

On the Almost Sure Location of the Singular Values of Certain Gaussian Block-Hankel Large Random Matrices

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Abstract This paper studies the almost sure location of the eigenvalues of matrices $\mathbf{W}_N \mathbf{W}_N^*$, where $\mathbf{W}_N = (\mathbf{W}_N^{(1)T}, \dots, \mathbf{W}_N^{(M)T})^T$ is a $ML \times N$ block-line matrix whose block-lines $(\mathbf{W}_N^{(m)})_{m=1, \dots, M}$ are independent identically distributed $L \times N$ Hankel matrices built from i.i.d. standard complex Gaussian sequences. It is shown that if $M \rightarrow +\infty$ and $\frac{ML}{N} \rightarrow c_*$ ($c_* \in (0, \infty)$), then the empirical eigenvalue distribution of $\mathbf{W}_N \mathbf{W}_N^*$ converges almost surely towards the Marcenko–Pastur distribution. More importantly, it is established using the Haagerup–Schultz–Thorbjørnsen ideas that if $L = O(N^\alpha)$ with $\alpha < 2/3$, then, almost surely, for N large enough, the eigenvalues of $\mathbf{W}_N \mathbf{W}_N^*$ are located in the neighbourhood of the Marcenko–Pastur distribution. It is conjectured that the condition $\alpha < 2/3$ is optimal.

Keywords Singular value limit distribution of random complex Gaussian large block-Hankel matrices · Almost sure location of the singular values · Marcenko–Pastur distribution · Poincaré–Nash inequality · Integration by parts formula

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1 Introduction

1.1 The Addressed Problem and the Results

In this paper, we consider independent identically distributed zero-mean complex-valued Gaussian random variables $(w_{m,n})_{m=1,\dots,M,n=1,\dots,N+L-1}$ such that $\mathbb{E}|w_{m,n}|^2 = \frac{\sigma^2}{N}$ and $\mathbb{E}(w_{m,n}^2) = 0$, where M, N, L are integers. We define the $L \times N$ matrices $(\mathbf{W}_N^{(m)})_{m=1,\dots,M}$ as the Hankel matrices whose entries are given by

$$\left(\mathbf{W}_N^{(m)}\right)_{i,j} = w_{m,i+j-1}, \quad 1 \leq i \leq L, 1 \leq j \leq N \tag{1.1}$$

and \mathbf{W}_N represents the $ML \times N$ matrix

$$\mathbf{W}_N = \begin{pmatrix} \mathbf{W}_N^{(1)} \\ \mathbf{W}_N^{(2)} \\ \vdots \\ \mathbf{W}_N^{(M)} \end{pmatrix} \tag{1.2}$$

In this paper, we establish that:

- the eigenvalue distribution of $ML \times ML$ matrix $\mathbf{W}_N \mathbf{W}_N^*$ converges towards the Marcenko–Pastur distribution when $M \rightarrow +\infty$ and when ML and N both converge towards $+\infty$ in such a way that $c_N = \frac{ML}{N}$ satisfies $c_N \rightarrow c_*$, where $0 < c_* < +\infty$
- more importantly that if $L = \mathcal{O}(N^\alpha)$ with $\alpha < 2/3$, then, almost surely, for N large enough, the eigenvalues $\mathbf{W}_N \mathbf{W}_N^*$ are located in the neighbourhood of the support of the Marcenko–Pastur distribution.

1.2 Motivation

This work is mainly motivated by detection/estimation problems of certain multivariate time series. Consider a M -variate time series $(\mathbf{y}_n)_{n \in \mathbb{Z}}$ given by

$$\mathbf{y}_n = \sum_{p=0}^{P-1} \mathbf{a}_p s_{n-p} + \mathbf{v}_n = \mathbf{x}_n + \mathbf{v}_n \tag{1.3}$$

where $(s_n)_{n \in \mathbb{Z}}$ represents a deterministic non-observable scalar signal, $(\mathbf{a}_p)_{p=0,\dots,P-1}$ are deterministic unknown M -dimensional vectors and $(\mathbf{v}_n)_{n \in \mathbb{Z}}$ represent i.i.d. zero-mean complex Gaussian M -variate random vectors such that $\mathbb{E}(\mathbf{v}_n \mathbf{v}_n^*) = \sigma^2 \mathbf{I}_M$ and $\mathbb{E}(\mathbf{v}_n \mathbf{v}_n^T) = 0$ for each n . The first term of the right-hand side of (1.3), which we denote by \mathbf{x}_n , represents a “useful” non-observable signal on which various kinds of information have to be inferred from the observation of N consecutive samples $(\mathbf{y}_n)_{n=1,\dots,N}$. Useful information on (\mathbf{x}_n) may include:

- Presence versus absence of (\mathbf{x}_n) , which is equivalent to a detection problem,
- Estimation of vectors $(\mathbf{a}_p)_{p=0,\dots,P-1}$,
- Estimation of sequence (s_n) from the observations.

The reader may refer, e.g. to [1,25,30,31] for more information. A number of existing detection/estimation schemes are based on the eigenvalues and eigenvectors of matrix $\frac{\mathbf{Y}_L \mathbf{Y}_L^*}{N}$ where \mathbf{Y}_L is the block-Hankel $ML \times (N - L + 1)$ matrix defined by

$$\mathbf{Y}_L = \begin{pmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \cdots & \mathbf{y}_{N-L+1} \\ \mathbf{y}_2 & \mathbf{y}_3 & \ddots & \ddots & \mathbf{y}_{N-L+2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{y}_L & \mathbf{y}_{L+1} & \cdots & \cdots & \mathbf{y}_N \end{pmatrix}$$

and where L is an integer usually chosen greater than P . We notice that matrix \mathbf{Y}_L is the sum of deterministic matrix \mathbf{X}_L and random matrix \mathbf{V}_L both defined as \mathbf{Y}_L . The behaviour of the above-mentioned detection/estimation schemes is easy to analyse when ML is fixed and $N \rightarrow +\infty$ because, in this asymptotic regime, it holds that

$$\left\| \frac{\mathbf{Y}_L \mathbf{Y}_L^*}{N} - \left(\frac{\mathbf{X}_L \mathbf{X}_L^*}{N} + \sigma^2 \mathbf{I}_{ML} \right) \right\| \rightarrow 0$$

where $\|\mathbf{A}\|$ represents the spectral norm of matrix \mathbf{A} . However, this asymptotic regime may be unrealistic because ML and N appear sometimes to be of the same order of magnitude. It is therefore of crucial interest to evaluate the behaviour of the eigenvalues of matrix $\frac{\mathbf{Y}_L \mathbf{Y}_L^*}{N}$ when ML and N converge to $+\infty$ at the same rate. Matrix $\mathbf{Y}_L = \mathbf{X}_L + \mathbf{V}_L$ can be interpreted as an Information plus Noise model (see [13]), but in which the noise and the information components are block-Hankel matrices. We believe that in order to understand the behaviour of the eigenvalues of $\frac{\mathbf{Y}_L \mathbf{Y}_L^*}{N}$, it is first quite useful to evaluate the eigenvalue distribution of the noise contribution, i.e. $\frac{\mathbf{V}_L \mathbf{V}_L^*}{N}$, and to check whether its eigenvalues tend to be located in a compact interval. Hopefully, the behaviour of the greatest eigenvalues of $\frac{\mathbf{Y}_L \mathbf{Y}_L^*}{N}$ may be obtained by adapting the approach of [7], at least if the rank of the ‘‘Information’’ component \mathbf{X}_L is small enough w.r.t. ML .

It is clear that if we replace N by $N + L - 1$ in the definition of matrix \mathbf{V}_L , matrix \mathbf{W}_N is obtained from $\frac{\mathbf{V}_L}{\sqrt{N}}$ by row permutations. Therefore, matrices $\frac{\mathbf{V}_L \mathbf{V}_L^*}{N}$ and $\mathbf{W}_N \mathbf{W}_N^*$ have the same eigenvalues. The problem we study in the paper is thus equivalent to the characterization of the eigenvalue distribution of the noise part of model \mathbf{Y}_L .

1.3 On the Literature

Matrix \mathbf{W}_N can be interpreted as a block-line matrix with i.i.d. $L \times N$ blocks $(\mathbf{W}_N^m)_{m=1,\dots,M}$. Such random block matrices have been studied in the past, e.g. by Girko ([15], Chapter 16) as well as in [14] in the Gaussian case. Using these results, it is easy to check that the eigenvalue distribution of $\mathbf{W}_N \mathbf{W}_N^*$ converges towards the Marcenko–Pastur distribution when L is fixed. However, the case $L \rightarrow +\infty$ and the almost sure location of the eigenvalues of $\mathbf{W}_N \mathbf{W}_N^*$ around the support of the Marcenko–Pastur

distribution cannot be addressed using the results of [15] and [14]. We note that the $L \times N$ blocks $(\mathbf{W}^m)_{m=1,\dots,M}$ are Hankel matrices. We therefore also mention the works [21] and [6] that are equivalent to the study of the eigenvalue distribution of symmetric $ML \times ML$ block matrices, each block being a Toeplitz or a Hankel $L \times L$ matrix built from i.i.d. (possibly non-Gaussian) entries. When $L \rightarrow +\infty$ while M remains fixed, it has been shown using the moments method that the eigenvalue distribution of the above matrices converges towards a non-bounded limit distribution. This behaviour generalizes the results of [9] obtained when $M = 1$. When M and L both converge on $+\infty$, it is shown in [6] that the eigenvalue distribution converges towards the semicircle law. We, however, note that the almost sure location of the eigenvalues in the neighbourhood of the support of the semicircle law is not addressed in [6]. The behaviour of the singular value distribution of random block-Hankel matrix (1.2) was addressed in [5] when $M = 1$ and $\frac{L}{N} \rightarrow c_*$ but when the $w_{1,n}$ for $N < n < N+L$ are forced to 0. The random variables $w_{1,n}$ are also non-Gaussian and are possibly dependent in [5]. It is shown using the moments method that the singular value distribution converges towards a non-bounded limit distribution. The case of block-Hankel matrices where both M and L converge towards ∞ considered in this paper thus appears simpler because we show that the eigenvalue distribution of $\mathbf{W}_N \mathbf{W}_N^*$ converges towards the Marcenko–Pastur distribution. This behaviour is not surprising in view of the convergence towards the semicircle law proved in [6] when both the number and the size of the blocks converge to ∞ . As mentioned above, the main result of the present paper concerns the almost sure location of the eigenvalues of $\mathbf{W}_N \mathbf{W}_N^*$ around the support of the Marcenko–Pastur distribution under the extra assumption that $L = O(N^\alpha)$ with $\alpha < 2/3$. This kind of result is known for a long time for $L = 1$ in more general conditions (correlated non-Gaussian entries, see, e.g. [4] and the references therein). Haagerup and Thorbjørnsen introduced in [17] an efficient approach to addressing these issues in the context of random matrices built on non-commutative polynomials of complex Gaussian matrices. The approach of [17] has been generalized to the real Gaussian case in [29] and used in [11, 12, 22] to address certain nonzero-mean random matrix models. We also mention that the results of [17] have been recently generalized in [24] to polynomials of complex Gaussian random matrices and deterministic matrices.

To our best knowledge, the existing literature does not allow to prove that the eigenvalues of $\mathbf{W}_N \mathbf{W}_N^*$ are located in the neighbourhood of the bulk of the Marcenko–Pastur distribution. We finally notice that the proof of our main result would have been quite standard if L was assumed fixed, and rather easy if it was assumed that $L \rightarrow +\infty$ and $\frac{L}{M} \rightarrow 0$, a condition very close from $L = O(N^\alpha)$ for $\alpha < 1/2$. However, the case $1/2 \leq \alpha < 2/3$ needs much more efforts. As explained below, we feel that $2/3$ is the optimal limit.

1.4 Overview of the Paper

We first state the main result of this paper.

Theorem 1.1 *When $M \rightarrow +\infty$ and ML and N converge towards ∞ in such a way that $c_N = \frac{ML}{N}$ converges towards $c_* \in (0, +\infty)$, the eigenvalue distribution of $\mathbf{W}_N \mathbf{W}_N^*$ converges weakly almost surely towards the Marcenko–Pastur distribution with para-*

mers σ^2, c_* . If, moreover,

$$L = O(N^\alpha) \tag{1.4}$$

where $\alpha < 2/3$, then, for each $\epsilon > 0$, almost surely for N large enough, all the eigenvalues of $\mathbf{W}_N \mathbf{W}_N^*$ are located in the interval $[\sigma^2(1 - \sqrt{c_*})^2 - \epsilon, \sigma^2(1 + \sqrt{c_*})^2 + \epsilon]$ if $c_* \leq 1$. If $c_* > 1$, almost surely for N large enough, 0 is eigenvalue of $\mathbf{W}_N \mathbf{W}_N^*$ with multiplicity $ML - N$, and the N nonzero eigenvalues of $\mathbf{W}_N \mathbf{W}_N^*$ are located in the interval $[\sigma^2(1 - \sqrt{c_*})^2 - \epsilon, \sigma^2(1 + \sqrt{c_*})^2 + \epsilon]$

In order to prove the almost sure location of the eigenvalues of $\mathbf{W}_N \mathbf{W}_N^*$, we follow the approach of [17] and [29]. We denote by $t_N(z)$ the Stieltjes transform associated with the Marcenko–Pastur distribution μ_{σ^2, c_N} with parameters σ^2, c_N , i.e. the unique Stieltjes transform solution of the equation

$$t_N(z) = \frac{1}{-z + \frac{\sigma^2}{1 + \sigma^2 c_N t_N(z)}} \tag{1.5}$$

or equivalently of the system

$$t_N(z) = \frac{-1}{z(1 + \sigma^2 \tilde{t}_N(z))} \tag{1.6}$$

$$\tilde{t}_N(z) = \frac{-1}{z(1 + \sigma^2 c_N t_N(z))} \tag{1.7}$$

where $\tilde{t}_N(z)$ coincides with the Stieltjes transform of $\mu_{\sigma^2 c_N, 1/c_N} = c_N \mu_{\sigma^2, c_N} + (1 - c_N) \delta_0$ where δ_0 represents the Dirac distribution at point 0. We denote by $\mathcal{S}_N^{(0)}$ the interval

$$\mathcal{S}_N^{(0)} = \left[\sigma^2(1 - \sqrt{c_N})^2, \sigma^2(1 + \sqrt{c_N})^2 \right] \tag{1.8}$$

and by \mathcal{S}_N the support of μ_{σ^2, c_N} . It is well known that \mathcal{S}_N is given by

$$\mathcal{S}_N = \mathcal{S}_N^{(0)} \quad \text{if } c_N \leq 1 \tag{1.9}$$

$$\mathcal{S}_N = \mathcal{S}_N^{(0)} \cup \{0\} \quad \text{if } c_N > 1 \tag{1.10}$$

Theorem 1.1 appears to be a consequence of the following identity:

$$\mathbb{E} \left[\frac{1}{ML} \text{Tr} \left((\mathbf{W}_N \mathbf{W}_N^* - z \mathbf{I}_{ML})^{-1} \right) \right] - t_N(z) = \frac{L}{MN} \left(\hat{s}_N(z) + \frac{L^{3/2}}{MN} \hat{r}_N(z) \right) \tag{1.11}$$

where $\hat{s}_N(z)$ coincides with the Stieltjes transform of a distribution whose support is included in $\mathcal{S}_N^{(0)}$ and where $\hat{r}_N(z)$ is a function holomorphic in \mathbb{C}^+ satisfying

$$|\hat{r}_N(z)| \leq P_1(|z|) P_2(1/\text{Im}(z)) \tag{1.12}$$

for $z \in F_N^{(2)}$ where $F_N^{(2)}$ is a subset of \mathbb{C}^+ defined by

$$F_N^{(2)} = \left\{ z \in \mathbb{C}^+, \frac{L^2}{MN} Q_1(|z|) Q_2(1/\text{Im}(z)) \leq 1 \right\} \tag{1.13}$$

where P_1, P_2, Q_1, Q_2 are polynomials independent of the dimensions L, M, N with positive coefficients. We note that (1.4) is nearly equivalent to $\frac{L^2}{MN} \rightarrow 0$ or $\frac{L}{M^2} \rightarrow 0$ (in the sense that if $\alpha \geq 2/3$, then $\frac{L^2}{MN}$ does not converge towards 0) and that $F_N^{(2)}$ appears arbitrary close from \mathbb{C}^+ when N increases. The present paper is essentially devoted to the proof of (1.11) under the assumption (1.4). For this, we study in the various sections the behaviour of the resolvent $\mathbf{Q}_N(z)$ of matrix $\mathbf{W}_N \mathbf{W}_N^*$ defined by

$$\mathbf{Q}_N(z) = (\mathbf{W}_N \mathbf{W}_N^* - z \mathbf{I}_{ML})^{-1} \tag{1.14}$$

when $z \in \mathbb{C}^+$. We use Gaussian tools (integration by parts formula and Poincaré–Nash inequality) as in [27] and [28] for that purpose.

In Sect. 2, we present some properties of certain useful operators which map matrices \mathbf{A} into band Toeplitz matrices whose elements depend on the sum of the elements of \mathbf{A} on each diagonal. Using Poincaré–Nash inequality, we evaluate in Sect. 3 the variance of certain functional of $\mathbf{Q}_N(z)$ (normalized trace, quadratic forms, and quadratic forms of the $L \times L$ matrix $\hat{\mathbf{Q}}_N(z)$ obtained as the mean of the $ML \times L$ diagonal blocks of $\mathbf{Q}_N(z)$). In Sect. 4, we use the integration by parts formula in order to express $\mathbb{E}(\mathbf{Q}_N(z))$ as

$$\mathbb{E}(\mathbf{Q}_N(z)) = \mathbf{I}_M \otimes \mathbf{R}_N(z) + \mathbf{\Delta}_N(z)$$

where $\mathbf{R}_N(z)$ is a certain holomorphic $\mathbb{C}^{L \times L}$ valued function depending on a Toeplitz-ified version of $\mathbb{E}(\mathbf{Q}_N(z))$, and where $\mathbf{\Delta}_N(z)$ is an error term. The goal of Sect. 5 is to control functionals of the error term $\mathbf{\Delta}_N(z)$. We prove that for each $z \in \mathbb{C}^+$,

$$\left| \frac{1}{ML} \text{Tr}(\mathbf{\Delta}_N(z)) \right| \leq \frac{L}{MN} P_1(|z|) P_2(1/\text{Im}(z)) \tag{1.15}$$

for some polynomials P_1 and P_2 independent of L, M, N and that if $\hat{\mathbf{\Delta}}_N(z)$ represents the $L \times L$ matrix $\hat{\mathbf{\Delta}}_N(z) = \frac{1}{M} \sum_{m=1}^M \mathbf{\Delta}_N^{m,m}(z)$, then it holds that

$$\left| \mathbf{b}_1^* \hat{\mathbf{\Delta}}_N(z) \mathbf{b}_2 \right| \leq \frac{L^{3/2}}{MN} P_1(|z|) P_2(1/\text{Im}(z)) \tag{1.16}$$

for deterministic unit norm L -dimensional vectors \mathbf{b}_1 and \mathbf{b}_2 . In Sect. 6, we prove that

$$\mathbb{E} \left[\frac{1}{ML} \text{Tr}(\mathbf{Q}_N(z)) \right] - t_N(z) \rightarrow 0 \tag{1.17}$$

for each $z \in \mathbb{C}^+$, a property which implies that the eigenvalue distribution of $\mathbf{W}_N \mathbf{W}_N^*$ converges towards the Marcenko–Pastur distribution. We note that (1.17) holds as soon as $M \rightarrow +\infty$. At this stage, however, the convergence rate of the left-hand side of (1.17) is not precise. Under the condition $\frac{L^{3/2}}{MN} \rightarrow 0$ (which implies that quadratic forms of $\hat{\mathbf{A}}_N(z)$ converge towards 0, see (1.16)), we prove in Sect. 8 that

$$\mathbb{E} \left[\frac{1}{ML} \text{Tr} \left((\mathbf{W}_N \mathbf{W}_N^* - z \mathbf{I}_{ML})^{-1} \right) \right] - t_N(z) = \frac{L}{MN} \tilde{r}_N(z) \tag{1.18}$$

where $\tilde{r}_N(z)$ is holomorphic in \mathbb{C}^+ and satisfies

$$|\tilde{r}_N(z)| \leq P_1(|z|) P_2(1/\text{Im}z)$$

for each $z \in F_N^{(3/2)}$, where $F_N^{(3/2)}$ is defined as $F_N^{(2)}$ (see (1.13)), but when $\frac{L^2}{MN}$ is replaced by $\frac{L^{3/2}}{MN}$. In order to establish (1.18), it is proved in Sect. 7 that the spectral norm of a Toeplitzified version of matrix $\mathbf{R}_N(z) - t_N(z) \mathbf{I}_L$ is upperbounded by a term such as $\frac{L^{3/2}}{MN} P_1(|z|) P_2(1/\text{Im}(z))$. (1.18) and Lemma 5.5.5 of [2] would allow to establish quite easily the almost sure location of the eigenvalues of $\mathbf{W}_N \mathbf{W}_N^*$ under the hypothesis $\frac{L}{M} \rightarrow 0$. However, this condition is very restrictive, and, at least intuitively, somewhat similar to L fixed. In Sect. 9, we establish that under condition (1.4), which is very close from the condition $\frac{L^2}{MN} \rightarrow 0$, or $\frac{L}{M^2} \rightarrow 0$, function $\tilde{r}_N(z)$ can be written as $\tilde{r}_N(z) = \hat{s}_N(z) + \frac{L^{3/2}}{MN} \hat{r}_N(z)$, where $\hat{s}_N(z)$ and $\hat{r}_N(z)$ verify the conditions of (1.11). We first prove that

$$\mathbb{E} \left[\frac{1}{ML} \text{Tr} (\mathbf{Q}_N(z) - \mathbf{I}_M \otimes \mathbf{R}_N(z)) \right] = \frac{L}{MN} \left(s_N(z) + \frac{L}{MN} r_N(z) \right) \tag{1.19}$$

where $s_N(z)$ and $r_N(z)$ satisfy the same properties than $\hat{s}_N(z)$ and $\hat{r}_N(z)$. For this, we compute explicitly $s_N(z)$ and verify that it coincides with the Stieltjes transform of a distribution whose support is included into $S_N^{(0)}$. The most technical part of the paper is to establish that

$$\mathbb{E} \left[\frac{1}{ML} \text{Tr} (\mathbf{Q}_N(z) - \mathbf{I}_M \otimes \mathbf{R}_N(z)) \right] - \frac{L}{MN} s_N(z) \tag{1.20}$$

converges towards 0 at rate $\left(\frac{L}{MN}\right)^2$. For this, the condition $\frac{L^2}{MN} \rightarrow 0$ appears to be fundamental because it allows, among others, to control the behaviour of the solutions of L -dimensional linear systems obtained by inverting the sum of a diagonal matrix with a matrix with $O\left(\frac{L}{MN}\right)$ entries. Using the results of Sect. 7 concerning the spectral norm of a Toeplitzified version of $\mathbf{R}_N(z) - t_N(z) \mathbf{I}_L$, we obtain easily (1.11) from (1.19). Theorem 1.1 is finally established in Sect. 10. For this, we follow [17, 29] and [2] (Lemma 5-5-5). We consider a smooth approximation ϕ of $\mathbb{1}_{[\sigma^2(1-\sqrt{c_*})^2-\epsilon, \sigma^2(1+\sqrt{c_*})^2+\epsilon]^{(c)}}$ that vanishes on $S_N^{(0)}$ for each N large enough and establish that almost surely,

$$\text{Tr}(\phi(\mathbf{W}_N \mathbf{W}_N^*)) = NO\left(\frac{L^{5/2}}{(MN)^2}\right) + [ML - N]_+ = O\left(\left(\frac{L}{M^2}\right)^{3/2}\right) + [ML - N]_+ \tag{1.21}$$

(1.4) implies that $\frac{L}{M^2} \rightarrow 0$ and that the right-hand side of (1.21) converges towards $[ML - N]_+$ almost surely. This, in turn, establishes that the number of nonzero eigenvalues of $\mathbf{W}_N \mathbf{W}_N^*$ that are located outside $[\sigma^2(1 - \sqrt{c_*})^2 - \epsilon, \sigma^2(1 + \sqrt{c_*})^2 + \epsilon]$ converges towards zero almost surely and is thus equal to 0 for N large enough as expected. We have not proved that this property does not hold if $L = O(N^\alpha)$ with $\alpha \geq 2/3$. We, however, mention that the hypothesis $\alpha < 2/3$ is used at various crucial independent steps:

- it is used extensively to establish that (1.20) converges towards 0 at rate $(\frac{L}{MN})^2$
- it is nearly equivalent to the condition $\frac{L^2}{MN} \rightarrow 0$ or $\frac{L}{M^2} \rightarrow 0$ which implies
 - that the set $F_N^{(2)}$ defined by (1.13) is arbitrarily close from \mathbb{C}^+ , a property that appears necessary to generalize Lemma 5-5-5 of [2]
 - that the right-hand side of (1.21) converges towards $[ML - N]_+$

We therefore suspect that the almost sure location of the eigenvalues of $\mathbf{W}_N \mathbf{W}_N^*$ cannot be established using the approach of [17] and [29] if $\alpha \geq 2/3$. It would be interesting to study the potential of combinatorial methods in order to be fully convinced that the almost sure location of the eigenvalues of $\mathbf{W}_N \mathbf{W}_N^*$ does not hold if $\alpha \geq 2/3$. We finally mention that we have performed numerical simulations to check whether it is reasonable to conjecture that the almost sure location property of the eigenvalues of $\mathbf{W}_N \mathbf{W}_N^*$ holds if and only if $\alpha < 2/3$. For this, we have generated 10.000 independent realizations of the largest eigenvalue $\lambda_{1,N}$ of $\mathbf{W}_N \mathbf{W}_N^*$ for $\sigma^2 = 1, N = 2^{14}, c_N = ML/N = 1/2$ and for the following values of (M, L) that seem to be in accordance with the asymptotic regime considered in this paper: $(M, L) = (2^8, 2^5), (M, L) = (2^7, 2^6), (M, L) = (2^6, 2^7), (M, L) = (2^5, 2^8)$, corresponding to ratios $\frac{L}{M^2}$ equal, respectively, to $2^{-11}, 2^{-8}, 2^{-5}$, and $1/4$. As condition $\alpha < 2/3$ is nearly equivalent to $\frac{L}{M^2} \rightarrow 0$, the first 3 values of (M, L) are in accordance with the asymptotic regime $L = O(N^\alpha)$ with $\alpha < 2/3$ while it is of course not the case for the last configuration. The almost sure location property of course implies that the largest eigenvalue converges towards $(1 + \sqrt{c_*})^2$. In order to check this property, we have evaluated the empirical mean $\bar{\lambda}_{1,N}$ of the 10.000 realizations of $\lambda_{1,N}$ and compared $\bar{\lambda}_{1,N}$ with $(1 + \sqrt{1/2})^2 \simeq 2.91$.

The values of $\bar{\lambda}_{1,N}$ in terms of $\frac{L}{M^2}$ are presented in Table 1. It is seen that the difference between $\bar{\lambda}_{1,N}$ and $(1 + \sqrt{1/2})^2 \simeq 2.91$ increases significantly with the ratio $\frac{L}{M^2}$, thus suggesting that $\lambda_{1,N}$ does not converge almost surely towards $(1 + \sqrt{c_*})^2$ when $\frac{L}{M^2}$ does not converge towards 0.

Table 1 Empirical mean of the largest eigenvalue versus L/M^2

L/M^2	2^{-11}	2^{-8}	2^{-5}	$1/4$
$\bar{\lambda}_{1,N}$	2.91	2.92	2.94	3

1.5 General Notations and Definitions

Assumptions on L, M, N

Assumption 1.1 – All along the paper, we assume that L, M, N satisfy $M \rightarrow +\infty, N \rightarrow +\infty$ in such a way that $c_N = \frac{ML}{N} \rightarrow c_*$, where $0 < c_* < +\infty$. In order to short the notations, $N \rightarrow +\infty$ should be understood as the above asymptotic regime.

- In Sects. 7 and 8, L, M, N also satisfy $\frac{L^{3/2}}{MN} \rightarrow 0$ or equivalently $\frac{L}{M^4} \rightarrow 0$.
- In Sects. 9 and 10, the extra condition $L = O(N^\alpha)$ with $\alpha < 2/3$ holds.

In the following, we will often drop the index N and will denote $\mathbf{W}_N, t_N, \mathbf{Q}_N, \dots$ by $\mathbf{W}, t, \mathbf{Q}, \dots$ in order to short the notations. The N columns of matrix \mathbf{W} are denoted $(\mathbf{w}_j)_{j=1, \dots, N}$. For $1 \leq l \leq L, 1 \leq m \leq M$, and $1 \leq j \leq N$, $\mathbf{W}_{i,j}^m$ represents the entry $(i + (m - 1)L, j)$ of matrix \mathbf{W} .

$C^\infty(\mathbb{R})$ (resp. $C_b^\infty(\mathbb{R}), C_c^\infty(\mathbb{R})$) denotes the space of all real-valued smooth functions (resp. bounded smooth functions, smooth functions with compact support) defined on \mathbb{R} .

If \mathbf{A} is a $ML \times ML$ matrix, we denote by $\mathbf{A}_{i_1, i_2}^{m_1, m_2}$ the entry $(i_1 + (m_1 - 1)L, i_2 + (m_2 - 1)L)$ of matrix \mathbf{A} , while \mathbf{A}^{m_1, m_2} represents the $L \times L$ matrix $(\mathbf{A}_{i_1, i_2}^{m_1, m_2})_{1 \leq (i_1, i_2) \leq L}$. We also denote by $\hat{\mathbf{A}}$ the $L \times L$ matrix defined by

$$\hat{\mathbf{A}} = \frac{1}{M} \sum_{m=1}^M \mathbf{A}^{m,m} \tag{1.22}$$

For each $1 \leq i \leq L$ and $1 \leq m \leq M$, \mathbf{f}_i^m represents the vector of the canonical basis of \mathbb{C}^{ML} whose nonzero component is located at index $i + (m - 1)L$. If $1 \leq j \leq N$, \mathbf{e}_j is the j^{th} -vector of the canonical basis of \mathbb{C}^N .

If \mathbf{A} and \mathbf{B} are 2 matrices, $\mathbf{A} \otimes \mathbf{B}$ represents the Kronecker product of \mathbf{A} and \mathbf{B} , i.e. the block matrix whose block (i, j) is $\mathbf{A}_{i,j} \mathbf{B}$. $\|\mathbf{A}\|$ represents the spectral norm of matrix \mathbf{A} .

If $x \in \mathbb{R}, [x]_+$ represents $\max(x, 0)$. \mathbb{C}^+ denotes the set of complex numbers with strictly positive imaginary parts. The conjugate of a complex number z is denoted z^* or \bar{z} depending on the context. Unless otherwise stated, z represents an element of \mathbb{C}^+ . If \mathbf{A} is a square matrix, $\text{Re}(\mathbf{A})$ and $\text{Im}(\mathbf{A})$ represent the Hermitian matrices $\text{Re}(\mathbf{A}) = \frac{\mathbf{A} + \mathbf{A}^*}{2}$ and $\text{Im}(\mathbf{A}) = \frac{\mathbf{A} - \mathbf{A}^*}{2i}$, respectively.

If $(\mathbf{A}_N)_{N \geq 1}$ (resp. $(\mathbf{b}_N)_{N \geq 1}$) is a sequence of matrices (resp. vectors) whose dimensions increase with N , $(\mathbf{A}_N)_{N \geq 1}$ (resp. $(\mathbf{b}_N)_{N \geq 1}$) is said to be uniformly bounded if $\sup_{N \geq 1} \|\mathbf{A}_N\| < +\infty$ (resp. $\sup_{N \geq 1} \|\mathbf{b}_N\| < +\infty$).

If x is a complex-valued random variable, the variance of x , denoted by $\text{Var}(x)$, is defined by

$$\text{Var}(x) = \mathbb{E}(|x|^2) - |\mathbb{E}(x)|^2$$

The zero-mean random variable $x - \mathbb{E}(x)$ is denoted x° .

Nice constants and nice polynomials A nice constant is a positive constant independent of the dimensions L, M, N and complex variable z . A nice polynomial is a polynomial whose degree is independent from L, M, N and whose coefficients are nice constants. In the following, P_1 and P_2 will represent generic nice polynomials whose values may change from one line to another, and $C(z)$ is a generic term of the form $C(z) = P_1(|z|)P_2(1/\text{Im}z)$.

Properties of matrix $\mathbf{Q}(z)$. We recall that $\mathbf{Q}(z)$ verifies the so-called resolvent identity

$$\mathbf{Q}(z) = -\frac{\mathbf{I}_{ML}}{z} + \frac{1}{z}\mathbf{Q}(z)\mathbf{W}\mathbf{W}^* \tag{1.23}$$

and that it holds that

$$\mathbf{Q}(z)\mathbf{Q}^*(z) \leq \frac{\mathbf{I}_{ML}}{(\text{Im}z)^2} \tag{1.24}$$

and that

$$\|\mathbf{Q}(z)\| \leq \frac{1}{\text{Im}(z)} \tag{1.25}$$

for $z \in \mathbb{C}^+$. We also mention that

$$\text{Im}(\mathbf{Q}(z)) > 0, \quad \text{Im}(z\mathbf{Q}(z)) > 0, \quad \text{if } z \in \mathbb{C}^+ \tag{1.26}$$

Gaussian tools We present the versions of the integration by parts formula (see Eq. (2.1.42) p. 40 in [28] for the real case and Eq. (17) in [19] for the present complex case) and of the Poincaré–Nash (see Proposition 2.1.6 in [28] for the real case and Eq. (18) in [19] for the complex case) that we use in this paper.

Proposition 1.1 (Integration by parts formula) *Let $\xi = [\xi_1, \dots, \xi_K]^T$ be a complex Gaussian random vector such that $\mathbb{E}[\xi] = \mathbf{0}$, $\mathbb{E}[\xi\xi^T] = \mathbf{0}$ and $\mathbb{E}[\xi\xi^*] = \mathbf{\Omega}$. If $\Gamma : (\xi) \mapsto \Gamma(\xi, \bar{\xi})$ is a C^1 complex function polynomially bounded together with its derivatives, then*

$$\mathbb{E}[\xi_p \Gamma(\xi)] = \sum_{m=1}^K \Omega_{pm} \mathbb{E} \left[\frac{\partial \Gamma(\xi)}{\partial \bar{\xi}_m} \right]. \tag{1.27}$$

Proposition 1.2 (Poincaré–Nash inequality) *Let $\xi = [\xi_1, \dots, \xi_K]^T$ be a complex Gaussian random vector such that $\mathbb{E}[\xi] = \mathbf{0}$, $\mathbb{E}[\xi\xi^T] = \mathbf{0}$ and $\mathbb{E}[\xi\xi^*] = \mathbf{\Omega}$. If $\Gamma : (\xi) \mapsto \Gamma(\xi, \bar{\xi})$ is a C^1 complex function polynomially bounded together with its derivatives, then, noting $\nabla_{\xi} \Gamma = [\frac{\partial \Gamma}{\partial \xi_1}, \dots, \frac{\partial \Gamma}{\partial \xi_K}]^T$ and $\nabla_{\bar{\xi}} \Gamma = [\frac{\partial \Gamma}{\partial \bar{\xi}_1}, \dots, \frac{\partial \Gamma}{\partial \bar{\xi}_K}]^T$,*

$$\text{Var}(\Gamma(\xi)) \leq \mathbb{E} \left[\nabla_{\xi} \Gamma(\xi)^T \mathbf{\Omega} \overline{\nabla_{\xi} \Gamma(\xi)} \right] + \mathbb{E} \left[\nabla_{\bar{\xi}} \Gamma(\xi)^* \mathbf{\Omega} \nabla_{\bar{\xi}} \Gamma(\xi) \right] \tag{1.28}$$

The above two propositions are used below in the case where ξ coincides with the LMN -dimensional vector $\text{vec}(\mathbf{W}_N)$. In the following, the particular structure $\mathbf{W}_{i,j}^m = w_{m,i+j-1}$ of \mathbf{W}_N is encoded by the correlation structure of the entries of \mathbf{W}_N :

$$\mathbb{E} \left(\mathbf{W}_{i_1, j_1}^{m_1} \overline{\mathbf{W}_{i_2, j_2}^{m_2}} \right) = \frac{\sigma^2}{N} \delta(i_1 - i_2 = j_2 - j_1) \delta(m_1 = m_2) \tag{1.29}$$

A useful property of the Stieltjes transform $t_N(z)$ of the Marcenko–Pastur μ_{σ^2, c_N} .

The following lemma is more or less known. A proof is provided in the Appendix of [23] for the reader’s convenience.

Lemma 1.1 *It holds that*

$$\sigma^4 c_N |z t_N(z) \tilde{t}_N(z)|^2 < 1 \tag{1.30}$$

for each $z \in \mathbb{C}^+$. Moreover, for each N and for each $z \in \mathbb{C}^+$, it holds that

$$1 - \sigma^4 c_N |z t_N(z) \tilde{t}_N(z)|^2 > C \frac{(\text{Im}z)^4}{(\eta^2 + |z|^2)^2} \tag{1.31}$$

for some nice constants C and η . Finally, for each N , it holds that

$$\left(1 - \sigma^4 c_N |z t(z) \tilde{t}(z)|^2\right)^{-1} \leq C \max\left(1, \frac{1}{\left(\text{dist}\left(z, \mathcal{S}_N^{(0)}\right)\right)^2}\right) \tag{1.32}$$

for some nice constant C and for each $z \in \mathbb{C} - \mathcal{S}_N^{(0)}$.

2 Preliminaries

In this section, we introduce certain Toeplitzification operators and establish some useful related properties.

Definition 2.1 – If \mathbf{A} is a $K \times K$ Toeplitz matrix, we denote by $(\mathbf{A}(k))_{k=-(K-1), \dots, K-1}$ the sequence such that $\mathbf{A}_{k,l} = \mathbf{A}(k-l)$.

- For any integer K , J_K is the $K \times K$ “shift” matrix defined by $(J_K)_{i,j} = \delta(j-i-1)$. In order to short the notations, matrix J_K^* is denoted J_K^{-1} , although J_K is of course not invertible.
- For any $PK \times PK$ block matrix \mathbf{A} with $K \times K$ blocks $(\mathbf{A}^{p_1, p_2})_{1 \leq (p_1, p_2) \leq P}$, we define $(\tau^{(P)}(\mathbf{A})(k))_{k=-(K-1), \dots, K-1}$ as the sequence

$$\begin{aligned} \tau^{(P)}(\mathbf{A})(k) &= \frac{1}{PK} \text{Tr} \left[\mathbf{A}(\mathbf{I}_P \otimes \mathbf{J}_K^k) \right] = \frac{1}{PK} \sum_{i-j=k}^P \sum_{p=1}^P \mathbf{A}_{i,j}^{(p,p)} \\ &= \frac{1}{PK} \sum_{p=1}^P \sum_{u=1}^K \mathbf{A}_{k+u,u}^{p,p} \mathbb{1}_{1 \leq k+u \leq K} \end{aligned} \tag{2.1}$$

- For any $PK \times PK$ block matrix \mathbf{A} and for 2 integers R and Q such that $R \geq Q$ and $Q \leq K$, matrix $\mathcal{T}_{R,Q}^{(P)}(\mathbf{A})$ represents the $R \times R$ Toeplitz matrix given by

$$\mathcal{T}_{R,Q}^{(P)}(\mathbf{A}) = \sum_{q=-(Q-1)}^{Q-1} \tau^{(P)}(\mathbf{A})(q) \mathbf{J}_R^{*q} \tag{2.2}$$

In other words, for $(i, j) \in \{1, 2, \dots, R\}$, it holds that

$$\left(\mathcal{T}_{R,Q}^{(P)}(\mathbf{A})\right)_{i,j} = \tau^{(P)}(\mathbf{A})(i-j) \mathbb{1}_{|i-j| \leq Q-1} \tag{2.3}$$

When $P = 1$, sequence $(\tau^{(1)}(\mathbf{A})(k))_{k=-(K-1), \dots, K-1}$ and matrix $\mathcal{T}_{R,Q}^{(1)}(\mathbf{A})$ are denoted $(\tau(\mathbf{A})(k))_{k=-(K-1), \dots, K-1}$ and matrix $\mathcal{T}_{R,Q}(\mathbf{A})$ in order to simplify the notations. We note that if \mathbf{A} is a $PK \times PK$ block matrix, then sequence $(\tau^{(P)}(\mathbf{A})(k))_{k=-(K-1), \dots, K-1}$ coincides with sequence $(\tau(\hat{\mathbf{A}})(k))_{k=-(K-1), \dots, K-1}$ where we recall that $\hat{\mathbf{A}} = \frac{1}{P} \sum_{p=1}^P \mathbf{A}^{p,p}$; matrix $\mathcal{T}_{R,Q}^{(P)}(\mathbf{A})$ is equal to $\mathcal{T}_{R,Q}(\hat{\mathbf{A}})$.

The reader may check that the following straightforward identities hold:

- If \mathbf{A} is a $R \times R$ Toeplitz matrix, for any $R \times R$ matrix \mathbf{B} , it holds that

$$\frac{1}{R} \text{Tr}(\mathbf{A}\mathbf{B}) = \sum_{k=-(R-1)}^{R-1} \mathbf{A}(-k) \tau(\mathbf{B})(k) = \frac{1}{R} \text{Tr}(\mathbf{A}\mathcal{T}_{R,R}(\mathbf{B})) \tag{2.4}$$

- If \mathbf{A} and \mathbf{B} are both $R \times R$ matrices, and if $Q \leq R$, then

$$\frac{1}{R} \text{Tr}(\mathcal{T}_{R,Q}(\mathbf{A})\mathbf{B}) = \sum_{q=-(Q-1)}^{Q-1} \tau(\mathbf{A})(-q) \tau(\mathbf{B})(q) = \frac{1}{R} \text{Tr}(\mathbf{A}\mathcal{T}_{R,Q}(\mathbf{B})) \tag{2.5}$$

- If \mathbf{A} is a $PK \times PK$ matrix, if \mathbf{B} is a $R \times R$ matrix, and if $R \geq Q$ and $Q \leq K$, then it holds that

$$\begin{aligned} \frac{1}{R} \text{Tr}(\mathbf{B}\mathcal{T}_{R,Q}^{(P)}(\mathbf{A})) &= \sum_{k=-(Q-1)}^{Q-1} \tau(\mathbf{B})(k) \tau^{(P)}(\mathbf{A})(-k) \\ &= \frac{1}{PK} \text{Tr}((\mathbf{I}_M \otimes \mathcal{T}_{K,Q}(\mathbf{B})) \mathbf{A}) \end{aligned} \tag{2.6}$$

- If \mathbf{C} is a $PK \times PK$ matrix, \mathbf{B} is a $K \times K$ matrix and $\mathbf{D}, \mathbf{E} R \times R$ matrices with $K \leq R$, then it holds that

$$\frac{1}{K} \text{Tr}[\mathbf{B}\mathcal{T}_{K,K}(\mathbf{D}\mathcal{T}_{R,K}^{(P)}(\mathbf{C})\mathbf{E})] = \frac{1}{PK} \text{Tr}[\mathbf{C}(\mathbf{I}_P \otimes \mathcal{T}_{K,K}[\mathbf{E}\mathcal{T}_{R,K}(\mathbf{B})\mathbf{D}])] \tag{2.7}$$

We now establish useful properties of matrix $\mathcal{T}_{R,Q}^{(P)}(\mathbf{A})$.

Proposition 2.1 *If \mathbf{A} is a $PK \times PK$ matrix, then, for each integer $R \geq K$, it holds that*

$$\left\| \mathcal{T}_{R,K}^{(P)}(\mathbf{A}) \right\| \leq \sup_{v \in [0,1]} \left| \mathbf{a}_K(v)^* \left(\frac{1}{P} \sum_{p=1}^P \mathbf{A}^{p,p} \right) \mathbf{a}_K(v) \right| \leq \|\mathbf{A}\| \tag{2.8}$$

where $\mathbf{a}_K(v)$ represents the K -dimensional vector defined by

$$\mathbf{a}_K(v) = \frac{1}{\sqrt{K}} \left(1, e^{2i\pi v}, \dots, e^{2i\pi(K-1)v} \right)^T \tag{2.9}$$

If \mathbf{A} is a $K \times K$ matrix and if $R \leq K$, then it holds that

$$\|\mathcal{T}_{R,R}(\mathbf{A})\| \leq \sup_{v \in [0,1]} |\mathbf{a}_K(v)^* \mathbf{A} \mathbf{a}_K(v)| \leq \|\mathbf{A}\| \tag{2.10}$$

Proof We first establish (2.8). As $R \geq K$, matrix $\mathcal{T}_{R,K}^{(P)}(\mathbf{A})$ is a submatrix of the infinite band Toeplitz matrix with (i, j) elements $\tau^{(P)}(\mathbf{A})(i-j)\mathbb{1}_{|i-j| \leq K-1}$. The norm of this matrix is known to be equal to the \mathbb{L}_∞ norm of the corresponding symbol (see [8], Eq. (1-14), p. 10). Therefore, it holds that

$$\|\mathcal{T}_{R,K}^{(P)}(\mathbf{A})\| \leq \sup_{v \in [0,1]} \left| \sum_{k=-(K-1)}^{K-1} \tau^{(P)}(\mathbf{A})(k) e^{-2i\pi kv} \right|$$

We now verify the following useful identity:

$$\sum_{k=-(K-1)}^{K-1} \tau^{(P)}(\mathbf{A})(k) e^{-2i\pi kv} = \mathbf{a}_K(v)^* \left(\frac{1}{P} \sum_{p=1}^P \mathbf{A}^{(p,p)} \right) \mathbf{a}_K(v) \tag{2.11}$$

Using the definition (2.1) of $\tau^{(P)}(\mathbf{A})(k)$, the term $\sum_{k=-(K-1)}^{K-1} \tau^{(P)}(\mathbf{A})(k) e^{-2i\pi kv}$ can also be written as

$$\sum_{k=-(K-1)}^{K-1} \tau^{(P)}(\mathbf{A})(k) e^{-2i\pi kv} = \frac{1}{K} \sum_{k=-(K-1)}^{K-1} \text{Tr} \left(\left(\frac{1}{P} \sum_{p=1}^P \mathbf{A}^{(p,p)} \right) e^{-2i\pi kv} \mathbf{J}_K^k \right)$$

or equivalently as

$$\text{Tr} \left(\left(\frac{1}{P} \sum_{p=1}^P \mathbf{A}^{(p,p)} \right) \frac{1}{K} \left(\sum_{k=-(K-1)}^{K-1} e^{-2i\pi kv} \mathbf{J}_K^k \right) \right)$$

It is easily seen that

$$\frac{1}{K} \left(\sum_{k=-(K-1)}^{K-1} e^{-2i\pi kv} \mathbf{J}_K^k \right) = \mathbf{a}_K(v) \mathbf{a}_K(v)^*$$

from which (2.11) and (2.8) follow immediately.

