PERFORMANCE ANALYSIS OF SPATIAL SMOOTHING SCHEMES IN THE CONTEXT OF LARGE ARRAYS.

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ABSTRACT

This paper addresses the statistical behaviour of spatial smoothing subspace DoA estimation schemes using a sensor array in the case where the number of observations N is significantly smaller than the number of sensors M, and that the number of virtual arrays L is such that M and NL are of the same order of magnitude. This context is modelled by an asymptotic regime in which NL and M both converge towards ∞ at the same rate. As in recent works devoted to the study of (unsmoothed) subspace methods in the case where M and N are of the same order of magnitude, it is shown that it is still possible to derive improved DoA estimators termed as Generalized-MUSIC (G-MUSIC). The key ingredient of this work is a technical result showing that the largest singular values and corresponding singular vectors of low rank deterministic perturbation of certain Gaussian block-Hankel large random matrices behave as if the entries of the latter random matrices were independent identically distributed.

1. INTRODUCTION

The statistical analysis of subspace DoA estimation methods using an array of sensors is a topic that has received a lot of attention since the seventies. Most of the works were devoted to the case where the number of available samples N of the observed signal is much larger than the number of sensors M of the array (see e.g. [11] and the references therein), The case where M and N are large and of the same order of magnitude was addressed for the first time in [9] using large random matrix theory. [9] was followed by various works such as [5], [14], [7]. In this paper, the number of observations may also be much smaller than the number of sensors. In this context, it is well established that spatial smoothing schemes, originally developed to address coherent sources ([1], [13], [12]), can be used to artificially increase the number of snapshots (see e.g. [11] and the references therein, see also the recent related contributions [3], [4] devoted to the case where $N = 1$). Spatial smoothing consists in considering $L < M$ overlapping arrays with $M - L + 1$ sensors, and allows to generate artificially NL snapshots observed on a virtual array of $M - L + 1$ sensors. $(M - L + 1) \times NL$ matrix $\mathbf{Y}_N^{(L)}$ collecting the observations is the sum of a low rank component generated by $M - L + 1$–dimensional steering vectors with a noise matrix having a block-Hankel structure. Subspace methods can therefore still be developed. The statistical analysis of the corresponding DoA estimators is standard in the regime where $M-L+1$ remains fixed while NL converges towards ∞. When M is large, this regime appears relevant when L is chosen in such a way that $M-L+1 << M$, thus limiting the aperture of the virtual array and the statistical performance of the subspace estimates. In this paper, we study spatial smoothing subspace DoA estimators in asymptotic regimes modelling contexts in which M and NL are large and of the same order of magnitude and where L is much less than M in order to not affect the aperture of the virtual array. The number of sources K is moreover assumed small enough w.r.t. M. As in [9] and [14], we derive improved subspace DoA estimates (called G-MUSIC estimates), and establish their consistency. For this, we evaluate the behaviour of the K largest eigenvalues and corresponding eigenvectors of the empirical covariance matrix $\frac{\mathbf{Y}_N^{(L)}\mathbf{Y}_N^{(L)*}}{NL}$. To address this issue, we prove that the above eigenvalues and eigenvectors have the same asymptotic behaviour as if the noise contribution $\mathbf{V}_N^{(L)}$ to matrix $\mathbf{Y}_N^{(L)}$, a block-Hankel random matrix, was a Gaussian random matrix with independent identically distributed entries.

This paper is organized as follows. In section 2, we precise the signal models and the underlying assumptions In section 3, we present our main results concerning the behaviour of the K greatest eigenvectors and eigenvalues of the empirical covariance, and deduce the G-MUSIC estimates and its properties. Finally, section 4 present numerical experiments sustaining our theoretical results.

