0,1)$,

we can show that we get a potential νth order scheme for L_1.
Second and third order schemes for the CIR can be found in A. 2008.
Here are some matching-moment variables for $\mathcal{N}(0,1)$ for $\nu = 2,\ 3$:

$$\mathbb{P}(\hat{G}^i = \sqrt{3}) = \mathbb{P}(\hat{G}^i = -\sqrt{3}) = \frac{1}{6} \text{ and } \mathbb{P}(\hat{G}^i = 0) = \frac{2}{3}$$

$$(\text{resp. } \mathbb{P}\left(\hat{G}^i = \varepsilon\sqrt{3+\sqrt{6}}\right) = \frac{\sqrt{6}-2}{4\sqrt{6}},\ \mathbb{P}\left(\hat{G}^i = \varepsilon\sqrt{3-\sqrt{6}}\right) = \frac{1}{2} - \frac{\sqrt{6}-2}{4\sqrt{6}},\ \varepsilon \in \{-1,1\}).$$

A third order scheme for Wishart processes

We use once again the splitting given by Proposition 1.

- We have a third order scheme for L_1.
- By a permutation of the first and i^{th} coordinate, we get also a third order scheme for L_i.
- By Corollary 4, we get a third order scheme $\hat{X}_t^x$ for a Wishart process with $a = I_d^n$ and $b = 0$.
- Last, we can show from (5) (under some assumptions) that $\theta_t \hat{X}_t^{\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T} \theta_t^T$ is a third order scheme.

Second order scheme for general Affine processes I

A First remark :
Let $dX_t = (\bar{\alpha} + B(X_t))dt + \sqrt{X_t}dW_t a + a^T dW_t^T \sqrt{X_t}, X_0 = x \in \mathcal{S}_d(\mathbb{R})$.
There is $u \in \mathcal{G}_d(\mathbb{R})$ such that

$$(X_t)_{t\geq 0} \underset{Law}{=} (u^T \tilde{X}_t u)_{t\geq 0},$$

where

- $d\tilde{X}_t = (\bar{\delta} + B_u(\tilde{X}_t))dt + \sqrt{\tilde{X}_t}dW_t I_d^n + I_d^n dW_t^T \sqrt{\tilde{X}_t}, \tilde{X}_0 = (u^{-1})^T x u^{-1}$,
- $B_u(x) = (u^{-1})^T B(u^T x u) u^{-1}$,
- $\bar{\delta}$ is a diagonal matrix such that $\bar{\delta} - (d-1)I_d^n \in \mathcal{S}_d^+(\mathbb{R})$

$\Longrightarrow$ It is sufficient to get a scheme for $\tilde{X}_t$ (i.e. when $a = I_d^n$ and $\bar{\alpha}$ is a diagonal matrix).

Second order scheme for general Affine processes II

Let $\delta_{\min} = \min_{i=1,\dots,n} \bar{\delta_{i,i}} \geq d-1$. We split the generator of $\tilde{X}_t$:

$$\begin{aligned} L &= \mathrm{Tr}([\overline{\delta} + B(x)]D^{\mathcal{S}}) + 2\mathrm{Tr}(xD^{\mathcal{S}} I_d^n D^{\mathcal{S}}) \\ &= \underbrace{\mathrm{Tr}([\overline{\delta} - \delta_{\min} I_d^n + B_u(x)]D^{\mathcal{S}})}_{L_{ODE}} + \underbrace{\delta_{\min}\mathrm{Tr}(D^{\mathcal{S}}) + 2\mathrm{Tr}(xD^{\mathcal{S}} I_d^n D^{\mathcal{S}})}_{L_{WIS_d(x,\delta_{\min},0,I_d^n)}}, \end{aligned}$$

where L_{ODE} is associated to the affine ODE $x'(t) = \overline{\delta} - \delta_{\min} I_d^n + B_u(x(t))$ that can be solved explicitly and is such that $x(t) \in \mathcal{S}_d^+(\mathbb{R})$ for $t \geq 0$. By Corollary 5, we get a second order scheme for $\tilde{X}_t$ and thus for X_t.