In order to justify (2.10), we remark that $R \leq K$ implies that $\mathcal{T}_{R,R}(\mathbf{A})$ is a submatrix of $\mathcal{T}_{K,K}(\mathbf{A})$ whose norm is bounded by $\sup_v |\mathbf{a}_K(v)^* \mathbf{A} \mathbf{a}_K(v)|$ by (2.8). □

We also prove that the operators \mathcal{T} preserve the positivity of matrices.

Proposition 2.2 *If \mathbf{A} is a $PK \times PK$ positive definite matrix, then, for each integer $R \geq K$, it holds that*

$$\mathcal{T}_{R,K}^{(P)}(\mathbf{A}) > 0 \tag{2.12}$$

If \mathbf{A} is a $K \times K$ positive definite matrix and if $R \leq K$, then it holds that

$$\mathcal{T}_{R,R}(\mathbf{A}) > 0 \tag{2.13}$$

Proof We first prove (2.12). (2.11) implies that

$$\sum_{k=-(K-1)}^{K-1} \tau^{(P)}(\mathbf{A})(k)e^{-2i\pi kv} > 0$$

for each $v. (\tau^{(P)}(\mathbf{A})(k))_{k=-(K-1), \dots, K-1}$, thus coinciding the Fourier coefficients of a positive function. Elementary results related the trigonometric moment problem (see, e.g. [18], 1.11 (a)) imply that for each $R \geq K$, matrix $\mathcal{T}_{R,K}^{(P)}(\mathbf{A})$ is positive definite. We finally justify (2.13). As $R \leq K$, matrix $\mathcal{T}_{R,R}(\mathbf{A})$ is a submatrix of $\mathcal{T}_{K,K}(\mathbf{A})$ which is positive definite by (2.12). \square

We finally give the following useful result proved in the ‘‘Appendix’’.

Proposition 2.3 *If \mathbf{A} is a $K \times K$ matrix and if $R \geq K$, then it holds that*

$$\mathcal{T}_{R,K}(\mathbf{A}) (\mathcal{T}_{R,K}(\mathbf{A}))^* \leq \mathcal{T}_{R,K}(\mathbf{A}\mathbf{A}^*) \tag{2.14}$$

If \mathbf{A} is a $K \times K$ matrix and if $R \leq K$, then

$$\mathcal{T}_{R,R}(\mathbf{A}) (\mathcal{T}_{R,R}(\mathbf{A}))^* \leq \mathcal{T}_{R,R}(\mathbf{A}\mathbf{A}^*) \tag{2.15}$$

3 Poincaré–Nash Variance Evaluations

In this section, we take benefit of the Poincaré–Nash inequality to evaluate the variance of certain important terms. In particular, we prove the following useful result.

Proposition 3.1 *Let \mathbf{A} be a deterministic $ML \times ML$ matrix for which $\sup_N \|\mathbf{A}\| \leq \kappa$, and consider 2 ML -dimensional deterministic vectors $\mathbf{a}_1, \mathbf{a}_2$ such that $\sup_N \|\mathbf{a}_i\| \leq \kappa$ for $i = 1, 2$ as well as 2 L -dimensional deterministic vectors $\mathbf{b}_1, \mathbf{b}_2$ such that $\sup_N \|\mathbf{b}_i\| \leq \kappa$ for $i = 1, 2$. Then, for each $z \in \mathbb{C}^+$, it holds that*

$$\text{Var} \left(\frac{1}{ML} \text{Tr}(\mathbf{A}\mathbf{Q}(z)) \right) \leq C(z) \kappa^2 \frac{1}{MN} \tag{3.1}$$

$$\text{Var}(\mathbf{a}_1^* \mathbf{Q}(z) \mathbf{a}_2) \leq C(z) \kappa^4 \frac{L}{N} \tag{3.2}$$

$$\text{Var} \left(\mathbf{b}_1^* \left[\frac{1}{M} \sum_{m=1}^M (\mathbf{Q}(z))^{m,m} \right] \mathbf{b}_2 \right) \leq C(z) \kappa^4 \frac{L}{MN} \tag{3.3}$$

where $C(z)$ can be written as $C(z) = P_1(|z|)P_2\left(\frac{1}{\text{Im}(z)}\right)$ for some nice polynomials P_1 and P_2 . Moreover, if \mathbf{G} is a $N \times N$ deterministic matrix verifying $\sup_N \|\mathbf{G}\| \leq \kappa$, the following evaluations hold:

$$\text{Var} \left(\frac{1}{ML} \text{Tr} (\mathbf{A}\mathbf{Q}(z)\mathbf{W}\mathbf{G}\mathbf{W}^*) \right) \leq C(z) \kappa^4 \frac{1}{MN} \tag{3.4}$$

$$\text{Var} (\mathbf{a}_1^* \mathbf{Q}(z) \mathbf{W} \mathbf{G} \mathbf{W}^* \mathbf{a}_2) \leq C(z) \kappa^6 \frac{L}{N} \tag{3.5}$$

$$\text{Var} \left(\mathbf{b}_1^* \left[\frac{1}{M} \sum_{m=1}^M (\mathbf{Q}(z) \mathbf{W} \mathbf{G} \mathbf{W}^*)^{m,m} \right] \mathbf{b}_2 \right) \leq C(z) \kappa^6 \frac{L}{MN} \tag{3.6}$$

where $C(z)$ can be written as above.

Proof We first establish (3.1) and denote by ξ the random variable $\xi = \frac{1}{ML} \text{Tr} (\mathbf{A}\mathbf{Q}(z))$. As the various entries of 2 different blocks $\mathbf{W}^{m_1}, \mathbf{W}^{m_2}$ are independent, the Poincaré–Nash inequality can be written as

$$\text{Var} \xi \leq \sum_{m,i_1,i_2,j_1,j_2} \mathbb{E} \left[\left(\frac{\partial \xi}{\partial \overline{\mathbf{W}}_{i_1,j_1}^m} \right)^* \mathbb{E} (\mathbf{W}_{i_1,j_1}^m \overline{\mathbf{W}}_{i_2,j_2}^m) \frac{\partial \xi}{\partial \overline{\mathbf{W}}_{i_2,j_2}^m} \right] \tag{3.7}$$

$$+ \sum_{m,i_1,i_2,j_1,j_2} \mathbb{E} \left[\frac{\partial \xi}{\partial \overline{\mathbf{W}}_{i_1,j_1}^m} \mathbb{E} (\mathbf{W}_{i_1,j_1}^m \overline{\mathbf{W}}_{i_2,j_2}^m) \left(\frac{\partial \xi}{\partial \overline{\mathbf{W}}_{i_2,j_2}^m} \right)^* \right] \tag{3.8}$$

In the following, we just evaluate the right-hand side of (3.7), denoted by β , because the behaviour of the term defined by (3.8) can be established similarly. It is easy to check that

$$\frac{\partial \mathbf{Q}}{\partial \overline{\mathbf{W}}_{i,j}^m} = -\mathbf{Q} \mathbf{W} \mathbf{e}_j (\mathbf{f}_i^m)^T \mathbf{Q}$$

so that

$$\frac{\partial \xi}{\partial \overline{\mathbf{W}}_{i,j}^m} = -\frac{1}{ML} \text{Tr} (\mathbf{A} \mathbf{Q} \mathbf{W} \mathbf{e}_j (\mathbf{f}_i^m)^T \mathbf{Q})$$

which can also be written $-\frac{1}{ML} (\mathbf{f}_i^m)^T \mathbf{Q} \mathbf{A} \mathbf{Q} \mathbf{W} \mathbf{e}_j$. We recall that $\mathbb{E} (\mathbf{W}_{i_1,j_1}^m \overline{\mathbf{W}}_{i_2,j_2}^m) = \frac{\sigma^2}{N} \delta(i_1 - i_2 = j_2 - j_1)$ (see (1.29)). Therefore, β is equal to the mathematical expectation of the term

$$\frac{1}{(ML)^2} \frac{\sigma^2}{N} \sum_{m, i_1, i_2, j_1, j_2} \delta(j_2 - j_1 = i_1 - i_2) \mathbf{e}_{j_1}^T \mathbf{W}^* \mathbf{Q}^* \mathbf{A}^* \mathbf{Q}^* \mathbf{f}_{i_1}^m (\mathbf{f}_{i_2}^m)^T \mathbf{Q} \mathbf{A} \mathbf{Q} \mathbf{W} \mathbf{e}_{j_2}$$

We put $u = i_1 - i_2$ and remark that $\sum_{m, i_1 - i_2 = u} \mathbf{f}_{i_1}^m (\mathbf{f}_{i_2}^m)^T = \mathbf{I}_M \otimes \mathbf{J}_L^{*u}$. We thus obtain that

$$\beta = \frac{1}{(ML)^2} \frac{\sigma^2}{N} \mathbb{E} \left[\sum_{u=-(L-1)}^{L-1} \sum_{j_2 - j_1 = u} \mathbf{e}_{j_1}^T \mathbf{W}^* \mathbf{Q}^* \mathbf{A}^* \mathbf{Q}^* (\mathbf{I}_M \otimes \mathbf{J}_L^{*u}) \mathbf{Q} \mathbf{A} \mathbf{Q} \mathbf{W} \mathbf{e}_{j_2} \right]$$

Using that $\sum_{j_2 - j_1 = u} \mathbf{e}_{j_2} \mathbf{e}_{j_1}^T = \mathbf{J}_N^{*u}$, we get that

$$\beta = \frac{1}{ML} \frac{\sigma^2}{N} \mathbb{E} \left[\sum_{u=-(L-1)}^{L-1} \frac{1}{ML} \text{Tr} (\mathbf{Q} \mathbf{A} \mathbf{Q} \mathbf{W} \mathbf{J}_N^{*u} \mathbf{W}^* \mathbf{Q}^* \mathbf{A}^* \mathbf{Q}^* (\mathbf{I}_M \otimes \mathbf{J}_L^{*u})) \right]$$

If \mathbf{B} is a $ML \times N$ matrix, the Schwartz inequality as well as the inequality $(xy)^{1/2} \leq 1/2(x + y)$ leads to

$$\left| \frac{1}{ML} \text{Tr} (\mathbf{B} \mathbf{J}_N^{*u} \mathbf{B}^* (\mathbf{I}_M \otimes \mathbf{J}_L^{*u})) \right| \leq \frac{1}{2ML} \text{Tr} (\mathbf{B} \mathbf{J}_N^{*u} \mathbf{J}_N^{*u} \mathbf{B}^*) + \frac{1}{2ML} \text{Tr} (\mathbf{B}^* (\mathbf{I}_M \otimes \mathbf{J}_L^{*u} \mathbf{J}_L^{*u}) \mathbf{B})$$

It is clear that matrices $\mathbf{J}_N^{*u} \mathbf{J}_N^{*u}$ and $\mathbf{J}_L^{*u} \mathbf{J}_L^{*u}$ are less than \mathbf{I}_N and \mathbf{I}_L , respectively. Therefore,

$$\left| \frac{1}{ML} \text{Tr} (\mathbf{B} \mathbf{J}_N^{*u} \mathbf{B}^* (\mathbf{I}_M \otimes \mathbf{J}_L^{*u})) \right| \leq \frac{1}{ML} \text{Tr} (\mathbf{B} \mathbf{B}^*) \tag{3.9}$$

Using (3.9) for $\mathbf{B} = \mathbf{Q} \mathbf{A} \mathbf{Q} \mathbf{W}$ for each u leads to

$$\beta \leq \frac{\sigma^2}{MN} \mathbb{E} \left[\frac{1}{ML} \text{Tr} (\mathbf{Q} \mathbf{A} \mathbf{Q} \mathbf{W} \mathbf{W}^* \mathbf{Q}^* \mathbf{A}^* \mathbf{Q}^*) \right]$$

The resolvent identity (1.23) can also be written as $\mathbf{Q} \mathbf{W} \mathbf{W}^* = \mathbf{I} + z \mathbf{Q}$. This implies that the greatest eigenvalue of $\mathbf{Q} \mathbf{W} \mathbf{W}^* \mathbf{Q}^*$ coincides with the greatest eigenvalue of $(\mathbf{I} + z \mathbf{Q}) \mathbf{Q}^*$ which is itself less than $\|\mathbf{Q}\| + |z| \|\mathbf{Q}\|^2$. As $\|\mathbf{Q}\| \leq \frac{1}{\text{Im}z}$, we obtain that

$$\mathbf{Q} \mathbf{W} \mathbf{W}^* \mathbf{Q}^* \leq \frac{1}{\text{Im}z} \left(1 + \frac{|z|}{\text{Im}z} \right) \mathbf{I}. \tag{3.10}$$

Therefore, it holds that

$$\beta \leq \frac{1}{\text{Im}z} \left(1 + \frac{|z|}{\text{Im}z} \right) \frac{1}{MN} \mathbb{E} \left[\frac{1}{ML} \text{Tr} (\mathbf{Q} \mathbf{A} \mathbf{A}^* \mathbf{Q}^*) \right] \tag{3.11}$$

We eventually obtain that

$$\beta \leq \kappa^2 \frac{1}{MN} C(z) \frac{1}{(\operatorname{Im}z)^3} \left(1 + \frac{|z|}{\operatorname{Im}z}\right)$$

The conclusion follows from the observation that

$$\frac{1}{(\operatorname{Im}z)^3} \left(1 + \frac{|z|}{\operatorname{Im}z}\right) \leq \left[\frac{1}{(\operatorname{Im}z)^3} + \frac{1}{(\operatorname{Im}z)^4}\right] (|z| + 1)$$

In order to prove (3.2) and (3.3), we remark that

$$\begin{aligned} \mathbf{a}_1^* \mathbf{Q} \mathbf{a}_2 &= ML \frac{1}{ML} \operatorname{Tr} (\mathbf{Q} \mathbf{a}_2 \mathbf{a}_1^*) \\ \mathbf{b}_1^* \left[\frac{1}{M} \sum_{m=1}^M (\mathbf{Q}(z))^{m,m} \right] \mathbf{b}_2 &= L \frac{1}{ML} \operatorname{Tr} (\mathbf{Q} (\mathbf{I}_M \otimes \mathbf{b}_2 \mathbf{b}_1^*)) \end{aligned}$$

(3.2) and (3.3) follow immediately from this and inequality (3.11) used in the case $\mathbf{A} = \mathbf{a}_2 \mathbf{a}_1^*$ and $\mathbf{A} = \mathbf{I}_M \otimes \mathbf{b}_2 \mathbf{b}_1^*$, respectively. \square

We finally provide a sketch of proof of (3.4) and omit the proof of (3.6) and (3.5) which can be obtained as above. We still denote by ξ the random variable $\xi = \frac{1}{ML} \operatorname{Tr} (\mathbf{Q}(z) \mathbf{W} \mathbf{G} \mathbf{W}^*)$ and only evaluate the behaviour of the right-hand side β of (3.7). After easy calculations using tricks similar to those used in the course of the proof of (3.1), we obtain that

$$\beta \leq \frac{2\sigma^2}{MN} \mathbb{E} \left[\frac{1}{ML} \operatorname{Tr} (\mathbf{Q} \mathbf{W} \mathbf{G} \mathbf{W}^* \mathbf{A} \mathbf{Q} \mathbf{W} \mathbf{W}^* \mathbf{Q}^* \mathbf{A}^* \mathbf{W} \mathbf{G}^* \mathbf{W}^* \mathbf{Q}^*) \right] \tag{3.12}$$

$$+ \frac{2\sigma^2}{MN} \mathbb{E} \left[\frac{1}{ML} \operatorname{Tr} (\mathbf{G}^* \mathbf{W}^* \mathbf{Q}^* \mathbf{A}^* \mathbf{A} \mathbf{Q} \mathbf{W} \mathbf{G}) \right] \tag{3.13}$$

The term defined by (3.13) is easy to handle because $\mathbf{Q}^* \mathbf{A}^* \mathbf{A} \mathbf{Q} \leq \frac{\kappa^2}{(\operatorname{Im}(z))^2} \mathbf{I}$. Therefore, (3.13) is less than $\frac{2\sigma^2 \kappa^2}{(\operatorname{Im}(z))^2} \frac{1}{MN} \mathbb{E} \left[\frac{1}{ML} \operatorname{Tr} (\mathbf{W} \mathbf{G} \mathbf{G}^* \mathbf{W}^*) \right]$ which is itself lower bounded by $\frac{1}{MN} \frac{2\sigma^4 \kappa^4}{(\operatorname{Im}(z))^2}$ because $\mathbb{E} \left(\frac{1}{ML} \operatorname{Tr} (\mathbf{W} \mathbf{W}^*) \right) = \sigma^2$. To evaluate the right-hand side of (3.12), we use (3.10) twice and obtain immediately that is less than $\frac{C(z) \kappa^4}{MN}$.

4 Expression of Matrix $\mathbb{E}(\mathbf{Q})$ Obtained Using the Integration by Parts Formula

In this section, we use the integration by parts formula in order to express $\mathbb{E}(\mathbf{Q}(z))$ as a term which will appear to be close from $t(z) \mathbf{I}_{ML}$ where we recall that $t(z)$ represents the Stieltjes transform of the Marcenko–Pastur distribution μ_{σ^2, c_N} . For this, we have first to introduce useful matrix-valued functions of the complex variable z and to study their properties.

Lemma 4.1 For each $z \in \mathbb{C}^+$, matrix $\mathbf{I}_N + \sigma^2 c_N \mathcal{T}_{N,L}^{(M)}(\mathbb{E}(\mathbf{Q}(z)))$ is invertible. We denote by $\mathbf{H}(z)$ its inverse, i.e.

$$\mathbf{H}(z) = \left[\mathbf{I}_N + \sigma^2 c_N \mathcal{T}_{N,L}^{(M)}(\mathbb{E}(\mathbf{Q}(z))) \right]^{-1} \tag{4.1}$$

Then, function $z \rightarrow \mathbf{H}(z)$ is holomorphic in \mathbb{C}^+ and verifies

$$\mathbf{H}(z)\mathbf{H}(z)^* \leq \left(\frac{|z|}{\text{Im}z} \right)^2 \mathbf{I}_N \tag{4.2}$$

Moreover, for each $z \in \mathbb{C}^+$, matrix $-\mathbf{I} + \sigma^2 \mathcal{T}_{L,L}(\mathbf{H}(z))$ is invertible. We denote by $\mathbf{R}(z)$ its inverse, i.e.

$$\mathbf{R}(z) = \left[-\mathbf{I}_L + \sigma^2 \mathcal{T}_{L,L}(\mathbf{H}(z)) \right]^{-1} \tag{4.3}$$

Then, function $z \rightarrow \mathbf{R}(z)$ is holomorphic in \mathbb{C}^+ , and it exists a positive matrix-valued measure $\mu_{\mathbf{R}}$ carried by \mathbb{R}^+ , satisfying $\mu_{\mathbf{R}}(\mathbb{R}^+) = \mathbf{I}_L$, and for which

$$\mathbf{R}(z) = \int_{\mathbb{R}^+} \frac{d\mu_{\mathbf{R}}(\lambda)}{\lambda - z}$$

Finally, it holds that

$$\mathbf{R}(z)\mathbf{R}(z)^* \leq \left(\frac{1}{\text{Im}z} \right)^2 \mathbf{I}_L \tag{4.4}$$

Proof The proof is sketched in the ‘‘Appendix’’. □

In order to be able to perform the integration by parts formula, we use the identity (1.23) which implies that

$$\mathbb{E} \left[\mathbf{Q}_{i_1, i_2}^{m_1, m_2} \right] = -\frac{1}{z} \delta(i_1 - i_2) \delta(m_1 - m_2) + \frac{1}{z} \mathbb{E} \left[(\mathbf{Q}\mathbf{W}\mathbf{W}^*)_{i_1, i_2}^{m_1, m_2} \right] \tag{4.5}$$

We express $(\mathbf{Q}\mathbf{W}\mathbf{W}^*)_{i_1, i_2}^{m_1, m_2}$ as

$$(\mathbf{Q}\mathbf{W}\mathbf{W}^*)_{i_1, i_2}^{m_1, m_2} = \sum_{j=1}^N (\mathbf{Q}\mathbf{w}_j \mathbf{w}_j^*)_{i_1, i_2}^{m_1, m_2} = \sum_{j=1}^N (\mathbf{Q}\mathbf{w}_j)_{i_1}^{m_1} \overline{\mathbf{w}_{i_2, j}^{m_2}}$$

where we recall that $(\mathbf{w}_j)_{j=1, \dots, N}$ represent the columns of \mathbf{W} . In order to be able to evaluate $\mathbb{E} \left[(\mathbf{Q}\mathbf{w}_j \mathbf{w}_j^*)_{i_1, i_2}^{m_1, m_2} \right]$, it is necessary to express $\mathbb{E} \left[(\mathbf{Q}\mathbf{w}_k \mathbf{w}_j^*)_{i_1, i_2}^{m_1, m_2} \right] = \mathbb{E} \left[(\mathbf{Q}\mathbf{w}_k)_{i_1}^{m_1} (\mathbf{w}_j^*)_{i_2}^{m_2} \right]$ for each pair (k, j) . For this, we use the identity

$$\mathbb{E} \left[(\mathbf{Q}\mathbf{w}_k)_{i_1}^{m_1} (\mathbf{w}_j^*)_{i_2}^{m_2} \right] = \sum_{i_3, m_3} \mathbb{E} \left(\mathbf{Q}_{i_1, i_3}^{m_1, m_3} \mathbf{w}_{i_3, k}^{m_3} \overline{\mathbf{w}_{i_2, j}^{m_2}} \right)$$

and use the integration by parts formula

$$\mathbb{E} \left(\mathbf{Q}_{i_1, i_3}^{m_1, m_3} \mathbf{W}_{i_3, k}^{m_3} \overline{\mathbf{W}}_{i_2, j}^{m_2} \right) = \sum_{i', j'} \mathbb{E} \left(\mathbf{W}_{i_3, k}^{m_3} \overline{\mathbf{W}}_{i', j'}^{m_3} \right) \mathbb{E} \left[\frac{\partial \left(\mathbf{Q}_{i_1, i_3}^{m_1, m_3} \overline{\mathbf{W}}_{i_2, j}^{m_2} \right)}{\partial \overline{\mathbf{W}}_{i', j'}^{m_3}} \right]$$

It is easy to check that

$$\begin{aligned} \frac{\partial \left(\mathbf{Q}_{i_1, i_3}^{m_1, m_3} \overline{\mathbf{W}}_{i_2, j}^{m_2} \right)}{\partial \overline{\mathbf{W}}_{i', j'}^{m_3}} &= \mathbf{Q}_{i_1, i_3}^{m_1, m_3} \delta(m_2 = m_3) \delta(i' = i_2) \delta(j = j') \\ &\quad - \left(\mathbf{Q} \mathbf{w}_{j'} \right)_{i_1}^{m_1} \mathbf{Q}_{i', i_3}^{m_3, m_3} \overline{\mathbf{W}}_{i_2, j}^{m_2} \end{aligned}$$

(1.1) implies that $\mathbb{E} \left(\mathbf{W}_{i_3, k}^{m_3} \overline{\mathbf{W}}_{i', j'}^{m_3} \right) = \frac{\sigma^2}{N} \delta(i_3 - i' = j' - k)$. Therefore, we obtain that

$$\begin{aligned} \mathbb{E} \left(\mathbf{Q}_{i_1, i_3}^{m_1, m_3} \mathbf{W}_{i_3, k}^{m_3} \overline{\mathbf{W}}_{i_2, j}^{m_2} \right) &= \frac{\sigma^2}{N} \delta(i_3 - i_2 = j - k) \delta(m_2 = m_3) \mathbb{E} \left(\mathbf{Q}_{i_1, i_3}^{m_1, m_3} \right) \\ &\quad - \frac{\sigma^2}{N} \sum_{i', j'} \delta(i_3 - i' = j' - k) \mathbb{E} \left[\left(\mathbf{Q} \mathbf{w}_{j'} \right)_{i_1}^{m_1} \left(\mathbf{w}_j^* \right)_{i_2}^{m_2} \mathbf{Q}_{i', i_3}^{m_3, m_3} \right] \end{aligned}$$

and that

$$\begin{aligned} \mathbb{E} \left[\left(\mathbf{Q} \mathbf{w}_k \right)_{i_1}^{m_1} \left(\mathbf{w}_j^* \right)_{i_2}^{m_2} \right] &= \frac{\sigma^2}{N} \sum_{i_3, m_3} \delta(i_3 - i_2 = j - k) \delta(m_2 = m_3) \mathbb{E} \left(\mathbf{Q}_{i_1, i_3}^{m_1, m_3} \right) \\ &\quad - \frac{\sigma^2}{N} \sum_{i_3, m_3} \sum_{i', j'} \delta(i_3 - i' = j' - k) \mathbb{E} \left[\left(\mathbf{Q} \mathbf{w}_{j'} \right)_{i_1}^{m_1} \left(\mathbf{w}_j^* \right)_{i_2}^{m_2} \mathbf{Q}_{i', i_3}^{m_3, m_3} \right] \end{aligned}$$

We put $i = i' - i_3$ in the above sum and get that

$$\begin{aligned} \mathbb{E} \left[\left(\mathbf{Q} \mathbf{w}_k \right)_{i_1}^{m_1} \left(\mathbf{w}_j^* \right)_{i_2}^{m_2} \right] &= \frac{\sigma^2}{N} \mathbb{E} \left(\mathbf{Q}_{i_1, i_2 - (k-j)}^{m_1, m_2} \right) \mathbb{1}_{1 \leq i_2 - (k-j) \leq L} \\ &\quad - \sigma^2 c_N \sum_{i = -(L-1)}^{L-1} \mathbb{1}_{1 \leq k-i \leq N} \mathbb{E} \left[\left(\mathbf{Q} \mathbf{w}_{k-i} \right)_{i_1}^{m_1} \left(\mathbf{w}_j^* \right)_{i_2}^{m_2} \frac{1}{ML} \sum_{i' = i_3 = i} \sum_{m_3} \mathbf{Q}_{i', i_3}^{m_3, m_3} \right] \end{aligned}$$

or, using the definition (2.1),

$$\begin{aligned} \mathbb{E} \left[\left(\mathbf{Q} \mathbf{w}_k \right)_{i_1}^{m_1} \left(\mathbf{w}_j^* \right)_{i_2}^{m_2} \right] &= \frac{\sigma^2}{N} \mathbb{E} \left(\mathbf{Q}_{i_1, i_2 - (k-j)}^{m_1, m_2} \right) \mathbb{1}_{1 \leq i_2 - (k-j) \leq L} \\ &\quad - \sigma^2 c_N \sum_{i = -(L-1)}^{L-1} \mathbb{1}_{1 \leq k-i \leq N} \mathbb{E} \left[\tau^{(M)}(\mathbf{Q})(i) \left(\mathbf{Q} \mathbf{w}_{k-i} \right)_{i_1}^{m_1} \left(\mathbf{w}_j^* \right)_{i_2}^{m_2} \right] \end{aligned} \tag{4.6}$$

Setting $u = k - i$, the second term of the right-hand side of the above equation can also be written as

$$-\sigma^2 c_N \mathbb{E} \left[\sum_{u=1}^N \tau^{(M)}(\mathbf{Q})(k-u) \mathbb{1}_{-(L-1) \leq k-u \leq L-1} (\mathbf{Q}\mathbf{w}_u)_{i_1}^{m_1} (\mathbf{w}_j^*)_{i_2}^{m_2} \right]$$

or, using the observation that $\tau^{(M)}(\mathbf{Q})(k-u) \mathbb{1}_{-(L-1) \leq k-u \leq L-1} = \left(\mathcal{T}_{N,L}^{(M)}(\mathbf{Q}) \right)_{k,u}$ (see Eq. (2.3)), as

$$-\sigma^2 c_N \mathbb{E} \left[\mathbf{e}_k^T \mathcal{T}_{N,L}^{(M)}(\mathbf{Q}) \begin{pmatrix} (\mathbf{Q}\mathbf{w}_1)_{i_1}^{m_1} (\mathbf{w}_j^*)_{i_2}^{m_2} \\ (\mathbf{Q}\mathbf{w}_2)_{i_1}^{m_1} (\mathbf{w}_j^*)_{i_2}^{m_2} \\ \vdots \\ (\mathbf{Q}\mathbf{w}_N)_{i_1}^{m_1} (\mathbf{w}_j^*)_{i_2}^{m_2} \end{pmatrix} \right]$$

We express matrix \mathbf{Q} as $\mathbf{Q} = \mathbb{E}(\mathbf{Q}) + \mathbf{Q}^\circ$ and define the following $N \times N$ matrices $\mathbf{A}_{i_1, i_2}^{m_1, m_2}$, $\mathbf{B}_{i_1, i_2}^{m_1, m_2}$, $\mathbf{Y}_{i_1, i_2}^{m_1, m_2}$

$$\begin{aligned} \left(\mathbf{A}_{i_1, i_2}^{m_1, m_2} \right)_{k,j} &= \mathbb{E} \left[(\mathbf{Q}\mathbf{w}_k)_{i_1}^{m_1} (\mathbf{w}_j^*)_{i_2}^{m_2} \right] \\ \left(\mathbf{B}_{i_1, i_2}^{m_1, m_2} \right)_{k,j} &= \mathbb{E} \left[\mathbf{Q}_{i_1, i_2 - (k-j)}^{m_1, m_2} \mathbb{1}_{1 \leq i_2 - (k-j) \leq L} \right] \\ \mathbf{Y}_{i_1, i_2}^{m_1, m_2} &= -\sigma^2 c_N \mathbb{E} \left[\mathcal{T}_{N,L}^{(M)}(\mathbf{Q}^\circ) \begin{pmatrix} (\mathbf{Q}\mathbf{w}_1)_{i_1}^{m_1} \\ (\mathbf{Q}\mathbf{w}_2)_{i_1}^{m_1} \\ \vdots \\ (\mathbf{Q}\mathbf{w}_N)_{i_1}^{m_1} \end{pmatrix} \left((\mathbf{w}_1^*)_{i_2}^{m_2} (\mathbf{w}_2^*)_{i_2}^{m_2} \cdots (\mathbf{w}_N^*)_{i_2}^{m_2} \right) \right] \end{aligned}$$

We notice that matrix

$$\left[\begin{pmatrix} (\mathbf{Q}\mathbf{w}_1)_{i_1}^{m_1} \\ (\mathbf{Q}\mathbf{w}_2)_{i_1}^{m_1} \\ \vdots \\ (\mathbf{Q}\mathbf{w}_N)_{i_1}^{m_1} \end{pmatrix} \left((\mathbf{w}_1^H)_{i_2}^{m_2} (\mathbf{w}_2^H)_{i_2}^{m_2} \cdots (\mathbf{w}_N^H)_{i_2}^{m_2} \right) \right]$$

can also be written as

$$\begin{pmatrix} \mathbf{w}_1^T \mathbf{Q}^T \\ \vdots \\ \mathbf{w}_N^T \mathbf{Q}^T \end{pmatrix} \left(\mathbf{f}_{i_1}^{m_1} \right) \left(\mathbf{f}_{i_2}^{m_2} \right)^T (\bar{\mathbf{w}}_1, \dots, \bar{\mathbf{w}}_N)$$

or as

$$\mathbf{W}^T \mathbf{Q}^T \left(\mathbf{f}_{i_1}^{m_1} \right) \left(\mathbf{f}_{i_2}^{m_2} \right)^T \overline{\mathbf{W}}$$

Therefore,

$$\boldsymbol{\Upsilon}_{i_1, i_2}^{m_1, m_2} = -\sigma^2 c_N \mathbb{E} \left[\mathcal{T}_{N, L}^{(M)}(\mathbf{Q}^\circ) \mathbf{W}^T \mathbf{Q}^T \left(\mathbf{f}_{i_1}^{m_1} \right) \left(\mathbf{f}_{i_2}^{m_2} \right)^T \overline{\mathbf{W}} \right] \tag{4.7}$$

It is useful to notice that matrix $\mathbf{B}_{i_1, i_2}^{m_1, m_2}$ is a band Toeplitz matrix whose (k, l) element is zero if $|k - l| \geq L$. It is clear that Eq. (4.6) is equivalent to

$$\left[\mathbf{I}_N + \sigma^2 c_N \mathcal{T}_{N, L}^{(M)}(\mathbb{E}(\mathbf{Q})) \right] \mathbf{A}_{i_1, i_2}^{m_1, m_2} = \frac{\sigma^2}{N} \mathbf{B}_{i_1, i_2}^{m_1, m_2} + \boldsymbol{\Upsilon}_{i_1, i_2}^{m_1, m_2}$$

Lemma 4.1 implies that matrix $\left[\mathbf{I}_N + \sigma^2 c_N \mathcal{T}_{N, L}^{(M)}(\mathbb{E}(\mathbf{Q}(z))) \right]$ is invertible for each $z \in \mathbb{C}^+$, and we recall that its inverse is denoted $\mathbf{H}(z)$. We obtain that

$$\mathbf{A}_{i_1, i_2}^{m_1, m_2} = \frac{\sigma^2}{N} \mathbf{H} \mathbf{B}_{i_1, i_2}^{m_1, m_2} + \mathbf{H} \boldsymbol{\Upsilon}_{i_1, i_2}^{m_1, m_2} \tag{4.8}$$

The term $\mathbb{E}(\mathbf{Q} \mathbf{W} \mathbf{W}^*)_{i_1, i_2}^{m_1, m_2}$ coincides with $\text{Tr} \left(\mathbf{A}_{i_1, i_2}^{m_1, m_2} \right)$, so that

$$\mathbb{E}(\mathbf{Q} \mathbf{W} \mathbf{W}^*)_{i_1, i_2}^{m_1, m_2} = \sigma^2 \frac{1}{N} \text{Tr} \left(\mathbf{H} \mathbf{B}_{i_1, i_2}^{m_1, m_2} \right) + \text{Tr} \left(\mathbf{H} \boldsymbol{\Upsilon}_{i_1, i_2}^{m_1, m_2} \right) \tag{4.9}$$

As matrix $\mathbf{B}_{i_1, i_2}^{m_1, m_2}$ is Toeplitz, it holds that (see Eq. 2.4)

$$\frac{1}{N} \text{Tr} \left(\mathbf{H} \mathbf{B}_{i_1, i_2}^{m_1, m_2} \right) = \sum_{u=-(N-1)}^{N-1} \tau(\mathbf{H})(u) \mathbb{E} \left(\mathbf{Q}_{i_1, i_2+u}^{m_1, m_2} \right) \mathbb{1}_{1 \leq i_2+u \leq L}$$

which also coincides with

$$\frac{1}{N} \text{Tr} \left(\mathbf{H} \mathbf{B}_{i_1, i_2}^{m_1, m_2} \right) = \sum_{u=-(L-1)}^{L-1} \tau(\mathbf{H})(u) \mathbb{E} \left(\mathbf{Q}_{i_1, i_2+u}^{m_1, m_2} \right) \mathbb{1}_{1 \leq i_2+u \leq L}$$

because $\mathbb{1}_{1 \leq i_2+u \leq L} = 0$ if $|u| \geq L$. Setting $v = i_2 + u$, this term can be written as

$$\frac{1}{N} \text{Tr} \left(\mathbf{H} \mathbf{B}_{i_1, i_2}^{m_1, m_2} \right) = \sum_{v=1}^L \mathbb{E} \left(\mathbf{Q}_{i_1, v}^{m_1, m_2} \right) \tau(\mathbf{H})(v - i_2)$$

or, using definition (2.3), as

$$\begin{aligned} \frac{1}{N} \text{Tr} \left(\mathbf{H} \mathbf{B}_{i_1, i_2}^{m_1, m_2} \right) &= \sum_{v=1}^L \mathbb{E} \left(\mathbf{Q}_{i_1, v}^{m_1, m_2} \right) \left(\mathcal{T}_{L, L}(\mathbf{H}) \right)_{v, i_2} \\ &= \left(\mathbb{E}(\mathbf{Q}^{m_1, m_2}) \mathcal{T}_{L, L}(\mathbf{H}) \right)_{i_1, i_2} \end{aligned}$$

Equation (4.9) eventually leads to

$$\mathbb{E} \left[(\mathbf{QWW}^*)^{m_1, m_2} \right] = \sigma^2 \mathbb{E}(\mathbf{Q}^{m_1, m_2}) \mathcal{T}_{L, L}(\mathbf{H}) + \Upsilon(\mathbf{H})^{m_1, m_2} \tag{4.10}$$

where, for each $N \times N$ matrix \mathbf{F} , $\Upsilon(\mathbf{F})$ represents the $ML \times ML$ matrix defined by

$$\Upsilon(\mathbf{F})_{i_1, i_2}^{m_1, m_2} = \text{Tr} \left(\mathbf{F} \Upsilon_{i_1, i_2}^{m_1, m_2} \right) \tag{4.11}$$

(4.7) implies that matrix $\Upsilon(\mathbf{F})$ can be written as

$$\Upsilon(\mathbf{F}) = -\sigma^2 c_N \mathbb{E} \left[\mathbf{QW} \left(\mathcal{T}_{N, L}^{(M)}(\mathbf{Q}^\circ) \right)^T \mathbf{F}^T \mathbf{W}^* \right] \tag{4.12}$$

By (1.23), it holds that $(\mathbf{QWW}^*)^{m_1, m_2} = \delta(m_1 = m_2) \mathbf{I}_L + z \mathbf{Q}^{m_1, m_2}$. Therefore, we deduce from (4.10) that

$$\mathbb{E}(\mathbf{Q}^{m_1, m_2}) \left(-z \mathbf{I}_L + \sigma^2 \mathcal{T}_{L, L}(\mathbf{H}) \right) = \mathbf{I}_L \delta(m_1 = m_2) - \Upsilon(\mathbf{H})^{m_1, m_2} \tag{4.13}$$

By Lemma 4.1, $-z \mathbf{I}_L + \sigma^2 \mathcal{T}_{L, L}(\mathbf{H}(z))$ is invertible for $z \in \mathbb{C}^+$ and we recall that its inverse is denoted by \mathbf{R} . We thus obtain that

$$\mathbb{E}(\mathbf{Q}) = \mathbf{I}_M \otimes \mathbf{R} + \mathbf{\Delta} \tag{4.14}$$

where $\mathbf{\Delta}$ is the $ML \times ML$ matrix defined by

$$\mathbf{\Delta} = -\Upsilon(\mathbf{H}) (\mathbf{I}_M \otimes \mathbf{R}) \tag{4.15}$$

The above evaluations also allow to obtain a similar expression of matrix $\mathbb{E}(\mathbf{QWGW}^*)$ where \mathbf{G} is a $N \times N$ matrix. For this, we express $\mathbb{E} \left[(\mathbf{QWGW}^*)_{i_1, i_2}^{m_1, m_2} \right]$ as

$$\mathbb{E} \left[(\mathbf{QWGW}^*)_{i_1, i_2}^{m_1, m_2} \right] = \sum_{(k, j)=1}^N \mathbf{G}_{k, j} \mathbb{E} \left[(\mathbf{Qw}_k)_{i_1}^{m_1} (\mathbf{w}_j^*)_{i_2}^{m_2} \right]$$

or equivalently as

$$\mathbb{E} \left[(\mathbf{QWGW}^*)_{i_1, i_2}^{m_1, m_2} \right] = \text{Tr} \left(\mathbf{G}^T \mathbf{A}_{i_1, i_2}^{m_1, m_2} \right)$$

Therefore, using (4.8), it holds that

$$\mathbb{E} \left[(\mathbf{Q}\mathbf{W}\mathbf{G}\mathbf{W}^*)_{i_1, i_2}^{m_1, m_2} \right] = \frac{\sigma^2}{N} \text{Tr} \left(\mathbf{G}^T \mathbf{H} \mathbf{B}_{i_1, i_2}^{m_1, m_2} \right) + \text{Tr} \left(\mathbf{G}^T \mathbf{H} \mathbf{\Upsilon}_{i_1, i_2}^{m_1, m_2} \right)$$

Replacing matrix \mathbf{H} by matrix $\mathbf{G}^T \mathbf{H}$ in the above calculations, we obtain that

$$\mathbb{E} [\mathbf{Q}\mathbf{W}\mathbf{G}\mathbf{W}^*] = \sigma^2 \mathbb{E}(\mathbf{Q}) \left(\mathbf{I}_M \otimes \mathcal{T}_{L, L}(\mathbf{G}^T \mathbf{H}) \right) + \mathbf{\Upsilon}(\mathbf{G}^T \mathbf{H})$$

Using (4.14), we eventually get that

$$\mathbb{E} (\mathbf{Q}\mathbf{W}\mathbf{G}\mathbf{W}^*) = \sigma^2 \left(\mathbf{I}_M \otimes \mathbf{R} \mathcal{T}_{L, L}(\mathbf{G}^T \mathbf{H}) \right) + \sigma^2 \mathbf{\Delta} \left(\mathbf{I}_M \otimes \mathcal{T}_{L, L}(\mathbf{G}^T \mathbf{H}) \right) + \mathbf{\Upsilon}(\mathbf{G}^T \mathbf{H}) \tag{4.16}$$

5 Controls of the Error Term $\mathbf{\Delta}$

In this section, we evaluate the behaviour of various terms depending on $\mathbf{\Delta}$, i.e. normalized traces $\frac{1}{ML} \text{Tr} \mathbf{\Delta} \mathbf{A}$, quadratic forms $\mathbf{a}_1^* \mathbf{\Delta} \mathbf{a}_2$, and quadratic forms of matrix $\hat{\mathbf{\Delta}} = \frac{1}{M} \sum_{m=1}^M \mathbf{\Delta}^{m, m}$. Using rough estimates based on the results of Sect. 3 and the Schwartz inequality, we establish that the normalized traces are $O(\frac{L}{MN})$ and that two other terms are $O(\sqrt{\frac{L}{M} \frac{L}{N}})$ and $O(\frac{L^{3/2}}{MN})$, respectively. We first establish the following proposition.

Proposition 5.1 *Let \mathbf{A} be a $ML \times ML$ matrix satisfying $\sup_N \|\mathbf{A}\| \leq \kappa$. Then, it holds that*

$$\left| \frac{1}{ML} \text{Tr} \mathbf{\Delta} \mathbf{A} \right| \leq \kappa \frac{L}{MN} C(z) \tag{5.1}$$

where $C(z)$ can be written as $C(z) = P_1(|z|) P_2((\text{Im}z)^{-1})$ for some nice polynomials P_1 and P_2 .

Proof As matrix \mathbf{R} verifies $\|\mathbf{R}\| \leq (\text{Im}z)^{-1}$, it is sufficient to establish (5.1) when $\mathbf{\Delta}$ is replaced by $\mathbf{\Upsilon}(\mathbf{H})$. In order to simplify the notations, matrix $\mathbf{\Upsilon}(\mathbf{H})$ is denoted by $\mathbf{\Upsilon}$ in this section. We denote by γ the term $\gamma = \frac{1}{ML} \text{Tr} \mathbf{\Upsilon} \mathbf{A}$ which is given by

$$\gamma = \frac{1}{M} \sum_{m_1, m_2} \frac{1}{L} \sum_{i_1, i_2} \mathbf{\Upsilon}_{i_1, i_2}^{m_1, m_2} \mathbf{A}_{i_2, i_1}^{m_2, m_1}$$

Using the expression (4.12) of matrix $\mathbf{\Upsilon}$, we obtain that γ can be written as

$$\gamma = -\sigma^2 \mathbb{E} \left[\frac{1}{N} \text{Tr} \left(\left(\mathcal{T}_{N, L}^{(M)}(\mathbf{Q}^\circ) \right)^T \mathbf{H}^T \mathbf{W}^* \mathbf{A} \mathbf{Q} \mathbf{W} \right) \right]$$

Using Eq. (2.6) and the identity $\tau^{(M)}((\mathbf{Q}^\circ)^T)(-u) = \tau^{(M)}(\mathbf{Q}^\circ)(u)$, we get that

$$\gamma = -\sigma^2 c_N \mathbb{E} \left[\sum_{u=-(L-1)}^{L-1} \tau^{(M)}(\mathbf{Q}^\circ)(u) \frac{1}{ML} \text{Tr}(\mathbf{QWJ}_N^u \mathbf{H}^T \mathbf{W}^* \mathbf{A}) \right] \tag{5.2}$$

(3.1, 3.4) imply that $\mathbb{E} |\tau^{(M)}(\mathbf{Q}^\circ)(-u)|^2$ and $\text{Var}(\frac{1}{ML} \text{Tr}(\mathbf{QWJ}_N^u \mathbf{H}^T \mathbf{W} \mathbf{A}))$ are upper-bounded by terms of the form $\frac{C(z)}{MN}$ and $\kappa^2 \frac{C(z)}{MN}$, respectively. The Cauchy–Schwartz inequality thus implies immediately (5.1). \square

We now evaluate the behaviour of quadratic forms of matrix $\mathbf{\Delta}$ and of matrix $\hat{\mathbf{\Delta}}$.