2. PROBLEM FORMULATION.

We assume that K narrow-band and far-field source signals are impinging on a uniform linear array of M sensors, with $K < M$. In this context, the M–dimensional received signal $(\mathbf{y}_n)_{n\geq 1}$ can be written as

$$\mathbf{y}_n = \mathbf{A}\mathbf{s}_n + \mathbf{v}_n,$$

where $\mathbf{A} = [\mathbf{a}_M(\theta_1), \ldots, \mathbf{a}_M(\theta_K)]$ is the $M \times K$ matrix of M–dimensionals steering vectors $\mathbf{a}_M(\theta_1), \ldots, \mathbf{a}_M(\theta_K)$, with $\theta_1, \ldots, \theta_K$ the source signals DoA, and $\mathbf{a}_M(\theta) = \frac{1}{\sqrt{M}}[1, \ldots, \mathrm{e}^{\mathrm{i}(M-1)\theta}]^T$. $\mathbf{s}_n \in \mathbb{C}^K$ contains the source signals received at time n, considered as unknown deterministic and $(\mathbf{v}_n)_{n\geq 1}$ is a temporally and spatially white complex Gaussian noise with spatial covariance $\mathbb{E}[\mathbf{v}_n\mathbf{v}_n^*] = \sigma^2\mathbf{I}$. When the number of observations N is much less than the number of sensors M, the standard subspace DoA estimation scheme fails, and it is possible to use spatial smoothing schemes in order to increase artificially the number of snapshots (see e.g. [11] and the references therein). Roughly speaking, if $L < M$, spatial smoothing consists in considering L overlapping subarrays of dimension $M - L + 1$. At each time

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n, the snapshot available on subarray l is the $(M-L+1)$–dimensional vector $\mathbf{y}_n^{(l)} = (\mathbf{y}_{l,n}, \ldots, \mathbf{y}_{M-L+l-1,n})^T$. We collect the available NL observations $(\mathbf{y}_n^{(l)})_{l=1,\ldots,L,n=1,\ldots,N}$ into the block-Hankel $(M-L+1) \times NL$ matrix $\mathbf{Y}_N^{(L)}$ defined by $\mathbf{Y}_N^{(L)} = (\mathbf{y}_1^{(1)}, \ldots, \mathbf{y}_1^{(L)}, \ldots, \mathbf{y}_N^{(1)}, \ldots, \mathbf{y}_N^{(L)})$. If we define matrix $\mathcal{A}_L(\theta)$ as the rank 1 $(M-L+1) \times L$ Hankel matrix given by

$$\mathcal{A}^{(L)}(\theta) = \sqrt{L(M-L+1)/M}\ \mathbf{a}_{M-L+1}(\theta)\left(\mathbf{a}_L(\theta)\right)^T \quad (1)$$

and if $\mathbf{A}^{(L)}$ is the rank K $(M-L+1) \times KL$ matrix

$$\mathbf{A}^{(L)} = \left(\mathcal{A}^{(L)}(\theta_1), \mathcal{A}^{(L)}(\theta_2), \ldots, \mathcal{A}^{(L)}(\theta_K)\right) \quad (2)$$

then, matrix $\mathbf{Y}_N^{(L)}$ can be written as

$$\mathbf{Y}_N^{(L)} = \mathbf{A}^{(L)}\ (\mathbf{S}_N \otimes \mathbf{I}_L) + \mathbf{V}_N^{(L)} \quad (3)$$

where $\mathbf{S}_N$ is the $K \times N$ matrix $\mathbf{S}_N = (\mathbf{s}_1, \ldots, \mathbf{s}_N)$ assumed to be full row rank K. It is easily checked that matrix $\mathbf{A}^{(L)}\ (\mathbf{S}_N \otimes \mathbf{I}_L)$ has rank K, and that its range is the K–dimensional subspace generated by vectors $\mathbf{a}_{M-L+1}(\theta_1), \ldots,\ \mathbf{a}_{M-L+1}(\theta_K)$. When $M-L+1$ remains fixed while NL converges towards ∞, the law of large numbers implies that the empirical covariance matrix $\mathbf{Y}_N^{(L)}\mathbf{Y}_N^{(L)*}/NL$ has the same asymptotic behaviour than $\mathbf{A}^{(L)}\ (\mathbf{S}\mathbf{S}^* \otimes \mathbf{I}_L/NL)\ \ \mathbf{A}^{(L)*} + \sigma^2\mathbf{I}_{M-L+1}$. In this context, the orthogonal projection matrix $\hat{\mathbf{\Pi}}_N^{(L)}$ onto the eigenspace associated to the $M-L+1-K$ smallest eigenvalues of $\mathbf{Y}_N^{(L)}\mathbf{Y}_N^{(L)*}/NL$ is a consistent estimate of the orthogonal projection matrix $\mathbf{\Pi}^{(L)}$ on the noise subspace, i.e. the orthogonal complement of $\mathrm{sp}\{\mathbf{a}_{M-L+1}(\theta_1), \ldots, \mathbf{a}_{M-L+1}(\theta_K)\}$. The traditional pseudo-spectrum estimate $\hat{\eta}_N^{(t)}(\theta)$ defined by $\hat{\eta}_N^{(t)}(\theta) = \mathbf{a}_{M-L+1}(\theta)^*\hat{\mathbf{\Pi}}_N^{(L)}\mathbf{a}_{M-L+1}(\theta)$ thus verify