A faster second order scheme when $\bar{\alpha} - dI_d^n \in \mathcal{S}_d^+(\mathbb{R})$

All the previous schemes rely on the splitting given by Proposition 1 and require thus $O(d^4)$ operations.
Remark : We can check that if $c^T c = x$, $(c + W_t I_d^n)^T (c + W_t I_d^n)$ is a Wishart process with $\alpha = d$, $a = I_d^n$, $b = 0$ starting from x. Also, $(c + \sqrt{t}\hat{G} I_d^n)^T (c + \sqrt{t}\hat{G} I_d^n)$ is a potential second order scheme for $WIS_d(x, d, 0, I_d^n)$ where $\hat{G}$ is a matrix with independent elements sampled according to (10).
Consequence : By using the splitting :

$$L = \underbrace{\mathrm{Tr}([\overline{\delta} - dI_d^n + B_u(x)]D^{\mathcal{S}})}_{\tilde{L}_{ODE}} + \underbrace{d\mathrm{Tr}(D^{\mathcal{S}}) + 2\mathrm{Tr}(xD^{\mathcal{S}} I_d^n D^{\mathcal{S}})}_{L_{WIS_d(x,d,0,I_d^n)}},$$

we get a by Corollary 5 a second order scheme for $\tilde{X}_t$ in $O(d^3)$ operations.

A modified Euler scheme

The Euler scheme for the Affine diffusion (3) is :

$$\hat{X}_{t_{i+1}} = \hat{X}_{t_i} + (\bar{\alpha} + B(\hat{X}_{t_i}))(t_{i+1} - t_i) + \sqrt{\hat{X}_{t_i}}(W_{t_{i+1}} - W_{t_i})a + a^T(W_{t_{i+1}} - W_{t_i})^T\sqrt{\hat{X}_{t_i}}.$$

It is not well-defined since $\hat{X}_{t_{i+1}}$ may not be nonnegative.
Corrected Euler scheme :

$$\begin{aligned} \hat{X}_{t_{i+1}} &= \hat{X}_{t_i} + (\bar{\alpha} + B(\hat{X}_{t_i}))(t_{i+1} - t_i) \\ &\quad + \sqrt{(\hat{X}_{t_i})^+}(W_{t_{i+1}} - W_{t_i})a + a^T(W_{t_{i+1}} - W_{t_i})^T\sqrt{(\hat{X}_{t_i})^+}, \end{aligned}$$

where $\sqrt{(}x^+) := odiag(\sqrt{\lambda_1^+}, \dots, \sqrt{\lambda_d^+})o^{-1}$ for $x \in \mathcal{S}_d(\mathbb{R})$ and $x = odiag(\lambda_1, \dots, \lambda_d^+)o^{-1}$.

A time comparison (10^6 samples, N time-steps)

	$N = 10$			$N = 30$		
Schemes	R. value	Im. value	Time	R. value	Im. value	Time
Exact (1 step)	-0.526852	-0.227962	12			
2^{nd} order bis	-0.526229	-0.228663	41	-0.526486	-0.229078	125
2^{nd} order	-0.526577	-0.228923	76	-0.526574	-0.228133	229
3^{rd} order	-0.527021	-0.227286	82	-0.527613	-0.228376	244
Exact (N steps)	-0.526963	-0.228303	123	-0.526891	-0.227729	369
Corrected Euler	-0.525627^*	-0.233863^*	225	-0.525638^*	-0.231449^*	687

$\alpha = 3.5$, $d = 3$, $\Delta_R = 10^{-3}$, $\Delta_{Im} = 10^{-3}$, exact value R. = -0.527090 and Im.= -0.228251

	$N = 10$			$N = 30$		
Exact (1 step)	-0.591579	-0.037651	12			
2^{nd} order	-0.590444	-0.037024	77	-0.590808	-0.036487	229
3^{rd} order	-0.591234	-0.034847	82	-0.590818	-0.036210	246
Exact (N steps)	-0.591169	-0.036618	174	-0.592145	-0.037411	920
Corrected Euler	-0.589735^*	-0.042002^*	223	-0.590079^*	-0.039937^*	680

$\alpha = 2.2$, $d = 3$, $\Delta_R = 0.9 \times 10^{-3}$, $\Delta_{Im} = 1.3 \times 10^{-3}$, exact value R. = -0.591411 and Im.= -0.036346