Proposition 5.2 *Let \mathbf{a}_1 and \mathbf{a}_2 2 ML -dimensional vectors such that $\sup_N \|\mathbf{a}_i\| \leq \kappa$ for $i = 1, 2$. Then, it holds that*

$$\mathbf{a}_1^* \mathbf{\Delta} \mathbf{a}_2 \leq \kappa^2 C(z) \sqrt{\frac{L}{M} \frac{L}{N}} \tag{5.3}$$

for each $z \in \mathbb{C}^+$, where $C(z)$ is as in Proposition 5.1. Let $\mathbf{b}_i, i = 1, 2$ be 2 deterministic L -dimensional vectors such that $\sup_N \|\mathbf{b}_i\| < \kappa$. Then, it holds that

$$\left| \mathbf{b}_1^* \left(\frac{1}{M} \sum_{m=1}^M \mathbf{\Delta}^{m,m} \right) \mathbf{b}_2 \right| \leq \kappa^2 C(z) \frac{L^{3/2}}{MN} \tag{5.4}$$

Proof As above, it is sufficient to establish the proposition when $\mathbf{\Delta}$ is replaced by $\mathbf{\Upsilon}$. We first establish (5.3). We remark that $\mathbf{a}_1^* \mathbf{\Upsilon} \mathbf{a}_2 = ML \frac{1}{ML} \text{Tr}(\mathbf{\Upsilon} \mathbf{a}_2 \mathbf{a}_1^*)$. Using Eq. (5.2) in the case $\mathbf{A} = \mathbf{a}_2 \mathbf{a}_1^*$, we obtain that

$$\mathbf{a}_1^* \mathbf{\Upsilon} \mathbf{a}_2 = -\sigma^2 \mathbb{E} \left[\sum_{u=-(L-1)}^{L-1} \tau^{(M)}(\mathbf{Q}^\circ)(u) \mathbf{a}_1^* \mathbf{QWJ}_N^u \mathbf{H}^T \mathbf{W}^* \mathbf{a}_2 \right]$$

(3.5, 3.1) and the Schwartz inequality lead immediately to

$$|\mathbf{a}_1^* \mathbf{\Upsilon} \mathbf{a}_2| \leq \kappa^2 C(z) L \frac{1}{\sqrt{MN}} \sqrt{\frac{L}{N}} = \kappa^2 C(z) \sqrt{\frac{L}{M} \frac{L}{N}}.$$

We now establish (5.4). We remark that

$$\mathbf{b}_1^* \left(\frac{1}{M} \sum_{m=1}^M \mathbf{\Upsilon}^{m,m} \right) \mathbf{b}_2 = L \frac{1}{ML} \text{Tr}(\mathbf{\Upsilon}(\mathbf{I}_M \otimes \mathbf{b}_2 \mathbf{b}_1^*))$$

Using Eq. (5.2) in the case $\mathbf{A} = \mathbf{I}_M \otimes \mathbf{b}_2 \mathbf{b}_1^*$, we obtain immediately that

$$\begin{aligned} & \mathbf{b}_1^* \left(\frac{1}{M} \sum_{m=1}^M \Upsilon^{m,m} \right) \mathbf{b}_2 \\ &= \sum_{u=-(L-1)}^{L-1} \mathbb{E} \left[\tau^{(M)}(\mathbf{Q}^\circ)(u) \mathbf{b}_1^* \left(\frac{1}{M} \sum_{m=1}^M (\mathbf{Q} \mathbf{W} \mathbf{J}_N^u \mathbf{H}^T \mathbf{W}^*)^{m,m} \right) \mathbf{b}_2 \right] \end{aligned} \tag{5.5}$$

(5.4) thus appears as a direct consequence of (3.1), (3.6) and the Schwartz inequality. □

We finally mention a useful corollary of (5.4).

Corollary 5.1 *It holds that*

$$\left\| \mathcal{T}_{N,L}^{(M)}(\mathbb{E}(\mathbf{Q}) - (\mathbf{I}_M \otimes \mathbf{R})) \right\| \leq C(z) \frac{L^{3/2}}{MN} \tag{5.6}$$

for each $z \in \mathbb{C}^+$ where $C(z)$ can be written as $C(z) = P_1(|z|) P_2((\text{Im}z)^{-1})$ for some nice polynomials P_1 and P_2 .

Taking into account Proposition 2.1, (5.6) follows immediately from (5.4) by considering the unit norm vector $\mathbf{b} = \mathbf{a}_L(v)$.

6 Convergence Towards the Marcenko–Pastur Distribution

In the following, we establish that

$$\frac{1}{ML} \text{Tr}(\mathbb{E}(\mathbf{Q}(z)) - t(z) \mathbf{I}_{ML}) \rightarrow 0 \tag{6.1}$$

for each $z \in \mathbb{C}^+$. (3.1) does not imply in general that $\frac{1}{ML} \text{Tr}(\mathbf{Q}(z) - \mathbb{E}(\mathbf{Q}(z)))$ converges towards 0 almost surely (this would be the case if M was of the same order of magnitude than N^κ for some $\kappa > 0$). However, the reader may check using the Poincaré–Nash inequality that the variance of $\left[\frac{1}{ML} \text{Tr}(\mathbf{Q}^\circ(z)) \right]^2$ is a $O\left(\frac{1}{(MN)^2}\right)$ term. As

$$\mathbb{E} \left| \frac{1}{ML} \text{Tr}(\mathbf{Q}^\circ(z)) \right|^4 = \mathbb{E} \left[\frac{1}{ML} \text{Tr}(\mathbf{Q}^\circ(z)) \right]^2 + \text{Var} \left[\frac{1}{ML} \text{Tr}(\mathbf{Q}^\circ(z)) \right]^2$$

(3.1) implies that the fourth-order moment of $\frac{1}{ML} \text{Tr}(\mathbf{Q}^\circ(z))$ is also a $O\left(\frac{1}{(MN)^2}\right)$ term and that $\frac{1}{ML} \text{Tr}(\mathbf{Q}(z) - \mathbb{E}(\mathbf{Q}(z)))$ converges towards 0 almost surely. Consequently, (6.1) allows us to prove that the eigenvalue value distribution of $\mathbf{W} \mathbf{W}^*$ has almost surely the same behaviour than the Marcenko–Pastur distribution μ_{σ^2, c_N} . As $c_N \rightarrow c_*$, this

of course establishes the almost sure convergence of the eigenvalue distribution of $\mathbf{W}_N \mathbf{W}_N^*$ towards the Marcenko–Pastur μ_{σ^2, c_*} .

In the following, we thus prove (6.1), (4.14) and Proposition 5.1 imply that for each uniformly bounded $L \times L$ matrix \mathbf{A} , then it holds that

$$\frac{1}{ML} \text{Tr} [(\mathbb{E}(\mathbf{Q}(z)) - \mathbf{I}_M \otimes \mathbf{R}(z)) (\mathbf{I}_M \otimes \mathbf{A})] = \mathcal{O} \left(\frac{L}{MN} \right) \tag{6.2}$$

for each $z \in \mathbb{C}^+$. We now establish that

$$\frac{1}{ML} \text{Tr} [(\mathbf{I}_M \otimes \mathbf{R}(z) - t(z)\mathbf{I}_{ML}) (\mathbf{I}_M \otimes \mathbf{A})] \rightarrow 0$$

or equivalently that

$$\frac{1}{L} \text{Tr} [(\mathbf{R}(z) - t(z)\mathbf{I}_L) \mathbf{A}] \rightarrow 0 \tag{6.3}$$

for each $z \in \mathbb{C}^+$. For this, we first mention that straightforward computations lead to

$$\mathbf{R} - t\mathbf{I} = -\sigma^4 c_N z t(z) \tilde{t}(z) \mathbf{R} \mathcal{T}_{L,L} \left(\mathbf{H} \mathcal{T}_{N,L}^{(M)} [\mathbb{E}(\mathbf{Q}) - t\mathbf{I}_{ML}] \right) \tag{6.4}$$

Therefore,

$$\frac{1}{L} \text{Tr} [(\mathbf{R} - t\mathbf{I}_L) \mathbf{A}] = -\sigma^4 c_N z t(z) \tilde{t}(z) \frac{1}{L} \text{Tr} \mathbf{A} \mathbf{R} \mathcal{T}_{L,L} \left(\mathbf{H} \mathcal{T}_{N,L}^{(M)} [\mathbb{E}(\mathbf{Q}) - t\mathbf{I}_{ML}] \right)$$

Direct application of (2.7) to the case $P = M, K = L, R = L, \mathbf{C} = \mathbb{E}(\mathbf{Q}) - t\mathbf{I}_{ML}, \mathbf{B} = \mathbf{A} \mathbf{R}$, and $\mathbf{D} = \mathbf{H}$ implies that

$$\begin{aligned} & \frac{1}{L} \text{Tr} ((\mathbf{R} - t\mathbf{I}_L) \mathbf{A}) \\ &= -\sigma^4 c_N z t(z) \tilde{t}(z) \frac{1}{ML} \text{Tr} [(\mathbb{E}(\mathbf{Q}) - t\mathbf{I}_{ML}) (\mathbf{I}_M \otimes \mathcal{T}_{L,L}(\mathcal{T}_{N,L}(\mathbf{A} \mathbf{R}) \mathbf{H}))] \end{aligned}$$

In the following, we denote by $\mathbf{G}(\mathbf{A})$ the $L \times L$ matrix defined by

$$\mathbf{G}(\mathbf{A}) = \mathcal{T}_{L,L}(\mathcal{T}_{N,L}(\mathbf{A} \mathbf{R}) \mathbf{H}) \tag{6.5}$$

Writing that $\mathbf{E}(\mathbf{Q}) - t\mathbf{I}_{ML} = \mathbf{E}(\mathbf{Q}) - \mathbf{I}_M \otimes \mathbf{R} + \mathbf{I}_M \otimes \mathbf{R} - t\mathbf{I}_{ML}$, we obtain that

$$\begin{aligned} \frac{1}{L} \text{Tr} [(\mathbf{R} - t\mathbf{I}_L) \mathbf{A}] &= -\sigma^4 c_N z t(z) \tilde{t}(z) \frac{1}{ML} \text{Tr} [(\mathbf{E}(\mathbf{Q}) - \mathbf{I}_M \otimes \mathbf{R}) (\mathbf{I}_M \otimes \mathbf{G}(\mathbf{A}))] \\ &\quad -\sigma^4 c_N z t(z) \tilde{t}(z) \frac{1}{L} \text{Tr} [(\mathbf{R} - t\mathbf{I}_L) \mathbf{G}(\mathbf{A})] \end{aligned} \tag{6.6}$$

We now prove that

$$\sup_{\|\mathbf{B}\| \leq 1} \left| \frac{1}{L} \text{Tr} ((\mathbf{R} - t\mathbf{I}_L) \mathbf{B}) \right| = \mathcal{O} \left(\frac{L}{MN} \right) \tag{6.7}$$

when z belongs to a certain domain. For this, we first remark that (2.8) implies that $\|\mathbf{G}(\mathbf{A})\| \leq \|\mathbf{H}\|\|\mathbf{R}\|\|\mathbf{A}\|$. By Lemma 4.1, it holds that $\|\mathbf{H}\|\|\mathbf{R}\| \leq \frac{|z|}{(\text{Im}(z))^2}$. Consequently, we obtain that

$$\|\mathbf{G}(\mathbf{A})\| < \frac{|z|}{(\text{Im}(z))^2} \|\mathbf{A}\| \tag{6.8}$$

This implies that for each $L \times L$ matrix \mathbf{A} such that $\|\mathbf{A}\| \leq 1$, then it holds that

$$\begin{aligned} & \left| \frac{1}{ML} \text{Tr} [(\mathbf{E}(\mathbf{Q}) - \mathbf{I}_M \otimes \mathbf{R}) (\mathbf{I}_M \otimes \mathbf{G}(\mathbf{A}))] \right| \\ & \leq \frac{|z|}{(\text{Im}(z))^2} \sup_{\|\mathbf{B}\| \leq 1} \left| \frac{1}{ML} \text{Tr} [(\mathbf{E}(\mathbf{Q}) - \mathbf{I}_M \otimes \mathbf{R}) \mathbf{B}] \right|, \\ & \left| \frac{1}{L} \text{Tr} [(\mathbf{R} - t\mathbf{I}_L)\mathbf{G}(\mathbf{A})] \right| \leq \frac{|z|}{(\text{Im}(z))^2} \sup_{\|\mathbf{B}\| \leq 1} \left| \frac{1}{L} \text{Tr} [(\mathbf{R} - t\mathbf{I}_L)\mathbf{B}] \right| \end{aligned}$$

Proposition 5.1 implies that

$$\sup_{\|\mathbf{B}\| \leq 1} \left| \frac{1}{ML} \text{Tr} [(\mathbf{E}(\mathbf{Q}) - \mathbf{I}_M \otimes \mathbf{R}) \mathbf{B}] \right| = O\left(\frac{L}{MN}\right)$$

This and Eq. (6.6) eventually imply that

$$\begin{aligned} & \sup_{\|\mathbf{B}\| \leq 1} \left| \frac{1}{L} \text{Tr} ((\mathbf{R} - t\mathbf{I}_L)\mathbf{B}) \right| \leq O\left(\frac{L}{MN}\right) \\ & + \sigma^4 c_N |z t(z) \tilde{t}(z)| \frac{|z|}{(\text{Im}(z))^2} \sup_{\|\mathbf{B}\| \leq 1} \left| \frac{1}{L} \text{Tr} ((\mathbf{R} - t\mathbf{I}_L)\mathbf{B}) \right| \end{aligned}$$

It also holds that $|z t(z) \tilde{t}(z)| \leq \frac{|z|}{(\text{Im}(z))^2}$. Therefore, if z belongs to the domain $\sigma^4 c_N \frac{|z|^2}{(\text{Im}(z))^4} < \frac{1}{2}$, we obtain that

$$\sup_{\|\mathbf{B}\| \leq 1} \left| \frac{1}{L} \text{Tr} [(\mathbf{R} - t\mathbf{I}_L)\mathbf{B}] \right| = O\left(\frac{L}{MN}\right) \tag{6.9}$$

This establishes (6.3) for each uniformly bounded $L \times L$ matrix \mathbf{A} whenever z is well chosen. Moreover, for these values of z , $\frac{1}{L} \text{Tr} ((\mathbf{R} - t\mathbf{I})\mathbf{A})$, and thus $\frac{1}{ML} \text{Tr} (\mathbb{E}(\mathbf{Q}(z) - t(z)\mathbf{I}_{ML})\mathbf{A})$ are $O(\frac{L}{MN})$ terms. A standard application of Montel’s theorem implies that (6.3) holds on \mathbb{C}^+ . This, in turn, establishes (6.1).

Remark 6.1 We have proved that for each uniformly bounded $L \times L$ matrix \mathbf{A} , then it holds that

$$\frac{1}{ML} \text{Tr} [(\mathbb{E}(\mathbf{Q}(z) - t(z)\mathbf{I}_{ML})) (\mathbf{I}_M \otimes \mathbf{A})] \rightarrow 0$$

for each $z \in \mathbb{C}^+$. It is easy to verify that matrix $\mathbf{I}_M \otimes \mathbf{A}$ can be replaced by any uniformly bounded $ML \times ML$ matrix \mathbf{B} . In effect, Proposition 5.1 implies that it is sufficient to establish that

$$\frac{1}{ML} \text{Tr} [(\mathbf{I}_M \otimes \mathbf{R}(z) - t(z)\mathbf{I}_{ML}) \mathbf{B}] \rightarrow 0$$

The above term can also be written as

$$\frac{1}{L} \text{Tr} \left[(\mathbf{R}(z) - t(z) \mathbf{I}_L) \left(\frac{1}{M} \sum_{m=1}^M \mathbf{B}^{m,m} \right) \right]$$

and converges towards 0 because matrix $\frac{1}{M} \sum_{m=1}^M \mathbf{B}^{m,m}$ is uniformly bounded.

7 Convergence of the Spectral Norm of $\mathcal{T}_{N,L}(\mathbf{R}(z) - t(z)\mathbf{I}_N)$

From now on, we assume that L, M, N satisfy the following extra assumption:

Assumption 7.1 $\frac{L^{3/2}}{MN} \rightarrow 0$ or equivalently, $\frac{L}{M^4} \rightarrow 0$.

The goal of this section is to prove Theorem 7.1 which will be used extensively in the following.

Theorem 7.1 *Under assumption 7.1, it exists 2 nice polynomials P_1 and P_2 for which*

$$\begin{aligned} \|\mathcal{T}_{N,L}(\mathbf{R}(z) - t(z)\mathbf{I}_N)\| &\leq \sup_{v \in [0,1]} |\mathbf{a}_L(v)^* (\mathbf{R}(z) - t(z)\mathbf{I}_L) \mathbf{a}_L(v)| \\ &\leq \frac{L^{3/2}}{MN} P_1(|z|) P_2\left(\frac{1}{\text{Im}(z)}\right) \end{aligned} \tag{7.1}$$

for each $z \in \mathbb{C}^+$.

Proof First step The first step consists in showing that

$$\sup_{v \in [0,1]} |\mathbf{a}_L(v)^* (\mathbf{R}(z) - t(z)\mathbf{I}_L) \mathbf{a}_L(v)| \rightarrow 0 \tag{7.2}$$

for each $z \in \mathbb{C}^+$, which implies that $\|\mathcal{T}_{N,L}(\mathbf{R} - t\mathbf{I}_L)\| \rightarrow 0$ for each $z \in \mathbb{C}^+$ (see (2.8)). We first establish that (7.2) holds for certain values of z and extend the property to \mathbb{C}^+ using Montel’s theorem. We take (6.4) as a starting point and write $\mathbb{E}(\mathbf{Q} - t\mathbf{I}_{ML})$ as

$$\mathbb{E}(\mathbf{Q} - t\mathbf{I}_{ML}) = \mathbb{E}(\mathbf{Q}) - \mathbf{I}_M \otimes \mathbf{R} + (\mathbf{I}_M \otimes \mathbf{R} - t\mathbf{I}_{ML})$$

(6.4) can thus be rewritten as

$$\mathbf{R} - t\mathbf{I}_L = -\sigma^4 c_{Nz} t(z) \tilde{t}(z) \mathbf{R} \mathcal{T}_{L,L} \left(\mathbf{H} \mathcal{T}_{N,L}^{(M)} [\mathbb{E}(\mathbf{Q}) - \mathbf{R}_M] \right)$$

$$-\sigma^4 c_N z t(z) \tilde{t}(z) \mathbf{R} \mathcal{T}_{L,L} (\mathbf{H} \mathcal{T}_{N,L} [\mathbf{R} - t \mathbf{I}_L]) \tag{7.3}$$

Therefore, for each deterministic uniformly bounded L -dimensional vector \mathbf{b} , then it holds that

$$\mathbf{b}^* (\mathbf{R} - t \mathbf{I}) \mathbf{b} = -z t(z) \tilde{t}(z) \sigma^4 c_N \mathbf{b}^* \mathbf{R} \mathcal{T}_{L,L} (\mathbf{H} \mathcal{T}_{N,L}^{(M)} [\mathbb{E}(\mathbf{Q}) - \mathbf{I}_M \otimes \mathbf{R}]) \mathbf{b} \tag{7.4}$$

$$-z t(z) \tilde{t}(z) \sigma^4 c_N \mathbf{b}^* \mathbf{R} \mathcal{T}_{L,L} (\mathbf{H} \mathcal{T}_{N,L} [\mathbf{R} - t \mathbf{I}]) \mathbf{b} \tag{7.5}$$

Proposition 2.1 implies that

$$\begin{aligned} \|\mathcal{T}_{L,L} (\mathcal{T}_{N,L} [\mathbf{R} - t \mathbf{I}] \mathbf{H})\| &\leq \|\mathbf{H}\| \|\mathcal{T}_{N,L} [\mathbf{R} - t \mathbf{I}]\| \\ &\leq \|\mathbf{H}\| \sup_v |\mathbf{a}_L(v)^* (\mathbf{R} - t \mathbf{I}) \mathbf{a}_L(v)| \end{aligned}$$

and that

$$\begin{aligned} \|\mathcal{T}_{L,L} (\mathbf{H} \mathcal{T}_{N,L}^{(M)} [\mathbb{E}(\mathbf{Q}) - \mathbf{I}_M \otimes \mathbf{R}])\| &\leq \|\mathbf{H}\| \|\mathcal{T}_{N,L}^{(M)} [\mathbb{E}(\mathbf{Q}) - \mathbf{I}_M \otimes \mathbf{R}]\| \\ &\leq \|\mathbf{H}\| \sup_v |\mathbf{a}_L(v)^* \hat{\Delta} \mathbf{a}_L(v)| \end{aligned}$$

where we recall that $\Delta = \mathbb{E}(\mathbf{Q}) - \mathbf{I}_M \otimes \mathbf{R}$ and that $\hat{\Delta} = \frac{1}{M} \sum_{m=1}^M \Delta^{(m,m)}$. We denote by β and δ the terms $\beta = \sup_v |\mathbf{a}_L(v)^* (\mathbf{R} - t \mathbf{I}) \mathbf{a}_L(v)|$ and $\delta = \sup_v |\mathbf{a}_L(v)^* \hat{\Delta} \mathbf{a}_L(v)|$.

We remark that $\delta = \mathcal{O} \left(\frac{L^{3/2}}{MN} \right)$ (see (5.4)). We choose $\mathbf{b} = \mathbf{a}_L(\mu)$ in (7.4), evaluate the modulus of the left-hand side of (7.4), and take the supremum over μ . This immediately leads to

$$\beta \leq |z t(z) \tilde{t}(z)| \sigma^4 c_N \|\mathbf{R}\| \|\mathbf{H}\| \delta + |z t(z) \tilde{t}(z)| \sigma^4 c_N \|\mathbf{R}\| \|\mathbf{H}\| \beta \tag{7.6}$$

Moreover, (see Lemma (4.1)), it holds that

$$|z t(z) \tilde{t}(z)| \sigma^4 c_N \|\mathbf{R}\| \|\mathbf{H}\| \leq \sigma^4 c_N \frac{|z|^2}{(\text{Im}(z))^4}$$

(7.6) implies that if z satisfies

$$\sigma^4 c_N \frac{|z|^2}{(\text{Im}(z))^4} \leq \frac{1}{2}, \tag{7.7}$$

then $\beta = \mathcal{O} \left(\frac{L^{3/2}}{MN} \right)$ and therefore converges towards 0. We now extend this property on \mathbb{C}^+ using Montel’s theorem. For this, we consider an integer sequence $K(N)$ for which $\frac{L(N)}{K(N)} \rightarrow 0$ and denote for each N and $0 \leq k \leq K(N)$ by $v_k^{(N)}$ the element of $[0, 1]$ defined by $v_k^{(N)} = \frac{k}{K(N)}$. We denote by $\phi(k, N)$ the one-to-one correspondence between the set of integer couples (k, N) , $k \leq K(N)$ and the set of

integers \mathbb{N} defined by $\phi(0, 0) = 0, \phi(k + 1, N) = \phi(k, N) + 1$ for $k < K(N)$ and $\phi(0, N + 1) = \phi(K(N), N) + 1$. Each integer n can therefore be written in a unique way as $n = \phi(k, N)$ for a certain couple $(k, N), 0 \leq k \leq K(N)$. We define a sequence of analytic functions $(g_n(z))_{n \in \mathbb{N}}$ defined on \mathbb{C}^+ by

$$g_{\phi(k, N)}(z) = \mathbf{a}_L \left(v_k^{(N)} \right)^* (\mathbf{R}(z) - t(z) \mathbf{I}_L) \mathbf{a}_L \left(v_k^{(N)} \right) \tag{7.8}$$

If z satisfies (7.7), the sequence $g_n(z)$ converges towards 0. Moreover, $(g_n(z))_{n \in \mathbb{N}}$ is a normal family of \mathbb{C}^+ . Consider a subsequence extracted from $(g_n)_{n \in \mathbb{Z}}$ converging uniformly on compact subsets of \mathbb{C}^+ towards an analytic function g_* . As $g_*(z) = 0$ if z satisfies (7.7), function g_* is zero. This shows that all convergent subsequences extracted from $(g_n)_{n \in \mathbb{N}}$ converges towards 0, so that the whole sequence $(g_n)_{n \in \mathbb{N}}$ converges towards 0. This immediately implies that

$$\lim_{N \rightarrow +\infty} \sup_{0 \leq k \leq K(N)} |g_{\phi(k, N)}(z)| = 0 \tag{7.9}$$

for each $z \in \mathbb{C}^+$. For each $v \in [0, 1]$, it exists an index $k, 0 \leq k \leq K(N)$ such that $|v - v_k^{(N)}| \leq \frac{1}{2K(N)}$. It is easily checked that

$$\left\| \mathbf{a}_L(v) - \mathbf{a}_L(v_k^{(N)}) \right\| = O \left(L(N) |v - v_k^{(N)}| \right) = O \left(\frac{L(N)}{K(N)} \right) = o(1)$$

and that

$$\left| \mathbf{a}_L(v)^* (\mathbf{R}(z) - t(z) \mathbf{I}_L) \mathbf{a}_L(v) - \mathbf{a}_L(v_k^{(N)})^* (\mathbf{R}(z) - t(z) \mathbf{I}_L) \mathbf{a}_L(v_k^{(N)}) \right| \rightarrow 0$$

for each $z \in \mathbb{C}^+$. We deduce from (7.9) that (7.2) holds for each $z \in \mathbb{C}^+$ as expected.

Second step The most difficult part of the proof consists in evaluating the rate of convergence of $\sup_v |\mathbf{a}_L(v)^* (\mathbf{R}(z) - t(z) \mathbf{I}_N) \mathbf{a}_L(v)|$.

By (2.11), the quadratic form $\mathbf{a}_L(v)^* (\mathbf{R}(z) - t(z) \mathbf{I}_N) \mathbf{a}_L(v)$ can also be written as

$$\mathbf{a}_L(v)^* (\mathbf{R}(z) - t(z) \mathbf{I}_N) \mathbf{a}_L(v) = \sum_{l=-(L-1)}^{L-1} \tau(\mathbf{R} - t \mathbf{I})(l) e^{-2i\pi lv}$$

where we recall that $\tau(\mathbf{R} - t \mathbf{I})(l) = \frac{1}{L} \text{Tr} \left((\mathbf{R} - t \mathbf{I}) \mathbf{J}_L^l \right)$. In order to study more thoroughly $\sup_v |\mathbf{a}_L(v)^* (\mathbf{R}(z) - t(z) \mathbf{I}_N) \mathbf{a}_L(v)|$, it is thus possible to evaluate the coefficients $(\tau(\mathbf{R} - t \mathbf{I})(l))_{l=-(L-1), \dots, L-1}$. In the following, for a $L \times L$ matrix \mathbf{X} , we denote by $\boldsymbol{\tau}(\mathbf{X})$ the $2L - 1$ -dimensional vector defined by

$$\boldsymbol{\tau}(\mathbf{X}) = (\tau(\mathbf{X})(-(L - 1)), \dots, \tau(\mathbf{X})(L - 1))^T$$

(7.3) can be associated with a linear equation whose unknown is vector $\boldsymbol{\tau}(\mathbf{R} - t \mathbf{I})$. Writing $\mathcal{T}_{N, L}[\mathbf{R} - t \mathbf{I}]$ as $\sum_{l=-(L-1)}^{L-1} \tau(\mathbf{R} - t \mathbf{I})(l) \mathbf{J}_N^{*l}$, multiplying (7.3) from both sides by \mathbf{J}_L^k , and taking the normalized trace, we obtain that

$$\tau(\mathbf{R} - t\mathbf{I}) = \tau(\mathbf{\Gamma}) + \mathbf{D}^{(0)} \tau(\mathbf{R} - t\mathbf{I}) \tag{7.10}$$

where $\mathbf{D}^{(0)}$ is the $(2L - 1) \times (2L - 1)$ matrix whose entries $\mathbf{D}_{k,l}^{(0)}$, $(k, l) \in \{-(L - 1), \dots, L - 1\}$ are defined by

$$\mathbf{D}_{k,l}^{(0)} = -\sigma^4 c_{Nz} t(z) \tilde{t}(z) \frac{1}{L} \text{Tr} \left[\mathbf{R} \mathcal{T}_{L,L} \left(\mathbf{H} \mathbf{J}_N^{*l} \right) \mathbf{J}_L^k \right]$$

and where matrix $\mathbf{\Gamma}$ represents the first term of the right-hand side of (7.3), i.e.

$$\mathbf{\Gamma} = -\sigma^4 c_{Nz} t(z) \tilde{t}(z) \mathbf{R} \mathcal{T}_{L,L} \left(\mathbf{H} \mathcal{T}_{N,L}^{(M)} [\mathbb{E}(\mathbf{Q}) - \mathbf{I}_M \otimes \mathbf{R}] \right) \tag{7.11}$$

Equation (7.10) should be inverted, and the effect of the inversion on vector $\tau(\mathbf{\Gamma})$ should be analysed in order to evaluate the behaviour of $\|\mathcal{T}_{N,L}(\mathbf{R}(z) - t(z)\mathbf{I}_N)\|$. The invertibility of matrix $\mathbf{I} - \mathbf{D}^{(0)}$ and the control of its inverse are, however, non-trivial and need some efforts.

In the following, we denote by $\Phi^{(0)}$ the operator defined on $\mathbb{C}^{L \times L}$ by

$$\Phi^{(0)}(\mathbf{X}) = -\sigma^4 c_{Nz} t(z) \tilde{t}(z) \mathbf{R} \mathcal{T}_{L,L} \left(\mathbf{H} \mathcal{T}_{N,L}(\mathbf{X}) \right) \tag{7.12}$$

for each $L \times L$ matrix \mathbf{X} . Eq. (7.3) can thus be written as

$$\mathbf{R} - t\mathbf{I}_L = \mathbf{\Gamma} + \Phi^{(0)}(\mathbf{R} - t\mathbf{I}_L)$$

We also remark that matrix $\mathbf{\Gamma}$ is given by

$$\mathbf{\Gamma} = \Phi^{(0)} \left(\mathbb{E}(\hat{\mathbf{Q}}) - \mathbf{R} \right) \tag{7.13}$$

Moreover, it is clear that vector $\tau(\Phi^{(0)}(\mathbf{X}))$ can be written as

$$\tau \left(\Phi^{(0)}(\mathbf{X}) \right) = \mathbf{D}^{(0)} \tau(\mathbf{X}) \tag{7.14}$$

In order to study the properties of operator $\Phi^{(0)}$ and of matrix $\mathbf{D}^{(0)}$, we introduce the operator Φ and the corresponding $(2L - 1) \times (2L - 1)$ matrix \mathbf{D} defined, respectively, by

$$\Phi(\mathbf{X}) = \sigma^4 c_N \mathbf{R} \mathcal{T}_{L,L} \left(\mathbf{H} \mathcal{T}_{N,L}(\mathbf{X}) \mathbf{H}^* \right) \mathbf{R}^* \tag{7.15}$$

and

$$\mathbf{D}_{k,l} = \sigma^4 c_N \frac{1}{L} \text{Tr} \left[\mathbf{R} \mathcal{T}_{L,L} \left(\mathbf{H} \mathbf{J}_N^{*l} \mathbf{H}^* \right) \mathbf{R}^* \mathbf{J}_L^k \right] \tag{7.16}$$

for $(k, l) \in \{-(L - 1), \dots, L - 1\}$. Matrix \mathbf{D} of course satisfies

$$\tau(\Phi(\mathbf{X})) = \mathbf{D} \tau(\mathbf{X}) \tag{7.17}$$

Before establishing the relationships between $(\Phi_0, \mathbf{D}^{(0)})$ and (Φ, \mathbf{D}) , we prove the following proposition.

Proposition 7.1 – *If \mathbf{X} is positive definite, then matrix $\Phi(\mathbf{X})$ is also positive definite.*

Moreover, if $\mathbf{X}_1 \geq \mathbf{X}_2$, then $\Phi(\mathbf{X}_1) \geq \Phi(\mathbf{X}_2)$.

– *It exists 2 nice polynomials P_1 and P_2 and an integer N_1 such that the spectral radius $\rho(\mathbf{D})$ of matrix \mathbf{D} verifies $\rho(\mathbf{D}) < 1$ for $N \geq N_1$ and for each $z \in E_N$, where E_N is the subset of \mathbb{C}^+ defined by*

$$E_N = \left\{ z \in \mathbb{C}^+, \frac{L^{3/2}}{MN} P_1(|z|) P_2(1/\text{Im}z) \leq 1 \right\}. \tag{7.18}$$

– *for $N \geq N_1$, matrix $\mathbf{I} - \mathbf{D}$ is invertible for $z \in E_N$. If we denote by $\mathbf{f} = (f_{-(L-1)}, \dots, f_0, \dots, f_{L-1})^T$, the $(2L - 1)$ -dimensional vector defined by*

$$\mathbf{f} = (\mathbf{I} - \mathbf{D})^{-1} \boldsymbol{\tau}(\mathbf{I}) = (\mathbf{I} - \mathbf{D})^{-1} \mathbf{e}_0 \tag{7.19}$$

where $\mathbf{e}_0 = (0, \dots, 0, 1, 0, \dots, 0)^T$, then, for each $v \in [0, 1]$, the term $\sum_{l=-(L-1)}^{L-1} \mathbf{f}_l e^{-2i\pi lv}$ is real and positive, and

$$\sup_{v \in [0,1]} \sum_{l=-(L-1)}^{L-1} \mathbf{f}_l e^{-2i\pi lv} \leq C \frac{(|\eta_1|^2 + |z|^2)^2}{(\text{Im}z)^4} \tag{7.20}$$

for some nice constants C and η_1 .

Proof The first item follows immediately from the basic properties of operators \mathcal{T} . The starting point of the proof of item 2 consists in writing matrix $\mathbb{E}(\hat{\mathbf{Q}}) = \frac{1}{M} \sum_{m=1}^M \mathbb{E}(\mathbf{Q}^{m,m})$ as $\mathbb{E}(\hat{\mathbf{Q}}) = \mathbf{R} + \hat{\mathbf{A}}$, and in expressing the imaginary part of $\mathbb{E}(\hat{\mathbf{Q}})$ as $\text{Im}(\mathbb{E}(\hat{\mathbf{Q}})) = \text{Im}(\mathbb{E}(\hat{\mathbf{A}})) + \text{Im}(\mathbf{R})$. Writing $\text{Im}(\mathbf{R})$ as

$$\text{Im}(\mathbf{R}) = \frac{\mathbf{R} - \mathbf{R}^*}{2i} = \frac{1}{2i} \mathbf{R} (\mathbf{R}^{-*} - \mathbf{R}^{-1}) \mathbf{R}^*$$

and expressing \mathbf{R}^{-1} in terms of \mathbf{H} , and using the same tricks for \mathbf{H} , we eventually obtain that

$$\text{Im}(\mathbb{E}(\hat{\mathbf{Q}})) = \text{Im}(\mathbb{E}(\hat{\mathbf{A}})) + \text{Im}z \mathbf{R} \mathbf{R}^* + \sigma^4 c_N \mathbf{R} \mathcal{T}_{L,L} \left[\mathbf{H} \mathcal{T}_{N,L} \left(\text{Im}(\mathbb{E}(\hat{\mathbf{Q}})) \right) \mathbf{H}^* \right] \mathbf{R}^* \tag{7.21}$$

In order to simplify the notations, we denote by \mathbf{X} and \mathbf{Y} the matrices $\text{Im}(\mathbb{E}(\hat{\mathbf{Q}}))$ and $\text{Im}(\mathbb{E}(\hat{\mathbf{A}})) + \text{Im}z \mathbf{R} \mathbf{R}^*$, respectively. (7.21) implies that for each $z \in \mathbb{C}^+$, then the positive definite matrix \mathbf{X} satisfies

$$\mathbf{X} = \mathbf{Y} + \Phi(\mathbf{X}) \tag{7.22}$$

Iterating this relation, we obtain that for each $n \geq 1$

$$\mathbf{X} = \mathbf{Y} + \sum_{k=1}^n \Phi^k(\mathbf{Y}) + \Phi^{n+1}(\mathbf{X}) \tag{7.23}$$

The general idea of the proof is to recognize that matrix $\mathcal{T}_{N,L}(\mathbf{Y})$ is positive definite if z belongs to a set E_N defined by (7.18). This implies that for $z \in E_N$, then $\Phi^k(\mathbf{Y}) > 0$ for each $k \geq 1$. Therefore, (7.23) and $\Phi^{n+1}(\mathbf{X}) > 0$ imply that for each n , the positive definite matrix $\sum_{k=1}^n \Phi^k(\mathbf{Y})$ satisfies

$$\sum_{k=1}^n \Phi^k(\mathbf{Y}) \leq \mathbf{X} - \mathbf{Y} \tag{7.24}$$

so that the series $\sum_{k=1}^{+\infty} \Phi^k(\mathbf{Y})$ appears to be convergent for $z \in E_N$. As shown below, this implies that $\rho(\mathbf{D}) < 1$. We begin to prove that $\mathcal{T}_{N,L}(\mathbf{Y})$ is positive definite on a set E_N . □

Lemma 7.1 *It exists 2 nice polynomials P_1 and P_2 , a nice constant η_1 and an integer N_1 such that*

$$\mathcal{T}_{N,L}(\mathbf{Y}) > \frac{(\text{Im}z)^3}{32(\eta_1^2 + |z|^2)^2} \mathbf{I} \tag{7.25}$$

for $N \geq N_1$ and $z \in E_N$ where E_N is defined by (7.18).

Proof We show that it exist a nice constant $\eta_1 > 0$ and 2 nice polynomials P_1 and P_2 such that for each $v \in [0, 1]$,

$$\mathbf{a}_L(v)^* \mathbf{Y} \mathbf{a}_L(v) > \frac{(\text{Im}z)^3}{16(\eta_1^2 + |z|^2)^2} - \frac{L^{3/2}}{MN} P_1(|z|) P_2(1/\text{Im}z) \tag{7.26}$$

For this, we first note that

$$\mathbf{a}_L(v)^* \mathbf{R} \mathbf{R}^* \mathbf{a}_L(v) \geq |\mathbf{a}_L(v)^* \mathbf{R} \mathbf{a}_L(v)|^2 \geq (\mathbf{a}_L(v)^* \text{Im}(\mathbf{R}) \mathbf{a}_L(v))^2$$

As $\mathbf{R}(z)$ is the Stieltjes transform of a positive matrix-valued measure $\mu_{\mathbf{R}}$ (see Lemma 4.1), it holds that

$$\mathbf{a}_L(v)^* \text{Im}(\mathbf{R}) \mathbf{a}_L(v) = \text{Im}z \int_{\mathbb{R}^+} \frac{\mathbf{a}_L(v)^* d\mu_{\mathbf{R}}(\lambda) \mathbf{a}_L(v)}{|\lambda - z|^2}$$

We claim that it exists $\eta_1 > 0$ and an integer N_0 such that

$$\mathbf{a}_L(v)^* \mu_{\mathbf{R}}([0, \eta_1]) \mathbf{a}_L(v) > \frac{1}{2} \tag{7.27}$$

for each $v \in [0, 1]$ and for each $N > N_0$. In effect, as $c_N \rightarrow c_*$, it exists a nice constant η_1 for which $\mu_{\sigma^2, c_N}([0, \eta_1]) > \frac{3}{4}$ for each N . We consider the sequence of analytic

functions $(g_n(z))_{n \in \mathbb{N}}$ defined by (7.8). If $n = \phi(k, N)$, $g_n(z)$ is the Stieltjes transform of measure μ_n defined by $\mu_n = \mathbf{a}_L(v_k^{(N)})^* \boldsymbol{\mu}_R \mathbf{a}_L(v_k^{(N)}) - \mu_{\sigma^2, c_N}$. Therefore, (7.9) implies that sequence $(\mu_n)_{n \in \mathbb{N}}$ converges weakly towards 0. As the Marcenko–Pastur distribution is absolutely continuous, this leads to

$$\lim_{N \rightarrow +\infty} \sup_{0 \leq k \leq K(N)} \left| \mathbf{a}_L(v_k^{(N)})^* \boldsymbol{\mu}_R([0, \eta_1]) \mathbf{a}_L(v_k^{(N)}) - \mu_{\sigma^2, c_N}([0, \eta_1]) \right| = 0$$

This implies the existence of $N'_0 \in \mathbb{N}$ such that

$$\sup_{0 \leq k \leq K(N)} \mathbf{a}_L(v_k^{(N)})^* \boldsymbol{\mu}_R([0, \eta_1]) \mathbf{a}_L(v_k^{(N)}) > \frac{5}{8}$$

for each $N \geq N'_0$. As mentioned above, for each $v \in [0, 1]$, it exists an index k , $0 \leq k \leq K(N)$ such that $|v - v_k^{(N)}| \leq \frac{1}{2K(N)}$. As

$$\left\| \mathbf{a}_L(v) - \mathbf{a}_L(v_k^{(N)}) \right\| = O\left(L(N)|v - v_k^{(N)}|\right) = o(1)$$

it is easy to check that

$$\mathbf{a}_L(v)^* \boldsymbol{\mu}_R([0, \eta_1]) \mathbf{a}_L(v) - \mathbf{a}_L(v_k^{(N)})^* \boldsymbol{\mu}_R([0, \eta_1]) \mathbf{a}_L(v_k^{(N)}) \rightarrow 0$$

which implies the existence of an integer $N_0 \geq N'_0$ for which

$$\sup_{v \in [0, 1]} \mathbf{a}_L(v)^* \boldsymbol{\mu}_R([0, \eta_1]) \mathbf{a}_L(v) > \frac{1}{2}$$

for each $N \geq N_0$, as expected.

It is clear that

$$\mathbf{a}_L(v)^* \text{Im}(\mathbf{R}) \mathbf{a}_L(v) \geq \text{Im}z \int_0^{\eta_1} \frac{\mathbf{a}_L(v)^* d\boldsymbol{\mu}_R(\lambda) \mathbf{a}_L(v)}{|\lambda - z|^2}$$

As $|\lambda - z|^2 \leq 2(\lambda^2 + |z|^2) \leq 2(\eta_1^2 + |z|^2)$ if $\lambda \in [0, \eta_1]$, it holds that

$$\mathbf{a}_L(v)^* \text{Im}(\mathbf{R}) \mathbf{a}_L(v) \geq \frac{\text{Im}z}{4(\eta_1^2 + |z|^2)}$$

and that

$$\mathbf{a}_L(v)^* \mathbf{R} \mathbf{R}^* \mathbf{a}_L(v) \geq \frac{(\text{Im}z)^2}{16(\eta_1^2 + |z|^2)^2}$$

for each $v \in [0, 1]$. (5.4) implies that for each v ,

$$\left| \mathbf{a}_L(v)^* \operatorname{Im} \hat{\mathbf{A}} \mathbf{a}_L(v) \right| \leq \frac{L^{3/2}}{MN} P_1(|z|) P_2\left(\frac{1}{\operatorname{Im} z}\right) \tag{7.28}$$

for some nice polynomials P_1 and P_2 , which, in turn, leads to (7.26). If we denote by E_N the subset of \mathbb{C}^+ defined by $\frac{L^{3/2}}{MN} P_1(|z|) P_2\left(\frac{1}{\operatorname{Im} z}\right) < \frac{1}{2} \frac{(\operatorname{Im} z)^3}{16(\eta_1^2 + |z|^2)^2}$, then $\mathbf{Y} = \operatorname{Im}(\hat{\mathbf{A}}) + \operatorname{Im} z \mathbf{R} \mathbf{R}^*$ verifies

$$\inf_{v \in [0, 1]} \mathbf{a}_L(v)^* \mathbf{Y} \mathbf{a}_L(v) > \frac{(\operatorname{Im} z)^3}{32(\eta_1^2 + |z|^2)^2} \tag{7.29}$$

for each $z \in E_N$. As

$$\mathbf{a}_L(v)^* \mathbf{Y} \mathbf{a}_L(v) = \sum_{l=-(L-1)}^{L-1} \tau(\mathbf{Y})(l) e^{-2i\pi lv}$$

we obtain that

$$\inf_{v \in [0, 1]} \sum_{l=-(L-1)}^{L-1} \tau(\mathbf{Y})(l) e^{-2i\pi lv} > \frac{(\operatorname{Im} z)^3}{32(\eta_1^2 + |z|^2)^2}$$

for $z \in E_N$. If we denote $\alpha(z) = \frac{(\operatorname{Im} z)^3}{32(\eta_1^2 + |z|^2)^2}$, this implies that $(\tau(\mathbf{Y})(l) - \alpha \delta(l=0))_{l=-(L-1)}^{L-1}$ coincide with Fourier coefficients of a positive function. Therefore, matrix $\mathcal{T}_{N,L}(\mathbf{Y}) - \alpha \mathbf{I}$ is positive definite (see [18], 1.11 (a)), which implies that (7.25) holds. Lemma 7.1 follows from the observation that the set E_N can be written as (7.18) for some other pair of nice polynomials P_1, P_2 . \square

We now complete the proof of item 2 of Proposition (7.1). We establish that for N fixed and large enough and $z \in E_N$, then for each L -dimensional vector \mathbf{b} , $\mathbf{D}^n \mathbf{b} \rightarrow 0$ when $n \rightarrow +\infty$, a property equivalent to $\rho(\mathbf{D}) < 1$. We emphasize that in the forthcoming analysis, N and, therefore, L are assumed to be fixed parameters. As matrix $\mathcal{T}_{N,L}(\mathbf{Y}) > \alpha(z) \mathbf{I}_N > 0$ on the set E_N for N large enough, (7.24) is valid there. This implies that the positive definite matrix-valued series $\sum_{n=1}^{+\infty} \Phi^n(\mathbf{Y})$ is convergent, in the sense that for each unit norm L -dimensional vector \mathbf{u} , then $\sum_{n=1}^{+\infty} \mathbf{u}^* \Phi^n(\mathbf{Y}) \mathbf{u} < +\infty$. Using the polarization identity, we obtain that the series $\sum_{n=1}^{+\infty} \mathbf{u}_1^* \Phi^n(\mathbf{Y}) \mathbf{u}_2$ is convergent for each pair of unit norm vectors $(\mathbf{u}_1, \mathbf{u}_2)$. This implies that each entry of $\Phi^n(\mathbf{Y})$ converges towards 0 when $n \rightarrow +\infty$ and that the same property holds true for each component of vector $\tau(\Phi^n(\mathbf{Y}))$. This vector of course coincides with $\mathbf{D}^n \tau(\mathbf{Y})$. We have thus shown that $\mathbf{D}^n \tau(\mathbf{Y}) \rightarrow 0$ when $n \rightarrow +\infty$. We now establish that this property holds, not only for vector $\tau(\mathbf{Y})$, but also for each $(2L - 1)$ -dimensional vector. We consider any positive hermitian $L \times L$ matrix \mathbf{Z} such that $\mathcal{T}_{N,L}(\mathbf{Y}) - \mathcal{T}_{N,L}(\mathbf{Z}) \geq 0$. Then, it is clear that for each $n \geq 1, 0 \leq \Phi^n(\mathbf{Z}) \leq \Phi^n(\mathbf{Y})$ and that the series

$\sum_{n=1}^{\infty} \Phi^n(\mathbf{Z})$ is convergent. As above, this implies that $\mathbf{D}^n \boldsymbol{\tau}(\mathbf{Z}) \rightarrow 0$ when $n \rightarrow +\infty$. If now \mathbf{Z} is any positive hermitian matrix, it holds that $0 \leq \mathcal{T}_{N,L} \left(\frac{\alpha(z)}{\|\mathbf{Z}\|} \mathbf{Z} \right) \leq \mathcal{T}_{N,L}(\mathbf{Y})$ because $\mathcal{T}_{N,L}(\mathbf{Z}) \leq \|\mathcal{T}_{N,L}(\mathbf{Z})\| \mathbf{I} \leq \|\mathbf{Z}\| \mathbf{I}$. This implies that $\mathbf{D}^n \left(\frac{\alpha(z)}{\|\mathbf{Z}\|} \boldsymbol{\tau}(\mathbf{Z}) \right) \rightarrow 0$, or equivalently that $\mathbf{D}^n \boldsymbol{\tau}(\mathbf{Z}) \rightarrow 0$ for each positive hermitian matrix \mathbf{Z} . This property holds in particular for positive rank one matrices $\mathbf{h}\mathbf{h}^*$, and thus for linear combination (with complex coefficients) of such matrices, and in particular for hermitian (non-necessarily positive) matrices. We now consider any $L \times L$ matrix \mathbf{B} . It can be written as $\mathbf{B} = \text{Re}(\mathbf{B}) + i \text{Im}(\mathbf{B})$, i.e. as a linear combination of hermitian matrices. Therefore, it holds that $\mathbf{D}^n \boldsymbol{\tau}(\mathbf{B}) \rightarrow 0$ for any $L \times L$ matrix. The conclusion follows from the obvious observation that any $(2L-1)$ -dimensional vector \mathbf{b} can be written as $\mathbf{b} = \boldsymbol{\tau}(\mathbf{B})$ for some $L \times L$ matrix \mathbf{B} . This completes the proof of item 2 of Proposition (7.1).