$$\sup_{\theta\in[-\pi,\pi]} \left|\hat{\eta}_N^{(t)}(\theta) - \eta(\theta)\right| \xrightarrow[N\to\infty]{a.s.} 0. \quad (4)$$

where $\eta(\theta) = \mathbf{a}_{M-L+1}(\theta)^*\mathbf{\Pi}^{(L)}\mathbf{a}_{M-L+1}(\theta)$ is the MUSIC pseudo-spectrum. Moreover, the K MUSIC traditional DoA estimates, defined formally, for $k = 1, \ldots, K$, by

$$\hat{\theta}_{k,N}^{(t)} = \underset{\theta\in\mathcal{I}_k}{\mathrm{argmax}}\, \hat{\eta}_N^{(t)}(\theta),$$

where $\mathcal{I}_k$ is a compact interval containing θ_k and such that $\mathcal{I}_k \cap \mathcal{I}_l = \emptyset$ for $k \neq l$, are consistent, i.e.

$$\hat{\theta}_{k,N}^{(t)} \xrightarrow[N\to\infty]{a.s.} \theta_k. \quad (5)$$

However, when M is large, this regime is not very interesting in practice because it appears relevant when the size $M-L+1$ of the subarrays is much smaller that the number of antennas M, thus reducing the resolution of the method. We therefore study spatial smoothing schemes in regimes where the dimensions $M-L+1$ and NL of matrix $\mathbf{Y}_N^{(L)}$ are of the same order of magnitude and where $L << M$ in order to not affect significantly the aperture of the array. More precisely, we assume that integers N and L depend on M and that

$$M \to +\infty, N = \mathcal{O}(M^\beta),\ 1/3 < \beta \leq 1,\ \frac{M-L+1}{NL} \to c \quad (6)$$

where $0 < c < \infty$. In regime (6), N thus converges towards ∞ but at a rate that may be much lower than M, thus modelling contexts in which N may be much smaller than M. As $\frac{M}{NL} \to c$, it is clear that $L = \mathcal{O}(M^\alpha)$ where $\alpha = 1-\beta$ verifies $0 \leq \alpha < 2/3$. Therefore, L may converge towards ∞ but in any case, $\frac{L}{M} \to 0$. We finally notice that L may converge towards ∞ faster than N when $\beta < 1/2$. While in this regime, N and L depend in principle on M, $N \to +\infty$ should be understood as the asymptotic regime (6) in order to short the notations.

In regime (6), (4) is no more valid because ratio $(M-L+1)/NL$ does not converge towards 0. Hence, (5) is questionable, and the aim of the following section is to derive consistent improved subspace DoA estimates.