	$N = 10$			$N = 30$		
Exact (1 step)	0.062712	-0.063757	181			
2^{nd} order bis	0.064237	-0.063825	921	0.064573	-0.062747	2762
2^{nd} order	0.064922	-0.064103	1431	0.063534	-0.063280	4283
3^{rd} order	0.064620	-0.064543	1446	0.064120	-0.063122	4343
Exact (N steps)	0.063418	-0.064636	1806	0.063469	-0.064380	5408
Corrected Euler	0.068298^*	-0.058491^*	2312	0.061732^*	-0.056882^*	7113

$\alpha = 10.5$, $d = 10$, $\Delta_R = 1.4 \times 10^{-3}$, $\Delta_{Im} = 1.3 \times 10^{-3}$, exact value R. = 0.063960 and Im.= -0.063544

	$N = 10$			$N = 30$		
Exact (1 step)	-0.036869	-0.094156	177			
2^{nd} order	-0.036246	-0.094196	1430	-0.035944	-0.092770	4285
3^{rd} order	-0.035408	-0.093479	1441	-0.036277	-0.093178	4327
Exact (N steps)	-0.036478	-0.092860	1866	-0.036145	-0.093003	6385
Corrected Euler	-0.028685^*	-0.094281^*	2321	-0.030118^*	-0.088988^*	7144

$\alpha = 9.2$, $d = 10$, $\Delta_R = 1.4 \times 10^{-3}$, $\Delta_{Im} = 1.4 \times 10^{-3}$, exact value R. = -0.036064 and Im.= -0.093275

Weak convergence I

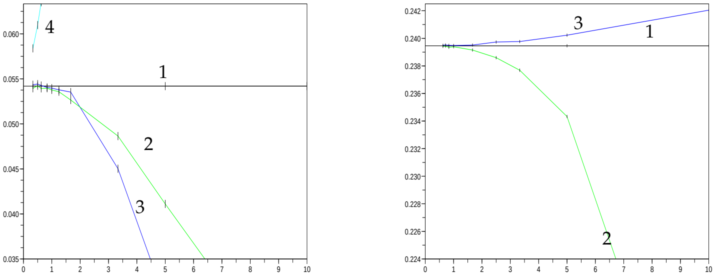
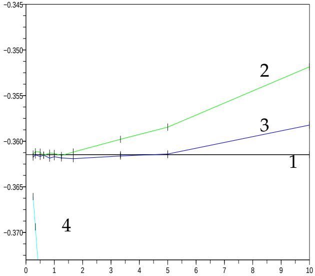


FIGURE: $d = 3$, 10^7 MC samples, $T = 10$. $\Re(\mathbb{E}[\exp(-\mathrm{Tr}(iv\hat{X}^N_{t^N_N}))])$ in fct of T/N. Left : $v = 0.05I_d$, $x = 0.4I_d$, $\alpha = 4.5$, $a = I_d$ and $b = 0$. Exact value : 0.054277. Right : $v = 0.2I_d + 0.04q$ $x = 0.4I_d + 0.2q$, $\alpha = 2.22$, $a = I_d$ and $b = -0.5I_d$. Exact value : 0.239836. $q_{i,j} = \mathbf{1}_{i \neq j}$.

Weak convergence II



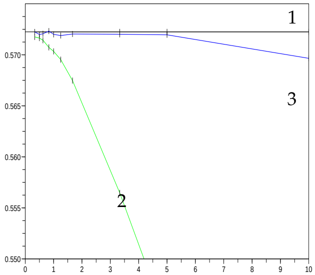
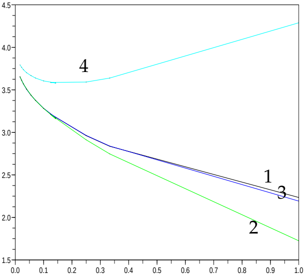


FIGURE: $d = 10$, 10^7 MC samples, $T = 10$. Left : $\Im(\mathbb{E}[\exp(-\mathrm{Tr}(iv\hat{X}^N_{t^N_N}))])$ with $v = 0.009I_d$ in fct of T/N. $x = 0.4I_d$, $\alpha = 12.5$, $b = 0$ and $a = I_d$. Ex. value : -0.361586. Right : $\Re(\mathbb{E}[\exp(-\mathrm{Tr}(iv\hat{X}^N_{t^N_N}))])$ with $v = 0.009I_d$ in fct of T/N. $x = 0.4I_d$, $\alpha = 9.2$, $b = -0.5I_d$ and $a = I_d$. Ex. value 0.572241.