We finally establish item 3. We assume that $z \in E_N$ and that N is large enough. We first remark that as $\mathcal{T}_{N,L}(\mathbf{Y}) \geq \alpha(z)\mathbf{I}_N$, then, for each $n \geq 1$, it holds that $\Phi^n(\mathbf{Y}) \geq \alpha(z) \Phi^n(\mathbf{I})$. We also note that $\Phi^n(\mathbf{I}) > 0$ for each n which implies that

$$\mathbf{a}_L(v)^* \Phi^n(\mathbf{Y}) \mathbf{a}_L(v) \geq \alpha(z) \mathbf{a}_L(v)^* \Phi^n(\mathbf{I}) \mathbf{a}_L(v) > 0$$

for each v . We also remark that this inequality also holds for $n = 0$ (see (7.29)). We recall that for each $L \times L$ matrix \mathbf{B} , then

$$\mathbf{a}_L(v)^* \mathbf{B} \mathbf{a}_L(v) = \sum_{l=-(L-1)}^{L-1} \boldsymbol{\tau}(\mathbf{B})(l) e^{-2i\pi lv} \tag{7.30}$$

Using this identity for $\mathbf{B} = \Phi^n(\mathbf{Y})$ and $\mathbf{B} = \Phi^n(\mathbf{I})$ and using that $\boldsymbol{\tau}(\mathbf{I}) = \mathbf{e}_0$, we obtain that

$$\sum_{l=-(L-1)}^{L-1} (\mathbf{D}^n \boldsymbol{\tau}(\mathbf{Y}))(l) e^{-2i\pi lv} \geq \alpha(z) \sum_{l=-(L-1)}^{L-1} (\mathbf{D}^n \mathbf{e}_0)(l) e^{-2i\pi lv} > 0$$

As $(\mathbf{I} - \mathbf{D})^{-1} = \sum_{n=0}^{+\infty} \mathbf{D}^n$, we finally obtain that

$$0 < \sum_{l=-(L-1)}^{L-1} \mathbf{f}_l e^{-2i\pi lv} \leq \frac{1}{\alpha(z)} \sum_{l=-(L-1)}^{L-1} \left((\mathbf{I} - \mathbf{D})^{-1} \boldsymbol{\tau}(\mathbf{Y}) \right) (l) e^{-2i\pi lv}$$

The conclusion follows from the observation that $\boldsymbol{\tau}(\mathbf{X}) = \boldsymbol{\tau}(\mathbf{Y}) + \mathbf{D} \boldsymbol{\tau}(\mathbf{X})$ and that $\boldsymbol{\tau}(\mathbf{X}) = (\mathbf{I} - \mathbf{D})^{-1} \boldsymbol{\tau}(\mathbf{Y})$. Therefore,

$$\sum_{l=-(L-1)}^{L-1} \left((\mathbf{I} - \mathbf{D})^{-1} \boldsymbol{\tau}(\mathbf{Y}) \right) (l) e^{-2i\pi lv}$$

coincides with $\mathbf{a}_L(v)^* \mathbf{X} \mathbf{a}_L(v)$, a term which is upperbounded by $\frac{1}{\text{Im}z}$ on \mathbb{C}^+ .

We now make the appropriate connections between $(\Phi_0, \mathbf{D}^{(0)})$ and (Φ, \mathbf{D}) and establish the following Proposition.

Proposition 7.2 *If N is large enough and if z belongs to the set E_N defined by (7.18), matrix $\mathbf{I} - \mathbf{D}^{(0)}$ is invertible, and for each matrix $L \times L$ matrix \mathbf{X} , it holds that*

$$\begin{aligned} & \sup_{v \in [0,1]} \left| \sum_{l=-(L-1)}^{L-1} \left((\mathbf{I} - \mathbf{D}^{(0)})^{-1} \boldsymbol{\tau}(\mathbf{X}) \right)_l e^{-2i\pi lv} \right| \\ & \leq \frac{\|\mathcal{T}_{N,L}(\mathbf{X})\|}{2} \left(\frac{1}{1 - \sigma^4 c_N |z t(z) \tilde{t}(z)|^2} + \sum_{l=-(L-1)}^{L-1} \mathbf{f}_l e^{-2i\pi lv} \right) \end{aligned} \tag{7.31}$$

Proof We first establish by induction that

$$\left(\Phi^{(0)} \right)^n (\mathbf{X}) \left(\Phi^{(0)} \right)^n (\mathbf{X})^* \leq \|\mathcal{T}_{N,L}(\mathbf{X})\|^2 \left(\sigma^4 c_N |z t(z) \tilde{t}(z)|^2 \right)^n \Phi^n (\mathbf{I}) \tag{7.32}$$

for each $n \geq 1$. We first verify that (7.32) holds for $n = 1$. Using Proposition (2.3), we obtain that

$$\mathcal{T}_{L,L}(\mathbf{H}\mathcal{T}_{N,L}(\mathbf{X})) \left[\mathcal{T}_{L,L}(\mathbf{H}\mathcal{T}_{N,L}(\mathbf{X})) \right]^* \leq \mathcal{T}_{L,L}(\mathbf{H}\mathcal{T}_{N,L}(\mathbf{X})\mathcal{T}_{N,L}(\mathbf{X})^* \mathbf{H}^*)$$

Remarking that $\mathcal{T}_{N,L}(\mathbf{X})\mathcal{T}_{N,L}(\mathbf{X})^* \leq \|\mathcal{T}_{N,L}(\mathbf{X})\|^2 \mathbf{I}$, we get that

$$\mathcal{T}_{L,L}(\mathbf{H}\mathcal{T}_{N,L}(\mathbf{X})) \left[\mathcal{T}_{L,L}(\mathbf{H}\mathcal{T}_{N,L}(\mathbf{X})) \right]^* \leq \|\mathcal{T}_{N,L}(\mathbf{X})\|^2 \mathcal{T}_{L,L}(\mathbf{H}\mathbf{H}^*)$$

This and the identity $\Phi(\mathbf{I}) = \sigma^4 c_N \mathbf{R}\mathcal{T}_{L,L}(\mathbf{H}\mathbf{H}^*)\mathbf{R}^*$ imply immediately (7.32) for $n = 1$. We assume that (7.32) holds until integer $n - 1$. By Proposition 2.3, we get that

$$\begin{aligned} & \left(\Phi^{(0)} \right)^n (\mathbf{X}) \left(\left(\Phi^{(0)} \right)^n (\mathbf{X}) \right)^* \\ & \leq \left| \sigma^4 c_N z t(z) \tilde{t}(z) \right|^2 \mathbf{R}\mathcal{T}_{L,L} \left[\mathbf{H}\mathcal{T}_{N,L} \left(\left(\Phi^{(0)} \right)^{n-1} (\mathbf{X}) \right) \right. \\ & \quad \left. \times \left(\mathcal{T}_{N,L} \left(\left(\Phi^{(0)} \right)^{n-1} (\mathbf{X}) \right) \right)^* \mathbf{H}^* \right] \mathbf{R}^* \end{aligned} \tag{7.33}$$

Using again Proposition (2.3), we obtain that

$$\begin{aligned} & \mathcal{T}_{N,L} \left(\left(\Phi^{(0)} \right)^{n-1} (\mathbf{X}) \right) \left(\mathcal{T}_{N,L} \left(\left(\Phi^{(0)} \right)^{n-1} (\mathbf{X}) \right) \right)^* \\ & \leq \mathcal{T}_{N,L} \left(\left(\Phi^{(0)} \right)^{n-1} (\mathbf{X}) \left[\left(\Phi^{(0)} \right)^{n-1} (\mathbf{X}) \right]^* \right) \end{aligned}$$

(7.32) for integer $n - 1$ yields to

$$\begin{aligned} & \left(\Phi^{(0)} \right)^n (\mathbf{X}) \left(\left(\Phi^{(0)} \right)^n (\mathbf{X}) \right)^* \\ & \leq \| \mathcal{T}_{N,L}(\mathbf{X}) \|^2 (\sigma^4 c_N)^{n+1} |z t(z) \tilde{t}(z)|^{2n} \mathbf{R} \mathcal{T}_{N,L} \left(\mathbf{H} \Phi^{n-1}(\mathbf{I}) \mathbf{H}^* \right) \mathbf{R}^* \end{aligned}$$

(7.32) for integer n directly follows from $\Phi^n(\mathbf{I}) = \sigma^4 c_N \mathbf{R} \mathcal{T}_{N,L}(\mathbf{H} \Phi^{n-1}(\mathbf{I}) \mathbf{H}^*) \mathbf{R}^*$. □

We now prove that if $z \in E_N$ defined by (7.18) and if N is large enough, then, for each $(2L - 1)$ -dimensional vector \mathbf{x} , it holds that $(\mathbf{D}^{(0)})^n \mathbf{x} \rightarrow 0$, a condition which is equivalent to $\rho(\mathbf{D}^{(0)}) < 1$. For this, we observe that each vector \mathbf{x} can be written as $\mathbf{x} = \boldsymbol{\tau}(\mathbf{X})$ for some $L \times L$ matrix \mathbf{X} . The entries of Toeplitz matrix $\mathcal{T}_{L,L} \left(\left(\Phi^{(0)} \right)^n (\mathbf{X}) \right)$ are the components of vector $(\mathbf{D}^{(0)})^n \boldsymbol{\tau}(\mathbf{X})$. Therefore, condition $(\mathbf{D}^{(0)})^n \mathbf{x} \rightarrow 0$ is equivalent to $\| \mathcal{T}_{L,L} \left(\left(\Phi^{(0)} \right)^n (\mathbf{X}) \right) \| \rightarrow 0$. We now prove that

$$\sup_{v \in [0,1]} \left| \mathbf{a}_L(v)^* \left(\Phi^{(0)} \right)^n (\mathbf{X}) \mathbf{a}_L(v) \right| \rightarrow 0$$

a condition which implies $\| \mathcal{T}_{L,L} \left(\left(\Phi^{(0)} \right)^n (\mathbf{X}) \right) \| \rightarrow 0$ by Proposition 2.1 and thus that $\rho(\mathbf{D}^{(0)}) < 1$. It is clear that

$$\left| \mathbf{a}_L(v)^* \left(\Phi^{(0)} \right)^n (\mathbf{X}) \mathbf{a}_L(v) \right|^2 \leq \mathbf{a}_L(v)^* \left(\Phi^{(0)} \right)^n (\mathbf{X}) \left(\left(\Phi^{(0)} \right)^n (\mathbf{X}) \right)^* \mathbf{a}_L(v) \tag{7.34}$$

Inequality (7.32) implies that

$$\begin{aligned} & \mathbf{a}_L(v)^* \left(\Phi^{(0)} \right)^n (\mathbf{X}) \left(\left(\Phi^{(0)} \right)^n (\mathbf{X}) \right)^* \mathbf{a}_L(v) \\ & \leq \| \mathcal{T}_{N,L}(\mathbf{X}) \|^2 \left(\sigma^4 c_N |z t(z) \tilde{t}(z)|^2 \right)^n \mathbf{a}_L(v)^* \Phi^n(\mathbf{I}) \mathbf{a}_L(v) \end{aligned} \tag{7.35}$$

By (1.31), it exists 2 nice constants C and $\eta > 0$ such that

$$\sigma^4 c_N |z t(z) \tilde{t}(z)|^2 \leq 1 - C \frac{(\eta^2 + |z|^2)^2}{(\text{Im}(z))^4} \tag{7.36}$$

for N large enough. Moreover, it has been shown before that each entry of matrix $\Phi^n(\mathbf{I})$ converges towards 0, which implies that $\sup_{v \in [0,1]} \mathbf{a}_L(v)^* \Phi^n(\mathbf{I}) \mathbf{a}_L(v) \rightarrow 0$ (we recall that L is assumed fixed in the present analysis). Therefore,

$$\sup_{v \in [0,1]} \mathbf{a}_L(v)^* \left(\Phi^{(0)} \right)^n (\mathbf{X}) \left(\left(\Phi^{(0)} \right)^n (\mathbf{X}) \right)^* \mathbf{a}_L(v) \rightarrow 0$$

which implies that $\|\mathcal{T}_{L,L} \left((\Phi^{(0)})^n(\mathbf{X}) \right)\|$ and $(\mathbf{D}^{(0)})^n \boldsymbol{\tau}(\mathbf{X})$ converge towards 0. We have thus established that $\rho(\mathbf{D}^{(0)}) < 1$ and that matrix $\mathbf{I} - \mathbf{D}^{(0)}$ is invertible.

We finally establish Eq. (7.31). Using $(\mathbf{I} - \mathbf{D}^{(0)})^{-1} = \sum_{n=0}^{+\infty} (\mathbf{D}^{(0)})^n$ and

$$\sum_{l=-(L-1)}^{L-1} \left((\mathbf{D}^{(0)})^n \boldsymbol{\tau}(\mathbf{X}) \right) (l) e^{-2i\pi lv} = \mathbf{a}_L(v)^* \left(\Phi^{(0)} \right)^n (\mathbf{X}) \mathbf{a}_L(v)$$

we first remark that

$$\left| \sum_{l=-(L-1)}^{L-1} \left((\mathbf{I} - \mathbf{D}^{(0)})^{-1} \boldsymbol{\tau}(\mathbf{X}) \right) (l) e^{-2i\pi lv} \right| \leq \sum_{n=0}^{+\infty} \left| \mathbf{a}_L(v)^* \left(\Phi^{(0)} \right)^n (\mathbf{X}) \mathbf{a}_L(v) \right|$$

Inequalities (7.34, 7.35) imply that

$$\left| \mathbf{a}_L(v)^* \left(\Phi^{(0)} \right)^n (\mathbf{X}) \mathbf{a}_L(v) \right| = \left| \sum_{l=-(L-1)}^{L-1} \left((\mathbf{D}^{(0)})^n \boldsymbol{\tau}(\mathbf{X}) \right) (l) e^{-2i\pi lv} \right|$$

is less than $\|\mathcal{T}_{N,L}(\mathbf{X})\| (\sigma^4 c_N |zt(z)\tilde{t}(z)|^2)^{n/2} (\mathbf{a}_L(v)^* \Phi^n(\mathbf{I}) \mathbf{a}_L(v))^{1/2}$. Using the inequality $|ab| \leq \frac{(a^2+b^2)}{2}$, we obtain that

$$\begin{aligned} & \left| \mathbf{a}_L(v)^* \left(\Phi^{(0)} \right)^n (\mathbf{X}) \mathbf{a}_L(v) \right| \\ & \leq \frac{\|\mathcal{T}_{N,L}(\mathbf{X})\|}{2} \left[(\sigma^4 c_N |zt(z)\tilde{t}(z)|^2)^n + \mathbf{a}_L(v)^* \Phi^n(\mathbf{I}) \mathbf{a}_L(v) \right] \end{aligned}$$

Summing over n eventually leads to (7.32).

We are now in position to establish the main result of this section, which, eventually, implies (7.1).

Proposition 7.3 *It exists 2 nice polynomials P_1 and P_2 for which*

$$\sup_{v \in [0,1]} \left| \mathbf{a}_L(v)^* (\mathbf{R}(z) - t(z) \mathbf{I}_L) \mathbf{a}_L(v) \right| \leq \frac{L^{3/2}}{MN} P_1(|z|) P_2 \left(\frac{1}{\text{Im}(z)} \right) \tag{7.37}$$

for N large enough and for each $z \in \mathbb{C}^+$

Proof We recall that $\mathbf{a}_L(v)^* (\mathbf{R}(z) - t(z) \mathbf{I}_L) \mathbf{a}_L(v)$ coincides with $\sum_{l=-(L-1)}^{L-1} \tau(\mathbf{R} - t\mathbf{I})(l) e^{-2i\pi lv}$ (see (2.11)) and recall that by Eq. (7.3), vector $\boldsymbol{\tau}(\mathbf{R} - t\mathbf{I})$ satisfies the equation

$$\boldsymbol{\tau}(\mathbf{R} - t\mathbf{I}) = \boldsymbol{\tau}(\Gamma) + \mathbf{D}^{(0)} \boldsymbol{\tau}(\mathbf{R} - t\mathbf{I})$$

where matrix \mathbf{F} is defined by (7.11). Proposition 7.1 and Proposition 7.2 used in the case $\mathbf{X} = \mathbf{F}$ as well as (7.36) imply that for N large and $z \in E_N$, it holds that

$$\left| \sum_{l=-(L-1)}^{L-1} \tau(\mathbf{R} - t\mathbf{I})(l)e^{-2i\pi lv} \right| \leq C \frac{(|z|^2 + \eta_2^2)^2}{(\text{Im}(z))^4} \|\mathcal{T}_{N,L}(\mathbf{F})\| \tag{7.38}$$

for some nice constant C and for $\eta_2 = \max(\eta, \eta_1)$. It is clear that

$$\|\mathcal{T}_{N,L}(\mathbf{F})\| \leq P_1(|z|)P_2\left(\frac{1}{\text{Im}(z)}\right) \left\| \mathcal{T}_{N,L}^{(M)}(\mathbb{E}(\mathbf{Q}) - \mathbf{R}_M) \right\| \tag{7.39}$$

Corollary 5.1 thus implies that (7.37) holds for N large enough and $z \in E_N$. It remains to establish that (7.37) also holds on the complementary E_N^c of E_N . For this, we remark that on E_N^c , $1 < \frac{L^{3/2}}{MN} P_1(|z|)P_2\left(\frac{1}{\text{Im}(z)}\right)$. As $\sup_{v \in [0,1]} |\mathbf{a}_L(v)^* (\mathbf{R}(z) - t(z)\mathbf{I}_L) \mathbf{a}_L(v)| \leq \frac{2}{\text{Im}(z)}$ on \mathbb{C}^+ , we obtain that

$$\sup_{v \in [0,1]} |\mathbf{a}_L(v)^* (\mathbf{R}(z) - t(z)\mathbf{I}_L) \mathbf{a}_L(v)| \leq \frac{1}{\text{Im}(z)} \frac{L^{3/2}}{MN} P_1(|z|)P_2\left(\frac{1}{\text{Im}(z)}\right)$$

for $z \in E_N^c$. This, in turn, shows that (7.37) holds for N large enough and for each $z \in \mathbb{C}^+$. □

Remark 7.1 We note that this property also implies that any quadratic form of $\mathbf{R} - t\mathbf{I}$ converges towards 0 at rate $\frac{L^{3/2}}{MN}$. Using the polarization identity, it is sufficient to prove that $\mathbf{b}^* (\mathbf{R} - t\mathbf{I}) \mathbf{b}$ is a $O(\frac{L^{3/2}}{MN})$ term for each uniformly bounded deterministic vector \mathbf{b} . We consider Eqs. (7.4, 7.5) and note that the right-hand side of (7.4) and (7.5) are bounded, up to constant terms depending on z (and not on the dimensions L, M, N) by $\|\mathcal{T}_{N,L}^{(M)}[\mathbb{E}(\mathbf{Q}) - \mathbf{R}_M]\|$ and $\|\mathcal{T}_{N,L}(\mathbf{R} - t\mathbf{I})\|$, respectively.

8 Proof of (1.18)

The purpose of this section is to establish the identity (1.18). For this, we have essentially to control the term $\frac{1}{L} \text{Tr}(\mathbf{R} - t\mathbf{I})$. More precisely, we prove the following proposition.

Proposition 8.1 *It exists nice polynomials P_1 and P_2 such that*

$$\sup_{\|\mathbf{A}\| \leq 1} \left| \frac{1}{L} \text{Tr}[(\mathbf{R} - t\mathbf{I}_L)\mathbf{A}] \right| \leq \frac{L}{MN} P_1(z)P_2(1/\text{Im}z) \tag{8.1}$$

for each $z \in F_N^{(3/2)}$ where $F_N^{(3/2)}$ is a subset of \mathbb{C}^+ defined by

$$F_N^{(3/2)} = \left\{ z \in \mathbb{C}^+, \frac{L^{3/2}}{MN} Q_1(z)Q_2(1/\text{Im}z) \leq 1 \right\} \tag{8.2}$$

for some nice polynomials Q_1 and Q_2 .

Proof In the following, we denote by $\beta(\mathbf{A})$ the term $\frac{1}{L} \text{Tr}[(\mathbf{R} - t\mathbf{I}_L)\mathbf{A}]$. We write (6.6) as

$$\begin{aligned} \frac{1}{L} \text{Tr}[(\mathbf{R} - t\mathbf{I}_L)\mathbf{A}] &= -\sigma^4 c_N z t(z) \tilde{t}(z) \frac{1}{ML} \text{Tr}[(\mathbf{E}(\mathbf{Q}) - \mathbf{I}_M \otimes \mathbf{R})(\mathbf{I}_M \otimes \mathbf{G}(\mathbf{A}))] \\ &\quad - \sigma^4 c_N z t(z) \tilde{t}(z) \frac{1}{L} \text{Tr}(\mathbf{R} - t\mathbf{I}) \mathcal{T}_{L,L}[(\mathcal{T}_{N,L}(\mathbf{A}\mathbf{R})) \mathbf{H}] \end{aligned} \tag{8.3}$$

We denote by $\epsilon(\mathbf{A})$ the first term of the right-hand side of (8.3). (6.8) and Proposition 5.1 imply that $\sup_{\|\mathbf{A}\| \leq 1} |\epsilon(\mathbf{A})| \leq \frac{L}{MN} P_1(|z|) P_2(1/\text{Im}z)$ for some nice polynomials P_1 and P_2 . In order to evaluate the contribution of the second term of the right-hand side of (8.3), we remark that matrices $\mathbf{R}(z)$ and $\mathbf{H}(z)$ should be “close” from $t(z)\mathbf{I}_L$ and $-z\tilde{t}(z)\mathbf{I}_N$, respectively. It is thus appropriate to rewrite (8.3) as

$$\begin{aligned} \frac{1}{L} \text{Tr}((\mathbf{R} - t\mathbf{I})\mathbf{A}) &= -z t(z) \tilde{t}(z) \sigma^4 c_N \frac{1}{ML} \text{Tr}[(\mathbb{E}(\mathbf{Q}) - \mathbf{I}_M \otimes \mathbf{R}) \mathbf{I}_M \otimes \mathbf{G}(\mathbf{A})] \\ &\quad + (z t(z) \tilde{t}(z))^2 \sigma^4 c_N \frac{1}{L} \text{Tr}[(\mathbf{R} - t\mathbf{I}) \mathcal{T}_{L,L}(\mathcal{T}_{N,L}(\mathbf{A}))] \\ &\quad + (z \tilde{t}(z))^2 t(z) \sigma^4 c_N \frac{1}{L} \text{Tr}[(\mathbf{R} - t\mathbf{I}) \mathcal{T}_{L,L}(\mathcal{T}_{N,L}[\mathbf{A}(\mathbf{R} - t\mathbf{I})])] \\ &\quad - z(t(z))^2 \tilde{t}(z) \sigma^4 c_N \frac{1}{L} \text{Tr}[(\mathbf{R} - t\mathbf{I}) \mathcal{T}_{L,L}(\mathcal{T}_{N,L}(\mathbf{A})(\mathbf{H} + z\tilde{t}(z)\mathbf{I}))] \\ &\quad - z t(z) \tilde{t}(z) \sigma^4 c_N \frac{1}{L} \text{Tr}[(\mathbf{R} - t\mathbf{I}) \mathcal{T}_{L,L}(\mathcal{T}_{N,L}[\mathbf{A}(\mathbf{R} - t\mathbf{I})](\mathbf{H} + z\tilde{t}(z)\mathbf{I}))] \end{aligned} \tag{8.4}$$

We denote by $\alpha_1(\mathbf{A})$, $\alpha_2(\mathbf{A})$, $\alpha_3(\mathbf{A})$, and $\alpha_4(\mathbf{A})$ the second, third, fourth, and fifth terms of the right-hand side of the above equation, respectively. \square

We first study the term $\alpha_1(\mathbf{A})$. We first recall that for each $z \in \mathbb{C}^+$ and N large enough, it holds that

$$\sigma^4 c_N |z t(z) \tilde{t}(z)|^2 < 1 - C \frac{(\text{Im}z)^4}{(\eta^2 + |z|^2)^2}$$

where C and η are nice constants (see Eq. (1.31)). Moreover, for each \mathbf{A} , $\|\mathbf{A}\| \leq 1$, it is clear that

$$\left| \frac{1}{L} \text{Tr}[(\mathbf{R} - t\mathbf{I}) \mathcal{T}_{L,L}(\mathcal{T}_{N,L}(\mathbf{A}))] \right| \leq \sup_{\|\mathbf{B}\| \leq 1} |\beta(\mathbf{B})| \|\mathcal{T}_{L,L}(\mathcal{T}_{N,L}(\mathbf{A}))\| \leq \sup_{\|\mathbf{B}\| \leq 1} |\beta(\mathbf{B})|$$

because $\|\mathcal{T}_{L,L}(\mathcal{T}_{N,L}(\mathbf{A}))\| \leq \|\mathbf{A}\| \leq 1$ (see Proposition 2.1). This shows that

$$\sup_{\|\mathbf{A}\| \leq 1} |\alpha_1(\mathbf{A})| \leq \left(1 - C \frac{(\text{Im}z)^4}{(\eta^2 + |z|^2)^2} \right) \sup_{\|\mathbf{A}\| \leq 1} |\beta(\mathbf{A})|$$

We now evaluate the behaviour of $\alpha_2(\mathbf{A})$. We first use (2.7) to obtain that

$$\alpha_2(\mathbf{A}) = (z\tilde{t}(z))^2 t(z) \sigma^4 c_N \frac{1}{L} \text{Tr} \left[\mathbf{A} (\mathbf{R} - t \mathbf{I}) \mathcal{T}_{L,L} (\mathcal{T}_{N,L}(\mathbf{R} - t \mathbf{I})) \right]$$

We remark that for each matrix \mathbf{A} , $\|\mathbf{A}\| \leq 1$, it holds that

$$\left| \frac{1}{L} \text{Tr} \left[(\mathbf{R} - t \mathbf{I}) \mathcal{T}_{L,L} (\mathcal{T}_{N,L}(\mathbf{R} - t \mathbf{I})) \mathbf{A} \right] \right| \leq \sup_{\|\mathbf{B}\| \leq 1} \beta(\mathbf{B}) \|\mathbf{A}\| \|\mathcal{T}_{N,L}(\mathbf{R} - t \mathbf{I})\|$$

(7.1) implies that

$$\sup_{\|\mathbf{A}\| \leq 1} |\alpha_2(\mathbf{A})| < \sup_{\|\mathbf{A}\| \leq 1} \beta(\mathbf{A}) \frac{L^{3/2}}{MN} P_1(|z|) P_2(1/\text{Im}z)$$

for each $z \in \mathbb{C}^+$. The terms $\alpha_3(\mathbf{A})$ and $\alpha_4(\mathbf{A})$ can be handled similarly by writing $\mathbf{H} + z\tilde{t}(z)\mathbf{I}$ as

$$\mathbf{H} + z\tilde{t}(z)\mathbf{I} = \sigma^2 c_N z\tilde{t}(z) \mathbf{H} \mathcal{T}_{N,L}^{(M)} (\mathbb{E}(\mathbf{Q}) - \mathbf{I}_M \otimes \mathbf{R}) + \sigma^2 c_N z\tilde{t}(z) \mathbf{H} \mathcal{T}_{N,L} (\mathbf{R} - t \mathbf{I})$$

In particular, it can be shown that for $i = 3, 4$ and N large enough, it holds that

$$\sup_{\|\mathbf{A}\| \leq 1} |\alpha_i(\mathbf{A})| < \sup_{\|\mathbf{A}\| \leq 1} \beta(\mathbf{A}) \frac{L^{3/2}}{MN} P_1(|z|) P_2(1/\text{Im}z)$$

Therefore, it holds that

$$\begin{aligned} \sup_{\|\mathbf{A}\| \leq 1} |\beta(\mathbf{A})| &\leq \sup_{\|\mathbf{A}\| \leq 1} |\epsilon(\mathbf{A})| \\ &+ \sup_{\|\mathbf{A}\| \leq 1} \beta(\mathbf{A}) \left[\left(1 - C \frac{(\text{Im}z)^4}{(\eta + |z|^2)^2} \right) + \frac{L^{3/2}}{MN} P_1(|z|) P_2(1/\text{Im}z) \right] \end{aligned}$$

We define the set $F_N^{(3/2)}$ as

$$F_N^{(3/2)} = \left\{ z \in \mathbb{C}^+, \frac{L^{3/2}}{MN} P_1(|z|) P_2(1/\text{Im}z) \leq C/2 \frac{(\text{Im}z)^4}{(\eta^2 + |z|^2)^2} \right\}$$

which can also be written as

$$F_N^{(3/2)} = \left\{ z \in \mathbb{C}^+, \frac{L^{3/2}}{MN} Q_1(|z|) Q_2(1/\text{Im}z) \leq 1 \right\}$$

for some nice polynomials Q_1 and Q_2 . Then, it is clear that for each $z \in F_N^{(3/2)}$, then it holds that

$$\sup_{\|\mathbf{A}\| \leq 1} |\beta(\mathbf{A})| \leq 2/C \frac{(\eta^2 + |z|^2)^2}{(\text{Im}z)^4} \sup_{\|\mathbf{A}\| \leq 1} |\epsilon(\mathbf{A})| \leq \frac{L}{MN} P_1(|z|)P_2(1/\text{Im}z)$$

for some nice polynomials P_1 and P_2 . This completes the proof of Proposition 8.1.

We conclude this section by the corollary:

Corollary 8.1 *The mathematical expectation of the Stieltjes transform $\frac{1}{ML} \text{Tr}(\mathbf{Q}(z))$ of the empirical eigenvalue distribution of $\mathbf{W}\mathbf{W}^*$ can be written for $z \in \mathbb{C}^+$ as*

$$\mathbb{E} \left[\frac{1}{ML} \text{Tr}(\mathbf{Q}(z)) \right] = t(z) + \frac{L}{MN} \tilde{r}(z) \tag{8.5}$$

where $\tilde{r}(z)$ is holomorphic in \mathbb{C}^+ and satisfies

$$|\tilde{r}(z)| \leq P_1(|z|)P_2 \left(\frac{1}{\text{Im}(z)} \right) \tag{8.6}$$

for each $z \in F_N^{(3/2)}$ defined by (8.2).

Proof In order to establish (8.5), we have to prove that

$$\left| \frac{1}{ML} \text{Tr}(\mathbb{E}(\mathbf{Q}(z))) - t(z) \right| \leq P_1(|z|)P_2 \left(\frac{1}{\text{Im}(z)} \right) \frac{L}{MN}$$

for $z \in F_N^{(3/2)}$. $\mathbb{E}(\mathbf{Q}(z)) - t(z)\mathbf{I}$ can be written as

$$\mathbb{E}(\mathbf{Q}(z)) - t(z)\mathbf{I}_{ML} = \mathbf{\Delta}(z) + \mathbf{I}_M \otimes \mathbf{R}(z) - t(z)\mathbf{I}_{ML}$$

Therefore, Proposition 5.1 implies that we have just to verify that

$$\left| \frac{1}{L} \text{Tr}(\mathbf{R} - t\mathbf{I}_L) \right| \leq P_1(|z|)P_2 \left(\frac{1}{\text{Im}(z)} \right) \frac{L}{MN}$$

for $z \in F_N^{(3/2)}$, a consequence of Proposition 8.1. □

9 Expansion of $\frac{1}{ML} \text{Tr}(\mathbb{E}(\mathbf{Q}_N(z))) - t_N(z)$

Notations and definitions used in section 9. In order to simplify the exposition of the results presented in this section, we define the following simplified notations:

- Let $(\beta_N)_{N \geq 1}$ be a sequence depending on N . A term $\phi_N(z)$ depending on N defined for $z \in \mathbb{C}^+$ will be said to be a $O(\beta_N)$ term if it exists 2 nice polynomials P_1 and P_2 such that

$$|\phi_N(z)| \leq \beta_N P_1(|z|)P_2(1/\text{Im}z)$$

for N large enough and for each z belonging to a set defined as $F_N^{(2)}$, but possibly with other nice polynomials.

- $C_N(z, u_1, \dots, u_k)$ will represent a generic term depending on N, z and on indices $u_1, \dots, u_k \in \{-(L-1), \dots, L-1\}$, and satisfying $\sup_{u_1, \dots, u_k} |C_N((z, u_1, \dots, u_k))| = O(1)$ in the sense of the above definition of operator $O(\cdot)$. Very often, we will not mention the dependency of $C_N(z, u_1, \dots, u_k)$ w.r.t. N and z and use the notation $C(u_1, \dots, u_k)$.
- By a real distribution, we mean a real-valued continuous (in an appropriate sense) linear form D defined on the space $\mathbb{C}_c^\infty(\mathbb{R})$ of all real-valued compactly supported smooth functions defined on \mathbb{R} . Such a distribution can of course be extended to complex-valued smooth functions defined on \mathbb{R} by setting $\langle D, \phi_1 + i\phi_2 \rangle = \langle D, \phi_1 \rangle + i\langle D, \phi_2 \rangle$ for $\phi_1, \phi_2 \in \mathbb{C}_c^\infty(\mathbb{R})$. We also recall that a compactly supported distribution D can be extended to a continuous linear form to the space $\mathbb{C}_b^\infty(\mathbb{R})$ of all bounded smooth functions. In particular, $\langle D, \mathbb{1} \rangle$ represents $\langle D, \phi \rangle$ where ϕ is any function of $\mathbb{C}_c^\infty(\mathbb{R})$ that is equal to 1 on the support of D .

From now on, we assume that L satisfies the condition

$$L = O(N^\alpha), \text{ where } \alpha < \frac{2}{3} \tag{9.1}$$

which implies that

$$\frac{L^2}{MN} \rightarrow 0, \text{ i.e. } \frac{L}{M^2} \rightarrow 0 \tag{9.2}$$

The goal of this section is to establish the following theorem.

Theorem 9.1 Under (9.1), $\frac{1}{ML} \text{Tr}(\mathbb{E}(\mathbf{Q}_N(z))) - t_N(z)$ can be expanded as

$$\frac{1}{ML} \text{Tr}(\mathbb{E}(\mathbf{Q}_N(z))) - t_N(z) = \frac{L}{MN} \left(\hat{s}_N(z) + \frac{L^{3/2}}{MN} \hat{r}_N(z) \right) \tag{9.3}$$

where $\hat{s}_N(z)$ coincides with the Stieltjes transform of a distribution \hat{D}_N whose support is included into $\mathcal{S}_N^{(0)} = [\sigma^2(1 - \sqrt{c_N})^2, \sigma^2(1 + \sqrt{c_N})^2]$ and which verifies $\langle \hat{D}_N, \mathbb{1} \rangle$, and where $|\hat{r}_N(z)| \leq P_1(|z|)P_2(\frac{1}{\text{Im}z})$ when z belongs to a set $F_N^{(2)}$ defined by

$$F_N^{(2)} = \left\{ z \in \mathbb{C}^+, \frac{L^2}{MN} Q_1(|z|)Q_2(1/\text{Im}z) \leq 1 \right\} \tag{9.4}$$

for some nice polynomials Q_1 and Q_2 .

As shown below in Sect. 10, (9.3) provides the desired almost sure location of the eigenvalues of $\mathbf{W}_N \mathbf{W}_N^*$. In order to establish (9.3), we express $\frac{1}{ML} \text{Tr} (\mathbb{E}(\mathbf{Q}_N(z))) - t_N(z)$ as

$$\frac{1}{ML} \text{Tr} (\mathbb{E}(\mathbf{Q}_N(z))) - t_N(z) = \frac{1}{ML} \text{Tr} \mathbf{\Delta}_N(z) + \frac{1}{L} \text{Tr} (\mathbf{R}_N(z) - t_N(z) \mathbf{I})$$

and study the 2 terms separately. We first establish that if (9.1) holds, then

$$\frac{1}{ML} \text{Tr} \mathbf{\Delta}_N(z) = \frac{L}{MN} s_N(z) + \left(\frac{L}{MN} \right)^2 r_N(z) \tag{9.5}$$

where $s_N(z)$ is the Stieltjes transform of a distribution whose support is included in $\mathcal{S}_N^{(0)}$, and where

$$|r_N(z)| \leq P_1(|z|) P_2(1/\text{Im}z)$$

for some nice polynomials P_1 and P_2 and for $z \in F_N^{(2)}$. Using Theorem 7.1, (9.3) will follow easily from (9.5).

The proof of (9.5) is quite demanding. It needs to establish a number of intermediate results that are presented in Sect. 9.2 and used in Sect. 9.3.

9.1 Useful Results Concerning the Stieltjes Transforms of Compactly Supported Distributions

Before establishing (9.5), we need to recall some results concerning the Stieltjes transform of compactly supported real distributions and to establish that the so-called Hellfer–Sjöstrand formula, valid for probability measures, can be generalized to compactly supported distributions.

The following useful result was used in [29], Theorem 5.4, and Lemma 5.6 (see also Theorem 4.3 in [11]).

Lemma 9.1 *If D is a real distribution with compact support $\text{Supp}(D)$, its Stieltjes transform $s(z)$ is defined for each $z \in \mathbb{C} - \text{Supp}(D)$ by*

$$s(z) = \langle D, \frac{1}{\lambda - z} \rangle.$$

Then, s is analytic on $\mathbb{C} - \text{Supp}(D)$ and verifies the following properties:

- (a) $s(z) \rightarrow 0$ if $|z| \rightarrow +\infty$
- It exists a compact $\mathcal{K} \subset \mathbb{R}$ containing $\text{Supp}(D)$ such that
 - (b) $s(z^*) = (s(z))^*$ for each $z \in \mathbb{C} - \mathcal{K}$
 - (c) It exists an integer n_0 and a constant C such that for each $z \in \mathbb{C} - \mathcal{K}$,

$$|s(z)| \leq C \text{Max} \left(\frac{1}{(\text{Dist}(z, \mathcal{K}))^{n_0}}, 1 \right) \tag{9.6}$$

– If ϕ is an element of $C_c^\infty(\mathbb{R})$, then the following inversion formula holds

$$\frac{1}{\pi} \lim_{y \rightarrow 0^+} \int \phi(\lambda) \operatorname{Im}(s(\lambda + iy)) d\lambda = \langle D, \phi \rangle \tag{9.7}$$

– If $\lim_{|z| \rightarrow +\infty} |zs(z)| = 0$, then it holds that

$$\langle D, \mathbb{1} \rangle = 0 \tag{9.8}$$

Conversely, if \mathcal{K} is a compact subset of \mathbb{R} , and if $s(z)$ is a function analytic on $\mathbb{C} - \mathcal{K}$ satisfying (a), (b), (c), then $s(z)$ is the Stieltjes transform of a compactly supported real distribution D such that $\operatorname{Supp}(D) \subset \mathcal{K}$. In this case, $\operatorname{Supp}(D)$ is the set of singular points of $s(z)$.

Remark 9.1 – We note that (9.6) of course implies that

$$|s(z)| \leq C \operatorname{Max} \left(\frac{1}{(\operatorname{Im}z)^{n_0}}, 1 \right) \leq C \left(1 + \frac{1}{(\operatorname{Im}z)^{n_0}} \right) \tag{9.9}$$

for each $z \in \mathbb{C} - \mathbb{R}$.

– We have chosen to present Lemma 9.1 as it is stated in [29]. However, we mention that (b) and (c) hold for each compact subset \mathcal{K} of \mathbb{R} containing $\operatorname{Supp}(D)$. n_0 does not depend on the compact \mathcal{K} and is related to the order of D . However, the constant C does depend on \mathcal{K} .

We now provide a useful example of such functions $s(z)$.

Lemma 9.2 *If $p \geq 1$, then function $s_N(z)$ defined by*

$$s_N(z) = (t_N(z))^p (z\tilde{t}_N(z))^q \frac{1}{(1 - a_N \sigma^4 c_N(z) t_N(z) \tilde{t}_N(z))^n}$$

for $|a_N| \leq 1$ coincides with the Stieltjes transform of a real bounded distribution D_N whose support is included in \mathcal{S}_N for each integer $q \geq 0$ and $n \geq 0$. Moreover, D_N satisfies (9.8) as soon as $p \geq 2$.

Proof It is clear that $s_N(z^*) = (s_N(z))^*$ and that $s_N(z) \rightarrow 0$ if $|z| \rightarrow +\infty$ because $p \geq 1$ and that $z\tilde{t}(z) \rightarrow -1$. We use Lemma 1.1 to manage the term

$$\frac{1}{(1 - a_N \sigma^4 c_N(z) t_N(z) \tilde{t}_N(z))^n}$$

and use that $|t_N(z)| \leq \frac{1}{\operatorname{dist}(z, \mathcal{S}_N)}$ for $z \in \mathbb{C} - \mathcal{S}_N$. We also remark that

$$z\tilde{t}_N(z) = c_N \int_{\mathcal{S}_N} \frac{z}{\lambda - z} d\mu_{\sigma^2, c_N}(\lambda) - (1 - c_N)$$

or equivalently that

$$z\tilde{t}_N(z) = c_N \int_{\mathcal{S}_N} \frac{\lambda}{\lambda - z} d\mu_{\sigma^2, c_N}(\lambda) - 1$$

Therefore,

$$|z\tilde{t}_N(z)| \leq C \left(1 + \frac{1}{\text{dist}(z, \mathcal{S}_N)} \right) \leq C \max \left(1, \frac{1}{\text{dist}(z, \mathcal{S}_N)} \right)$$

for each $z \in \mathbb{C} - \mathcal{S}_N$. Moreover, it holds that $zs(z) \rightarrow 0$ if $|z| \rightarrow +\infty$ as soon as $p \geq 2$. □

We now briefly justify that the Hellfer–Sjöstrand formula can be generalized to compactly supported distributions. In order to introduce this formula, used in the context of large random matrices in [2, 3] and [26], we have to define some notations. χ is a function of $C_c^\infty(\mathbb{R})$ with support $[-1, 1]$, which is equal to 1 in a neighbourhood of 0. If $\phi(x) \in C_c^\infty(\mathbb{R})$, we denote by $\bar{\phi}_k$ the function of $C_c^\infty(\mathbb{R}^2, \mathbb{C})$ defined for $z = x + iy$ by

$$\bar{\phi}_k(z) = \sum_{l=0}^k \phi^{(l)}(x) \frac{(iy)^l}{l!} \chi(y)$$

Function $\partial\bar{\phi}_k$ is the “derivative”

$$\partial\bar{\phi}_k(z) = \frac{\partial\bar{\phi}_k(z)}{\partial x} + i \frac{\partial\bar{\phi}_k(z)}{\partial y}$$

and is given by

$$\partial\bar{\phi}_k(z) = \phi^{(k+1)}(x) \frac{(iy)^k}{k!} \tag{9.10}$$

in the neighbourhood of 0 in which $\chi(y) = 1$. If $s(z)$ is the Stieltjes transform of a probability measure μ , $s(z)$ verifies $|s(z)| \leq \frac{1}{\text{Im}z}$ on \mathbb{C}^+ . Therefore, (9.10) implies that if $k \geq 1$, then function $\partial\bar{\phi}_k(z) s(z)$ is well defined near the real axis. The Hellfer–Sjöstrand allows us to reconstruct $\int \phi(\lambda) d\mu(\lambda)$ as:

$$\int \phi(\lambda) d\mu(\lambda) = \frac{1}{\pi} \text{Re} \left(\int_{\mathbb{C}^+} \partial\bar{\phi}_k(z) s(z) dx dy \right) \tag{9.11}$$

The following Lemma extends formula (9.11) to real compactly supported distributions.