3. DERIVATION OF A CONSISTENT G-MUSIC METHOD.

In order to simplify the notations, we denote by $\mathbf{X}_N$, $\mathbf{W}_N$ and $\mathbf{B}_N$ the matrices defined by $\mathbf{X}_N = \mathbf{Y}_N^{(L)}/\sqrt{NL}$, $\mathbf{W}_N = \mathbf{V}_N^{(L)}/\sqrt{NL}$ and $\mathbf{B}_N = \frac{1}{\sqrt{NL}}\mathbf{A}^{(L)}(\mathbf{S}_N \otimes \mathbf{I}_L)$. This paper is based on a technical result which establishes that, in a certain sense, the eigenvalues of matrix $\mathbf{W}_N\mathbf{W}_N^*$ behave as if the entries of $\mathbf{W}_N$ were i.i.d. In order to state the corresponding result, we recall that the Marcenko-Pastur distribution μ with parameters (σ^2, c) is the probability distribution defined by

$$d\mu(x) = \delta_0[1-c^{-1}]_+ + \frac{\sqrt{(x-x^-)(x^+-x)}}{2\sigma^2 c\pi x}\mathbb{1}_{[x^-,x^+]}(x)\,dx$$

with $x^- = \sigma^2(1-\sqrt{c})^2$ and $x^+ = \sigma^2(1+\sqrt{c})^2$. We denote by $m(z)$ its Stieltjes transform defined by $m(z) = \int_{\mathbb{R}} \frac{d\mu(\lambda)}{\lambda - z}$ and by $\tilde{m}(z)$ the function $\tilde{m}(z) = cm(z) - (1-c)/z$. We denote by $\mathbf{Q}_N(z)$ and $\tilde{\mathbf{Q}}_N(z)$ the so-called resolvent of matrices $\mathbf{W}_N\mathbf{W}_N^*$ and $\mathbf{W}_N^*\mathbf{W}_N$ defined by $\mathbf{Q}_N(z) = \left(\mathbf{W}_N\mathbf{W}_N^* - z\mathbf{I}_{M-L+1}\right)^{-1}$ and $\tilde{\mathbf{Q}}_N(z) = \left(\mathbf{W}_N^*\mathbf{W}_N - z\mathbf{I}_{NL}\right)^{-1}$ Then, in regime (6), the following result holds.

Proposition 1. *The eigenvalue distribution of matrix $\mathbf{W}_N\mathbf{W}_N^*$ converges almost surely towards the Marcenko-Pastur distribution μ. Moreover, for each $\epsilon > 0$, almost surely, for N large enough, all the eigenvalues of $\mathbf{W}_N\mathbf{W}_N^*$ belong to $[x^- - \epsilon, x^+ + \epsilon]$ if $c \leq 1$, and to $[x^- - \epsilon, x^+ + \epsilon] \cup \{0\}$ if $c > 1$. Moreover, if $\mathbf{a}_N, \mathbf{b}_N$ are 2 unit norm $(M-L+1)$–dimensional deterministic vectors, then it holds that for each $z \in \mathbb{C}^+$*

$$\mathbf{a}_N^*\left(\mathbf{Q}_N(z) - m(z)\mathbf{I}\right)\mathbf{b}_N \to 0\ a.s. \quad (7)$$

Similarly, if $\tilde{\mathbf{a}}_N, \tilde{\mathbf{b}}_N$ are 2 unit norm NL–dimensional deterministic vectors, then for each $z \in \mathbb{C}^+$, it holds that

$$\tilde{\mathbf{a}}_N^*\left(\tilde{\mathbf{Q}}_N(z) - \tilde{m}(z)\mathbf{I}\right)\tilde{\mathbf{b}}_N \to 0\ a.s. \quad (8)$$

Moreover, for each $z \in \mathbb{C}^+$, it holds that

$$\mathbf{a}_N^*\left(\mathbf{Q}_N(z)\mathbf{W}_N\right)\tilde{\mathbf{b}}_N \to 0\ a.s. \quad (9)$$

Finally, for each $\epsilon > 0$, convergence properties (7, 8, 9) hold uniformly w.r.t. z on each compact subset of $\mathbb{C} - [0, x^+ + \epsilon]$.