Convergence on pathwise expectations



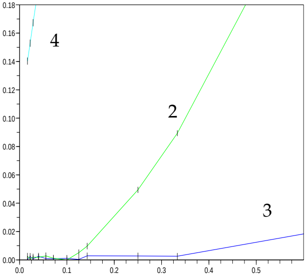


FIGURE: $d = 3$, 10^7 MC samples, $T = 1$. $x = 0.4I_d + 0.2q$ with $q_{i,j} = \mathbf{1}_{i \neq j}$, $\alpha = 2.2$, $b = 0$ and $a = I_d$. Left, $\mathbb{E}[\max_{0 \leq k \leq N} \mathrm{Tr}(\hat{X}^N_{t^N_k})]$, right : $\mathbb{E}[\max_{0 \leq k \leq N} \mathrm{Tr}(\hat{X}^N_{t^N_k})] - \mathbb{E}[\max_{0 \leq k \leq N} \mathrm{Tr}(X^x_{t^N_k})]$ in fct of T/N.

A scheme for the Gourieroux-Sufana model

In (4), the joint operator of (S_t, X_t) is

$$L = L^S + L^X, \text{ where } L^S = \sum_{i=1}^{d} r s_i \partial_{s_i} + \frac{1}{2} \sum_{i,j=1}^{d} s_i s_j x_{i,j} \partial_{s_i} \partial_{s_j},$$

and L^X is the generator of a Wishart process. We can solve explicitly the SDE associated to L^S : $S^l_t = S^l_0 \exp[(r - x_{l,l}/2)t + (\sqrt{x} Z_t)_l]$.
By using a second order scheme for L^X, we get a second order scheme for L by Corollary 5.

Put option in the Gourieroux-Sufana model

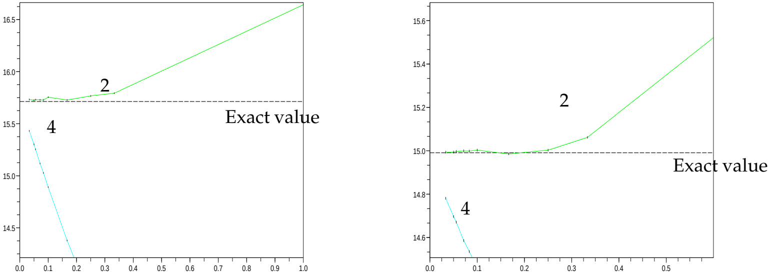


FIGURE: $\mathbb{E}[e^{-rT}(K - \max(\hat{S}^{1,N}_{t^N_N}, \hat{S}^{2,N}_{t^N_N}))^+]$ in fct of T/N. $d = 2$, $T = 1$, $K = 120$, $S^1_0 = S^2_0 = 100$, and $r = 0.02$. $x = 0.04 I_d + 0.02 q$ with $q_{i,j} = \mathbf{1}_{i \neq j}$, $a = 0.2 I_d$, $b = 0.5 I_d$ and $\alpha = 4.5$ (left), $\alpha = 1.05$ (right). 10^6 Monte-Carlo samples.

Summary

We have obtained using splitting methods :

- an exact simulation algorithm for Wishart processes,
- second and third order schemes for Wishart processes,
- second order scheme for affine processes on nonnegative matrices.

The discretization schemes are much more accurate and less time-consuming than the modified Euler scheme.
Which scheme to use ? We recommend the exact scheme to calculate expectations that depends on one or few dates. For pathwise expectations, we recommend instead to use discretization schemes : the second order scheme “bis” if it is defined and the second/third order scheme otherwise.