Lemma 9.3 *We consider a compactly supported distribution D and $s(z)$ is Stieltjes transform. Then, if k is greater than the index n_0 defined by (9.9), then $\partial\bar{\phi}_k(z) s(z)$ is well defined near the real axis, and*

$$\langle D, \phi \rangle = \frac{1}{\pi} \text{Re} \left(\int_{\mathbb{C}^+} \partial\bar{\phi}_k(z) s(z) dx dy \right) \tag{9.12}$$

Sketch of proof. It is clear that $\partial \bar{\phi}_k(z) s(z)$ is well defined near the real axis. Therefore, the integral at the right-hand side of (9.12) exists. By linearity, it is sufficient to establish (9.12) if D coincides with a derivative of a Dirac distribution $D = \delta_{\lambda_0}^{(p)}$ for $p \leq n_0 - 1$, i.e. $s(z) = \frac{1}{(\lambda_0 - z)^{p+1}}$. Using the integration by parts formula and the analyticity of $s(z)$ on \mathbb{C}^+ , we obtain that

$$\frac{1}{\pi} \operatorname{Re} \left(\int_{\mathbb{C}^+} \partial \bar{\phi}_k(z) s(z) \, dx dy \right) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \operatorname{Re} \left(-i \int_{\mathbb{R}} \bar{\phi}_k(x + i\epsilon) s(x + i\epsilon) \, dx \right)$$

$\langle D, \phi \rangle$ is of course equal to

$$\langle D, \phi \rangle = (-1)^p \langle \delta_{\lambda_0}, \phi^{(p)} \rangle$$

As the Hellfer–Sjöstrand formula is valid for measure δ_{λ_0} and that the Stieltjes transform of δ_{λ_0} is $\frac{1}{\lambda_0 - z}$, it holds that

$$\langle \delta_{\lambda_0}, \phi^{(p)} \rangle = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \operatorname{Re} \left(-i \int_{\mathbb{R}} \left(\overline{\phi^{(p)}} \right)_k (x + i\epsilon) \frac{1}{\lambda_0 - (x + i\epsilon)} \, dx \right)$$

It is clear that $\left(\overline{\phi^{(p)}} \right)_k (x + i\epsilon) = \frac{d^p}{dx^p} \bar{\phi}_k(x + i\epsilon)$. Therefore, the integration by parts leads to

$$\int_{\mathbb{R}} \left(\overline{\phi^{(p)}} \right)_k (x + i\epsilon) \frac{1}{\lambda_0 - (x + i\epsilon)} \, dx = (-1)^p \int_{\mathbb{R}} \bar{\phi}_k(x + i\epsilon) \frac{1}{(\lambda_0 - (x + i\epsilon))^{p+1}} \, dx$$

from which (9.12) follows immediately. □

9.2 Some Useful Evaluations

Equations (4.15) and (5.2) imply that $\frac{1}{ML} \operatorname{Tr} (\mathbf{\Delta}(z))$ is given by

$$\frac{1}{ML} \operatorname{Tr} (\mathbf{\Delta}(z)) = \sigma^2 c_N \sum_{l_1 = -(L-1)}^{L-1} \mathbb{E} \left(\tau^{(M)}(\mathbf{Q}^\circ)(l_1) \frac{1}{ML} \operatorname{Tr} \left(\mathbf{QWJ}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R}) \right)^\circ \right)$$

In order to establish (9.5), it is necessary to evaluate the right-hand side of the above equation up to $O\left(\frac{L}{MN}\right)^2$ terms using the integration by parts formula. If we denote by $\kappa^{(2)}(l_1, l_2)$ the term defined by $\kappa^{(2)}(l_1, l_2) = \mathbb{E} \left(\tau^{(M)}(\mathbf{Q}^\circ)(l_1) \tau^{(M)}(\mathbf{Q}^\circ)(l_2) \right)$, then we establish in the following that

$$\begin{aligned} \frac{1}{ML} \operatorname{Tr} (\mathbf{\Delta}(z)) &= (\sigma^2 c_N)^2 \sum_{l_1, l_2 = -(L-1)}^{L-1} \kappa^{(2)}(l_1, l_2) \\ &\times \mathbb{E} \left[\frac{1}{ML} \operatorname{Tr} \left(\mathbf{QWJ}_N^{l_2} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \sigma^2 \mathbf{R}_{\mathcal{T}_{L,L}} (\mathbf{HJ}_N^{*l_1} \mathbf{H} \mathbf{R})) \right) \right] \end{aligned}$$

$$\begin{aligned}
 & - (\sigma^2 c_N)^2 \sum_{l_1, l_2 = -(L-1)}^{L-1} \kappa^{(2)}(l_1, l_2) \\
 & \quad \times \mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_2} \mathbf{H}^T \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R}) \right) \right] \\
 & + \frac{\sigma^4 c_N}{MLN} \sum_{l_1, i = -(L-1)}^{L-1} \mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\mathbf{Q} (\mathbf{I}_M \otimes \mathbf{J}_L^i) \mathbf{Q} (\mathbf{I}_M \otimes \mathbf{J}_L^i) \mathbf{Q} \mathbf{W} \mathbf{J}_N^i \mathbf{H}^T \mathbf{W}^* \right. \right. \\
 & \quad \left. \left. \times (\mathbf{I}_M \otimes \sigma^2 \mathbf{R} \mathcal{T}_{L,L} (\mathbf{H} \mathbf{J}_N^{*l_1} \mathbf{H} \mathbf{R})) \right) \right] \\
 & - \frac{\sigma^4 c_N}{MLN} \sum_{l_1, i = -(L-1)}^{L-1} \mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\mathbf{Q} (\mathbf{I}_M \otimes \mathbf{J}_L^i) \mathbf{Q} (\mathbf{I}_M \otimes \mathbf{J}_L^i) \mathbf{Q} \mathbf{W} \mathbf{J}_N^i \mathbf{H}^T \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R}) \right) \right] \\
 & + (\sigma^2 c_N)^2 \sum_{l_1, l_2 = -(L-1)}^{L-1} \mathbb{E} \left[\tau^{(M)}(\mathbf{Q}^\circ)(l_1) \tau^{(M)}(\mathbf{Q}^\circ)(l_2) \right. \\
 & \quad \left. \times \frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_2} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \sigma^2 \mathbf{R} \mathcal{T}_{L,L} (\mathbf{H} \mathbf{J}_N^{*l_1} \mathbf{H} \mathbf{R})) \right) \right] \\
 & - (\sigma^2 c_N)^2 \sum_{l_1, l_2 = -(L-1)}^{L-1} \mathbb{E} \left[\tau^{(M)}(\mathbf{Q}^\circ)(l_1) \tau^{(M)}(\mathbf{Q}^\circ)(l_2) \frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_2} \mathbf{H}^T \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R}) \right) \right]
 \end{aligned} \tag{9.13}$$

We evaluate in closed form the third and the fourth term of the right-hand side of (9.13) up to $O(\frac{L}{MN})^2$, prove that $\kappa^{(2)}(u_1, u_2) = \frac{1}{MN} C(z, u_1) \delta(u_1 + u_2 = 0) + O(\frac{L}{(MN)^2})$, and establish that the 2 last terms of (9.13) are $O(\frac{L}{MN})^2$. In Paragraph 9.2.1, we calculate useful quantities similar to the third and the fourth term of the right-hand side of (9.13), and in Paragraph 9.2.2, we evaluate $\kappa^{(2)}(u_1, u_2)$.

9.2.1 Evaluation of the Third and Fourth Terms of the Right-Hand Side of (9.13)

We first state 2 technical Lemmas.

Lemma 9.4 *We consider uniformly bounded $ML \times ML$ matrices $(\mathbf{C}^s)_{s=1, \dots, r}$ and \mathbf{A} , and a uniformly bounded $N \times N$ matrix \mathbf{G} . Then, for each $p \geq 2$, it holds that*

$$\mathbb{E} \left[\frac{1}{ML} \text{Tr} \left((\Pi_{s=1}^r \mathbf{Q} \mathbf{C}^s)^\circ \right)^p \right] = O \left(\frac{1}{(MN)^{p/2}} \right) \tag{9.14}$$

$$\mathbb{E} \left[\frac{1}{ML} \text{Tr} \left((\Pi_{s=1}^r \mathbf{Q} \mathbf{C}^s) \mathbf{W} \mathbf{G} \mathbf{W}^* \mathbf{A} \right)^p \right] = O \left(\frac{1}{(MN)^{p/2}} \right) \tag{9.15}$$

Proof We just provide a sketch of proof. We first establish (9.14) and (9.15) by induction for even integers $p = 2q$. For $q = 1$, we use the Poincaré–Nash inequality, and for $q \geq 1$, we take benefit of the identity

$$\mathbb{E}|x|^{2q} = |\mathbb{E}(x^q)|^2 + \text{Var}(x^q)$$

and of the Poincaré–Nash inequality. We obtain (9.14) and (9.15) for odd integers using the Schwartz inequality. \square

We now evaluate the expectation of normalized traces of matrices such as $\Pi_{s=1}^r \mathbf{Q}\mathbf{C}^s$. Proposition 9.1 is used in the sequel in the case $r = 2$ and $r = 3$.

Proposition 9.1 *For each $ML \times ML$ deterministic uniformly bounded matrices $(\mathbf{C}^s)_{s=1,\dots,r+1}$ and \mathbf{A} , it holds that*

$$\begin{aligned} \mathbb{E} \left(\frac{1}{ML} \text{Tr} \left(\Pi_{s=1}^{r+1} \mathbf{Q}\mathbf{C}^s \right) \right) &= \mathbb{E} \left(\frac{1}{ML} \text{Tr} \left[\left(\Pi_{s=1}^r \mathbf{Q}\mathbf{C}^s \right) \left(\mathbf{I}_M \otimes \mathbf{R} \right) \mathbf{C}^{r+1} \right] \right) \\ &+ \mathcal{O} \left(\frac{L}{MN} \right) + \sigma^2 c_N \sum_{s=1}^r \sum_{i=-(L-1)}^{L-1} \mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\left(\Pi_{t=s}^r \mathbf{Q}\mathbf{C}^s \right) \mathbf{Q} \left(\mathbf{I}_M \otimes \mathbf{J}_L^i \right) \right) \right] \\ &\times \mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\left(\Pi_{t=1}^{s-1} \mathbf{Q}\mathbf{C}^s \right) \mathbf{Q}\mathbf{W}\mathbf{J}_N^i \mathbf{H}^T \mathbf{W}^* \left(\mathbf{I}_M \otimes \mathbf{R} \right) \mathbf{C}^{r+1} \right) \right] \end{aligned} \tag{9.16}$$

and that

$$\begin{aligned} &\mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\left(\Pi_{s=1}^r \mathbf{Q}\mathbf{C}^s \right) \mathbf{Q}\mathbf{W}\mathbf{G}\mathbf{W}^* \mathbf{A} \right) \right] \\ &= \mathbb{E} \left(\frac{1}{ML} \text{Tr} \left[\Pi_{s=1}^r \mathbf{Q}\mathbf{C}^s \left(\mathbf{I}_M \otimes \sigma^2 \mathbf{R}\mathcal{T}_{L,L} \left(\mathbf{G}^T \mathbf{H} \right) \right) \mathbf{A} \right] \right) + \mathcal{O} \left(\frac{L}{MN} \right) \\ &+ \sigma^2 c_N \sum_{s=1}^r \sum_{i=-(L-1)}^{L-1} \mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\left(\Pi_{t=s}^r \mathbf{Q}\mathbf{C}^s \right) \mathbf{Q} \left(\mathbf{I}_M \otimes \mathbf{J}_L^i \right) \right) \right] \\ &\times \mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\Pi_{t=1}^{s-1} \mathbf{Q}\mathbf{C}^s \right) \mathbf{Q}\mathbf{W}\mathbf{J}_N^i \mathbf{H}^T \mathbf{W}^* \left(\mathbf{I}_M \otimes \sigma^2 \mathbf{R}\mathcal{T}_{L,L} \left(\mathbf{G}^T \mathbf{H} \right) \right) \mathbf{A} \right] \\ &- \sigma^2 c_N \sum_{s=1}^r \sum_{i=-(L-1)}^{L-1} \mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\left(\Pi_{t=s}^r \mathbf{Q}\mathbf{C}^s \right) \left(\mathbf{Q} \left(\mathbf{I}_M \otimes \mathbf{J}_L^i \right) \right) \right) \right] \\ &\times \mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\left(\Pi_{t=1}^{s-1} \mathbf{Q}\mathbf{C}^s \right) \mathbf{Q}\mathbf{W}\mathbf{J}_N^i \mathbf{H}^T \mathbf{G}\mathbf{W}^* \mathbf{A} \right) \right] \end{aligned} \tag{9.17}$$

The proof of this result is similar to the proof of (4.14) and (4.16), but is of course more tedious. To establish (9.16) and (9.17), it is sufficient to evaluate matrix $\mathbb{E} \left[\Pi_{s=1}^r \mathbf{Q}_{l_s, l'_s}^{n_s, n'_s} \mathbf{Q}\mathbf{W}\mathbf{G}\mathbf{W}^* \right]$ using the integration by parts formula for each multi-indices (l'_1, \dots, l'_r) and (n'_1, \dots, n'_r) . A proof is provided in [23].

We now use Proposition 9.1 to study the behaviour of certain useful terms. For this, it is first necessary to give the following lemma. If \mathbf{A} is a matrix, $|||\mathbf{A}|||_\infty$ is defined as

$$|||\mathbf{A}|||_\infty = \sup_i \sum_j |A_{i,j}|$$

Lemma 9.5 *We consider the $(2L - 1) \times (2L - 1)$ diagonal matrix $\mathbf{D}(z) = \text{Diag}(d(-(L - 1), z), \dots, d(0), \dots, d(L - 1, z))$ where for each $l \in \mathbb{Z}$, $d(l, z)$ is defined as*

$$d(l, z) = \sigma^4 c_N (z t(z) \tilde{t}(z))^2 (1 - |l|/L)_+ (1 - |l|/N)_+ \tag{9.18}$$

We consider a $(2L - 1) \times (2L - 1)$ deterministic matrix \mathbf{Y} whose entries $(\epsilon_{k,l})_{-(L-1) \leq k, l \leq L-1}$ depend on z, L, M, N and satisfy

$$|\epsilon_{k,l}| \leq \frac{L}{MN} P_1(|z|) P_2\left(\frac{1}{\text{Im}(z)}\right) \tag{9.19}$$

for some nice polynomials P_1 and P_2 for each $z \in \mathbb{C}^+$. Then, for each z belonging to a set E_N defined by

$$E_N = \left\{ z \in \mathbb{C}^+, \frac{L^2}{MN} Q_1(|z|) Q_2\left(\frac{1}{\text{Im}(z)}\right) < 1 \right\} \tag{9.20}$$

for some nice polynomials Q_1, Q_2 , matrix $(\mathbf{I} - (\mathbf{D} + \mathbf{Y}))$ is invertible and for each L, M, N , and for each $z \in E_N$, it holds that

$$\sup_{L, M, N} ||| (\mathbf{I} - (\mathbf{D} + \mathbf{Y}))^{-1} |||_\infty < C \frac{(\eta^2 + |z|^2)^2}{(\text{Im}(z))^4} \tag{9.21}$$

for some nice constants η and C .

Proof It is well known (see, e.g. [20], Corollary 6.1.6 p. 390) that

$$\rho(\mathbf{D} + \mathbf{Y}) \leq |||\mathbf{D} + \mathbf{Y}|||_\infty$$

Therefore, we obtain that

$$\rho(\mathbf{D} + \mathbf{Y}) \leq \sigma^4 c_N |z t(z) \tilde{t}(z)|^2 + \frac{L^2}{MN} P_1(|z|) P_2\left(\frac{1}{\text{Im}(z)}\right)$$

As $\sigma^4 c_N |z t(z) \tilde{t}(z)|^2 \leq 1 - C \frac{(\text{Im}(z))^4}{(\eta^2 + |z|^2)^2}$ for some nice constants C and η (see Eq. 1.31), we get that

$$\rho(\mathbf{D} + \mathbf{Y}) < 1 - \frac{C}{2} \frac{(\text{Im}(z))^4}{(\eta^2 + |z|^2)^2}$$

if z satisfies

$$C \frac{(\text{Im}(z))^4}{(\eta^2 + |z|^2)^2} - \frac{L^2}{MN} P_1(|z|) P_2\left(\frac{1}{\text{Im}(z)}\right) > \frac{C}{2} \frac{(\text{Im}(z))^4}{(\eta^2 + |z|^2)^2}$$

a condition that can be written as $z \in E_N$ for well chosen nice polynomials Q_1, Q_2 . We note that a similar result holds for $\rho(|\mathbf{D}| + |\mathbf{Y}|)$ where for any matrix \mathbf{A} , $|\mathbf{A}|$ is the matrix defined by $(|\mathbf{A}|)_{i,j} = |\mathbf{A}|_{i,j}$. This implies that for $z \in E_N$, matrices $\mathbf{I} - \mathbf{D} - \mathbf{Y}$ and $\mathbf{I} - |\mathbf{D}| - |\mathbf{Y}|$ are invertible and that $(\mathbf{I} - \mathbf{D} - \mathbf{Y})^{-1} = \sum_{n=0}^{+\infty} (\mathbf{D} + \mathbf{Y})^n$ and $(\mathbf{I} - |\mathbf{D}| - |\mathbf{Y}|)^{-1} = \sum_{n=0}^{+\infty} (|\mathbf{D}| + |\mathbf{Y}|)^n$. We note that for each k, l , $|((\mathbf{D} + \mathbf{Y})^n)_{k,l}| \leq ((|\mathbf{D}| + |\mathbf{Y}|)^n)_{k,l}$. Therefore,

$$\left| \left((\mathbf{I} - \mathbf{D} - \mathbf{Y})^{-1} \right)_{k,l} \right| \leq \left((\mathbf{I} - |\mathbf{D}| - |\mathbf{Y}|)^{-1} \right)_{k,l} \tag{9.22}$$

We denote by $\mathbf{1}$ the $2L - 1$ dimensional vector with all components equal to 1, and by \mathbf{b} the vector $\mathbf{b} = (\mathbf{I} - |\mathbf{D}| - |\mathbf{Y}|)^{-1} \mathbf{1}$. It is clear that for each $l \in \{-(L - 1), \dots, L - 1\}$, \mathbf{b}_l is equal to

$$\mathbf{b}_l = 1 - \sigma^4 c_N |z t(z) \tilde{t}(z)|^2 (1 - |l|/L)(1 - |l|/N) - \sum_k |\epsilon_{l,k}|$$

which is greater than $\frac{C}{2} \frac{(\text{Im}(z))^4}{(\eta^2 + |z|^2)^2}$ if $z \in E_N$. Therefore, for each l , for $z \in E_N$, it holds that

$$1 = \sum_k (\mathbf{I} - |\mathbf{D}| - |\mathbf{Y}|)^{-1}_{l,k} \mathbf{b}_k > \frac{C}{2} \frac{(\text{Im}(z))^4}{(\eta^2 + |z|^2)^2} \sum_k (\mathbf{I} - |\mathbf{D}| - |\mathbf{Y}|)^{-1}_{l,k}$$

which implies that

$$\| (\mathbf{I} - (|\mathbf{D}| + |\mathbf{Y}|))^{-1} \|_\infty < \frac{2}{C} \frac{(\eta^2 + |z|^2)^2}{(\text{Im}(z))^4}$$

(9.21) follows immediately from (9.22). □

We now introduce $\omega(u_1, u_2, z)$ defined for $-(L - 1) \leq u_i \leq (L - 1)$ for $i = 1, 2$ by

$$\omega(u_1, u_2, z) = \frac{1}{ML} \text{Tr} \left(\mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^{u_1}) \mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^{u_2}) \right) \tag{9.23}$$

and prove the following result.

Proposition 9.2 $\mathbb{E}(\omega(u_1, u_2, z))$ can be expressed as

$$\mathbb{E}(\omega(u_1, u_2, z)) = \delta(u_1 + u_2 = 0) \bar{\omega}(u_1, z) + O\left(\frac{L}{MN}\right) \tag{9.24}$$

for each $z \in E_N$ where E_N is defined by (9.20), and where $\bar{\omega}(u_1, z)$ is defined by

$$\bar{\omega}(u_1, z) = \frac{(1 - |u_1|/L) t^2(z)}{1 - \sigma^4 c_N (z t(z) \tilde{t}(z))^2 (1 - |u_1|/L)(1 - |u_1|/N)}$$

Proof We use (9.16) for $r = 1$, $\mathbf{C}^1 = (\mathbf{I}_M \otimes \mathbf{J}_L^{u_1})$, $\mathbf{C}^2 = (\mathbf{I}_M \otimes \mathbf{J}_L^{u_2})$. Using that

$$\begin{aligned} \mathbb{E} \left(\frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{C}^1 (\mathbf{I}_M \otimes \mathbf{R}) \mathbf{C}^2 \right) \right) &= \frac{1}{ML} \text{Tr} \left((\mathbf{I}_M \otimes \mathbf{R}) \mathbf{C}^1 (\mathbf{I}_M \otimes \mathbf{R}) \mathbf{C}^2 \right) \\ &\quad + O \left(\frac{L}{MN} \right) \end{aligned}$$

we obtain that

$$\begin{aligned} \mathbb{E}(\omega(u_1, u_2)) &= \frac{1}{L} \text{Tr} (\mathbf{R} \mathbf{J}_L^{u_1} \mathbf{R} \mathbf{J}_L^{u_2}) \\ &\quad + \sigma^2 c_N \sum_{i=-(L-1)}^{L-1} \mathbb{E} \left(\frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^i \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_2}) \right) \right) \mathbb{E}(\omega(u_1, i)) \\ &\quad + O \left(\frac{L}{MN} \right) \end{aligned} \tag{9.25}$$

For each u_1 fixed, this equation can be interpreted as a linear system whose unknowns are the $(\mathbb{E}(\omega(u_1, u_2)))_{u_2=-(L-1), \dots, L-1}$. (4.16) implies that

$$\mathbb{E} \left(\frac{1}{ML} \text{Tr} (\mathbf{Q} \mathbf{W} \mathbf{J}_N^i \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_2})) \right) = \frac{\sigma^2}{L} \text{Tr} \mathbf{R} \mathcal{T}_{L,L} (\mathbf{H} \mathbf{J}_L^{*i} \mathbf{H}) \mathbf{R} \mathbf{J}_L^{u_2} + O \left(\frac{L}{MN} \right)$$

Moreover, we check that up to a $O \left(\frac{L}{MN} \right)$ term, matrices \mathbf{R} and \mathbf{H} can be replaced into the right-hand side of the above equation by $t(z) \mathbf{I}_L$ and $-z \tilde{t}(z) \mathbf{I}_L$, respectively. In other words,

$$\begin{aligned} \mathbb{E} \left(\frac{1}{ML} \text{Tr} (\mathbf{Q} \mathbf{W} \mathbf{J}_N^i \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_2})) \right) &= \sigma^2 (z t(z) \tilde{t}(z))^2 \frac{1}{L} \text{Tr} \left(\mathcal{T}_{L,L} (\mathbf{J}_L^{*i}) \mathbf{J}_L^{u_2} \right) \\ &\quad + O \left(\frac{L}{MN} \right) = \delta(i - u_2) \sigma^2 (z t(z) \tilde{t}(z))^2 (1 - |u_2|/L)(1 - |u_2|/N) + O \left(\frac{L}{MN} \right) \end{aligned}$$

We write $\mathbf{R} \mathcal{T}_{L,L} (\mathbf{H} \mathbf{J}_N^{*i} \mathbf{H}) \mathbf{R} \mathbf{J}_L^{u_2}$ as

$$\begin{aligned} \mathbf{R} \mathcal{T}_{L,L} (\mathbf{H} \mathbf{J}_N^{*i} \mathbf{H}) \mathbf{R} \mathbf{J}_L^{u_2} &= (\mathbf{R} - t \mathbf{I}) \mathcal{T}_{L,L} (\mathbf{H} \mathbf{J}_N^{*i} \mathbf{H}) \mathbf{R} \mathbf{J}_L^{u_2} \\ &\quad + t \mathcal{T}_{L,L} \left((\mathbf{H} + z \tilde{t} \mathbf{I}) \mathbf{J}_N^{*i} \mathbf{H} \right) \mathbf{R} \mathbf{J}_L^{u_2} - z t \tilde{t} \mathcal{T}_{L,L} \left(\mathbf{J}_N^{*i} (\mathbf{H} + z \tilde{t} \mathbf{I}) \right) \mathbf{R} \mathbf{J}_L^{u_2} \\ &\quad + t (z \tilde{t})^2 \mathcal{T}_{L,L} \left(\mathbf{J}_N^{*i} \right) (\mathbf{R} - t \mathbf{I}) \mathbf{J}_L^{u_2} + t^2 (z \tilde{t})^3 \mathcal{T}_{L,L} \left(\mathbf{J}_N^{*i} \right) \mathbf{J}_L^{u_2} \end{aligned}$$

The terms $\frac{1}{L} \text{Tr} ((\mathbf{R} - t \mathbf{I}) \mathcal{T}_{L,L} (\mathbf{H} \mathbf{J}_N^{*i} \mathbf{H}) \mathbf{R} \mathbf{J}_L^{u_2})$ and $\frac{1}{L} \text{Tr} (\mathcal{T}_{L,L} (\mathbf{J}_N^{*i}) (\mathbf{R} - t \mathbf{I}) \mathbf{J}_L^{u_2})$ are $\mathcal{O}(\frac{L}{MN})$ by Proposition 8.1. We just study the term $\frac{1}{L} \text{Tr} (t \mathcal{T}_{L,L} ((\mathbf{H} + z\tilde{\mathbf{I}}) \mathbf{J}_N^{*i} \mathbf{H}) \mathbf{R} \mathbf{J}_L^{u_2})$ and omit $\frac{1}{L} \text{Tr} (\mathcal{T}_{L,L} (\mathbf{J}_N^{*i} (\mathbf{H} + z\tilde{\mathbf{I}})) \mathbf{R} \mathbf{J}_L^{u_2})$ because it can be handled similarly. We express $\mathbf{H} + z\tilde{\mathbf{I}}$ as

$$\begin{aligned} \mathbf{H} + z\tilde{\mathbf{I}} &= \sigma^2 c_N z\tilde{\mathbf{I}} \mathbf{H} \mathcal{T}_{N,L}^{(M)} (\mathbb{E}(\mathbf{Q}) - t \mathbf{I}) \\ &= \sigma^2 c_N z\tilde{\mathbf{I}} \mathbf{H} \mathcal{T}_{N,L}^{(M)} (\mathbb{E}(\mathbf{Q}) - \mathbf{I}_M \otimes \mathbf{R}) + \sigma^2 c_N z\tilde{\mathbf{I}} \mathbf{H} \mathcal{T}_{N,L} (\mathbf{R} - t \mathbf{I}) \end{aligned}$$

Property (2.7) and Proposition 8.1 imply that $\frac{1}{L} \text{Tr} (t \mathcal{T}_{L,L} ((\mathbf{H} + z\tilde{\mathbf{I}}) \mathbf{J}_N^{*i} \mathbf{H}) \mathbf{R} \mathbf{J}_L^{u_2})$ is a $\mathcal{O}(\frac{L}{MN})$. We have thus shown that for $i, u_2 \in -(L - 1), \dots, L - 1$, then it holds that

$$\sigma^2 c_N \mathbb{E} \left(\frac{1}{ML} \text{Tr} (\mathbf{Q} \mathbf{W} \mathbf{J}_N^i \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_2})) \right) = \delta(i + u_2 = 0) d(i, z) + \mathcal{O} \left(\frac{L}{MN} \right) \tag{9.26}$$

Similarly, it holds that

$$\begin{aligned} \frac{1}{L} \text{Tr} (\mathbf{R} \mathbf{J}_L^{u_1} \mathbf{R} \mathbf{J}_L^{u_2}) &= t(z)^2 \frac{1}{L} \text{Tr} (\mathbf{J}_L^{u_1} \mathbf{J}_L^{u_2}) + \mathcal{O} \left(\frac{L}{MN} \right) \\ &= \delta(u_1 + u_2 = 0) (t(z))^2 (1 - |u_1|/L) + \mathcal{O} \left(\frac{L}{MN} \right) \end{aligned}$$

We denote by $\omega(u_1)$ the $(2L - 1)$ dimension vector $(\omega(u_1, u_2))_{u_2=-(L-1), \dots, L-1}$, and by $\bar{\gamma}(u_1)$ the vector such that

$$\bar{\gamma}(u_1)_{u_2} = \delta(u_1 + u_2 = 0) (t(z))^2 (1 - |u_1|/L)$$

The linear system (9.25) can be written as

$$\mathbb{E}(\omega(u_1)) = (\mathbf{D} + \mathbf{Y}) \mathbb{E}(\omega(u_1)) + \bar{\gamma}(u_1) + \epsilon$$

where the elements of matrix \mathbf{Y} and the components of vector ϵ are $\mathcal{O}(\frac{L}{MN})$ terms. Matrices \mathbf{D} and \mathbf{Y} verify the assumptions of Lemma 9.5. Therefore, it holds that

$$\mathbb{E}(\omega(u_1)) = (\mathbf{I} - \mathbf{D} - \mathbf{Y})^{-1} (\bar{\gamma}(u_1) + \epsilon)$$

when z belongs to a set E_N defined as in (9.20). Writing matrix $(\mathbf{I} - \mathbf{D} - \mathbf{Y})^{-1}$ as

$$(\mathbf{I} - \mathbf{D} - \mathbf{Y})^{-1} = (\mathbf{I} - \mathbf{D})^{-1} + (\mathbf{I} - \mathbf{D} - \mathbf{Y})^{-1} \mathbf{Y} (\mathbf{I} - \mathbf{D})^{-1}$$

we obtain that

$$\begin{aligned} \mathbb{E}(\omega(u_1)) &= (\mathbf{I} - \mathbf{D})^{-1} \bar{\gamma}(u_1) + (\mathbf{I} - \mathbf{D} - \mathbf{Y})^{-1} \mathbf{Y} (\mathbf{I} - \mathbf{D})^{-1} \bar{\gamma}(u_1) \\ &\quad + (\mathbf{I} - \mathbf{D} - \mathbf{Y})^{-1} \epsilon \end{aligned}$$

(9.21) implies that for each u_2 ,

$$\left((\mathbf{I} - \mathbf{D} - \mathbf{\Upsilon})^{-1} \boldsymbol{\epsilon} \right)_{u_2} = O\left(\frac{L}{MN}\right)$$

Moreover, as vector $\bar{\boldsymbol{\gamma}}(u_1)$ has only 1 nonzero component, it is clear that each component of vector $\mathbf{\Upsilon} (\mathbf{I} - \mathbf{D})^{-1} \bar{\boldsymbol{\gamma}}(u_1)$ is a $O\left(\frac{L}{MN}\right)$ term. Hence, (9.21) leads to

$$\left((\mathbf{I} - \mathbf{D} - \mathbf{\Upsilon})^{-1} \mathbf{\Upsilon} (\mathbf{I} - \mathbf{D})^{-1} \bar{\boldsymbol{\gamma}}(u_1) \right)_{u_2} = O\left(\frac{L}{MN}\right)$$

This establishes (9.24). We notice that Lemma 9.5 plays an important role in the above calculations. The control of $\|(\mathbf{I} - (|\mathbf{D}| + |\mathbf{\Upsilon}|))^{-1}\|_\infty$ allows in particular to show that $\mathbb{E}(\omega(u_1, u_2)) = O\left(\frac{L}{MN}\right)$ if $u_1 + u_2 \neq 0$, instead of $O\left(\frac{L^2}{MN}\right)$ in the absence of control on $\|(\mathbf{I} - (|\mathbf{D}| + |\mathbf{\Upsilon}|))^{-1}\|_\infty$. As Lemma 9.5 is a consequence of $\frac{L^2}{MN} \rightarrow 0$, this discussion confirms the importance of condition (9.1) and strongly suggests that it is a necessary condition to obtain positive results. \square

It is also necessary to evaluate $\mathbb{E}(\omega(u_1, u_2, u_3, z))$ where $\omega(u_1, u_2, u_3, z)$ is defined by

$$\omega(u_1, u_2, u_3, z) = \mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^{u_1}) \mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^{u_2}) \mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^{u_3}) \right) \right] \tag{9.27}$$

It holds that for $z \in E_N$ defined as in (9.20)

Proposition 9.3 $\mathbb{E}(\omega(u_1, u_2, u_3, z))$ can be expressed as

$$\mathbb{E}(\omega(u_1, u_2, u_3, z)) = \delta(u_1 + u_2 + u_3 = 0) \bar{\omega}(u_1, u_2, z) + O\left(\frac{L}{MN}\right) \tag{9.28}$$

where $\bar{\omega}(u_1, u_2, z)$ is given by

$${}_{(z)} \frac{\frac{1}{L} \text{Tr} \left(\mathbf{J}_L^{u_2} \mathbf{J}_L^{u_1} \mathbf{J}_L^{*(u_1+u_2)} \right) + \sigma^6 c_N^2 (z(z) \bar{z}(z))^3 (1 - |u_1|/L)(1 - |u_2|/L)(1 - |u_1 + u_2|/L) + \frac{1}{N} \text{Tr} \left(\mathbf{J}_N^{u_1} \mathbf{J}_N^{u_2} \mathbf{J}_N^{*(u_1+u_2)} \right)}{(1 - d(u_1, z))(1 - d(u_2, z))(1 - d(u_1 + u_2, z))} \tag{9.29}$$

Proof The proof is somewhat similar to the proof of Proposition 9.2, but it needs rather tedious calculations. We just provide the main steps and omit the straightforward details. We use again (9.16), but for $r = 2$, and $\mathbf{C}^s = (\mathbf{I}_M \otimes \mathbf{J}_L^{u_s})$ for $s = 1, 2, 3$. We obtain immediately that

$$\begin{aligned} \mathbb{E}(\omega(u_1, u_2, u_3)) &= \frac{1}{ML} \mathbb{E} \left[\text{Tr} \left(\mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^{u_1}) \mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^{u_2}) \mathbf{R} \mathbf{J}_L^{u_3} \right) \right] \\ &+ \sigma^2 c_N \sum_{i=-(L-1)}^{L-1} \mathbb{E} \left[\frac{1}{ML} \left(\text{Tr}(\mathbf{Q} \mathbf{W} \mathbf{J}_N^i \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_3})) \right) \right] \mathbb{E}(\omega(u_1, u_2, i)) \end{aligned}$$

$$\begin{aligned}
 & + \sigma^2 c_N \sum_{i=-(L-1)}^{L-1} \frac{1}{ML} \mathbb{E} \left[\text{Tr} \left(\mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^{u_1}) \mathbf{Q} \mathbf{W} \mathbf{J}_N^i \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_3}) \right) \right] \\
 & \times \mathbb{E}(\omega(u_2, i)) + O\left(\frac{L}{MN}\right) \tag{9.30}
 \end{aligned}$$

(9.30) can still be interpreted as a linear system whose unknown are the $(\mathbb{E}(\omega(u_1, u_2, u_3)))_{u_3 \in \{-(L-1), \dots, L-1\}}$. The matrix governing the system is the same matrix $\mathbf{D} + \mathbf{Y}$ as in the proof of Proposition 9.2 (but for a different matrix \mathbf{Y}). In order to use the same arguments, it is sufficient to establish that

$$\begin{aligned}
 \frac{1}{ML} \mathbb{E} \left[\text{Tr} \left(\mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^{u_1}) \mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^{u_2} \mathbf{R} \mathbf{J}_L^{u_3}) \right) \right] & = C(u_1, u_2, z) \delta(u_1 + u_2 + u_3 = 0) \\
 & + O\left(\frac{L}{MN}\right) \tag{9.31}
 \end{aligned}$$

and

$$\begin{aligned}
 & \sum_{i=-(L-1)}^{L-1} \frac{1}{ML} \mathbb{E} \left[\text{Tr} \left(\mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^{u_1}) \mathbf{Q} \mathbf{W} \mathbf{J}_N^i \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_3}) \right) \right] \mathbb{E}(\omega(u_2, i)) \\
 & = C(u_1, u_2, z) \delta(u_1 + u_2 + u_3 = 0) + O\left(\frac{L}{MN}\right) \tag{9.32}
 \end{aligned}$$

To check (9.31), we use (9.16) for $r = 1$, $\mathbf{C}^1 = \mathbf{I}_M \otimes \mathbf{J}_L^{u_1}$, $\mathbf{C}^2 = \mathbf{I}_M \otimes \mathbf{J}_L^{u_2} \mathbf{R} \mathbf{J}_L^{u_3}$. This leads to

$$\begin{aligned}
 \frac{1}{ML} \mathbb{E} \left[\text{Tr} \left(\mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^{u_1}) \mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^{u_2} \mathbf{R} \mathbf{J}_L^{u_3}) \right) \right] & = \frac{1}{L} \text{Tr} \left(\mathbf{R} \mathbf{J}_L^{u_1} \mathbf{R} \mathbf{J}_L^{u_2} \mathbf{R} \mathbf{J}_L^{u_3} \right) \\
 & + \sigma^2 c_N \sum_{i=-(L-1)}^{L-1} \mathbb{E}(\omega(u_1, i)) \mathbb{E} \left[\frac{1}{ML} \left(\text{Tr}(\mathbf{Q} \mathbf{W} \mathbf{J}_N^i \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R}^2 \mathbf{J}_L^{u_2} \mathbf{R} \mathbf{J}_L^{u_3})) \right) \right] \\
 & + O\left(\frac{L}{MN}\right)
 \end{aligned}$$

Up to a $O\left(\frac{L}{MN}\right)$ term, it is possible to replace $\mathbf{R}(z)$ by $t(z)\mathbf{I}$ into the first term of the right-hand side of the above equation. This leads to

$$\begin{aligned}
 \frac{1}{L} \text{Tr} \left(\mathbf{R} \mathbf{J}_L^{u_1} \mathbf{R} \mathbf{J}_L^{u_2} \mathbf{R} \mathbf{J}_L^{u_3} \right) & = (t(z))^3 \frac{1}{L} \text{Tr} \mathbf{J}_L^{u_1} \mathbf{J}_L^{u_2} \mathbf{J}_L^{u_3} + O\left(\frac{L}{MN}\right) \\
 & = (t(z))^3 \frac{1}{L} \text{Tr} \mathbf{J}_L^{u_1} \mathbf{J}_L^{u_2} \mathbf{J}_L^{*(u_1+u_2)} \delta(u_1 + u_2 + u_3 = 0) \\
 & + O\left(\frac{L}{MN}\right)
 \end{aligned}$$

Similarly, it is easy to check that

$$\mathbb{E} \left[\frac{1}{ML} \left(\text{Tr}(\mathbf{QWJ}_N^i \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R}^2 \mathbf{J}_L^{u_2} \mathbf{R} \mathbf{J}_L^{u_3})) \right) \right] = C(u_2, u_3, z) \delta(i = u_2 + u_3) + O\left(\frac{L}{MN}\right)$$

As $\mathbb{E}(\omega(u_1, i, z)) = \bar{\omega}(u_1, z) \delta(i + u_1 = 0) + O\left(\frac{L}{MN}\right)$, we get immediately that if $u_1 + u_2 + u_3 \neq 0$, then,

$$\begin{aligned} & \sigma^2 c_N \sum_{i=-(L-1)}^{L-1} \mathbb{E}(\omega(u_1, i)) \mathbb{E} \left[\frac{1}{ML} \left(\text{Tr}(\mathbf{QWJ}_N^i \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R}^2 \mathbf{J}_L^{u_2} \mathbf{R} \mathbf{J}_L^{u_3})) \right) \right] \\ & = O\left(\frac{L}{MN}\right) + L O\left(\left(\frac{L}{MN}\right)^2\right) \end{aligned}$$

(9.31) follows from the observation that as $\frac{L^2}{MN} \rightarrow 0$, then $L \left(\frac{L}{MN}\right)^2 = \frac{L^2}{MN} \frac{L}{MN} = o\left(\frac{L}{MN}\right)$.

Finally, (9.32) holds because, using (9.17) for $r = 1$, $\mathbf{C}^1 = \mathbf{I}_M \otimes \mathbf{J}_L^{u_1}$, $\mathbf{G} = \mathbf{J}_N^i \mathbf{H}^T$, $\mathbf{A} = \mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_3}$, it can be shown that

$$\begin{aligned} & \frac{1}{ML} \mathbb{E} \left[\text{Tr} \left(\mathbf{Q} (\mathbf{I}_M \otimes \mathbf{J}_L^{u_1}) \mathbf{QWJ}_N^i \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_3}) \right) \right] \\ & = C(u_1, u_3, z) \delta(i = u_1 + u_3) + O\left(\frac{L}{MN}\right) \end{aligned}$$

As $\mathbb{E}(\omega_2(i, u_2, z)) = \delta(i + u_2 = 0) \bar{\omega}(u_2, z) + O\left(\frac{L}{MN}\right)$, $\frac{L^2}{MN} \rightarrow 0$ implies (9.32). \square

The calculation of $\bar{\omega}(u_1, u_2, z)$ is omitted.

We now define and evaluate the following useful terms. If $p \geq 1$ and $q \geq 1$, for each integers $i, u_1, u_2, l_1, \dots, l_p, k_1, \dots, k_q$ belonging to $\{-(L-1), \dots, L-1\}$, we define

$$\beta_{p,q}(i, u_1, l_1, \dots, l_p, k_1, \dots, k_q, u_2, z)$$

as

$$\begin{aligned} & \frac{1}{ML} \text{Tr} \left(\mathbf{Q} (\mathbf{I}_M \otimes \mathbf{J}_L^i) \mathbf{Q} (\mathbf{I}_M \otimes \mathbf{J}_L^{u_1}) \mathbf{QWJ}_N^i \mathbf{H}^T \prod_{j=1}^p (\mathbf{J}_N^{l_j} \mathbf{H}^T) \mathbf{W}^* \right. \\ & \left. \times \left(\mathbf{I}_M \otimes \prod_{n=1}^q (\mathbf{R} \mathcal{T}_{L,L} (\mathbf{H} \mathbf{J}_N^{*k_n} \mathbf{H})) \mathbf{R} \mathbf{J}_L^{u_2} \right) \right) \end{aligned} \tag{9.33}$$

We also define $\beta_{p,0}(i, u_1, l_1, \dots, l_p, u_2, z)$ as

$$\frac{1}{ML} \text{Tr} \left(\mathbf{Q} \left(\mathbf{I}_M \otimes \mathbf{J}_L^i \right) \mathbf{Q} \left(\mathbf{I}_M \otimes \mathbf{J}_L^{u_1} \right) \mathbf{Q} \mathbf{W} \mathbf{J}_N^i \mathbf{H}^T \prod_{j=1}^p \left(\mathbf{J}_N^{l_j} \mathbf{H}^T \right) \mathbf{W}^* \left(\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_2} \right) \right) \tag{9.34}$$

and $\beta_{0,q}(i, u_1, k_1, \dots, k_q, u_2, z)$ is defined similarly. We finally denote by $\beta(i, u_1, u_2, z)$ the term $\beta_{0,0}(i, u_1, u_2, z)$, i.e.

$$\beta(i, u_1, u_2, z) = \frac{1}{ML} \text{Tr} \left(\mathbf{Q} \left(\mathbf{I}_M \otimes \mathbf{J}_L^i \right) \mathbf{Q} \left(\mathbf{I}_M \otimes \mathbf{J}_L^{u_1} \right) \mathbf{Q} \mathbf{W} \mathbf{J}_N^i \mathbf{H}^T \mathbf{W}^* \left(\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_2} \right) \right) \tag{9.35}$$

Proposition 9.4 For $p \geq 0$ and $q \geq 0$, it holds that

$$\begin{aligned} &\mathbb{E} \left(\beta_{p,q}(i, u_1, l_1, \dots, l_p, k_1, \dots, k_q, u_2, z) \right) \\ &= \delta \left(u_1 + u_2 = \sum_j l_j + \sum_n k_n \right) \bar{\beta}_{p,q}(i, u_1, l_1, \dots, l_p, k_1, \dots, k_q, z) \\ &\quad + O \left(\frac{L}{MN} \right) \end{aligned} \tag{9.36}$$

where for each $i, u_1, l_1, \dots, l_p, k_1, \dots, k_q$, function $z \rightarrow \bar{\beta}_{p,q}(i, u_1, l_1, \dots, l_p, k_1, \dots, k_q, z)$ is the Stieltjes transform of a distribution D whose support is included into \mathcal{S}_N and such that $\langle D, \mathbb{1} \rangle = 0$. Moreover, if $c_N > 1$, for each i, l_1 , function $z \rightarrow \bar{\beta}_{1,0}(i, l_1, l_1, z)$ is analytic in a neighbourhood of 0, while 0 is pole of multiplicity 1 of functions $z \rightarrow z \bar{\beta}(i, l_1, z)$ and $z \rightarrow \bar{\beta}_{0,1}(i, l_1, z)$ where we denote $\bar{\beta}_{0,0}(i, l_1, z)$ by $\bar{\beta}(i, l_1, z)$ in order to simplify the notations. Finally, function $s(i, l_1, z)$ defined by

$$\begin{aligned} s(i, l_1, z) &= -\sigma^2 \bar{\beta}_{1,0}(i, l_1, l_1, z) + \sigma^2 \bar{\beta}_{0,1}(i, l_1, l_1, z) \\ &\quad + \sigma^6 c_N \left(z t(z) \tilde{t}(z) \right)^2 z \tilde{t}(z) \left(1 + \sigma^2 z t(z) \tilde{t}(z) (1 - |l_1|/L) (1 - |l_1|/N) \right) \\ &\quad \times \left(\frac{1 - |l_1|/N}{1 - d(l_1, z)} \bar{\beta}(i, l_1, z) \right) \end{aligned} \tag{9.37}$$

is the Stieltjes transform of a distribution D whose support is included in $\mathcal{S}_N^{(0)}$ and verifying $\langle D, \mathbb{1} \rangle = 0$.