Wright-Fisher (or Jacobi) processes

Up to our knowledge, there is no literature on particular diffusions defined on correlation matrices in dimension d.
With 2 assets, X_t is a correlation matrix in dimension 2 iff $X_t = \begin{bmatrix} 1 & \rho_t \\ \rho_t & 1 \end{bmatrix}$, with $\rho_t \in [-1,1]$. A frequent choice (e.g Jun Ma 2009) is to consider a Wright-Fisher (or Jacobi) process

$$d\rho_t = \kappa(\bar{\rho} - \rho_t)dt + \sigma\sqrt{1-\rho_t^2}dB_t,$$

$\kappa > 0, \bar{\rho} \in [-1,1], \sigma \geq 0$.
Properties : mean-reversion, explicit calculation of moments, ergodic law.
The processes that we present extend naturally in dimension d these processes.

Correlation processes in higher dimension

Up to our knowledge, the existing processes in the literature rely on a parametrization of a subset of correlation matrices $\mathfrak{C}_d(\mathbb{R})$. For example, Driessen and Maenhout (2006) consider :

$$(C_t)_{i,j} = \mathbf{1}_{i=j} + \rho_t \mathbf{1}_{i \neq j},\ 1 \leq i,j \leq d,$$

with

$$d\rho_t = \kappa(\bar{\rho} - \rho_t)dt + \sigma\sqrt{\rho_t(1-\rho_t)}dB_t,$$

$\kappa > 0, \bar{\rho} \in [0,1], \sigma \geq 0$. Other choices can be found in the working paper of Christopher Kaya Boortz (2008).

Motivations to get an SDE on correlation matrices

- As far as Index options are concerned, one factor correlation may be sufficient since the Index has a "mean effect" on correlations.
- However, one may want to incorporate some views on the market (correlation between some companies or sectors...) in order to price more exotic options.
- A one factor correlation or parametrization of subsets of correlation matrices include (at least theoretically !) some easy detectable arbitrage.
- Up to our knowledge, there is no literature on particular diffusions defined on correlation matrices.

The MRC process

$$
\begin{aligned}
X_t &= x + \int_0^t \left(\kappa(c - X_s) + (c - X_s)\kappa\right) ds \\
&\quad + \sum_{n=1}^{d} a_n \int_0^t \left(\sqrt{X_s - X_s e_d^n X_s}\, dW_s e_d^n + e_d^n dW_s^T \sqrt{X_s - X_s e_d^n X_s} \right),
\end{aligned} \tag{10}
$$

- $(e_d^n)_{i,j} = \mathbf{1}_{i=j=n}$, $1 \leq i, j \leq d$,
- W is a d-square matrix made with independent Brownian motions,
- $x, c \in \mathfrak{C}_d(\mathbb{R})$,
- $\kappa = diag(\kappa_1, \ldots, \kappa_d)$ and $a = diag(a_1, \ldots, a_d)$ are nonnegative diagonal matrices.

Intuitive parameters : mean-reversion towards c with a speed and a noise respectively tuned by κ, a.
Notations : $MRC_d(x, \kappa, c, a)$ law of $(X_t)_{t \geq 0}$.

Infinitesimal generator

Quadratic variation :

$$\begin{aligned}
&\langle d(X_t)_{i,j}, d(X_t)_{k,l} \rangle \\
= \quad &\Big[a_i^2(\mathbf{1}_{i=k}((X_t)_{j,l} - (X_t)_{i,j}(X_t)_{i,l}) + \mathbf{1}_{i=l}((X_t)_{j,k} - (X_t)_{i,j}(X_t)_{i,k})) \\
&+a_j^2(\mathbf{1}_{j=k}((X_t)_{i,l} - (X_t)_{j,i}(X_t)_{j,l}) + \mathbf{1}_{j=l}((X_t)_{i,k} - (X_t)_{j,i}(X_t)_{j,k}))\Big] dt.
\end{aligned}$$

$$\begin{aligned}
L \quad = \quad &\sum_{i=1}^{d} \Bigg(\sum_{\substack{1 \le j \le d \\ j \ne i}} \kappa_i (c_{\{i,j\}} - x_{\{i,j\}}) \partial_{\{i,j\}} \\
&+ \frac{1}{2} \sum_{\substack{1 \le j,k \le d \\ j \ne i, k \ne i}} a_i^2 (x_{\{j,k\}} - x_{\{i,j\}} x_{\{i,k\}}) \partial_{\{i,j\}} \partial_{\{i,k\}} \Bigg).
\end{aligned}$$

Existence and uniqueness results

Theorem 6

Let $x \in \mathfrak{C}_d^*(\mathbb{R})$. *If* $\kappa c + c\kappa - da^2 \in \mathcal{S}_d^+(\mathbb{R})$, *there is a unique strong solution of the SDE* (10) *that is such that* $\forall t \geq 0, X_t \in \mathfrak{C}_d^*(\mathbb{R})$.

Theorem 7

If $\kappa c + c\kappa - (d-2)a^2 \in \mathcal{S}_d^+(\mathbb{R})$ *or* $d = 2$, *there is a unique weak solution* $(X_t, t \geq 0)$ *to SDE* (10) *such that* $\mathbb{P}(\forall t \geq 0, X_t \in \mathfrak{C}_d(\mathbb{R})) = 1$.

Some properties I

- Each cross correlation follows a 1D WF process on $[-1,1]$:

$$d(X_t)_{i,j} = (\kappa_i + \kappa_j)(c_{i,j} - (X_t)_{i,j})dt + \sqrt{a_i^2 + a_j^2}\sqrt{1 - (X_t)_{i,j}^2}d\beta_t^{i,j}.$$

- Any principal sub-matrix of X_t follows a MRC process : Let $I = \{k_1 < \cdots < k_{d'}\} \subset \{1, \ldots, d\}$ and denote for $x \in \mathcal{M}_d(\mathbb{R})$, $(x^I)_{i,j} = x_{k_i,k_j}$ for $1 \leq i, j \leq d'$. We have :

$$(X_t^I)_{t\geq 0} \overset{law}{=} MRC_{d'}(x^I, \kappa^I, c^I, a^I).$$

Some properties II

- Explicit calculation of moments ($\implies$ weak uniqueness). Let $m \in \mathcal{S}_d(\mathbb{N})$ such that $m_{i,i} = 0$ for $1 \leq i \leq d$. $x^m = \prod_{1\leq i \leq j \leq d} x_{\{i,j\}}^{m_{\{i,j\}}}$ and $|m| = \sum_{1 \leq i \leq j \leq d} m_{\{i,j\}}$. We have

$$\mathbb{E}\left[X_t^m\right] = x^m \exp(-tK_m) + \exp(-tK_m) \int_0^t \exp(sK_m)\mathbb{E}[f_m(X_s)]ds,$$

with $K_m = \sum_{i=1}^d \sum_{j=1}^d \kappa_i m_{\{i,j\}} + \frac{1}{2} \sum_{i=1}^d a_i^2 \sum_{j,k=1}^d m_{\{i,j\}} m_{\{i,k\}}$ and
$f_m(x) = \sum_{i=1}^d \sum_{j=1}^d \kappa_i c_{\{i,j\}} m_{\{i,j\}} x^{m-e_d^{\{i,j\}}} + \frac{1}{2} \sum_{i=1}^d a_i^2 \sum_{j,k=1}^d m_{\{i,j\}} m_{\{i,k\}} x^{m-e_d^{\{i,j\}}-e_d^{\{i,k\}}+e_d^{\{j,k\}}}$
is a polynomial function of degree smaller than $|m| - 1$.

- Ergodic law :

$$\begin{aligned} \mathbb{E}[X_\infty^m] &= x^m \text{ if } m \in \mathcal{S}_d(\mathbb{N}) \text{ is such that } m_{\{i,j\}} > 0 \iff \kappa_i = \kappa_j = 0, \\ \mathbb{E}[X_\infty^m] &= \mathbb{E}[f_m(X_\infty)]/K_m \text{ otherwise.} \end{aligned}$$

A Girsanov Theorem

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left(\int_0^T \mathrm{Tr}(H_s^T dW_s) - \frac{1}{2}\int_0^T \mathrm{Tr}(H_s^T H_s)ds\right),$$