Proof In order to simplify the notations, we just establish the first part of the proposition when $p = q = 0$, i.e. for the term $\beta(i, u_1, u_2, z) = \beta_{0,0}(i, u_1, u_2, z)$. Then, we check that

$$\mathbb{E} \left(\beta(i, u_1, u_2, z) \right) = \delta(u_1 + u_2 = 0) \bar{\beta}(i, u_1, z) + O \left(\frac{L}{MN} \right) \tag{9.38}$$

where $\bar{\beta}(i, u, z)$ is given by

$$\bar{\beta}(i, u, z) = \sum_{j=1}^5 \bar{\beta}_j(i, u, z)$$

with

$$\begin{aligned} \bar{\beta}_1(i, u, z) &= \frac{\sigma^2 t(z)^4 (z \tilde{t}(z))^2 (1 - |i|/N) \frac{1}{L} \text{Tr}(\mathbf{J}_L^i \mathbf{J}_L^u \mathbf{J}_L^{*i} \mathbf{J}_L^{*u})}{1 - d(i, z)}, \\ \bar{\beta}_2(i, u, z) &= \sigma^6 c_N t(z)^3 (z \tilde{t}(z))^4 \bar{\omega}(i, u) (1 - |i+u|/N) (1 - |i|/N) \frac{1}{L} \text{Tr}(\mathbf{J}_L^{u+i} \mathbf{J}_L^{*i} \mathbf{J}_L^{*u}), \\ \bar{\beta}_3(i, u, z) &= \sigma^4 c_N t(z)^2 (z \tilde{t}(z))^3 \bar{\omega}(i, u) \mathbb{1}_{|i+u| \leq L-1} (1 - |u_1|/L) \frac{1}{N} \text{Tr}(\mathbf{J}_N^{u+i} \mathbf{J}_N^{*u} \mathbf{J}_N^{*i}), \\ \bar{\beta}_4(i, u, z) &= \sigma^6 c_N t(z)^4 (z \tilde{t}(z))^4 \bar{\omega}(u) (1 - |u|/N) (1 - |i|/N) \frac{1}{L} \text{Tr}(\mathbf{J}_L^i \mathbf{J}_L^u \mathbf{J}_L^{*i} \mathbf{J}_L^{*u}) \\ &\quad + \sigma^{10} c_N^2 t(z)^4 (z \tilde{t}(z))^6 \bar{\omega}(u) \bar{\omega}(i) (1 - |i|/N)^2 (1 - |u|/N) \frac{1}{L} \text{Tr}(\mathbf{J}_L^i \mathbf{J}_L^u \mathbf{J}_L^{*i} \mathbf{J}_L^{*u}) \\ &\quad - \sigma^8 c_N^2 t(z)^3 (z \tilde{t}(z))^5 \bar{\omega}(u) \bar{\omega}(i) (1 - |i|/N) \frac{1}{N} \text{Tr}(\mathbf{J}_N^u \mathbf{J}_N^i \mathbf{J}_N^{*(i+u)}) \frac{1}{L} \text{Tr}(\mathbf{J}_L^{u+i} \mathbf{J}_L^{*i} \mathbf{J}_L^{*u}), \\ \bar{\beta}_5(i, u, z) &= \sigma^4 c_N t(z)^3 (z \tilde{t}(z))^3 \bar{\omega}(u) \frac{1}{N} \text{Tr}(\mathbf{J}_N^{*i} \mathbf{J}_N^u \mathbf{J}_N^{i-u}) \frac{1}{L} \text{Tr}(\mathbf{J}_L^u \mathbf{J}_L^{u-i} \mathbf{J}_L^{*u}) \\ &\quad + \sigma^8 c_N^2 t(z)^3 (z \tilde{t}(z))^5 \bar{\omega}(u) \bar{\omega}(i) (1 - |i|/N) \frac{1}{N} \text{Tr}(\mathbf{J}_N^{*i} \mathbf{J}_N^u \mathbf{J}_N^{i-u}) \frac{1}{L} \text{Tr}(\mathbf{J}_L^u \mathbf{J}_L^{u-i} \mathbf{J}_L^{*u}) \\ &\quad + \sigma^6 c_N^2 t(z)^2 (z \tilde{t}(z))^4 \bar{\omega}(i) \bar{\omega}(u) (1 - |u|/L) \frac{1}{N} \text{Tr}(\mathbf{J}_N^{*i} \mathbf{J}_N^u \mathbf{J}_N^i \mathbf{J}_N^{*u}) \end{aligned}$$

The proof is based on (9.17) for $r = 2$, with $\mathbf{C}^1 = \mathbf{I}_M \otimes \mathbf{J}_L^i$, $\mathbf{C}^2 = \mathbf{I}_M \otimes \mathbf{J}_L^{u_1}$, $\mathbf{G} = \mathbf{J}_N^i \mathbf{H}^T$, $\mathbf{A} = \mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_2}$. It holds that

$$\begin{aligned} \mathbb{E}(\beta(i, u_1, u_2)) &= \frac{1}{ML} \mathbb{E} \left[\text{Tr} \left(\mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^i) \mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^{u_1}) \right. \right. \\ &\quad \left. \left. \times (\mathbf{I}_M \otimes \sigma^2 \mathbf{R} \mathcal{T}_{L,L}(\mathbf{H} \mathbf{J}_N^{*i} \mathbf{H}) \mathbf{R} \mathbf{J}_L^{u_2}) \right) \right] + \sigma^2 c_N \sum_{j=-(L-1)}^{L-1} \mathbb{E}(\omega(i, u_1, j)) \\ &\quad \times \frac{1}{ML} \mathbb{E} \left[\text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^j \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \sigma^2 \mathbf{R} \mathcal{T}_{L,L}(\mathbf{H} \mathbf{J}_N^{*i} \mathbf{H}) \mathbf{R} \mathbf{J}_L^{u_2}) \right) \right] \\ &\quad + \sigma^2 c_N \sum_{j=-(L-1)}^{L-1} \mathbb{E}(\omega(u_1, j)) \\ &\quad \times \frac{1}{ML} \mathbb{E} \left[\text{Tr} \left(\mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^i) \mathbf{Q} \mathbf{W} \mathbf{J}_N^j \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \sigma^2 \mathbf{R} \mathcal{T}_{L,L}(\mathbf{H} \mathbf{J}_N^{*i} \mathbf{H}) \mathbf{R} \mathbf{J}_L^{u_2}) \right) \right] \\ &\quad - \sigma^2 c_N \sum_{j=-(L-1)}^{L-1} \mathbb{E}(\omega(i, u_1, j)) \frac{1}{ML} \mathbb{E} \left[\text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^j \mathbf{H}^T \mathbf{J}_N^i \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_2}) \right) \right] \end{aligned}$$

$$\begin{aligned}
 & -\sigma^2 c_N \sum_{j=-(L-1)}^{L-1} \mathbb{E}(\omega(u_1, j)) \\
 & \times \frac{1}{ML} \mathbb{E} \left[\text{Tr} \left(\mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^i) \mathbf{Q} \mathbf{W} \mathbf{J}_N^j \mathbf{H}^T \mathbf{J}_N^i \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_2}) \right) \right]
 \end{aligned}$$

Using (9.16), it is easy to check that

$$\begin{aligned}
 & \frac{1}{ML} \mathbb{E} \left[\text{Tr} \left(\mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^i) \mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^{u_1}) (\mathbf{I}_M \otimes \sigma^2 \mathbf{R} \mathcal{T}_{L,L}(\mathbf{H} \mathbf{J}_N^{*i} \mathbf{H}) \mathbf{R} \mathbf{J}_L^{u_2}) \right) \right] \\
 & = \delta(u_1 + u_2 = 0) C(i, u_1, z) + O\left(\frac{L}{MN}\right), \\
 & \frac{1}{ML} \mathbb{E} \left[\text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^j \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \sigma^2 \mathbf{R} \mathcal{T}_{L,L}(\mathbf{H} \mathbf{J}_N^{*i} \mathbf{H}) \mathbf{R} \mathbf{J}_L^{u_2}) \right) \right] \\
 & = \delta(j = u_2 - i) C(i, u_2, z) + O\left(\frac{L}{MN}\right), \\
 & \frac{1}{ML} \mathbb{E} \left[\text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^j \mathbf{H}^T \mathbf{J}_N^i \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_2}) \right) \right] \\
 & = \delta(j = u_2 - i) C(i, u_2, z) + O\left(\frac{L}{MN}\right), \\
 & \frac{1}{ML} \mathbb{E} \left[\text{Tr} \left(\mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^i) \mathbf{Q} \mathbf{W} \mathbf{J}_N^j \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \sigma^2 \mathbf{R} \mathcal{T}_{L,L}(\mathbf{H} \mathbf{J}_N^{*i} \mathbf{H}) \mathbf{R} \mathbf{J}_L^{u_2}) \right) \right] \\
 & = \delta(j = u_2) C(i, u_2, z) + O\left(\frac{L}{MN}\right), \\
 & \frac{1}{ML} \mathbb{E} \left[\text{Tr} \left(\mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^i) \mathbf{Q} \mathbf{W} \mathbf{J}_N^j \mathbf{H}^T \mathbf{J}_N^i \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_2}) \right) \right] \\
 & = \delta(j = u_2) C(i, u_2, z) + O\left(\frac{L}{MN}\right)
 \end{aligned}$$

Proposition 9.2 and Proposition 9.3 immediately imply that $\mathbb{E}(\beta(i, u_1, u_2))$ can be written as (9.38). We omit the proof of the expression of $\bar{\beta}(i, u, z)$. Moreover, Lemma 9.2 implies that function $z \rightarrow \bar{\beta}(i, u, z)$ is the Stieltjes transform of a distribution D whose support is included in \mathcal{S}_N and which verifies $\langle D, 1 \rangle = 0$. \square

We now establish the second part of the proposition and assume that $c_N > 1$. In this case, 0 is pole of multiplicity 1 of $t(z)$ and $\tilde{t}(z)$ is analytic at 0. It is easy to check that for each $j = 1, \dots, 5$, 0 is pole with multiplicity 1 of function $z \rightarrow z \bar{\beta}_j(i, l_1, z)$ and thus of function $z \rightarrow z \bar{\beta}(i, l_1, z)$. As for function $z \rightarrow \bar{\beta}_{0,1}(i, l_1, l_1, z)$, it can be shown that

$$\bar{\beta}_{0,1}(i, l_1, l_1, z) = \sigma^2 (1 - |l_1|/N) t(z) (z \tilde{t}(z))^2 \bar{\beta}(i, l_1, z) \tag{9.39}$$

from which we deduce immediately that 0 is pole with multiplicity 1 of $\bar{\beta}_{0,1}(i, l_1, l_1, z)$. The analytic expression of $\bar{\beta}_{1,0}(i, l_1, l_1, z)$ (not provided) allows us to conclude

immediately that 0 may be pole with multiplicity 1, but it can be checked that the corresponding residue vanishes. Therefore, function $z \rightarrow \bar{\beta}_{1,0}(i, l_1, l_1, z)$ appears to be analytic in a neighbourhood of 0 and thus coincides with the Stieltjes transform of a distribution whose support is included into $\mathcal{S}_N^{(0)}$. In order to complete the proof of the proposition, it remains to check that function $z \rightarrow s(i, l_1, z)$ is analytic in a neighbourhood of 0. As 0 is pole of $z\bar{\beta}(i, l_1, z)$ and $\bar{\beta}_{0,1}(i, l_1, l_1, z)$ with multiplicity 1, it is sufficient to verify that

$$\lim_{z \rightarrow 0} z \left[\bar{\beta}_{0,1}(i, l_1, l_1, z) + \sigma^4 c_N (zt(z)\tilde{t}(z))^2 z\tilde{t}(z) \times \left(1 + \sigma^2 zt(z)\tilde{t}(z)(1 - |l_1|/L)(1 - |l_1|/N) \right) \left(\frac{1 - |l_1|/N}{1 - d(l_1, z)} \bar{\beta}(i, l_1, z) \right) \right] = 0$$

This property follows immediately from (9.39).

9.2.2 Evaluation of $\kappa^{(2)}(l_1, l_2)$

The treatment of the terms $\kappa^{(2)}(l_1, l_2)$ appears to be difficult and also needs a sharp evaluation for each r of the term of $\kappa^{(r)}(u_1, \dots, u_r)$ defined for $u_1, \dots, u_r \in \{-(L - 1), \dots, L - 1\}$ by

$$\kappa^{(r)}(u_1, \dots, u_r) = \mathbb{E} \left(\prod_{s=1}^r \tau^{(M)}(\mathbf{Q}^\circ)(u_s) \right) \tag{9.40}$$

Lemma 9.4 and the Hölder inequality immediately lead to $\kappa^{(r)}(u_1, \dots, u_r) = \mathcal{O}(\frac{1}{(MN)^{r/2}})$, but this evaluation is not optimal and has to be refined, in particular if $r = 2$. More precisely, the following result holds.

Proposition 9.5 *If z belongs to a set E_N defined as in (9.20), then, for $r = 2$, it holds that*

$$\kappa^{(2)}(u_1, u_2) = \frac{1}{MN} C(z, u_1) \delta(u_1 + u_2 = 0) + \mathcal{O} \left(\frac{L}{(MN)^2} \right) \tag{9.41}$$

More generally, if $r \geq 2$, and if (u_1, u_2, \dots, u_r) are integers such that $-(L - 1) \leq u_i \leq (L - 1)$ for $i = 1, \dots, r$ for which $u_k + u_l \neq 0$ for each $k, l, k \neq l$, then it holds that

$$\kappa^{(r)}(u_1, \dots, u_r) = \frac{1}{\sqrt{MN}} \mathcal{O} \left(\frac{1}{(MN)^{r/2}} \right) \tag{9.42}$$

The proof of this result is quite intricate. The goal of paragraph 9.2.2 is to establish Proposition 9.5.

In order to evaluate $\kappa^{(r)}(u_1, \dots, u_r)$, we state the following result. It can be proved by calculating, for each integers $(l_1, l'_1, n_1, n'_1, \dots, l_r, l'_r, n_r, n'_r)$, matrix

$$\mathbb{E} \left[\prod_{s=1}^r (\mathbf{Q}^\circ)_{l_s, l'_s}^{n_s, n'_s} \mathbf{Q} \mathbf{W} \mathbf{G} \mathbf{W}^* \right]$$

by the integration by parts formula. This calculation is provided in [23].

Proposition 9.6 We consider integers $(u_1, u_2, \dots, u_r), (v_1, v_2, \dots, v_r)$ such that $-(L - 1) \leq u_i \leq (L - 1), -(L - 1) \leq v_i \leq (L - 1)$ for $i = 1, \dots, r$. Then, it holds that

$$\begin{aligned}
 \mathbb{E} \left[\prod_{s=1}^r \tau^{(M)}(\mathbf{Q}^\circ)(u_s) \right] &= -\mathbb{E} \left[\prod_{s=1}^{r-1} \tau^{(M)}(\mathbf{Q}^\circ)(u_s) \right] \frac{1}{ML} \text{Tr} \left(\mathbf{A}(\mathbf{I}_M \otimes \mathbf{J}_L^{u_r}) \right) \\
 &+ \sigma^2 c_N \sum_{l_1=-(L-1)}^{L-1} \mathbb{E} \left(\prod_{s=1}^{r-1} \tau^{(M)}(\mathbf{Q}^\circ)(u_s) \tau^{(M)}(\mathbf{Q}^\circ)(l_1) \right) \\
 &\times \mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_r}) \right) \right] \\
 &+ \sigma^2 c_N \sum_{l_1=-(L-1)}^{L-1} \mathbb{E} \left(\prod_{s=1}^{r-1} \tau^{(M)}(\mathbf{Q}^\circ)(u_s) \tau^{(M)}(\mathbf{Q}^\circ)(l_1) \right. \\
 &\times \left. \left[\frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_r}) \right) \right]^\circ \right) \\
 &+ \frac{\sigma^2}{MLN} \sum_{s=1}^{r-1} \sum_{i=-(L-1)}^{L-1} \mathbb{E} \left[\prod_{t \neq s, r} \tau^{(M)}(\mathbf{Q}^\circ)(u_t) \right] \mathbb{E} \left(\beta(i, u_s, u_r) \right) \\
 &+ \frac{\sigma^2}{MLN} \sum_{s=1}^{r-1} \sum_{i=-(L-1)}^{L-1} \mathbb{E} \left[\prod_{t \neq s, r} \tau^{(M)}(\mathbf{Q}^\circ)(u_t) \beta(i, u_s, u_r)^{(0)} \right] \tag{9.43}
 \end{aligned}$$

and that

$$\begin{aligned}
 \mathbb{E} \left[\prod_{s=1}^r \tau^{(M)}(\mathbf{Q}^\circ)(v_s) \left(\frac{1}{ML} \text{Tr}(\mathbf{Q} \mathbf{W} \mathbf{G} \mathbf{W}^* \mathbf{A}) \right)^\circ \right] &= \kappa^{(r)}(v_1, \dots, v_r) \in (\mathbf{G}, \mathbf{A}) \\
 &+ \sigma^2 c_N \sum_{l_2=-(L-1)}^{L-1} \mathbb{E} \left[\prod_{s=1}^r \tau^{(M)}(\mathbf{Q}^\circ)(v_s) \tau^{(M)}(\mathbf{Q}^\circ)(l_2) \right. \\
 &\times \left. \frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_2} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \sigma^2 \mathbf{R} \mathcal{T}_{L,L}(\mathbf{G}^T \mathbf{H})) \mathbf{A} \right) \right] \\
 &- \sigma^2 c_N \sum_{l_2=-(L-1)}^{L-1} \mathbb{E} \left[\prod_{s=1}^r \tau^{(M)}(\mathbf{Q}^\circ)(v_s) \tau^{(M)}(\mathbf{Q}^\circ)(l_2) \right. \\
 &\times \left. \frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_2} \mathbf{H}^T \mathbf{G} \mathbf{W}^* \mathbf{A}^* \right) \right] \\
 &+ \frac{\sigma^2}{MLN} \sum_{s < r, |i| \leq L-1} \mathbb{E} \left[\prod_{t \neq s} \tau^{(M)}(\mathbf{Q}^\circ)(v_t) \right. \\
 &\times \left. \frac{1}{ML} \text{Tr} \left(\mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^i) \mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^{v_s}) \mathbf{Q} \mathbf{W} \mathbf{J}_N^i \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \sigma^2 \mathbf{R} \mathcal{T}_{L,L}(\mathbf{G}^T \mathbf{H})) \mathbf{A} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{\sigma^2}{MLN} \sum_{s \leq r, |i| \leq L-1} \mathbb{E} \left[\Pi_{t \neq s} \tau^{(M)}(\mathbf{Q}^\circ)(v_t) \right. \\
 & \left. \times \frac{1}{ML} \text{Tr} \left(\mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^i) \mathbf{Q}(\mathbf{I}_M \otimes \mathbf{J}_L^{\mathcal{U}_s}) \mathbf{Q} \mathbf{W} \mathbf{J}_N^i \mathbf{H}^T \mathbf{G} \mathbf{W}^* \mathbf{A} \right) \right] \tag{9.44}
 \end{aligned}$$

where we recall that $\beta(i, u_s, u_r)$ is defined by (9.33), and where $\epsilon(\mathbf{G}, \mathbf{A})$ is defined by

$$\begin{aligned}
 \epsilon(\mathbf{G}, \mathbf{A}) &= \sigma^2 c_N \mathbb{E} \left(\frac{1}{ML} \left(\mathbf{Q} \mathbf{W} \mathbf{T}_{N,L}^{(M)}(\mathbf{Q}^\circ)^T \mathbf{H}^T (\mathbf{G} \mathbf{W}^* \mathbf{A} - \mathbf{W}^* \right. \right. \\
 & \left. \left. \times (\mathbf{I}_M \otimes \sigma^2 \mathbf{R} \mathbf{T}_{L,L}(\mathbf{G}^T \mathbf{H})) \mathbf{A} \right) \right)
 \end{aligned}$$

In order to evaluate $\kappa^{(r)}(u_1, \dots, u_{r-1}, u_r)$, we interpret (9.43) as a linear system whose unknowns are the $(\kappa^{(r)}(u_1, \dots, u_{r-1}, u_r))_{u_r = -(L-1), \dots, L-1}$, the integers $(u_s)_{s=1, \dots, r-1}$ being considered as fixed.

Structure of the linear system We now precise the structure of this linear system. We denote by $\kappa^{(r)} = (\kappa^{(r)}(u_1, \dots, u_{r-1}, u_r))_{u_r = -(L-1), \dots, L-1}$ the corresponding $2L - 1$ -dimensional vector. We remark that the second term of the right-hand side of (9.43) coincides with component u_r of the action of vector $\kappa^{(r)}$ on the matrix whose entry (u_r, l_1) is

$$\sigma^2 c_N \mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_r}) \right) \right]$$

This matrix appears to be close from a diagonal matrix because

$$\begin{aligned}
 \sigma^2 c_N \mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_r}) \right) \right] &= \delta(l_1 + u_r = 0) d(u_r, z) \\
 &+ O\left(\frac{L}{MN}\right)
 \end{aligned}$$

(see (9.26)). We now study the fourth and the fifth term of the right-hand side of (9.43). We introduce y_{1,u_r} and y_{2,u_r} defined by

$$y_{1,u_r} = \frac{\sigma^2}{MLN} \sum_{s=1}^{r-1} \sum_{i=-(L-1)}^{L-1} \mathbb{E}(\beta(i, u_s, u_r)) \mathbb{E} \left[\Pi_{t \neq s,r} \tau^{(M)}(\mathbf{Q}^\circ)(u_t) \right]$$

and

$$y_{2,u_r} = \frac{\sigma^2}{MLN} \sum_{s=1}^{r-1} \sum_{i=-(L-1)}^{L-1} \mathbb{E} \left[\Pi_{t \neq s,r} \tau^{(M)}(\mathbf{Q}^\circ)(u_t) \beta(i, u_s, u_r)^\circ \right] \tag{9.45}$$

and denote by \mathbf{y}_1 and \mathbf{y}_2 the corresponding $2L - 1$ -dimensional-related vectors. We first evaluate the behaviour of \mathbf{y}_1 . (9.38) and the rough evaluation $\mathbb{E} [\Pi_{t \neq (s,r)} \tau^{(M)}(\mathbf{Q}^\circ)(u_t)] = \mathcal{O}(\frac{1}{(MN)^{r/2-1}})$ based on Lemma 9.4 and the Hölder inequality imply that vector \mathbf{y}_1 can be written as

$$\mathbf{y}_1 = \mathbf{y}_1^* + \mathbf{z}_1 \tag{9.46}$$

where all the components of \mathbf{z}_1 are $\frac{L}{MN} \mathcal{O}(\frac{1}{(MN)^{r/2}})$ terms, or equivalently $\frac{L}{\sqrt{MN}} \mathcal{O}(\frac{1}{(MN)^{(r+1)/2}}) = o(\frac{1}{(MN)^{(r+1)/2}})$, and where \mathbf{y}_1^* is defined by

$$\mathbf{y}_{1,u_r}^* = \frac{\sigma^2}{MN} \left(\frac{1}{L} \sum_{s=1}^{r-1} \sum_{i=-(L-1)}^{L-1} \bar{\beta}(i, u_s) \delta(u_s + u_r = 0) \right) \kappa^{(r-2)}((u_t)_{t \neq (s,r)}) \tag{9.47}$$

so that

$$\mathbf{y}_{1,u_r}^* = 0 \text{ if } u_r \neq -u_s \text{ for each } s = 1, \dots, r - 1 \tag{9.48}$$

Hence, (9.47,9.48) imply that

$$y_{1,u_r} = \mathcal{O} \left(\frac{1}{(MN)^{r/2}} \right) \mathbb{1}_{u_r \in \{-u_1, \dots, -u_{r-1}\}} + \frac{L}{\sqrt{MN}} \mathcal{O} \left(\frac{1}{(MN)^{(r+1)/2}} \right) \mathbb{1}_{u_r \in \{-u_1, \dots, -u_{r-1}\}^c} \tag{9.49}$$

We note that if $r = 3$, $y_{1,u_3} = 0$ for each u_3 because for each $s = 1, 2$, the term $\mathbb{E} [\Pi_{t \neq s,3} \tau^{(M)}(\mathbf{Q}^\circ)(u_t)]$ is identically zero. Therefore, for $r = 3$, it holds that $\mathbf{y}_1^* = 0$.

As for \mathbf{y}_2 , we notice that Lemma 9.4 and the Hölder inequality lead to

$$y_{2,u_r} = \mathcal{O} \left(\frac{1}{(MN)^{(r+1)/2}} \right) \tag{9.50}$$

We remark that if $r = 2$, then $y_{2,u} = 0$ for each u because the term $\Pi_{t \neq s,r} \tau^{(M)}(\mathbf{Q}^\circ)(u_s)$ disappears and that $y_{2,u}$ represents the mathematical expectation of a zero-mean term.

In order to evaluate the third term of the right-hand side of (9.43), we define $\tilde{x}(u_r, l_1)$ by

$$\begin{aligned} &\tilde{x}(u_r, l_1) \\ &= \mathbb{E} \left(\Pi_{s=1}^{r-1} \tau^{(M)}(\mathbf{Q}^\circ)(u_s) \tau^{(M)}(\mathbf{Q}^\circ)(l_1) \left[\frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_r}) \right) \right]^\circ \right), \end{aligned} \tag{9.51}$$

and $\tilde{x}(u_r)$ by

$$\tilde{x}(u_r) = \sum_{l_1=-(L-1)}^{L-1} \tilde{x}(u_r, l_1) \tag{9.52}$$

In order to have a better understanding of $\tilde{x}(u_r)$, we expand $\tilde{x}(u_r, l_1)$ for each l_1 using (9.44). We define (v_1, \dots, v_r) by $v_s = u_s$ for $s \leq r - 1$ and $v_r = l_1$, while \mathbf{G} and \mathbf{A} represent the matrices $\mathbf{G} = \mathbf{J}_N^{l_1} \mathbf{H}^T$ and $\mathbf{A} = (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_r})$. We denote by $(s_i(u_r, l_1))_{i=1, \dots, 5}$ the i -th term of the right-hand side of (9.44) and denote by $(s_i(u_r))_{i=1, \dots, 5}$ the term

$$s_i(u_r) = \sum_{l_1=-(L-1)}^{L-1} s_i(u_r, l_1)$$

and by \mathbf{s}_i vector $\mathbf{s}_i = (s_i(u_r))_{u_r=-(L-1), \dots, L-1}$. Vector \mathbf{s}_1 plays a particular role because $s_1(u_r, l_1)$ is equal to

$$\begin{aligned} s_1(u_r, l_1) &= \kappa^{(r)}(u_1, \dots, u_{r-1}, l_1) \in (\mathbf{J}_N^{l_1} \mathbf{H}^T, \mathbf{I} \otimes \mathbf{R} \mathbf{J}_L^{u_r}) \\ &= \kappa^{(r)}(u_1, \dots, u_{r-1}, l_1) O\left(\frac{L}{MN}\right) \end{aligned}$$

We remark that vector \mathbf{s}_1 coincides with the action of vector $\kappa^{(r)}$ on matrix $(\in (\mathbf{J}_N^{l_1}, \mathbf{I} \otimes \mathbf{R} \mathbf{J}_L^{u_r}))_{-(L-1) \leq u_r, l_1 \leq (L-1)}$. We define by $x(u_r, l_1)$ and $x(u_r)$ the terms

$$x(u_r, l_1) = \sum_{i=2}^5 s_i(u_r, l_1), \quad x(u_r) = \sum_{l_1=-(L-1)}^{L-1} x(u_r, l_1) \tag{9.53}$$

and vector \mathbf{x} represents the $2L - 1$ -dimensional vector $(x(u_r))_{u_r=-(L-1), \dots, L-1}$.

We finally consider the first term of the right-hand side of (9.43) and denote by ϵ the $2L - 1$ -dimensional vector whose components $(\epsilon_{u_r})_{u_r=-(L-1), \dots, L-1}$ are given by

$$\epsilon_{u_r} = -\mathbb{E} \left[\prod_{s=1}^{r-1} \tau^{(M)}(\mathbf{Q}^\circ)(u_s) \right] \frac{1}{ML} \text{Tr} (\mathbf{A} (\mathbf{I}_M \otimes \mathbf{J}_L^{u_r}))$$

We notice that if $r = 2$, vector ϵ is reduced to 0.

This discussion and (9.26) imply that (9.43) can be written as

$$\kappa^{(r)} = (\mathbf{D} + \mathbf{Y}) \kappa^{(r)} + \mathbf{y}_{1,*} + \mathbf{z}_1 + \mathbf{y}_2 + \epsilon + \sigma^2 c_N \mathbf{x} \tag{9.54}$$

where we recall that \mathbf{D} represents the diagonal matrix $\mathbf{D} = \text{Diag}(d(-(L - 1), z), \dots, d((L - 1), z))$ and where the entries of matrix \mathbf{Y} are defined by

$$\begin{aligned} \mathbf{Y}_{u_r, l_1} &= \sigma^2 c_N \mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_r}) \right) \right] \\ &\quad - \mathbf{D}_{u_r, l_1} + \sigma^2 c_N \in (\mathbf{J}_N^{l_1} \mathbf{H}^T, (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_r})) \end{aligned}$$

It is clear the each entry of \mathbf{Y} is a $O\left(\frac{L}{MN}\right)$ term.

Overview of the proof of Proposition 9.5 We now present informally the various steps of the proof of Proposition 9.5 and concentrate on the proof of Eq. (9.42) in order to simplify the presentation. The particular case $r = 2$ is, however, briefly considered at the end of the overview, but it is of course detailed in the course of the proof. \square

First step: inversion of the linear system (9.54). Lemma 9.5 implies that if z belongs to a set E_N defined as (9.20), matrix $(\mathbf{I} - \mathbf{D} - \mathbf{\Upsilon})$ is invertible. Therefore, vector $\kappa^{(r)}$ can be written as

$$\kappa^{(r)} = (\mathbf{I} - \mathbf{D} - \mathbf{\Upsilon})^{-1} \left(\mathbf{y}_{1,*} + \mathbf{z}_1 + \mathbf{y}_2 + \boldsymbol{\epsilon} + \sigma^2 c_N \mathbf{x} \right)$$

Using (9.21) and the properties of the components of vectors \mathbf{z}_1 , \mathbf{y}_2 , and $\boldsymbol{\epsilon}$, we obtain easily that

$$\begin{aligned} \left((\mathbf{I} - \mathbf{D} - \mathbf{\Upsilon})^{-1} \mathbf{y}_{1,*} \right)_{u_r} &= \frac{1}{1 - d(u_r, z)} \mathbf{y}_{1,u_r}^* + \frac{L}{\sqrt{MN}} O \left(\frac{1}{(MN)^{(r+1)/2}} \right), \\ \left((\mathbf{I} - \mathbf{D} - \mathbf{\Upsilon})^{-1} \mathbf{y}_2 \right)_{u_r} &= O \left(\frac{1}{(MN)^{(r+1)/2}} \right), \end{aligned}$$

and that

$$\begin{aligned} \left| \kappa^{(r)}(u_1, \dots, u_r) - \frac{1}{1 - d(u_r, z)} \mathbf{y}_{1,u_r}^* \right| &\leq |\kappa^{(r-1)}(u_1, \dots, u_{r-1})| O \left(\frac{L}{MN} \right) \\ &\quad + C \sup_u |x(u)| + O \left(\frac{1}{(MN)^{(r+1)/2}} \right) \end{aligned} \tag{9.55}$$

If multi-index (u_1, \dots, u_r) satisfies $u_k + u_l \neq 0$ for $k \neq l$, then $\mathbf{y}_{1,u_r}^* = 0$ (see Eq. (9.48)). Therefore, in order to establish (9.42), it is necessary to evaluate $\sup_u |x(u)|$.

Second step: evaluation of $\sup_u |x(u)|$ In order to evaluate $\sup_u |x(u)|$, we express $x(u_r, l_1)$ as $x(u_r, l_1) = \sum_{i=2}^5 s_i(u_r, l_1)$ (see Eq. (9.53)) and study each term $s_i(u_r) = \sum_{l_1} s_i(u_r, l_1)$ for $i = 2, 3, 4, 5$. $s_4(u_r)$ and $s_5(u_r)$ can be written as $\kappa^{(r-1)}(u_1, \dots, u_{r-1}) O \left(\frac{L}{MN} \right) \delta(u_r = 0) + o \left(\frac{1}{(MN)^{(r+1)/2}} \right)$. The terms $s_2(u_r)$ and $s_3(u_r)$ have a more complicated structure. We just address $s_3(u_r)$ because the behaviour of $s_2(u_r)$ is similar. $s_3(u_r, l_1)$ can be written as $s_3(u_r, l_1) = \sum_{l_2} s_3(u_r, l_1, l_2)$ where

$$\begin{aligned} s_3(u_r, l_1, l_2) &= -\sigma^2 c_N \mathbb{E} \left[\prod_{s=1}^{r-1} \tau^{(M)}(\mathbf{Q}^\circ)(u_s) \tau^{(M)}(\mathbf{Q}^\circ)(l_1) \tau^{(M)}(\mathbf{Q}^\circ)(l_2) \right. \\ &\quad \left. \times \frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_2} \mathbf{H}^T \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I} \otimes \mathbf{R} \mathbf{J}_L^{u_r}) \right) \right] \end{aligned}$$

We define $\bar{s}_3(u_r, l_1, l_2)$ and $\tilde{x}_3^{(1)}(u_r, l_1, l_2)$ by

$$\begin{aligned} \bar{s}_3(u_r, l_1, l_2) &= -\sigma^2 c_N \kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, l_2) \\ &\quad \times \mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_2} \mathbf{H}^T \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I} \otimes \mathbf{R} \mathbf{J}_L^{u_r}) \right) \right] \end{aligned} \tag{9.56}$$

and

$$\begin{aligned} \tilde{x}_3^{(1)}(u_r, l_1, l_2) &= \\ &= -\sigma^2 c_N \mathbb{E} \left[\prod_{s=1}^{r-1} \tau^{(M)}(\mathbf{Q}^\circ)(u_s) \tau^{(M)}(\mathbf{Q}^\circ)(l_1) \tau^{(M)}(\mathbf{Q}^\circ)(l_2) \right. \\ &\quad \left. \times \frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_2} \mathbf{H}^T \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I} \otimes \mathbf{R} \mathbf{J}_L^{u_r}) \right)^\circ \right] \end{aligned} \tag{9.57}$$

Then, it holds that

$$s_3(u_r, l_1, l_2) = \bar{s}_3(u_r, l_1, l_2) + \tilde{x}_3^{(1)}(u_r, l_1, l_2)$$

and obtain that $s_3(u_r) = \bar{s}_3(u_r) + \tilde{x}_3^{(1)}(u_r)$ where $\bar{s}_3(u_r)$ and $\tilde{x}_3^{(1)}(u_r)$ are defined as the sum over l_1, l_2 of $\bar{s}_3(u_r, l_1, l_2)$ and $\tilde{x}_3^{(1)}(u_r, l_1, l_2)$. Similarly, $s_2(u_r)$ can be expressed as $s_2(u_r) = \bar{s}_2(u_r) + \tilde{x}_2^{(1)}(u_r)$ where $\bar{s}_2(u_r)$ and $\tilde{x}_2^{(1)}(u_r)$ are defined in the same way than $\bar{s}_3(u_r)$ and $\tilde{x}_3^{(1)}(u_r)$. The behaviour of $(\bar{s}_j(u_r))_{j=2,3}$ is easy to analyse because it can be shown that

$$\begin{aligned} \bar{s}_j(u_r) &= \sum_{l_1} C_j(u_r, l_1) \kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, u_r - l_1) \\ &\quad + \sum_{l_1, l_2} \kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, l_2) O\left(\frac{L}{MN}\right) \end{aligned}$$

Therefore, (9.55) implies that

$$\begin{aligned} &\left| \kappa^{(r)}(u_1, \dots, u_r) - \frac{1}{1 - d(u_r, z)} \mathbf{y}_{1, u_r}^* \right| \\ &\leq |\kappa^{(r-1)}(u_1, \dots, u_{r-1})| O\left(\frac{L}{MN}\right) \\ &\quad + C \sup_u \sum_{l_1} |\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, u - l_1)| \\ &\quad + \sum_{l_1, l_2} |\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, l_2)| O\left(\frac{L}{MN}\right) \\ &\quad + \sup_u \tilde{x}^{(1)}(u) + O\left(\frac{1}{(MN)^{(r+1)/2}}\right) \end{aligned} \tag{9.58}$$

where $\tilde{x}^{(1)}(u)$ is the positive term defined by

$$\tilde{x}^{(1)}(u) = \left| \tilde{x}_2^{(1)}(u) \right| + \left| \tilde{x}_3^{(1)}(u) \right|$$

Therefore, if $u_r + u_s \neq 0$ for $s = 1, \dots, r - 1$, then $\mathbf{y}_{1,u_r}^* = 0$ and it holds that

$$\begin{aligned} \left| \kappa^{(r)}(u_1, \dots, u_r) \right| &\leq \left| \kappa^{(r-1)}(u_1, \dots, u_{r-1}) \right| O\left(\frac{L}{MN}\right) \\ &+ C \sup_u \sum_{l_1} \left| \kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, u - l_1) \right| \\ &+ \sum_{l_1, l_2} \left| \kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, l_2) \right| O\left(\frac{L}{MN}\right) \\ &+ \sup_u \tilde{x}^{(1)}(u) + O\left(\frac{1}{(MN)^{(r+1)/2}}\right) \end{aligned} \tag{9.59}$$

In order to manage $\sup_u \tilde{x}^{(1)}(u)$, we expand $\tilde{x}_j^{(1)}(u, l_1, l_2)$ using (9.44) when r is exchanged by $r + 1$. In the same way than $\tilde{x}(u)$ defined by (9.52), it holds that

$$\tilde{x}_j^{(1)}(u) = \sum_{i=1}^5 s_{j,i}^{(1)}(u)$$

where the terms $(s_{j,i}^{(1)}(u))_{i=1,\dots,5}$ are defined in the same way than $(s_i(u))_{i=1,\dots,5}$. We define $\tilde{x}_{j,i}^{(2)}(u)$ for $i = 2, 3$ by the fact that

$$s_{j,i}^{(1)}(u) = \bar{s}_{j,i}^{(1)}(u) + \tilde{x}_{j,i}^{(2)}(u)$$

We define $\tilde{x}^{(2)}(u)$ as the positive term given by

$$\tilde{x}^{(2)}(u) = \sum_{(i,j)=(2,3)} \left| \tilde{x}_{j,i}^{(2)}(u) \right|$$

The terms $\tilde{x}_{j,i}^{(2)}(u)$ can be developed similarly, and pursuing the iterative process, we are able to define for each $q \geq 3$ the positive terms $\tilde{x}^{(q)}(u)$ which are the analogues of $\tilde{x}^{(1)}(u)$ and $\tilde{x}^{(2)}(u)$. In order to characterize the behaviour of $\sup_u \tilde{x}^{(1)}(u)$, we express $\tilde{x}^{(1)}(u)$ as

$$\tilde{x}^{(1)}(u) = \sum_{q=1}^{p-1} \left(\tilde{x}^{(q)}(u) - \tilde{x}^{(q+1)}(u) \right) + \tilde{x}^{(p)}(u)$$

where the choice of p depends on the context. The term $\tilde{x}^{(p)}(u)$ is easy to control because the Hölder inequality leads immediately to $\tilde{x}^{(p)}(u) = \left(\frac{L}{\sqrt{MN}}\right)^{p+1} O\left(\frac{1}{(MN)^{r/2}}\right)$.

Moreover, it is shown that

$$\begin{aligned} & \tilde{x}^{(q)}(u) - \tilde{x}^{(q+1)}(u) \\ & \leq \sum_{l_i, i=1, \dots, q+1} \left| \kappa^{(r+q)}(u_1, \dots, u_{r-1}, l_i, i = 1, \dots, q + 1) \right| O\left(\frac{L}{MN}\right) \\ & + C \sum_{l_i, i=1, \dots, q+1} \left| \kappa^{(r+q+1)}(u_1, \dots, u_{r-1}, l_i, i = 1, \dots, q + 1, u - \sum_{i=1}^{q+1} l_i) \right| \\ & + \sum_{l_i, i=1, \dots, q+2} \left| \kappa^{(r+q+1)}(u_1, \dots, u_{r-1}, l_i, i = 1, \dots, q + 2) \right| O\left(\frac{L}{MN}\right) \\ & + o\left(\frac{1}{(MN)^{(r+1)/2}}\right) \tag{9.60} \end{aligned}$$

This allows us to evaluate $\sum_{q=1}^{p-1} (\tilde{x}^{(q)}(u) - \tilde{x}^{(q+1)}(u))$ in the course of the proof.

Third step: establishing (9.42) (9.59) suggests that the rough evaluation $\kappa^{(r)}(u_1, \dots, u_r) = O\left(\frac{1}{(MN)^{r/2}}\right)$ can be improved when $u_k + u_l \neq 0$ for $k \neq l$. The first term of the right-hand side of (9.59) can also be written as

$$\frac{L}{\sqrt{MN}} \frac{1}{\sqrt{MN}} \left| \kappa^{(r-1)}(u_1, \dots, u_{r-1}) \right|$$

Even if we evaluate $\kappa^{(r-1)}(u_1, \dots, u_{r-1})$ as $O\left(\frac{1}{(MN)^{(r-1)/2}}\right)$, it is clear the first term of the right-hand side of (9.59) appears as a $\frac{L}{\sqrt{MN}} O\left(\frac{1}{(MN)^{r/2}}\right)$. A factor $\frac{L}{\sqrt{MN}}$ is thus obtained w.r.t. the rate $O\left(\frac{1}{(MN)^{r/2}}\right)$. One may imagine that using the information that $u_i + u_j \neq 0$ for $1 \leq i, j \leq r - 1, i \neq j$, should allow to improve the above rough evaluation of $\kappa^{(r-1)}(u_1, \dots, u_{r-1})$ and thus the evaluation of the first term of the right-hand side of (9.59). A similar phenomenon is observed for the second and the third terms of the right-hand side of (9.59). We just consider the second term. If each term $\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, u - l_1)$ is roughly evaluated as $O\left(\frac{1}{(MN)^{(r+1)/2}}\right)$, taking into account the sum over l_1 , the second term of the right-hand side of (9.59) is decreased by a factor $\frac{L}{\sqrt{MN}}$ w.r.t. the rough evaluation $O\left(\frac{1}{(MN)^{r/2}}\right)$.

In order to formalize the above discussion, it seems reasonable to be able to prove (9.42) from (9.59) using induction technics. However, this needs some care because $|\kappa^{(r)}(u_1, \dots, u_r)|$ is controlled by $|\kappa^{(r-1)}(u_1, \dots, u_{r-1})|$ and by similar terms of orders greater than r . In order to establish (9.42), it is proved in Proposition 9.10 that if (u_1, \dots, u_r) satisfy $u_t + u_s \neq 0$ for $1 \leq t, s \leq r$ and $t \neq s$, then, for each $q \geq 1$, for each $r \geq 2$, it holds that

which is proved in Lemma 9.8.