Let $x \in \mathfrak{C}_d^*(\mathbb{R})$, $(X_t, t \geq 0) \sim MRC_d(x, \kappa^1, c^1, a)$ s.t. $\kappa^1 c^1 + c^1\kappa^1 - da^2 \in \mathcal{S}_d^+(\mathbb{R})$. Let c^2, κ^2 such that $a_i = 0 \implies \kappa_i^2 = 0$ and $\kappa^1 c^1 + c^1\kappa^1 + \kappa^2 c^2 + c^2\kappa^2 - da^2 \in \mathcal{S}_d^+(\mathbb{R})$. We set :

$$\lambda = diag(\lambda_1, \dots, \lambda_d) \text{ with } \lambda_i = \begin{cases} \kappa_i^2/a_i \text{ if } a_i > 0 \\ 0 \text{ otherwise} \end{cases} \quad \text{and } H_t = (\sqrt{X_t})^{-1} c^2 \lambda$$

Then, $(X_t, t \geq 0) \sim MRC_d(x, \kappa, c, a)$ under $\mathbb{Q}$, where $\kappa = \kappa^1 + \kappa^2$

$$c \in \mathfrak{C}_d(\mathbb{R}) \text{ s.t. } c_{i,j} = \begin{cases} \frac{(\kappa_i^1+\kappa_j^1)c_{i,j}^1 + (\kappa_i^2+\kappa_j^2)c_{i,j}^2}{\kappa_i+\kappa_j} \text{ if } \kappa_i + \kappa_j > 0 \\ 0 \text{ if } \kappa_i + \kappa_j = 0. \end{cases}.$$

Connection with Wishart processes I

For $x \in \mathcal{S}_d^+(\mathbb{R})$ such that $x_{i,i} > 0$ for all $1 \leq i \leq d$, we define $\mathbf{p}(x) \in \mathfrak{C}_d(\mathbb{R})$ by

$$(\mathbf{p}(x))_{i,j} = \frac{x_{i,j}}{\sqrt{x_{i,i}x_{j,j}}}, \ 1 \leq i,j \leq d. \quad (11)$$

A natural idea to construct a process on $\mathfrak{C}_d(\mathbb{R})$ is to consider $X_t = \mathbf{p}(Y_t)$, where $(Y_t)_{t \geq 0}$ is a Wishart process $WIS_d(x, \alpha, b, a)$, i.e.

$$\begin{aligned} dY_t &= (\alpha a^T a + bY_t + Y_t b^T)dt + \sqrt{Y_t} dW_t a + a^T dW_t^T \sqrt{Y_t}, \ t \geq 0, \\ Y_0 &= x \in \mathcal{S}_d^+(\mathbb{R}). \end{aligned}$$

Problem : this does not lead in general to an autonomous SDE... unless in special cases !

Connection with Wishart processes II

Result : Let $\alpha \geq \max(1, d-2)$ and $y \in \mathcal{S}_d^+(\mathbb{R})$ such that $y_{i,i} > 0$ for $1 \leq i \leq d$. Let $(Y_t^y)_{t\geq 0} \sim WIS_d(y, \alpha+1, 0, e_d^1)$. Then, $(Y_t^y)_{i,i} = y_{i,i}$ for $2 \leq i \leq d$ and $(Y_t^y)_{1,1}$ follows a squared Bessel process of dimension $\alpha + 1$ and a.s. never vanishes. We set

$$X_t = \mathbf{p}(Y_t^y),\ \phi(t) = \int_0^t \frac{1}{(Y_s^y)_{1,1}} ds.$$

The function ϕ is a.s. one-to-one on $\mathbb{R}_+$ and defines a time-change such that :

$$(X_{\phi^{-1}(t)}, t \geq 0) \overset{law}{=} MRC_d(\mathbf{p}(y), \frac{\alpha}{2} e_d^1, I_d, e_d^1).$$

Besides, the processes $(X_{\phi^{-1}(t)}, t \geq 0)$ and $((Y_t^y)_{1,1}, t \geq 0)$ are independent.

Connection with Wishart processes III

Sketch of the proof : We have $d(Y_t^y)_{i,j} = 0$ for $2 \leq i,j \leq d$ and

$$d(Y_t^y)_{1,1} = (\alpha+1)dt + 2\sum_{k=1}^d (\sqrt{Y_t^y})_{1,k}(dW_t)_{k,1},\ d(Y_t^y)_{1,i} = \sum_{k=1}^d (\sqrt{Y_t^y})_{i,k}(dW_t)_{k,1}.$$

$d\langle (Y_t^y)_{1,1}\rangle = 4(Y_t^y)_{1,1}dt$ and $(Y_t^y)_{1,1}$ is a square Bessel process that never vanishes. We have :

$$d(X_t)_{1,i} \quad = \quad -\frac{\alpha}{2}(X_t)_{1,i}\frac{dt}{(Y_t^y)_{1,1}} + \sum_{k=1}^d \left(\frac{(\sqrt{Y_t^y})_{i,k}}{\sqrt{(Y_t^y)_{1,1}y_{i,i}}} - (X_t)_{1,i}\frac{(\sqrt{Y_t^y})_{1,k}}{(Y_t^y)_{1,1}} \right)(dW_t)_{k,1}$$

$$d\langle (X_t)_{1,i}, (X_t)_{1,j}\rangle = \frac{1}{(Y_t^y)_{1,1}}[(X_t)_{i,j} - (X_t)_{1,i}(X_t)_{1,j}]dt \text{ and } d\langle (X_t)_{1,i}, (Y_t^y)_{1,1}\rangle = 0.$$

Consequences

- There is a weak solution to $MRC_d(x, \frac{\alpha}{2} e_d^1, I_d, e_d^1)$.
- Up to a permutation of the first and lth coordinate, there is also a weak solution to $MRC_d(x, \frac{\alpha}{2} e_d^l, I_d, e_d^l)$, where $(e_d^l)_{i,j} = \mathbf{1}_{i=j=l}$.

We will denote by $L^{l,C}$ the infinitesimal generator of $MRC_d(x, \frac{d-2}{2} e_d^l, I_d, e_d^l)$.
Remark : the operators $L^{l,C}$ and $L^{k,C}$ commute for $1 \leq k, l \leq d$.

Linear ODEs on correlation matrices

Let $b \in \mathcal{S}_d(\mathbb{R})$, $\kappa \in \mathcal{M}_d(\mathbb{R})$ and consider the following linear ODE

$$x'(t) = b - (\kappa x(t) + x(t)\kappa^T),\ x(0) = x \in \mathfrak{C}_d(\mathbb{R}),$$

What are necessary and sufficient conditions on κ and b such that

$$\forall x \in \mathfrak{C}_d(\mathbb{R}), \forall t \geq 0, x(t) \in \mathfrak{C}_d(\mathbb{R})?$$

Necessary condition :$\exists c \in \mathfrak{C}_d(\mathbb{R}), \exists \kappa_1, \dots, \kappa_d \in \mathbb{R}$,

$$\forall i \neq j, \kappa_i + \kappa_j \geq 0, \kappa = diag(\kappa_1, \dots, \kappa_d) \text{ and } b = \kappa c + c\kappa.$$

Sufficient condition :$\kappa = diag(\kappa_1, \dots, \kappa_d)$ and $b = \kappa c + c\kappa \in \mathcal{S}_d^+(\mathbb{R})$

Putting the pieces together

Let $L^{i,C}$ the infinitesimal generator of $MRC_d(\mathbf{p}(y), \frac{d-2}{2}e_d^i, I_d, e_d^i)$. The SDE (10) is nothing but the one associated to generator :

$$\sum_{i=1}^{d} a_i^2 L^{i,C} + L^{ODE},$$

where L^{ODE} is the operator associated to

$$\xi'(t,x) = \kappa(c-x)+(c-x)\kappa-\frac{d-2}{2}[a^2(I_d-x)+(I_d-x)a^2],\ \xi(0,x) = x \in \mathfrak{C}_d(\mathbb{R}).$$

This (linear) ODE can be solved explicitly and such that $\forall t \geq 0, x \in \mathfrak{C}_d(\mathbb{R}), \xi(t,x) \in \mathfrak{C}_d(\mathbb{R})$ if $\kappa c + c\kappa - (d-2)a^2 \in \mathcal{S}_d^+(\mathbb{R})$.