Proof of Proposition 9.5 We now complete the proof of Proposition 9.5. In order to evaluate $\kappa^{(r)}(u_1, \dots, u_r)$, we use (9.54) and Lemma 9.5. We write that

$$\kappa^{(r)} = (\mathbf{I} - \mathbf{D} - \mathbf{\Upsilon})^{-1} \left(\mathbf{y}_{1,*} + \mathbf{z}_1 + \mathbf{y}_2 + \boldsymbol{\epsilon} + \sigma^2 c_N \mathbf{x} \right)$$

We first evaluate each component of the first 3 terms of the right-hand side of the above equation. Vector $(\mathbf{I} - \mathbf{D} - \mathbf{\Upsilon})^{-1} \mathbf{y}_{1,*}$ can also be written as

$$(\mathbf{I} - \mathbf{D} - \mathbf{\Upsilon})^{-1} \mathbf{y}_{1,*} = (\mathbf{I} - \mathbf{D})^{-1} \mathbf{y}_{1,*} + (\mathbf{I} - \mathbf{D} - \mathbf{\Upsilon})^{-1} \mathbf{\Upsilon} (\mathbf{I} - \mathbf{D})^{-1} \mathbf{y}_{1,*}$$

As vector $\mathbf{y}_{1,*}$ has at most $r - 1$ nonzero components which are $\mathcal{O}\left(\frac{1}{(MN)^{r/2}}\right)$ terms and that the entries of $\mathbf{\Upsilon}$ are $\mathcal{O}\left(\frac{L}{MN}\right)$ terms, the entries of vector $\mathbf{\Upsilon}(\mathbf{I} - \mathbf{D})^{-1} \mathbf{y}_{1,*}$ are $\frac{L}{\sqrt{MN}} \mathcal{O}\left(\frac{1}{(MN)^{(r+1)/2}}\right) = o\left(\frac{1}{(MN)^{(r+1)/2}}\right)$ terms. (9.21) implies that the entries of $(\mathbf{I} - \mathbf{D} - \mathbf{\Upsilon})^{-1} \mathbf{\Upsilon} (\mathbf{I} - \mathbf{D})^{-1} \mathbf{y}_{1,*}$ are $\frac{L}{\sqrt{MN}} \mathcal{O}\left(\frac{1}{(MN)^{(r+1)/2}}\right)$ terms as well. Therefore, it holds that

$$\left((\mathbf{I} - \mathbf{D} - \mathbf{\Upsilon})^{-1} \mathbf{y}_{1,*} \right)_{u_r} = \frac{1}{1 - d(u_r, z)} \mathbf{y}_{1,u_r}^* + \frac{L}{\sqrt{MN}} \mathcal{O}\left(\frac{1}{(MN)^{(r+1)/2}}\right)$$

and that this term is reduced to a $\frac{L}{\sqrt{MN}} \mathcal{O}\left(\frac{1}{(MN)^{(r+1)/2}}\right)$ if u_r does not belong to $\{-u_1, \dots, -u_{r-1}\}$. (9.21) implies that

$$\left((\mathbf{I} - \mathbf{D} - \mathbf{\Upsilon})^{-1} \mathbf{z}_1 \right)_{u_r} = \frac{L}{\sqrt{MN}} \mathcal{O}\left(\frac{1}{(MN)^{(r+1)/2}}\right)$$

and that

$$\left((\mathbf{I} - \mathbf{D} - \mathbf{\Upsilon})^{-1} \mathbf{y}_2 \right)_{u_r} = \mathcal{O}\left(\frac{1}{(MN)^{(r+1)/2}}\right)$$

for $r \geq 3$, while this term is zero for $r = 2$ because $\mathbf{y}_2 = 0$ in this case. If u_r does not belong to $\{-u_1, \dots, -u_{r-1}\}$, the contributions of the above 3 terms to $\kappa^{(r)}(u_1, \dots, u_r)$ are at most $\mathcal{O}\left(\frac{1}{(MN)^{(r+1)/2}}\right)$ terms, which corresponds to what is expected because we recall that the goal of the subsection is to establish that $\kappa^{(r)}(u_1, \dots, u_r) = \mathcal{O}\left(\frac{1}{(MN)^{(r+1)/2}}\right)$ if $u_k + u_l \neq 0$ for $k \neq l$ (see (9.42)). Finally, (9.21) implies that

$$\left((\mathbf{I} - \mathbf{D} - \mathbf{\Upsilon})^{-1} \boldsymbol{\epsilon} \right)_{u_r} = \kappa^{(r-1)}(u_1, \dots, u_{r-1}) \mathcal{O}\left(\frac{L}{MN}\right) \tag{9.64}$$

and that

$$\sup_u \left| \left((\mathbf{I} - \mathbf{D} - \mathbf{\Upsilon})^{-1} \mathbf{x} \right)_u \right| \leq C \sup_u |x(u)| \tag{9.65}$$

Therefore, it holds that

$$\left| \kappa^{(r)}(u_1, \dots, u_r) - \frac{1}{1 - d(u_r, z)} \mathbf{y}_{1, u_r}^* \right| \leq |\kappa^{(r-1)}(u_1, \dots, u_{r-1})| O\left(\frac{L}{MN}\right) + C \sup_u |x(u)| + O\left(\frac{1}{(MN)^{(r+1)/2}}\right) \tag{9.66}$$

where we recall that $\mathbf{y}_{1, u_r}^* = 0$ if u_r does not belong to $\{-u_1, \dots, -u_{r-1}\}$. We note that if $r = 2$, (9.66) can be written as

$$\left| \kappa^{(2)}(u_1, u_2) - \frac{1}{1 - d(u_2, z)} \mathbf{y}_{1, u_2}^* \right| \leq C \sup_u |x(u)| + O\left(\frac{L}{(MN)^2}\right) \tag{9.67}$$

because $\mathbf{y}_2 = \boldsymbol{\epsilon} = 0$. □

In order to establish (9.42), it is necessary to study the behaviour of $\sup_u |x(u)|$. We express $x(u_r)$ as $x(u_r) = \sum_{l_1=-(L-1)}^{L-1} x(u_r, l_1)$ and evaluate the 4 terms $s_i(u_r) = \sum_{l_1=-(L-1)}^{L-1} s_i(u_r, l_1)$ for $i = 2, 3, 4, 5$. We just study $s_i(u_r)$ for $i = 3$ and $i = 5$ because $s_2(u_r)$ (resp. $s_4(u_r)$) has essentially the same behaviour than $s_3(u_r)$ (resp. $s_5(u_r)$). $s_3(u_r, l_1)$ is given by

$$s_3(u_r, l_1) = \sum_{l_2=-(L-1)}^{L-1} s_3(u_r, l_1, l_2)$$

where

$$s_3(u_r, l_1, l_2) = -\sigma^2 c_N \mathbb{E} \left[\prod_{s=1}^{r-1} \tau^{(M)}(\mathbf{Q}^\circ)(u_s) \tau^{(M)}(\mathbf{Q}^\circ)(l_1) \tau^{(M)}(\mathbf{Q}^\circ)(l_2) \times \frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_2} \mathbf{H}^T \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I} \otimes \mathbf{R} \mathbf{J}_L^{u_r}) \right) \right]$$

We define $\bar{s}_3(u_r, l_1, l_2)$ and $\tilde{x}_3^{(1)}(u_r, l_1, l_2)$ by

$$\bar{s}_3(u_r, l_1, l_2) = -\sigma^2 c_N \kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, l_2) \times \mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_2} \mathbf{H}^T \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I} \otimes \mathbf{R} \mathbf{J}_L^{u_r}) \right) \right] \tag{9.68}$$

and

$$\tilde{x}_3^{(1)}(u_r, l_1, l_2) = -\sigma^2 c_N \mathbb{E} \left[\prod_{s=1}^{r-1} \tau^{(M)}(\mathbf{Q}^\circ)(u_s) \tau^{(M)}(\mathbf{Q}^\circ)(l_1) \tau^{(M)}(\mathbf{Q}^\circ)(l_2) \times \frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_2} \mathbf{H}^T \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I} \otimes \mathbf{R} \mathbf{J}_L^{u_r}) \right) \right] \tag{9.69}$$

Then, it holds that

$$s_3(u_r, l_1, l_2) = \bar{s}_3(u_r, l_1, l_2) + \tilde{x}_3^{(1)}(u_r, l_1, l_2)$$

We also define $\bar{s}_3(u_r, l_1)$, $\bar{s}_3(u_r)$, $\tilde{x}_3^{(1)}(u_r, l_1)$, and $\tilde{x}_3^{(1)}(u_r)$ as $\bar{s}_3(u_r, l_1) = \sum_{l_2} \bar{s}_3(u_r, l_1, l_2)$, $\bar{s}_3(u_r) = \sum_{l_1} \bar{s}_3(u_r, l_1)$, $\tilde{x}_3^{(1)}(u_r, l_1) = \sum_{l_2} \tilde{x}_3^{(1)}(u_r, l_1, l_2)$, and $\tilde{x}_3^{(1)}(u_r) = \sum_{l_1} \tilde{x}_3^{(1)}(u_r, l_1)$. It is easy to check that

$$\mathbb{E} \frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_2} \mathbf{H}^T \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I} \otimes \mathbf{R} \mathbf{J}_L^{u_r}) \right) = C(u_r, l_1) \delta(l_2 = u_r - l_1) + O \left(\frac{L}{MN} \right)$$

Therefore, $\bar{s}_3(u_r)$ is equal to

$$\begin{aligned} \bar{s}_3(u_r) &= \sum_{l_1} C(u_r, l_1) \kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, u_r - l_1) \\ &\quad + \sum_{l_1, l_2} \kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, l_2) O \left(\frac{L}{MN} \right) \end{aligned}$$

We now evaluate $s_5(u_r)$. For this, we recall that we denote $\beta_{1,0}(i, u_s, l_1, u_r)$ the term

$$\begin{aligned} &\beta_{1,0}(i, u_s, l_1, u_r) \\ &= \frac{1}{ML} \text{Tr} \left(\mathbf{Q} (\mathbf{I}_M \otimes \mathbf{J}_L^i) \mathbf{Q} (\mathbf{I}_M \otimes \mathbf{J}_L^{u_s}) \mathbf{Q} \mathbf{W} \mathbf{J}_N^i \mathbf{H}^T \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R} \mathbf{J}_L^{u_r}) \right) \end{aligned}$$

We notice that

$$s_5(u_r, l_1) = s_{5,1}(u_r, l_1) + s_{5,2}(u_r, l_1)$$

where

$$\begin{aligned} s_{5,1}(u_r, l_1) &= -\frac{\sigma^2}{MLN} \sum_{s=1}^{r-1} \sum_{i=-(L-1)}^{L-1} \\ &\quad \times \mathbb{E} \left[\left(\prod_{t \neq (s,r)} \tau^{(M)}(\mathbf{Q}^\circ)(u_t) \tau^{(M)}(\mathbf{Q}^\circ)(l_1) \right) \beta_{1,0}(i, u_s, l_1, u_r) \right] \end{aligned}$$

and

$$s_{5,2}(u_r, l_1) = -\frac{\sigma^2}{MLN} \sum_{i=-(L-1)}^{L-1} \mathbb{E} \left[\prod_{t \leq r-1} \tau^{(M)}(\mathbf{Q}^\circ)(u_t) \beta_{1,0}(i, l_1, l_1, u_r) \right]$$

We first evaluate $s_{5,1}(u_r) = \sum_{l_1} s_{5,1}(u_r, l_1)$. We express $\beta_{1,0}(i, u_s, l_1, u_r)$ as

$$\beta_{1,0}(i, u_s, l_1, u_r) = \mathbb{E} \left(\beta_{1,0}(i, u_s, l_1, u_r) \right) + \beta_{1,0}(i, u_s, l_1, u_r)^\circ$$

and notice that $s_{5,1}(u_r, l_1) = \bar{s}_{5,1}(u_r, l_1) + \tilde{s}_{5,1}(u_r, l_1)$ where

$$\bar{s}_{5,1}(u_r, l_1) = -\frac{\sigma^2}{MLN} \sum_{i=-(L-1)}^{L-1} \sum_{s=1}^{r-1} \kappa^{(r-1)}((u_t)_{t \neq (s,r)}, l_1) \mathbb{E}(\beta_{1,0}(i, u_s, l_1, u_r))$$

and

$$\begin{aligned} \tilde{s}_{5,1}(u_r, l_1) = & -\frac{\sigma^2}{MLN} \sum_{s=1}^{r-1} \sum_{i=-(L-1)}^{L-1} \\ & \times \mathbb{E} \left[\Pi_{t \neq s,r} \tau^{(M)}(\mathbf{Q}^\circ)(u_t) \tau^{(M)}(\mathbf{Q}^\circ)(l_1) \beta_{1,0}(i, u_s, l_1, u_r)^\circ \right] \end{aligned}$$

It is clear that $\tilde{s}_{5,1}(u_r, l_1) = O(\frac{1}{(MN)^{(r+2)/2}})$ which implies that

$$\sum_{l_1} \tilde{s}_{5,1}(u_r, l_1) = \frac{L}{\sqrt{MN}} O\left(\frac{1}{(MN)^{(r+1)/2}}\right) = o\left(\frac{1}{(MN)^{(r+1)/2}}\right) \tag{9.70}$$

Proposition 9.4 implies that

$$\mathbb{E}(\beta_{1,0}(i, u_s, l_1, u_r)) = \bar{\beta}_{1,0}(i, u_s, l_1) \delta(l_1 = u_r + u_s) + O\left(\frac{L}{MN}\right) \tag{9.71}$$

Using the rough evaluation $\kappa^{(r-1)}((u_t)_{t \neq s,r}, l_1) = O(\frac{1}{(MN)^{(r-1)/2})}$, we get immediately that

$$\bar{s}_{5,1}(u_r) = \sum_{l_1} \bar{s}_{5,1}(u_r, l_1) = O\left(\frac{1}{(MN)^{(r+1)/2}}\right) \tag{9.72}$$

We finally notice that if $r = 2$, $\bar{s}_{5,1}(u_r)$ is reduced to 0.

We define $\tilde{s}_{5,2}(u_r, l_1)$ and $\bar{s}_{5,2}(u_r, l_1)$ in the same way and obtain easily that

$$\sum_{l_1} \tilde{s}_{5,2}(u_r, l_1) = \frac{L}{\sqrt{MN}} O\left(\frac{1}{(MN)^{(r+1)/2}}\right) = o\left(\frac{1}{(MN)^{(r+1)/2}}\right) \tag{9.73}$$

The behaviour of $\sum_{l_1} \bar{s}_{5,2}(u_r, l_1)$ is, however, different from the behaviour of $\sum_{l_1} \bar{s}_{5,1}(u_r, l_1)$ if $u_r = 0$. Indeed,

$$\mathbb{E}(\beta_{1,0}(i, l_1, l_1, u_r)) = \bar{\beta}_{1,0}(i, l_1, l_1) \delta(u_r = 0) + O\left(\frac{L}{MN}\right)$$

It is easy to check that the contribution of the $O\left(\frac{L}{MN}\right)$ terms to $\sum_{l_1} \bar{s}_{5,2}(u_r, l_1)$ is a $o\left(\frac{1}{(MN)^{(r+1)/2}\right)$ term. Therefore,

$$\begin{aligned} \bar{s}_{5,2}(u_r) &= \sum_{l_1} \bar{s}_{5,2}(u_r, l_1) \\ &= \sum_{l_1} \left(\frac{1}{L} \sum_{i=-(L-1)}^{L-1} \bar{\beta}_{1,0}(i, l_1, l_1) \right) \frac{1}{MN} \kappa^{(r-1)}(u_1, \dots, u_{r-1}) \delta(u_r = 0) \\ &\quad + o\left(\frac{1}{(MN)^{(r+1)/2}\right) \\ &= \kappa^{(r-1)}(u_1, \dots, u_{r-1}) O\left(\frac{L}{MN}\right) \delta(u_r = 0) + o\left(\frac{1}{(MN)^{(r+1)/2}\right) \end{aligned} \tag{9.74}$$

As above, $\bar{s}_{5,2}(u_r)$ is reduced to 0 if $r = 2$.

The reader may check that the terms $s_2(u_r) = \bar{s}_2(u_r) + \tilde{x}_2^{(1)}(u_r)$ and $s_4(u_r)$ have exactly the same behaviour than $s_3(u_r)$ and $s_5(u_r)$. For the reader’s convenience, we mention that $\tilde{x}_2^{(1)}(u_r)$ is defined as

$$\tilde{x}_2^{(1)}(u_r) = \sum_{l_1, l_2} \tilde{x}_2^{(1)}(u_r, l_1, l_2)$$

where $\tilde{x}_2^{(1)}(u_r, l_1, l_2)$ is the term given by

$$\begin{aligned} &\sigma^2 c_N \mathbb{E} \left[\prod_{s=1}^{r-1} \tau^{(M)}(\mathbf{Q}^\circ)(u_s) \tau^{(M)}(\mathbf{Q}^\circ)(l_1) \tau^{(M)}(\mathbf{Q}^\circ)(l_2) \right. \\ &\quad \left. \times \frac{1}{ML} \text{Tr} \left(\mathbf{Q} \mathbf{W} \mathbf{J}_N^{l_2} \mathbf{H}^T \mathbf{W}^* (\mathbf{I} \otimes \sigma^2 \mathbf{R} \mathcal{T}_{L,L} (\mathbf{H} \mathbf{J}_N^{*l_1} \mathbf{H}) \mathbf{R} \mathbf{J}_L^{u_r}) \right) \right] \end{aligned} \tag{9.75}$$

In sum, we have proved the following useful result.

Proposition 9.7 *If $r \geq 2$, for each u_r , it holds that*

$$\begin{aligned} x(u_r) &= \sum_{l_1} C(u_r, l_1) \kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, u_r - l_1) \\ &\quad + \sum_{l_1, l_2} \kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, l_2) O\left(\frac{L}{MN}\right) \\ &\quad + \kappa^{(r-1)}(u_1, \dots, u_{r-1}) O\left(\frac{L}{MN}\right) \delta(u_r = 0) + \tilde{x}_2^{(1)}(u_r) \\ &\quad + \tilde{x}_3^{(1)}(u_r) + O\left(\frac{1}{(MN)^{(r+1)/2}\right) \end{aligned} \tag{9.76}$$

We prove the proposition by induction on r . We first check (9.82) if $r = 2$. In this case, the integer p is necessarily equal to 1 and (9.81) reduces to $u_1 + u_2 \neq 0$. We use (9.80). Using the rough evaluations $\kappa^{(3)}(v_1, v_2, v_3) = O(\frac{1}{(MN)^{3/2}})$ and $\sup_u |\tilde{x}_j^{(1)}(u)| = O(\frac{L^2}{(MN)^2}) = (\frac{L}{\sqrt{MN}})^2 O(\frac{1}{MN})$ for $j = 2, 3$, we obtain immediately that (9.82) holds if $r = 2$.

We now assume that (9.82) holds until integer $r_0 - 1$ and prove that it is true for integer r_0 . For this, we establish that for each $r \geq r_0$ and for each u_1, \dots, u_r , (9.82) holds provided (9.81) is true until $p \leq r_0 - 1$. We first verify that (9.82) holds for each $r \geq r_0$ and for $p = 1$ as soon as $u_r + u_s \neq 0$ $s = 1, \dots, r - 1$. For this, we use (9.79). \mathbf{y}_{1, u_r}^* is of course equal to 0. Moreover, as $\kappa^{(r-1)}(u_1, \dots, u_{r-1}) = O(\frac{1}{(MN)^{(r-1)/2})}$, it is clear that

$$|\kappa^{(r-1)}(u_1, \dots, u_{r-1})| O\left(\frac{L}{MN}\right) = \frac{L}{\sqrt{MN}} O\left(\frac{1}{(MN)^{r/2}}\right)$$

as expected. Using that $\kappa^{(r+1)}(v_1, \dots, v_{r+1}) = O(\frac{1}{(MN)^{(r+1)/2})}$ for each (v_1, \dots, v_{r+1}) , we obtain immediately that

$$\sup_u \sum_{l_1} |\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, u - l_1)| = \frac{L}{\sqrt{MN}} O\left(\frac{1}{(MN)^{r/2}}\right)$$

and

$$\sum_{l_1, l_2} |\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, l_2)| O\left(\frac{L}{MN}\right) = \frac{L^2}{MN} \frac{L}{\sqrt{MN}} O\left(\frac{1}{(MN)^{r/2}}\right)$$

Finally, the Hölder inequality leads to

$$\sup_u |\tilde{x}_2^{(1)}(u)| + \sup_u |\tilde{x}_3^{(1)}(u)| = O\left(\frac{L^2}{(MN)^{(r+2)/2}}\right) = \left(\frac{L}{\sqrt{MN}}\right)^2 O\left(\frac{1}{(MN)^{r/2}}\right) \tag{9.83}$$

Next, we consider the case $p = 2$ for the reader's convenience. We consider $r \geq r_0$, and assume that $u_r + u_s \neq 0$ $s = 1, \dots, r - 1$ as well as $u_{r-1} + u_s \neq 0$ $s = 1, \dots, r - 2$. We again use (9.79) and remark that $\mathbf{y}_{1, u_r}^* = 0$. As $u_{r-1} + u_s \neq 0$ $s = 1, \dots, r - 2$, the use of (9.82) for integer $r - 1$, multi-index (u_1, \dots, u_{r-1}) and $p = 1$ (proved above) implies that $\kappa^{(r-1)}(u_1, \dots, u_{r-1}) = \frac{L}{\sqrt{MN}} O(\frac{1}{(MN)^{(r-1)/2})}$ and that

$$\kappa^{(r-1)}(u_1, \dots, u_{r-1}) O\left(\frac{L}{MN}\right) = \left(\frac{L}{\sqrt{MN}}\right)^2 O\left(\frac{1}{(MN)^{r/2}}\right)$$

We now evaluate $\sum_{l_1} |\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, u - l_1)|$. It is clear that

$$\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, u - l_1) = \kappa^{(r+1)}(l_1, u - l_1, u_1, \dots, u_{r-1})$$

As $u_{r-1} + u_s \neq 0$ $s = 1, \dots, r - 2$, the use of (9.82) for integer $r + 1$, multi-index $(l_1, u - l_1, u_1, \dots, u_{r-1})$, and $p = 1$ leads to

$$\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, u - l_1) = \frac{L}{\sqrt{MN}} O\left(\frac{1}{(MN)^{(r+1)/2}}\right)$$

as soon as $u_{r-1} + l_1 \neq 0$ and $u_{r-1} + u - l_1 \neq 0$, or equivalently if $l_1 \neq -u_{r-1}$ and $l_1 \neq u + u_{r-1}$. Therefore,

$$\sum_{l_1 \neq (-u_{r-1}, u + u_{r-1})} |\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, u - l_1)| = \left(\frac{L}{\sqrt{MN}}\right)^2 O\left(\frac{1}{(MN)^{r/2}}\right)$$

If $l_1 = -u_{r-1}$ or $l_1 = u + u_{r-1}$, we use the rough evaluation

$$\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, u - l_1) = \left(\frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r/2}}\right)$$

Therefore, we obtain that

$$\sum_{l_1} |\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, u - l_1)| = \max\left(\left(\frac{L}{\sqrt{MN}}\right)^2, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r/2}}\right)$$

We now consider $\sum_{l_1, l_2} |\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, l_2)| O\left(\frac{L}{MN}\right)$. We remark that $\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, l_2) = \kappa^{(r+1)}(l_1, l_2, u_1, \dots, u_{r-1})$. Therefore, if $u_{r-1} + l_1 \neq 0$ and $u_{r-1} + l_2 \neq 0$, (9.82) for integer $r + 1$, multi-index $(l_1, l_2, u_1, \dots, u_{r-1})$, and $p = 1$ implies that

$$\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, l_2) = \frac{L}{\sqrt{MN}} O\left(\frac{1}{(MN)^{(r+1)/2}}\right)$$

If $l_1 = -u_{r-1}$ or $l_2 = -u_{r-1}$, we use again that

$$\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, l_2) = \left(\frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r/2}}\right)$$

so that for $i, j = 1, 2, i \neq j$, it holds that

$$\begin{aligned} \sum_{l_i = -u_{r-1}, l_j} |\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_i, l_j)| O\left(\frac{L}{MN}\right) &= \frac{L^2}{MN} \left(\frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r/2}}\right) \\ &= o\left(\frac{1}{(MN)^{(r+1)/2}}\right) \end{aligned}$$

Therefore, (9.84) used for $t = r - 1$, $p = p_0$, and multi-index (v_1, \dots, v_{r-1}) with $v_s = u_s$ leads to

$$\kappa^{(r-1)}(u_1, \dots, u_{r-1}) = \max \left(\left(\frac{L}{\sqrt{MN}} \right)^{p_0}, \frac{1}{\sqrt{MN}} \right) O \left(\frac{1}{(MN)^{(r-1)/2}} \right)$$

Therefore,

$$\begin{aligned} &\kappa^{(r-1)}(u_1, \dots, u_{r-1}) O \left(\frac{L}{MN} \right) \\ &= \frac{L}{\sqrt{MN}} \max \left(\left(\frac{L}{\sqrt{MN}} \right)^{p_0}, \frac{1}{\sqrt{MN}} \right) O \left(\frac{1}{(MN)^{r/2}} \right) \end{aligned}$$

which, of course, also coincides with a $\max \left(\left(\frac{L}{\sqrt{MN}} \right)^{p_0+1}, \frac{1}{\sqrt{MN}} \right) O \left(\frac{1}{(MN)^{r/2}} \right)$ term.

We now study the term

$$\sum_{l_1} \left| \kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, u - l_1) \right|$$

Using (9.84) for $t = r + 1$ and multi-index $l_1, u - l_1, u_1, \dots, u_{r-1}$, we obtain that

$$\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, u - l_1) = \max \left(\left(\frac{L}{\sqrt{MN}} \right)^{p_0}, \frac{1}{\sqrt{MN}} \right) O \left(\frac{1}{(MN)^{(r+1)/2}} \right)$$

if l_1 is such that $u_{r-j} + l_1 \neq 0$ and $u_{r-j} + u - l_1 \neq 0$ for each $j = 1, \dots, p_0$. The sum of the terms $|\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, u - l_1)|$ over these values of l_1 is therefore a $L \max \left(\left(\frac{L}{\sqrt{MN}} \right)^{p_0}, \frac{1}{\sqrt{MN}} \right) O \left(\frac{1}{(MN)^{(r+1)/2}} \right)$ term, or equivalently a $\frac{L}{\sqrt{MN}} \left(\left(\frac{L}{\sqrt{MN}} \right)^{p_0}, \frac{1}{\sqrt{MN}} \right) O \left(\frac{1}{(MN)^{r/2}} \right)$ term, which, of course, is also a

$$\max \left(\left(\frac{L}{\sqrt{MN}} \right)^{p_0+1}, \frac{1}{\sqrt{MN}} \right) O \left(\frac{1}{(MN)^{r/2}} \right)$$

term. If l_1 is equal to $-u_{r-j_0}$ or to $u_{r-j_0} + u$ for some $j_0 = 1, \dots, p_0$, we use the rough evaluation

$$\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, u - l_1) = O \left(\frac{1}{(MN)^{(r+1)/2}} \right) = \frac{1}{\sqrt{MN}} O \left(\frac{1}{(MN)^{r/2}} \right)$$

This discussion implies that

$$\sum_{l_1} |\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, u - l_1)|$$

$$= \max \left(\left(\frac{L}{\sqrt{MN}} \right)^{p_0+1}, \frac{1}{\sqrt{MN}} \right) O \left(\frac{1}{(MN)^{r/2}} \right)$$

The evaluation of $\sum_{l_1, l_2} |\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, l_2)| O \left(\frac{L}{MN} \right)$ is similar and is thus omitted. □

In order to complete the proof of Lemma 9.6, it remains to prove that

$$\sup_u |\tilde{x}_j^{(1)}(u)| \leq \max \left(\left(\frac{L}{\sqrt{MN}} \right)^{(p_0+1)}, \frac{1}{\sqrt{MN}} \right) O \left(\frac{1}{(MN)^{r/2}} \right)$$

for $j = 2, 3$. For this, we study in more detail $\sup_u |\tilde{x}_j^{(1)}(u)|$ for $j = 2, 3$. We expand $\tilde{x}_j^{(1)}(u, l_1, l_2)$ using (9.44) when r is exchanged by $r + 1$. In the same way than $\tilde{x}(u)$ defined by (9.52), it holds that

$$\tilde{x}_j^{(1)}(u) = \sum_{i=1}^5 s_{j,i}^{(1)}(u)$$

where the terms $(s_{j,i}^{(1)}(u))_{i=1, \dots, 5}$ are defined in the same way than $(s_i(u))_{i=1, \dots, 5}$. We define $\tilde{x}_{j,i}^{(2)}(u)$ for $i = 2, 3$ by the fact that

$$s_{j,i}^{(1)}(u) = \bar{s}_{j,i}^{(1)}(u) + \tilde{x}_{j,i}^{(2)}(u)$$

We define $\tilde{x}^{(1)}(u)$ as the positive term

$$\tilde{x}^{(1)}(u) = \left| \tilde{x}_2^{(1)}(u) \right| + \left| \tilde{x}_3^{(1)}(u) \right|$$

and, similarly, $\tilde{x}^{(2)}(u)$ is given by

$$\tilde{x}^{(2)}(u) = \sum_{(i,j)=(2,3)} \left| \tilde{x}_{j,i}^{(2)}(u) \right|$$

A rough evaluation (based on the Hölder inequality and on (9.36)) of the various terms $s_{j,i}^{(1)}(u)$ for $i = 4, 5$ leads to $s_{j,i}^{(1)}(u) = \frac{L}{\sqrt{MN}} O \left(\frac{1}{(MN)^{(r+1)/2}} \right)$. After some calculations, we obtain that

$$\begin{aligned} \tilde{x}^{(1)}(u) &\leq \sum_{l_1, l_2} |\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, l_2)| O \left(\frac{L}{MN} \right) \\ &+ C \sum_{l_1, l_2} |\kappa^{(r+2)}(u_1, \dots, u_{r-1}, l_1, l_2, u - l_1 - l_2)| \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{l_1, l_2, l_3} |\kappa^{(r+2)}(u_1, \dots, u_{r-1}, l_1, l_2, l_3)| O\left(\frac{L}{MN}\right) + \tilde{x}^{(2)}(u) \\
 &+ \frac{L}{\sqrt{MN}} O\left(\frac{1}{(MN)^{(r+1)/2}}\right) \tag{9.87}
 \end{aligned}$$

The first term of the right-hand side of (9.87) corresponds to the contribution of $s_{j,1}^{(1)}(u)$, while the second and the third terms are due to $\bar{s}_{j,2}^{(1)}(u)$ and $\bar{s}_{j,3}^{(1)}(u)$. The term $\frac{L}{\sqrt{MN}} O\left(\frac{1}{(MN)^{(r+1)/2}}\right)$ is due to the $s_{j,i}^{(1)}(u)$ for $i = 4, 5$. The terms $\tilde{x}_{j,i}^{(2)}(u)$ can of course be also developed, and we obtain similarly

$$\begin{aligned}
 \tilde{x}^{(2)}(u) &\leq \sum_{l_1, l_2, l_3} |\kappa^{(r+2)}(u_1, \dots, u_{r-1}, l_1, l_2, l_3)| O\left(\frac{L}{MN}\right) \\
 &+ C \sum_{l_1, l_2, l_3} |\kappa^{(r+3)}(u_1, \dots, u_{r-1}, l_1, l_2, l_3, u - l_1 - l_2 - l_3)| \\
 &+ \sum_{l_1, l_2, l_3, l_4} |\kappa^{(r+3)}(u_1, \dots, u_{r-1}, l_1, l_2, l_3, l_4)| O\left(\frac{L}{MN}\right) + \tilde{x}^{(3)}(u) \\
 &+ \left(\frac{L}{\sqrt{MN}}\right)^2 O\left(\frac{1}{(MN)^{(r+1)/2}}\right) \tag{9.88}
 \end{aligned}$$

The term $\left(\frac{L}{\sqrt{MN}}\right)^2 O\left(\frac{1}{(MN)^{(r+1)/2}}\right)$ is due to the terms $(s_{k_1, k_2, i}^{(2)}(u))$ for $i = 4, 5$ and $k_1, k_2 = 2, 3$: it is easily seen using the Hölder inequality that their order of magnitude is $\frac{L}{\sqrt{MN}}$ smaller than the order of magnitude of the $(s_{k,i}^{(1)})_{i=4,5}$ for $k = 2, 3$. More generally, it holds that

$$\begin{aligned}
 \tilde{x}^{(q)}(u) &\leq \sum_{l_i, i=1, \dots, q+1} |\kappa^{(r+q)}(u_1, \dots, u_{r-1}, l_i, i = 1, \dots, q+1)| O\left(\frac{L}{MN}\right) \\
 &+ C \sum_{l_i, i=1, \dots, q+1} |\kappa^{(r+q+1)}(u_1, \dots, u_{r-1}, l_i, i = 1, \dots, q+1, u - \sum_{i=1}^{q+1} l_i)| \\
 &+ \sum_{l_i, i=1, \dots, q+2} |\kappa^{(r+q+1)}(u_1, \dots, u_{r-1}, l_i, i = 1, \dots, q+2)| O\left(\frac{L}{MN}\right) \\
 &+ \tilde{x}^{(q+1)}(u) + \left(\frac{L}{\sqrt{MN}}\right)^q O\left(\frac{1}{(MN)^{(r+1)/2}}\right) \tag{9.89}
 \end{aligned}$$

We remark that the Hölder inequality leads to

$$\sup_u \tilde{x}^{(p)}(u) = \left(\frac{L}{\sqrt{MN}}\right)^{p+1} O\left(\frac{1}{(MN)^{r/2}}\right) \tag{9.90}$$

for each p . We express $\tilde{x}^{(1)}(u)$ as

$$\tilde{x}^{(1)}(u) = \sum_{q=1}^{p_0-1} \left(\tilde{x}^{(q)}(u) - \tilde{x}^{(q+1)}(u) \right) + \tilde{x}^{(p_0)}(u) \tag{9.91}$$

We now prove that for each q , then it holds that

$$\tilde{x}^{(q)}(u) - \tilde{x}^{(q+1)}(u) \leq \left(\frac{L}{\sqrt{MN}} \right)^q \max \left(\left(\frac{L}{\sqrt{MN}} \right)^{p_0+1}, \frac{1}{\sqrt{MN}} \right) O \left(\frac{1}{(MN)^{r/2}} \right) \tag{9.92}$$

(9.89) implies that $\tilde{x}^{(q)}(u) - \tilde{x}^{(q+1)}(u)$ is upperbounded by the sum of 4 terms. We just study the second term, i.e.

$$\sum_{l_i, i=1, \dots, q+1} \left| \kappa^{(r+q+1)} \left(u_1, \dots, u_{r-1}, l_i, i = 1, \dots, q+1, u - \sum_{i=1}^{q+1} l_i \right) \right|$$

because, as the fourth term $\left(\frac{L}{\sqrt{MN}} \right)^q O \left(\frac{1}{(MN)^{(r+1)/2}} \right)$, it can be easily checked that the first and the third term are negligible w.r.t. the right-hand side of inequality (9.92). If the integers $l_1, \dots, l_{q+1}, u - \sum_{i=1}^{q+1} l_i$ do not belong $\{-u_{r-1}, \dots, -u_{r-p_0}\}$, (9.84) for $t = r + q + 1$ and for multi-index $(l_1, \dots, l_{q+1}, u - \sum_{i=1}^{q+1} l_i, u_1, \dots, u_{r-1})$ implies that

$$\begin{aligned} & \kappa^{(r+q+1)} \left(u_1, \dots, u_{r-1}, l_i, i = 1, \dots, q+1, u - \sum_{i=1}^{q+1} l_i \right) \\ &= \max \left(\left(\frac{L}{\sqrt{MN}} \right)^{(p_0)}, \frac{1}{\sqrt{MN}} \right) O \left(\frac{1}{(MN)^{(r+q+1)/2}} \right) \end{aligned}$$

Therefore, the sum over all these integers can be upperbounded by

$$\begin{aligned} & L^{q+1} \max \left(\left(\frac{L}{\sqrt{MN}} \right)^{(p_0)}, \frac{1}{\sqrt{MN}} \right) O \left(\frac{1}{(MN)^{(r+q+1)/2}} \right) \\ &= \left(\frac{L}{\sqrt{MN}} \right)^q \frac{L}{\sqrt{MN}} \max \left(\left(\frac{L}{\sqrt{MN}} \right)^{p_0}, \frac{1}{\sqrt{MN}} \right) O \left(\frac{1}{(MN)^{r/2}} \right) \end{aligned}$$

which, of course, is a $\left(\frac{L}{\sqrt{MN}} \right)^q \max \left(\left(\frac{L}{\sqrt{MN}} \right)^{p_0+1}, \frac{1}{\sqrt{MN}} \right) O \left(\frac{1}{(MN)^{r/2}} \right)$ term as expected.

If at least one of the indexes $l_1, \dots, l_{q+1}, u - \sum_{i=1}^{q+1} l_i$ is equal an integer $(-u_{r-i})_{i=1, \dots, p_0}$, we use the rough evaluation

$$\kappa^{(r+q+1)} \left(u_1, \dots, u_{r-1}, l_i, i = 1, \dots, q + 1, u - \sum_{i=1}^{q+1} l_i \right) = O \left(\frac{1}{(MN)^{(r+q+1)/2}} \right)$$

The sum over the corresponding multi-indices is thus a $L^q O(\frac{1}{(MN)^{(r+q+1)/2}}) = (\frac{L}{\sqrt{MN}})^q O(\frac{1}{(MN)^{(r+1)/2}})$. This completes the proof of (9.92). Therefore, (9.91) and (9.90) imply that

$$\sup_u \tilde{x}^{(1)}(u) = \max \left(\left(\frac{L}{\sqrt{MN}} \right)^{p_0+1}, \frac{1}{\sqrt{MN}} \right) O \left(\frac{1}{(MN)^{r/2}} \right)$$

as expected. This, in turn, completes the proof of Lemma 9.6.

We now improve the evaluation of Proposition 9.9 when (u_1, \dots, u_r) satisfy $u_t + u_s \neq 0$ for $1 \leq t, s \leq r$, and $t \neq s$, or equivalently if (u_1, \dots, u_r) verify (9.81) for $p = r - 1$. More precisely, we prove the following result.

Proposition 9.10 *Assume that (u_1, \dots, u_r) satisfy $u_t + u_s \neq 0$ for $1 \leq t, s \leq r$, and $t \neq s$. Then, for each $q \geq 1$, for each $r \geq 2$, it holds that*

$$\kappa^{(r)}(u_1, \dots, u_r) = \max \left(\left(\frac{L}{\sqrt{MN}} \right)^{r-1+q}, \frac{1}{\sqrt{MN}} \right) O \left(\frac{1}{(MN)^{r/2}} \right) \tag{9.93}$$

Proof We prove this result by induction on integer q . We first establish (9.93) for $q = 1$ by induction on integer r . If $r = 2$, we have to check that if $u_1 + u_2 \neq 0$, then it holds that

$$\kappa^{(2)}(u_1, u_2) = \max \left(\left(\frac{L}{\sqrt{MN}} \right)^2, \frac{1}{\sqrt{MN}} \right) O \left(\frac{1}{MN} \right) \tag{9.94}$$

For this, we use (9.80). We have already mentioned that the Hölder inequality leads to

$$\sup_u \tilde{x}^{(1)}(u) = \left(\frac{L}{\sqrt{MN}} \right)^2 O \left(\frac{1}{MN} \right)$$

We study the term

$$\sup_u \sum_{l_1} |\kappa^{(3)}(u_1, l_1, u - l_1)|$$

Proposition 9.9 in the case $r = 3$ and $p = 1$ implies that

$$\kappa^{(3)}(u_1, l_1, u - l_1) = \frac{L}{\sqrt{MN}} O\left(\frac{1}{(MN)^{3/2}}\right)$$

as soon as $l_1 \neq -u_1$ and $l_1 \neq u + u_1$. Therefore,

$$\sum_{l_1 \neq (-u_1, u+u_1)} |\kappa^{(3)}(u_1, l_1, u - l_1)| = \left(\frac{L}{\sqrt{MN}}\right)^2 O\left(\frac{1}{MN}\right)$$

If $l_1 = -u_1$ or $l_1 = u + u_1$, we use the rough evaluation $\kappa^{(3)}(u_1, l_1, u - l_1) = O\left(\frac{1}{(MN)^{3/2}}\right)$, and we finally obtain that

$$\sum_{l_1} |\kappa^{(3)}(u_1, l_1, u - l_1)| = \max\left(\left(\frac{L}{\sqrt{MN}}\right)^2, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{MN}\right)$$

as expected. The term

$$\sum_{l_1, l_2} |\kappa^{(3)}(u_1, l_1, l_2)| O\left(\frac{L}{MN}\right)$$

is evaluated similarly. We have thus established (9.94). □

We assume that (9.93) holds for $q = 1$ until index $r_0 - 1$ and prove that it also holds for index r_0 . We take (9.79) as a starting point. We consider (u_1, \dots, u_{r_0}) satisfying $u_t + u_s \neq 0$ for $1 \leq t, s \leq r_0$, or equivalently (9.81) for $r = r_0$ and $p = r_0 - 1$. (9.93) for $q = 1, r = r_0 - 1$, and multi-index (u_1, \dots, u_{r_0-1}) leads to

$$\kappa^{(r_0-1)}(u_1, \dots, u_{r_0-1}) = \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r_0-2+1}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{(r_0-1)/2}}\right)$$

and to

$$\begin{aligned} &\kappa^{(r_0-1)}(u_1, \dots, u_{r_0-1}) O\left(\frac{L}{MN}\right) \\ &= \frac{L}{\sqrt{MN}} \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r_0-1}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r_0/2}}\right) \end{aligned}$$

which, of course, is a $\max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r_0-1+1}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r_0/2}}\right)$ term as expected.

We now evaluate

$$\sum_{l_1} \left| \kappa^{(r_0+1)}(u_1, \dots, u_{r_0-1}, l_1, u - l_1) \right|$$

If $l_1 + u_s \neq 0$ and $u - l_1 + u_s \neq 0$ for $s = 1, \dots, r_0 - 1$, Proposition 9.9 for $r = r_0 + 1$, multi-index $(l_1, u - l_1, u_1, \dots, u_{r-1})$, and $p = r_0 - 1$ implies that

$$\begin{aligned} &\kappa^{(r_0+1)}(u_1, \dots, u_{r_0-1}, l_1, u - l_1) \\ &= \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r_0-1}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{(r_0+1)/2}}\right) \end{aligned}$$

and that the sum of the $|\kappa^{(r_0+1)}(u_1, \dots, u_{r_0-1}, l_1, u - l_1)|$ over these indices is a

$$\frac{L}{\sqrt{MN}} \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r_0-1}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r_0/2}}\right)$$

term. If $l_1 + u_s = 0$ or $u - l_1 + u_s = 0$ for some integer s , we use as previously that

$$\kappa^{(r_0+1)}(u_1, \dots, u_{r_0-1}, l_1, u - l_1) = \frac{1}{\sqrt{MN}} O\left(\frac{1}{(MN)^{r_0/2}}\right)$$

This, in turn, implies that

$$\begin{aligned} &\sup_u \left| \sum_{l_1} \kappa^{(r_0+1)}(u_1, \dots, u_{r_0-1}, l_1, u - l_1) \right| \\ &= \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r_0-1+1}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r_0/2}}\right) \end{aligned}$$

as expected.

The term

$$\sum_{l_1, l_2} |\kappa^{(r_0+1)}(u_1, \dots, u_{r_0-1}, l_1, l_2)| O\left(\frac{L}{MN}\right)$$

can be evaluated similarly. Finally, it is easy to show as in the proof of Lemma 9.6 that $\sup_u \tilde{x}^{(1)}(u)$ behaves as expected.

This completes the proof of (9.93) for each r and $q = 1$. In order to establish the proposition for each q , we assume that it is true until integer $q - 1$ and prove that it holds for integer q . We prove this statement by induction on integer r and begin to consider $r = 2$. We of course use (9.80) for $u_1 + u_2 \neq 0$. It is easy to check as previously that the term $\sup_u \tilde{x}^{(1)}(u)$ is as expected and that it is also the case for $\sum_{l_1, l_2} |\kappa^{(3)}(u_1, l_1, l_2)| O\left(\frac{L}{MN}\right)$. However, the term

$$\sup_u \sum_{l_1} |\kappa^{(3)}(u_1, l_1, u - l_1)|$$

appears more difficult to evaluate. If $u \neq 0$, it is easy to check that $\sum_{l_1} |\kappa^{(3)}(u_1, l_1, u - l_1)|$ is a

$$\max\left(\left(\frac{L}{\sqrt{MN}}\right)^{1+q}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{MN}\right)$$

term because, except if $u_1 + l_1 = 0$ or $u_1 + u - l_1 = 0$ (the contribution of these particular values to the sum is a $O\left(\frac{1}{(MN)^{3/2}}\right)$ term), (9.93) used for $r = 3$ and integer $q - 1$ implies that

$$\kappa^{(3)}(u_1, l_1, u - l_1) = \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{1+q}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{3/2}}\right)$$

and that

$$\begin{aligned} & \sum_{l_1 \neq -u_1, u_1+u} |\kappa^{(3)}(u_1, l_1, u - l_1)| \\ &= \frac{L}{\sqrt{MN}} \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{1+q}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)}\right) \end{aligned}$$

If $u = 0$, the sum becomes

$$\sum_{l_1} \left| \kappa^{(3)}(u_1, l_1, -l_1) \right|$$

(9.93) for $r = 3$ and integer $q - 1$ cannot be used to evaluate $\kappa^{(3)}(u_1, l_1, -l_1)$ because $l_1 - l_1 = 0$. We thus have to study separately this kind of term. For this, we prove the following lemma.

Lemma 9.7 *We consider an integer $r \geq 2$ and assume the following hypotheses:*

- for each integer s and for each v_1, \dots, v_s such that $v_{s_1} + v_{s_2} \neq 0, 1 \leq s_1, s_2 \leq s, s_1 \neq s_2$, it holds that

$$\kappa^{(s)}(v_1, \dots, v_s) = \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{s-1+q-1}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{s/2}}\right) \quad (9.95)$$

- for each $s \leq r - 1$, and each v_1, \dots, v_s such that $v_{s_1} + v_{s_2} \neq 0, 1 \leq s_1, s_2 \leq s, s_1 \neq s_2$, it holds that

$$\kappa^{(s)}(v_1, \dots, v_s) = \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{s-1+q}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{s/2}}\right) \quad (9.96)$$

Then, if u_1, \dots, u_{r-1} verify $u_{s_1} + u_{s_2} \neq 0, 1 \leq s_1, s_2 \leq r - 1, s_1 \neq s_2$, it holds that

$$\begin{aligned} & \sum_{l_1} |\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, -l_1)| \\ &= \max \left(\left(\frac{L}{\sqrt{MN}} \right)^{r-1+q}, \frac{1}{\sqrt{MN}} \right) O \left(\frac{1}{(MN)^{r/2}} \right) \end{aligned} \tag{9.97}$$

Proof We evaluate $\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, -l_1)$ using (9.79) when r is replaced by $r + 1$ and for multi-index $(u_1, \dots, u_{r-1}, l_1, -l_1)$. If $l_1 = \pm u_s$ for some s , the term $\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, -l_1)$ is a $O \left(\frac{1}{(MN)^{(r+1)/2} } \right)$. It is thus sufficient to prove (9.97) when the sum is over the integers l_1 that do not belong to $\{-u_1, \dots, -u_{r-1}\}$ and $\{u_1, \dots, u_{r-1}\}$. In order to simplify the notations, we do not mention in the following that the sum does not take into account $\{-u_1, \dots, -u_{r-1}\}$ and $\{u_1, \dots, u_{r-1}\}$. \square

If l_1 does not belong to $\{-u_1, \dots, -u_{r-1}\}$ and $\{u_1, \dots, u_{r-1}\}$, component $-l_1$ of vector \mathbf{y}_1^* corresponding to $\kappa = (\kappa^{r+1}(u_1, \dots, u_{r-1}, l_1, u))_{u=-(L-1), \dots, (L-1)}$ can be written as

$$\mathbf{y}_{1,-l_1}^* = \kappa^{(r-1)}(u_1, \dots, u_{r-1}) O \left(\frac{1}{MN} \right)$$

(see (9.47)). Therefore, for $l_1 \neq \pm u_s, s = 1, \dots, r - 1$, (9.79) implies that

$$\begin{aligned} & |\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, -l_1)| \leq |\kappa^{(r-1)}(u_1, \dots, u_{r-1})| O \left(\frac{1}{MN} \right) \\ & + |\kappa^{(r)}(u_1, \dots, u_{r-1}, l_1)| O \left(\frac{L}{MN} \right) \\ & + C \sup_u \sum_{l_2} |\kappa^{(r+2)}(u_1, \dots, u_{r-1}, l_1, l_2, u - l_2)| \\ & + \left| \sum_{l_2, l_3} \kappa^{(r+2)}(u_1, \dots, u_{r-1}, l_1, l_2, l_3) O \left(\frac{L}{MN} \right) \right| \\ & + \sup_u |\tilde{x}_{2,l_1}^{(1)}(u)| + \sup_u |\tilde{x}_{3,l_1}^{(1)}(u)| + O \left(\frac{1}{(MN)^{(r+2)/2} } \right) \end{aligned} \tag{9.98}$$

where we indicate that the terms $\tilde{x}_j^{(1)}(u)$ associated with $(u_1, \dots, u_{r-1}, l_1, u)$ depend on l_1 (these terms also depend on $(u_s)_{s \leq r-1}$ but it is not useful to mention this dependency). In the following, we denote by $\alpha^{(1)}(u_1, \dots, u_{r-1})$ the term

$$\alpha^{(1)}(u_1, \dots, u_{r-1}) = \sum_{l_1} |\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, -l_1)|$$

(9.98) implies that

$$\begin{aligned} \alpha^{(1)}(u_1, \dots, u_{r-1}) &\leq |\kappa^{(r-1)}(u_1, \dots, u_{r-1})| O\left(\frac{L}{MN}\right) \\ &+ \sum_{l_1} |\kappa^{(r)}(u_1, \dots, u_{r-1}, l_1)| O\left(\frac{L}{MN}\right) \\ &+ C \sup_u \sum_{l_1, l_2} |\kappa^{(r+2)}(u_1, \dots, u_{r-1}, l_1, l_2, u - l_2)| \\ &+ \left| \sum_{l_1, l_2, l_3} \kappa^{(r+2)}(u_1, \dots, u_{r-1}, l_1, l_2, l_3) O\left(\frac{L}{MN}\right) \right| \\ &+ \sup_u \sum_{l_1} |\tilde{x}_{2, l_1}^{(1)}(u)| + \sup_u \sum_{l_1} |\tilde{x}_{3, l_1}^{(1)}(u)| + O\left(\frac{L}{(MN)^{(r+2)/2}}\right) \end{aligned} \tag{9.99}$$

(9.96) for $s = r - 1$ implies that

$$|\kappa^{(r-1)}(u_1, \dots, u_{r-1})| = \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r-2+q}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{(r-1)/2}}\right)$$

and that

$$\begin{aligned} &|\kappa^{(r-1)}(u_1, \dots, u_{r-1})| O\left(\frac{L}{MN}\right) \\ &= \frac{L}{\sqrt{MN}} \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r-2+q}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r/2}}\right) \end{aligned}$$

which, of course, is also a $\max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r-1+q}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r/2}}\right)$ term as expected.

In order to evaluate the second term of the right-hand side of (9.99), we first notice that if $l_1 \in \{-u_1, \dots, -u_{r-1}\}$, the Hölder inequality leads to

$$|\kappa^{(r)}(u_1, \dots, u_{r-1}, l_1)| O\left(\frac{L}{MN}\right) = o\left(\frac{1}{(MN)^{(r+1)/2}}\right)$$

If $l_1 + u_s \neq 0$ for each $s = 1, \dots, r - 1$, we use (9.95) for $s = r$ and $(v_1, \dots, v_r) = (u_1, \dots, u_{r-1}, l_1)$. It holds that

$$\kappa^{(r)}(u_1, \dots, u_{r-1}, l_1) = \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r-1+q-1}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r/2}}\right)$$

so that

$$\begin{aligned} & \sum_{l_1 \neq -u_s, s=1, \dots, r-1} |\kappa^{(r)}(u_1, \dots, u_{r-1}, l_1)| O\left(\frac{L}{MN}\right) \\ &= \left(\frac{L}{\sqrt{MN}}\right)^2 \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r-1+q-1}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r/2}}\right) \end{aligned}$$

which is a $\max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r-1+q}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r/2}}\right)$ term. The fourth term of the right-hand side of (9.99) is evaluated similarly. Moreover, following the arguments used to establish Lemma 9.6, it can be shown that

$$\begin{aligned} & \sup_u \sum_{l_1} |\tilde{x}_{2,l_1}^{(1)}(u)| + \sup_u \sum_{l_1} |\tilde{x}_{3,l_1}^{(1)}(u)| \\ &= \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r-1+q}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r/2}}\right) \end{aligned}$$

It remains to evaluate the third term of the right-hand side of (9.99). The supremum over $u \neq 0$ is as expected, but the term corresponding to $u = 0$ has also to be evaluated. We denote $\alpha^{(2)}(u_1, \dots, u_{r-1})$ the term

$$\alpha^{(2)}(u_1, \dots, u_{r-1}) = \sum_{l_1, l_2} \left| \kappa^{(r+2)}(u_1, \dots, u_{r-1}, l_1, l_2, -l_2) \right|$$

The previous discussion implies that

$$\begin{aligned} \alpha^{(1)}(u_1, \dots, u_{r-1}) &\leq C \alpha^{(2)}(u_1, \dots, u_{r-1}) \\ &\quad + \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r-1+q}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r/2}}\right) \end{aligned}$$

It can be shown similarly that

$$\begin{aligned} \alpha^{(2)}(u_1, \dots, u_{r-1}) &\leq C \alpha^{(3)}(u_1, \dots, u_{r-1}) \\ &\quad + \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r-1+q}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r/2}}\right) \end{aligned}$$

where

$$\alpha^{(3)}(u_1, \dots, u_{r-1}) = \sum_{l_1, l_2, l_3} |\kappa^{(r+3)}(u_1, \dots, u_{r-1}, l_1, l_2, l_3, -l_3)|$$

More generally, if $\alpha^{(p)}(u_1, \dots, u_{r-1})$ is defined by

$$\alpha^{(p)}(u_1, \dots, u_{r-1}) = \sum_{l_i, i=1, \dots, p} |\kappa^{(r+p)}(u_1, \dots, u_{r-1}, (l_i, i = 1, \dots, p), -l_p)|$$

it holds that

$$\alpha^{(p-1)}(u_1, \dots, u_{r-1}) \leq C \alpha^{(p)}(u_1, \dots, u_{r-1}) + \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r-1+q}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r/2}}\right)$$

and consequently that

$$\alpha^{(1)}(u_1, \dots, u_{r-1}) \leq C \alpha^{(p)}(u_1, \dots, u_{r-1}) + \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r-1+q}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r/2}}\right) \tag{9.100}$$

The Hölder inequality leads immediately to

$$\alpha^{(p)}(u_1, \dots, u_{r-1}) = \left(\frac{L}{\sqrt{MN}}\right)^p O\left(\frac{1}{(MN)^{r/2}}\right)$$

and choosing $p = r - 1 + q$ provides (9.97).

We finally complete the proof of Proposition 9.10. The use of Lemma 9.7 for $r = 2$ establishes immediately that if (9.93) holds until integer $q - 1$ for each s , then it also holds for integer q and $r = 2$. We assume that (9.93) holds for integer q until integer $r - 1$, i.e. that both (9.95) and (9.96) hold, and prove that it also holds for integer r , i.e. that

$$\kappa^{(r)}(u_1, \dots, u_r) = \max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r-1+q}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r/2}}\right)$$

For this, we use (9.79). All the terms of the right-hand side of (9.79) are easily seen to be as expected, except the second one. However, Lemma 9.7 implies that the second term is also a $\max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r-1+q}, \frac{1}{\sqrt{MN}}\right) O\left(\frac{1}{(MN)^{r/2}}\right)$. This completes the proof of Proposition 9.10.

We are now in position to establish (9.42)

Corollary 9.1 *If (u_1, \dots, u_r) satisfy $u_t + u_s \neq 0$ for $t \neq s, 1 \leq t, s \leq r$, then (9.42) holds for $r \geq 2$.*

Proof As $L = N^\alpha$ with $\alpha < 2/3$, it exists an integer q_0 for which $(\frac{L}{\sqrt{MN}})^{r-1+q_0} = o(\frac{1}{\sqrt{MN}})$. Therefore,

$$\max\left(\left(\frac{L}{\sqrt{MN}}\right)^{r-1+q_0}, \frac{1}{\sqrt{MN}}\right) = \frac{1}{\sqrt{MN}}$$

(9.93) for $q = q_0$ thus implies (9.42). □

It remains to establish (9.41). For this, we take (9.80) as a starting point and prove that the right-hand side of (9.80) is a $O\left(\frac{L}{(MN)^2}\right)$ term. We first justify that:

$$\sup_u \tilde{x}^{(1)}(u) = O\left(\frac{L}{(MN)^2}\right) \tag{9.101}$$

We use the decomposition (9.91) of $\tilde{x}^{(1)}(u)$ for the following convenient value of p : we recall that the Hölder inequality implies that

$$\tilde{x}^{(p)}(u) = \left(\frac{L}{\sqrt{MN}}\right)^{p+1} O\left(\frac{1}{MN}\right)$$

As $L = N^\alpha$ with $\alpha < 2/3$, it exists p for which

$$\left(\frac{L}{\sqrt{MN}}\right)^{p+1} = o\left(\frac{L}{MN}\right)$$

For such a value of p , it holds that

$$\tilde{x}^{(p)}(u) = o\left(\frac{L}{(MN)^2}\right)$$

Using (9.87) for $r = 2$ as well as (9.42), it is easy to check that $\tilde{x}^{(1)}(u) - \tilde{x}^{(2)}(u)$ is a $O\left(\frac{L}{(MN)^2}\right)$ term and that the same holds true for $\tilde{x}^{(q)}(u) - \tilde{x}^{(q+1)}(u)$ for each $q \geq 1$. This establishes (9.101).

(9.42) implies that the second term of the right-hand side of (9.80) is a $O\left(\left(\frac{L}{MN}\right)^3\right) = o\left(\frac{L}{(MN)^2}\right)$ term. It remains to establish that

$$\sum_{l_1} |\kappa^3(u_1, l_1, -l_1)| = O\left(\frac{L}{(MN)^2}\right) \tag{9.102}$$

Lemma 9.7 for $q = q_0$ (where q_0 is defined in the proof of Corollary 9.1 for $r = 2$) implies that this term is $O\left(\frac{1}{(MN)^{3/2}}\right)$, but this evaluation is not sufficient to prove (9.41).

Using (9.42), we now evaluate $\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, -l_1)$ when $u_{s_1} + u_{s_2} \neq 0$ for $s_1 \neq s_2, -(L - 1) \leq s_1, s_2 \leq L - 1$ and for $r \geq 2$.

Lemma 9.8 We consider $r \geq 2$ and a multi-index $(u_1, \dots, u_{r-1}, l_1, -l_1)$ such that $u_{s_1} + u_{s_2} \neq 0$ for $s_1 \neq s_2, -(L - 1) \leq s_1, s_2 \leq L - 1$. Then,

– if $l_1 \pm u_s \neq 0$ for $s = 1, \dots, r - 1$, it holds that

$$\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, -l_1) = O\left(\frac{1}{(MN)^{(r+2)/2}}\right) \tag{9.103}$$

– if $l_1 \pm u_s = 0$ for some $s = 1, \dots, r - 1$,

$$\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, -l_1) = \frac{L}{\sqrt{MN}} O\left(\frac{1}{(MN)^{(r+1)/2}}\right) \tag{9.104}$$

Proof The proof is similar to the proof of Lemma 9.7. We take (9.98) as a starting point, but just evaluate $\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, -l_1)$ instead of $\alpha^{(1)}(u_1, \dots, u_{r-1})$ by iterating (9.98). Using (9.42), it is easy to check that for each l_1 ,

$$\sup_u |\tilde{x}_{j,l_1}^{(1)}(u)| = O\left(\frac{1}{(MN)^{(r+2)/2}}\right)$$

We first assume that $l_1 \pm u_s \neq 0$ for $s = 1, \dots, r - 1$. (9.42) implies that the first term of the right-hand side of (9.98) is $O\left(\frac{1}{(MN)^{(r+2)/2}\right)$ (and is identically 0 if $r = 2$). The second term is $\frac{L}{\sqrt{MN}} \mathcal{O}\left(\frac{1}{(MN)^{(r+2)/2}\right)$ while the fourth term is $\left(\frac{L}{\sqrt{MN}}\right)^2 \mathcal{O}\left(\frac{1}{(MN)^{(r+2)/2}\right)$. The supremum over $u \neq 0$ of the third term is $\mathcal{O}\left(\frac{1}{(MN)^{(r+2)/2}\right)$ which implies that

$$\begin{aligned} |\kappa^{(r+1)}(u_1, \dots, u_{r-1}, l_1, -l_1)| &\leq \sum_{l_2} \left| \kappa^{(r+2)}(u_1, \dots, u_{r-1}, l_1, l_2, -l_2) \right| \\ &+ \mathcal{O}\left(\frac{1}{(MN)^{(r+2)/2}\right) \end{aligned}$$

As in the proof of Lemma (9.7), we iterate this inequality until an index p for which

$$\sum_{l_2, \dots, l_p} \left| \kappa^{(r+p)}(u_1, \dots, u_{r-1}, l_1, l_2, \dots, l_p, -l_p) \right| = O\left(\frac{L^{p-1}}{(MN)^{(r+p)/2}\right)$$

is a $o\left(\frac{1}{(MN)^{(r+2)/2}\right)$ term. This, in turn, proves (9.103). (9.104) follows directly from the use of the Hölder inequality in (9.98). □

We now complete the proof of (9.102). For this, we remark that

$$\sum_{l_1} |\kappa^3(u_1, l_1, -l_1)| = \sum_{l_1 \neq \pm u_1} |\kappa^3(u_1, l_1, -l_1)| + 2|\kappa^3(u_1, -u_1, u_1)|$$

Lemma 9.8 implies that

$$\sum_{l_1 \neq \pm u_1} |\kappa^3(u_1, l_1, -l_1)| = O\left(\frac{L}{(MN)^2}\right)$$

and that

$$|\kappa^3(u_1, -u_1, u_1)| = O\left(\frac{L}{(MN)^2}\right)$$

This establishes (9.102) as well as (9.41).

9.3 Expansion of $\frac{1}{ML} \text{Tr}(\Delta(z))$

In the following, we establish (9.5). We recall that (5.2) implies that $\frac{1}{ML} \text{Tr}(\Delta(z))$ is given by

$$\begin{aligned} & \frac{1}{ML} \text{Tr}(\Delta(z)) \\ &= \sigma^2 c_N \sum_{l_1 = -(L-1)}^{L-1} \mathbb{E} \left(\tau^{(M)}(\mathbf{Q}^\circ)(l_1) \frac{1}{ML} \text{Tr} \left(\mathbf{QWJ}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R}) \right)^\circ \right) \end{aligned}$$

In the following, we denote by $\tilde{x}(l_1)$ and \tilde{x} the terms defined by

$$\tilde{x}(l_1) = \mathbb{E} \left(\tau^{(M)}(\mathbf{Q}^\circ)(l_1) \frac{1}{ML} \text{Tr} \left(\mathbf{QWJ}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R}) \right)^\circ \right)$$

and

$$\tilde{x} = \sum_{l_1 = -(L-1)}^{L-1} \mathbb{E} \left(\tau^{(M)}(\mathbf{Q}^\circ)(l_1) \frac{1}{ML} \text{Tr} \left(\mathbf{QWJ}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R}) \right)^\circ \right)$$

$\tilde{x}(l_1)$ and \tilde{x} appear to be formally similar to $\tilde{x}(0, l_1)$ and $\tilde{x}(0)$ defined by (9.51) and (9.52) in the particular case $r = 1$. While we have considered in the previous subsection the case $r \geq 2$, a number of evaluations and results can be adapted to the easier case $r = 1$. As in Sect. 9.2, we expand $\tilde{x}(l_1)$ and \tilde{x} using (9.44) in the case $r = 1$, $v_1 = l_1$, $\mathbf{G} = \mathbf{J}_N^{l_1} \mathbf{H}^T$, and $\mathbf{A} = (\mathbf{I}_M \otimes \mathbf{R})$. Using the same notations as in Sect. 9.2, we obtain that

$$\tilde{x}(l_1) = \sum_{j=2}^5 s_j(l_1)$$

and

$$\tilde{x} = \sum_{j=2}^5 s_j$$

where $s_j = \sum_{l_1} s_j(l_1)$. We note that the term s_1 is reduced to 0 in the present context. It is easy to check that

$$s_4(l_1) = \frac{\sigma^2}{MN} \sum_{i=-(L-1)}^{L-1} \mathbb{E}(\beta_{0,1}(i, l_1, l_1, 0))$$

and that

$$s_5(l_1) = -\frac{\sigma^2}{MN} \sum_{i=-(L-1)}^{L-1} \mathbb{E}(\beta_{1,0}(i, l_1, l_1, 0))$$

where the terms β are defined by (9.33). Proposition 9.4 immediately implies that

$$s_4 = \frac{\sigma^2}{MN} \sum_{l_1} \left(\frac{1}{L} \sum_i \bar{\beta}_{0,1}(i, l_1, l_1) \right) + O\left(\left(\frac{L}{MN} \right)^2 \right)$$

or equivalently,

$$s_4(z) = \sigma^2 \frac{L}{MN} \bar{\beta}_{0,1}(z) + O\left(\frac{L^2}{(MN)^2} \right)$$

where $\bar{\beta}_{0,1}(z)$ is defined as

$$\bar{\beta}_{0,1}(z) = \frac{1}{L^2} \sum_{l_1, i} \bar{\beta}_{0,1}(i, l_1, l_1)(z)$$

Similarly, it holds that

$$s_5(z) = -\sigma^2 \frac{L}{MN} \bar{\beta}_{1,0}(z) + O\left(\frac{L^2}{(MN)^2} \right)$$

where

$$\bar{\beta}_{1,0}(z) = \frac{1}{L^2} \sum_{l_1, i} \bar{\beta}_{1,0}(i, l_1, l_1)(z)$$

We have now to evaluate $s_2(z)$ and $s_3(z)$. For $j = 2, 3$, s_j can be written as

$$s_j = \bar{s}_j + \tilde{x}_j^{(1)}$$

We first evaluate \bar{s}_3 and $\bar{s}_2 \cdot \bar{s}_3$ is equal to

$$\bar{s}_3 = -\sigma^2 c_N \sum_{l_1, l_2} \kappa^{(2)}(l_1, l_2) \mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\mathbf{QWJ}_N^{l_2} \mathbf{H}^T \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R}) \right) \right]$$

We remark that

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{ML} \text{Tr} \left(\mathbf{QWJ}_N^{l_2} \mathbf{H}^T \mathbf{J}_N^{l_1} \mathbf{H}^T \mathbf{W}^* (\mathbf{I}_M \otimes \mathbf{R}) \right) \right] \\ &= -\sigma^2 t(z)^2 (z\tilde{t}(z))^3 (1 - |l_1|/N) \delta(l_1 + l_2 = 0) + o\left(\frac{L}{MN}\right) \end{aligned}$$

We also have to evaluate $\kappa^{(2)}(l_1, l_2)$. Using (9.47), (9.80), and the observation that the right-hand side of (9.80) is a $\mathcal{O}\left(\frac{L}{(MN)^2}\right)$ term (see (9.41)), we obtain that

$$\kappa^{(2)}(l_1, l_2) = \frac{\sigma^2}{MN} \frac{1}{1 - d(l_1, z)} \frac{1}{L} \sum_i \bar{\beta}(i, l_1) \delta(l_1 + l_2 = 0) + o\left(\frac{L}{(MN)^2}\right)$$

Therefore, \bar{s}_3 can be written as

$$\bar{s}_3(z) = \sigma^6 c_N t(z)^2 (z\tilde{t}(z))^3 \frac{1}{L^2} \sum_{i, l_1} \frac{1 - |l_1|/N}{1 - d(l_1, z)} \bar{\beta}(i, l_1) \frac{L}{MN} + o\left(\frac{L^2}{(MN)^2}\right)$$

Similar calculations lead to

$$\begin{aligned} \bar{s}_2 &= \sigma^8 c_N t(z)^3 (z\tilde{t}(z))^4 \frac{1}{L^2} \sum_{i, l_1} \frac{(1 - |l_1|/N)^2 (1 - |l_1|/L)}{1 - d(l_1, z)} \bar{\beta}(i, l_1) \frac{L}{MN} \\ &+ o\left(\frac{L^2}{(MN)^2}\right) \end{aligned}$$

Therefore, it holds that

$$\bar{s}_2(z) + \bar{s}_3(z) + s_4(z) + s_5(z) = \frac{L}{MN} \frac{1}{L^2} \sum_{i, l_1} s(i, l_1, z) + \tilde{x}_2^{(1)}(z) + \tilde{x}_3^{(1)}(z) + o\left(\frac{L^2}{(MN)^2}\right) \tag{9.105}$$

where $s(i, l_1, z)$ is defined by (9.37). Proposition 9.4 implies that function $s_N(z)$ defined by

$$s_N(z) = \sigma^2 c_N \frac{1}{L^2} \sum_{i, l_1} s_N(i, l_1, z)$$

coincides with the Stieltjes transform of a distribution whose support is included into $\mathcal{S}_N^{(0)}$ and satisfying (9.8) for $\mathcal{K} = \mathcal{S}_N^{(0)}$. In order to complete the proof of (9.5), we finally prove that $\tilde{x}^{(1)} = |\tilde{x}_2^{(1)}| + |\tilde{x}_3^{(1)}|$ is a $O\left(\frac{L^2}{(MN)^2}\right)$ term. For this, we remark that $\tilde{x}^{(1)}$ verifies (9.87) in the case $r = 1$ and $u = 0$. However, the term $\frac{L}{\sqrt{MN}} O\left(\frac{1}{(MN)^{(r+1)/2}\right)$ (for $r = 1$) is replaced by a $O\left(\frac{L^2}{(MN)^2}\right)$ term. This term corresponds to the contribution of the $s_{j,i}^{(1)}$ for $j = 2, 3$ and $i = 4, 5$. In the present context, $r = 1$ and it is easy to check that $\bar{s}_{j,i}^{(1)}$ is identically zero and that $s_{j,i}^{(1)}$ coincides with $\tilde{s}_{j,i}^{(1)}$, which, using the Hölder inequality, appears to be a $O\left(\frac{L^2}{(MN)^2}\right)$ term. In order to prove that $\tilde{x}^{(1)} = O\left(\frac{L^2}{(MN)^2}\right)$, we use (9.91) as in the proof of Lemma 9.6. The Hölder inequality implies that

$$\tilde{x}^{(p)} = \left(\frac{L}{\sqrt{MN}}\right)^{p+1} O\left(\frac{1}{\sqrt{MN}}\right) = \left(\frac{L}{\sqrt{MN}}\right)^p O\left(\frac{L}{MN}\right)$$

As $L = N^\alpha$ with $\alpha < 2/3$, it exists an integer p_1 such that

$$\left(\frac{L}{\sqrt{MN}}\right)^{p_1} = o\left(\frac{L}{MN}\right)$$

Therefore, using (9.91) for $p = p_1$, we obtain as in the proof of Lemma 9.6 that $\tilde{x}^{(1)} = O\left(\frac{L^2}{(MN)^2}\right)$ as expected. This, in turn, completes the proof of (9.5).

9.4 Evaluation of $\mathbb{E}\left(\frac{1}{ML} \text{Tr}(\mathbf{Q}_N(z))\right) - t_N(z)$

In order to establish (9.3), we evaluate $\frac{1}{L} \text{Tr}(\mathbf{R}_N(z)) - t_N(z)$. For this, we use (8.4) for $\mathbf{A} = \mathbf{I}$. We claim that the third, fourth, and fifth terms of the right-hand side of (8.4) are $O\left(\frac{L^{5/2}}{(MN)^2}\right)$. We just check the third term. It is clear that

$$\left| \frac{1}{L} \text{Tr} \left((\mathbf{R} - t\mathbf{I}) \mathcal{T}_{L,L} \left[\mathcal{T}_{N,L}(\mathbf{R} - t\mathbf{I}) \right] \right) \right| \leq \sup_{\|\mathbf{A}\| \leq 1} \left| \frac{1}{L} \text{Tr} \left((\mathbf{R} - t\mathbf{I}) \mathbf{A} \right) \right| \|\mathcal{T}_{N,L}(\mathbf{R} - t\mathbf{I})\|$$

Proposition 8.1 and (7.1) immediately implies that the third term of the right-hand side of (8.4) is a $O\left(\frac{L^{5/2}}{(MN)^2}\right)$ term. The fourth and the fifth term can be addressed similarly.

The first term is equal to

$$-\sigma^4 c_N(zt(z)\tilde{t}(z)) \frac{1}{ML} \text{Tr}(\Delta (\mathbf{I}_M \otimes \mathcal{T}_{L,L} [\mathcal{T}_{N,L}(\mathbf{R})\mathbf{H}]))$$

Writing that $\mathbf{R} = t\mathbf{I} + \mathbf{R} - t\mathbf{I}$ and $\mathbf{H} = -z\tilde{t}(z) + \mathbf{H} + z\tilde{t}(z)$, and using (5.1), Proposition 8.1 and (7.1), we obtain that

$$\frac{1}{ML} \text{Tr}(\Delta (\mathbf{I}_M \otimes \mathcal{T}_{L,L} [\mathcal{T}_{N,L}(\mathbf{R})\mathbf{H}])) = -zt(z)\tilde{t}(z) \frac{1}{ML} \text{Tr}(\Delta) + O\left(\frac{L^{5/2}}{(MN)^2}\right)$$

Therefore, we deduce from (8.4) that

$$\frac{1}{L} \text{Tr}(\mathbf{R}_N(z)) - t_N(z) = \frac{d_N(0, z)}{1 - d_N(0, z)} \frac{1}{ML} \text{Tr}(\Delta_N(z)) + O\left(\frac{L^{5/2}}{(MN)^2}\right)$$

This, in turn, implies that

$$\mathbb{E}\left(\frac{1}{ML} \text{Tr}(\mathbf{Q}_N(z))\right) - t_N(z) = \frac{L}{MN} \frac{s_N(z)}{1 - d_N(0, z)} + O\left(\frac{L^{5/2}}{(MN)^2}\right)$$

and that (9.3) holds with $\hat{s}_N(z) = \frac{s_N(z)}{1 - d_N(0, z)}$, which has the same properties that $s_N(z)$. This, in turn, establishes Theorem 9.1.

10 Almost Sure Location of the Eigenvalues of $\mathbf{W}\mathbf{W}^*$

Under condition (9.1), we finally establish that the eigenvalues of $\mathbf{W}_N \mathbf{W}_N^*$ lie almost surely in a neighbourhood of the support of the Marcenko–Pastur distribution.

Theorem 10.1 *If $c_* \leq 1$, for each $\epsilon > 0$, almost surely, it exists $N_0 \in \mathbb{N}$ such that all the eigenvalues of $\mathbf{W}_N \mathbf{W}_N^*$ belong to $[\sigma^2 (1 - \sqrt{c_*})^2 - \epsilon, \sigma^2 (1 + \sqrt{c_*})^2 + \epsilon]$ for $N > N_0$. If $c_* > 1$, for each $\epsilon > 0$, almost surely, it exists $N_0 \in \mathbb{N}$ such that the N nonzero eigenvalues of $\mathbf{W}_N \mathbf{W}_N^*$ belong to $[\sigma^2 (1 - \sqrt{c_*})^2 - \epsilon, \sigma^2 (1 + \sqrt{c_*})^2 + \epsilon]$ for $N > N_0$.*

The proof follows [17] and the Lemma 5.5.5 of [2] which needs to verify conditions that are less demanding than in [17].

We first establish the following lemma.

Lemma 10.1 *For all $\psi \in C_b^\infty(\mathbb{R})$ constant on the complementary of a compact interval, and vanishing on \mathcal{S}_N for each N large enough, it holds that:*

$$\mathbb{E}[\text{Tr}(\psi(\mathbf{W}_N \mathbf{W}_N^*))] = O\left(\left(\frac{L}{M^2}\right)^{3/2}\right) \tag{10.1}$$

$$\mathbb{E} \left| \text{Tr} (\psi (\mathbf{W}_N \mathbf{W}_N^*)) - \mathbb{E} (\psi (\mathbf{W}_N \mathbf{W}_N^*)) \right|^{2l} = o \left[\left(\frac{L^{3/2}}{M^4} \right)^l \right] \tag{10.2}$$

for each $l \geq 1$.

Proof In order to establish (10.1), we first justify that for each smooth compactly supported function ψ_c , then it holds that

$$\begin{aligned} & \mathbb{E} [\text{Tr} (\psi_c (\mathbf{W}_N \mathbf{W}_N^*))] - ML \int \psi_c(\lambda) d\mu_{\sigma^2, c_N}(\lambda) - ML \frac{L}{MN} \langle \hat{D}_N, \psi_c \rangle \\ & = o \left(\left(\frac{L}{M^2} \right)^{3/2} \right) \end{aligned} \tag{10.3}$$

(10.3) is a consequence of Theorem 9.1. In order to prove (10.3), we cannot use Theorem 6.2 of [17] because function $\hat{r}_N(z)$ defined by (9.3) does not satisfy $|\hat{r}_N(z)| \leq P_1(|z|)P_2(1/\text{Im}z)$ for each $z \in \mathbb{C}^+$, but when z belongs to the set $F_N^{(2)}$ defined by (9.4). To solve this issue, we use the approach of [2] based on the Helffer–Sjöstrand formula which is still valid when $|\hat{r}_N(z)|$ is controlled by $P_1(|z|)P_2(1/\text{Im}z)$ for $z \in F_N^{(2)}$. □

As we have proved in Lemma 9.3 that the Helffer–Sjöstrand formula is valid for compactly supported distributions, (10.3) follows directly from Lemma 5.5.5 of [2] provided we verify that for each nice constants C_0, C'_0 , it exist nice constants C_1, C_2, C_3 and an integer N_0 such that

$$\left| \frac{1}{ML} \mathbb{E} (\text{Tr} \mathbf{Q}_N(z)) - t_N(z) - \frac{L}{MN} \hat{s}_N(z) \right| \leq C_2 \frac{L^{5/2}}{(MN)^2} \frac{1}{(\text{Im}z)^{C_3}} \tag{10.4}$$

for each z in the domain $|\text{Re}(z)| \leq C_0, \frac{1}{N^{C_1}} \leq \text{Im}(z) \leq C'_0$ and for each $N > N_0$.

In order to check that (10.4) holds, we fix nice constants C_0, C'_0 , and first show that it exists C_1 such that the above domain, denoted E_{N, C_1} , is included in the set $F_N^{(2)}$ defined by (9.4) for N large enough. It is clear that for each $z \in E_{N, C_1}$, it holds that

$$Q_1(|z|)Q_2(1/\text{Im}z) \leq Q_1 \left((C_0^2 + C_0'^2)^{1/2} \right) Q_2(N^{C_1}) \leq C N^{q_2 C_1}$$

for some nice constant C , where $q_2 = \text{Deg}(Q_2)$. Hence,

$$\frac{L^2}{MN} Q_1(|z|)Q_2(1/\text{Im}z) \leq C \frac{L^2}{MN} N^{q_2 C_1}$$

Using that $N = \mathcal{O}(ML)$, we obtain immediately that

$$\frac{L^2}{MN} Q_1(|z|)Q_2(1/\text{Im}z) \leq C \frac{L^{1+q_2 C_1}}{M^{2-q_2 C_1}}$$

Condition (9.1) implies that

$$\frac{L^{1+q_2C_1}}{M^{2-q_2C_1}} = O\left(\frac{1}{N^{2-3\alpha-q_2C_1}}\right)$$

We choose $C_1 > (2 - 3\alpha)/q_2$ so that $\frac{L^{1+q_2C_1}}{M^{2-q_2C_1}} \rightarrow 0$. Therefore, $\frac{L^2}{MN} Q_1(|z|)Q_2(1/Imz)$ is less than 1 for N large enough. We have thus shown the existence of a nice constant C_1 for which $D_{N,C_1} \subset F_N^{(2)}$ for N large enough. Hence, for each $z \in E_{N,C_1}$,

$$\left| \frac{1}{ML} \mathbb{E}(\text{Tr} \mathbf{Q}_N(z)) - t_N(z) - \frac{L}{MN} \hat{s}_N(z) \right| \leq \frac{L^{5/2}}{(MN)^2} P_1(|z|)P_2(1/Imz)$$

We now prove that if $z \in E_{N,C_1}$, then $P_1(|z|)P_2(1/Imz) \leq C_2 \frac{1}{(Imz)^{C_3}}$ for some nice constants C_2 and C_3 . We remark that $P_1(|z|) \leq P_1\left((C_0^2 + C_0'^2)^{1/2}\right)$ and denote by p_2 and $(P_{2,i})_{i=0,\dots,p_2}$ the degree and the coefficients of P_2 , respectively. If $Imz \leq 1$, it is clear that $P_2(1/Imz) \leq \left(\sum_{i=0}^{p_2} P_{2,i}\right) \frac{1}{(Imz)^{p_2}}$. This completes the proof of (10.4) if $C_0' \leq 1$. If $C_0' > 1$, it remains to consider the case where $z \in D_{N,C_1}$ verifies $1 < Imz \leq C_0'$. It is clear that $\frac{1}{Imz} \leq \frac{C_0'}{Imz}$. Therefore,

$$P_2(1/Imz) \leq P_2\left(\frac{C_0'}{Imz}\right) \leq \left(\sum_{i=0}^{p_2} P_{2,i}\right) \frac{(C_0')^{p_2}}{(Imz)^{p_2}}$$

In sum, we have proved that $P_1(|z|)P_2(1/Imz) \leq C_2 \frac{1}{(Imz)^{p_2}}$ for some nice constant C_2 and for each $z \in E_{N,C_1}$, which, in turn, establishes (10.4).

Equation (9.12) allows us to follow the arguments of the proof of Lemma 5.5.5 of [2] and to establish (10.3). In order to prove (10.1), we follow [17]. We denote by κ the constant for which $\psi(\lambda) = \kappa$ outside a compact subset. Function $\psi_c = \psi - \kappa$ is thus compactly supported and is equal to $-\kappa$ on S_N for N large enough. Therefore,

$$\int \psi_c(\lambda) d\mu_{\sigma^2, c_N}(\lambda) = -\kappa \text{ and } \langle \hat{D}_N, \psi_c \rangle = 0$$

and (10.3) implies (10.1).

The proof of (10.2) is based on the Poincaré–Nash inequality and is rather standard. A proof is provided in [23].

As $\frac{L^{3/2}}{M^3} \rightarrow 0$, (10.1) and (10.2) for l large enough imply that

$$\text{Tr}(\psi(\mathbf{W}_N \mathbf{W}_N^*)) \rightarrow 0 \text{ a.s.} \tag{10.5}$$

Consider a function $\psi \in C_b^\infty(\mathbb{R})$ such that

$$-\psi(x) = 1 \text{ if } x \in \left([\sigma^2(1 - \sqrt{c_*})^2 - \epsilon, \sigma^2(1 + \sqrt{c_*})^2 + \epsilon\right] \cup [-\epsilon, \epsilon] \mathbb{1}_{c_* > 1} \Big)^c$$

- $\psi(x) = 0$ if $x \in \left([\sigma^2 (1 - \sqrt{c_*})^2 - \epsilon/2, \sigma^2 (1 + \sqrt{c_*})^2 + \epsilon/2] \cup [-\epsilon/2, \epsilon/2] \mathbb{1}_{c_* > 1} \right)$
- $0 \leq \psi(x) \leq 1$ elsewhere

Such a function ψ satisfies the hypotheses of Lemma 10.1. It is clear that the number of eigenvalues of $\mathbf{W}_N \mathbf{W}_N^*$ located into $\left([\sigma^2 (1 - \sqrt{c_*})^2 - \epsilon, \sigma^2 (1 + \sqrt{c_*})^2 + \epsilon] \cup [-\epsilon, \epsilon] \mathbb{1}_{c_* > 1} \right)^c$ is less than $\text{Tr}(\psi(\mathbf{W}_N \mathbf{W}_N^*))$, which, by (10.5), converges almost surely towards 0. This completes the proof of Theorem 10.1 if $c_* \leq 1$. If $c_* > 1$, we consider a function $\psi_c \in C_c^\infty(\mathbb{R})$ such that

- $\psi_c(x) = 1$ if $x \in [-\epsilon/2, \epsilon/2]$
- $\psi_c(x) = 0$ if $x \in [-\epsilon, \epsilon]^c$
- $0 \leq \psi_c(x) \leq 1$ elsewhere

As 0 does not belong to the support of \hat{D}_N , it holds that $\langle \hat{D}_N, \psi_c \rangle = 0$ for each N large enough. Using (10.3) and the observation that function ψ_c satisfies also (10.2), we obtain as above that almost surely, for N large enough, the interval $[-\epsilon, \epsilon]$ contains $ML - N$ eigenvalues of $\mathbf{W}_N \mathbf{W}_N^*$. As $ML - N$ coincides with the multiplicity of eigenvalue 0, this implies that the N remaining (nonzero) eigenvalues are located into $[\sigma^2 (1 - \sqrt{c_*})^2 - \epsilon, \sigma^2 (1 + \sqrt{c_*})^2 + \epsilon]$. This establishes Theorem 10.1 if $c_* > 1$.

Appendix 1: Proof of Proposition 2.3

We first establish (2.14). For this, we first remark that as K coincides with the size of square matrix \mathbf{A} , then, for $i, j \in \{1, 2, \dots, R\}$, it holds that $(\mathcal{T}_{R,K}(\mathbf{A}))_{i,j} = \tau(\mathbf{A})(i - j) \mathbb{1}_{|i-j| \leq (K-1)}$ is equal to

$$(\mathcal{T}_{R,K}(\mathbf{A}))_{i,j} = \frac{1}{K} \sum_{k=1}^K \mathbf{A}_{k+i-j,k} \mathbb{1}_{1 \leq k+i-j \leq K}$$

We establish that for each R -dimensional vector \mathbf{b} , then $\|\mathbf{b}^* \mathcal{T}_{R,K}(\mathbf{A})\|^2 \leq \mathbf{b}^* \mathcal{T}_{R,K}(\mathbf{A} \mathbf{A}^*) \mathbf{b}$. For this, we note that component r of $\mathbf{b}^* \mathcal{T}_{R,K}(\mathbf{A})$ is equal to

$$(\mathbf{b}^* \mathcal{T}_{R,K}(\mathbf{A}))_r = \sum_{i=1}^R \bar{\mathbf{b}}_i \frac{1}{K} \sum_{k=1}^K \mathbf{A}_{k+i-r,k} \mathbb{1}_{1 \leq k+i-r \leq K}$$

Therefore,

$$\|\mathbf{b}^* \mathcal{T}_{R,K}(\mathbf{A})\|^2 = \sum_{r=1}^R \left| \frac{1}{K} \sum_{k=1}^K \sum_{i=1}^R \bar{\mathbf{b}}_i \mathbf{A}_{k+i-r,k} \mathbb{1}_{1 \leq k+i-r \leq K} \right|^2$$

and is thus less than the term a defined by

$$a = \sum_{r=1}^R \frac{1}{K} \sum_{k=1}^K \left| \sum_{i=1}^R \bar{\mathbf{b}}_i \mathbf{A}_{k+i-r,k} \mathbb{1}_{1 \leq k+i-r \leq K} \right|^2$$

a can also be written as

$$a = \sum_{(i,j)=1,\dots,R} \bar{\mathbf{b}}_i \mathbf{b}_j \frac{1}{K} \sum_{r=1}^R \sum_{k=1}^K \mathbf{A}_{k+i-r} \bar{\mathbf{A}}_{k+j-r,k} \mathbb{1}_{1 \leq k+i-r \leq K, 1 \leq k+j-r \leq K}$$

We denote by u the index $u = k - r$ and rewrite a as

$$a = \sum_{(i,j)=1,\dots,R} \bar{\mathbf{b}}_i \mathbf{b}_j \frac{1}{K} \sum_{k=1}^K \sum_{u \in \mathbb{Z}} \mathbb{1}_{1 \leq k-u \leq R} \mathbf{A}_{u+i,k} \bar{\mathbf{A}}_{u+j,k} \mathbb{1}_{1 \leq u+i \leq K, 1 \leq u+j \leq K}$$

or equivalently as,

$$a = \sum_{k=1}^K \sum_{u \in \mathbb{Z}} \mathbb{1}_{1 \leq k-u \leq R} \frac{1}{K} \left| \sum_{i=1}^R \bar{\mathbf{b}}_i \mathbf{A}_{u+i,k} \mathbb{1}_{1 \leq u+i \leq K} \right|^2$$

Therefore, a satisfies

$$a \leq \sum_{k=1}^K \sum_{u \in \mathbb{Z}} \frac{1}{K} \left| \sum_{i=1}^R \bar{\mathbf{b}}_i \mathbf{A}_{u+i,k} \mathbb{1}_{1 \leq u+i \leq K} \right|^2$$

or equivalently

$$a \leq \sum_{(i,j)=1,\dots,R} \bar{\mathbf{b}}_i \mathbf{b}_j \frac{1}{K} \sum_{u \in \mathbb{Z}} (\mathbf{A}\mathbf{A}^*)_{u+i,u+j} \mathbb{1}_{1 \leq u+i \leq K, 1 \leq u+j \leq K}$$

We define index k as $k = u + j$ and remark that

$$\begin{aligned} & \frac{1}{K} \sum_{u \in \mathbb{Z}} (\mathbf{A}\mathbf{A}^*)_{u+i,u+j} \mathbb{1}_{1 \leq u+i \leq K, 1 \leq u+j \leq K} \\ &= \frac{1}{K} \sum_{k=1}^K (\mathbf{A}\mathbf{A}^*)_{k+i-j,k} \mathbb{1}_{1 \leq k+i-j \leq K} = (\mathcal{T}_{R,K}(\mathbf{A}\mathbf{A}^*))_{i,j} \end{aligned}$$

Therefore, we have shown that

$$\|\mathbf{b}^* \mathcal{T}_{R,K}(\mathbf{A})\|^2 \leq a \leq \mathbf{b}^* \mathcal{T}_{R,K}(\mathbf{A}\mathbf{A}^*) \mathbf{b}$$

In order to prove (2.15), it is sufficient to remark that the entry $(i, j), (i, j) \in \{1, 2, \dots, R\}$ of matrix $\mathcal{T}_{R,R}(\mathbf{A})$ is still equal to

$$(\mathcal{T}_{R,R}(\mathbf{A}))_{i,j} = \frac{1}{K} \sum_{k=1}^K \mathbf{A}_{k+i-j,k} \mathbb{1}_{1 \leq k+i-j \leq K}$$

because $R \leq K$ and to follow the proof of (2.14).

Appendix 2: Proof of Lemma 4.1

We use the same ingredients than in the proof of Lemma 5-1 of [16]. Therefore, we just provide a sketch of proof. The invertibility of $\mathbf{I}_N + \sigma^2 c_N \mathcal{T}_{N,L}^{(M)}(\mathbb{E}(\mathbf{Q}(z)))$ for $z \in \mathbb{C}^+$ is a direct consequence of $\text{Im}(\mathbf{Q}(z)) > 0$ on \mathbb{C}^+ (see (1.26)) as well as of Proposition 2.2. In order to prove (4.2), we first establish that function $\mathbf{G}(z)$ defined by

$$\mathbf{G}(z) = -\frac{\mathbf{H}(z)}{z}$$

coincides with the Stieltjes transform of a positive $\mathbb{C}^{N \times N}$ matrix-valued measure ν carried by \mathbb{R}^+ such that $\nu(\mathbb{R}^+) = \mathbf{I}_N$, i.e.

$$\mathbf{G}(z) = \int_{\mathbb{R}^+} \frac{d\nu(\lambda)}{\lambda - z}$$

For this, it is sufficient to check that $\text{Im}(\mathbf{G}(z))$ and $\text{Im}(z\mathbf{G}(z))$ are both positive on \mathbb{C}^+ and that $\lim_{y \rightarrow +\infty} -iy \mathbf{G}(iy) = \mathbf{I}_N$ (see proof of Lemma 5-1 of [16]). We omit the corresponding derivations. It is clear that

$$\text{Im}(\mathbf{G}(z)) = \text{Im}(z) \int_{\mathbb{R}^+} \frac{d\nu(\lambda)}{|\lambda - z|^2} \leq \frac{1}{\text{Im}(z)} \mathbf{I}_N$$

for $z \in \mathbb{C}^+$. $\text{Im}(\mathbf{G}(z))$ can also be written as

$$\text{Im}(\mathbf{G}(z)) = \frac{\mathbf{H}(z)}{z} \frac{1}{2i} \left[z\mathbf{H}^{-1}(z) - z^* \left(\mathbf{H}^{-1}(z) \right)^* \right] \frac{\mathbf{H}(z)^*}{z^*}$$

or equivalently as

$$\text{Im}(\mathbf{G}(z)) = \frac{\mathbf{H}(z)}{z} \left[\text{Im}(z) + \sigma^2 c_N \mathcal{T}_{N,L}^{(M)}(\text{Im}(z\mathbf{Q}(z))) \right] \frac{\mathbf{H}(z)^*}{z^*}$$

As $\text{Im}(z\mathbf{Q}(z)) > 0$ on \mathbb{C}^+ (see (1.26)), this implies that

$$\frac{1}{\text{Im}(z)} \mathbf{I}_N \geq \text{Im}(\mathbf{G}(z)) > \frac{\text{Im}(z)}{|z|^2} \mathbf{H}(z)\mathbf{H}(z)^*$$

which implies (4.2). The other statements of Lemma 4.1 are proved similarly.

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