We recall that, roughly speaking, the convergence of the eigenvalue distribution of $\mathbf{W}_N\mathbf{W}_N^*$ towards distribution μ means that the histograms of the eigenvalues of any realization of $\mathbf{W}_N\mathbf{W}_N^*$ tend to accumulate around the graph of the probability density of μ. The statements of Proposition 1 are well known when $L = 1$ and that M and N converge towards $+\infty$ at the same rate. Apart (7) and (9), Proposition appears as a consequence of the results of [8]. We

note that the convergence of the eigenvalue distribution of $\mathbf{W}_N\mathbf{W}_N^*$ towards the Marcenko-Pastur holds as soon as $N \to +\infty$, and does not need to assume that $N = \mathcal{O}(M^\beta)$ for $\beta > 1/3$. The latter assumption is necessary to ensure that the eigenvalues of $\mathbf{W}_N\mathbf{W}_N^*$ stay in the neighborhood of the support of μ, a crucial point to establish Theorem 1 below. We finally note that if M and L converge toward ∞ at the same rate and that N remains fixed, the convergence of the eigenvalue distribution $\mathbf{W}_N\mathbf{W}_N^*$ towards μ is no longer true. Intuitively, this is because $\mathbf{W}_N$ depends on MN independent random variables, and that if N is fixed, this number is not sufficient to ensure nice averaging effects. In particular, if $N = 1$, it is shown in [2] that the eigenvalue distribution of $\mathbf{W}_N\mathbf{W}_N^*$ converges towards an unbounded probability distribution that can be characterized by its moments.

In the following, we denote by $(\hat{\lambda}_{k,N})_{k=1,\ldots,M-L+1}$ and $(\hat{\mathbf{u}}_{k,N})_{k=1,\ldots,M-L+1}$ the eigenvalues and corresponding eigenvectors of $\mathbf{X}_N\mathbf{X}_N^*$, and by $\lambda_{1,N} \geq \lambda_{2,N} \ldots \geq \lambda_{K,N}$ and $(\mathbf{u}_{k,N})_{k=1,\ldots,K}$ the non zero eigenvalues and eigenvectors of $\mathbf{B}_N\mathbf{B}_N^*$. Proposition 1 allows to generalize immediately the approach used in [6] (see also [7]), and to prove that the K greatest eigenvalues and corresponding eigenvectors of $\mathbf{X}_N\mathbf{X}_N^*$ also behave as if the entries of $\mathbf{W}_N$ were i.i.d.

Theorem 1. *We assume that:*

Assumption 1. *The K non zero eigenvalues $(\lambda_{k,N})_{k=1,\ldots,K}$ of matrix $\mathbf{B}_N\mathbf{B}_N^*$ converge towards $\lambda_1 > \lambda_2 > \ldots > \lambda_K$ when $N \to +\infty$.*

We denote by s, $0 \leq s \leq K$, the largest integer for which $\lambda_s > \sigma^2\sqrt{c}$. Then, for $k = 1, \ldots, s$, it holds that

$$\hat{\lambda}_{k,N} \xrightarrow[N\to\infty]{a.s.} \rho_k = \phi(\lambda_k) = \frac{(\lambda_k + \sigma^2)(\lambda_k + \sigma^2 c)}{\lambda_k} > x^+.$$

while for $k = s+1, \ldots, K$, $\hat{\lambda}_{k,N} \to x^+$ a.s. Moreover, for all deterministic sequences of unit norm vectors $(\mathbf{a}_N)$, $(\mathbf{b}_N)$, we have for $k = 1, \ldots, s$

$$\mathbf{a}_N^* \left(\hat{\mathbf{u}}_{k,N}\hat{\mathbf{u}}_{k,N}^* - h(\rho_k)\mathbf{u}_{k,N}\mathbf{u}_{k,N}^* \right) \mathbf{b}_N \to 0 \ a.s. \qquad (10)$$

where function $h(z)$ is defined by $h(z) = \frac{zm(z)^2\tilde{m}(z)}{(zm(z)\tilde{m}(z))'}$

Here, for ease of exposition, we have assumed that $\lambda_k \neq \lambda_l$ for $k \neq l$. However, Theorem 1 and the forthcoming results still hold true if some $(\lambda_k)_{k=1,\ldots,K}$ coincide (see [7]). Theorem 1 leads immediately to the derivation of the following improved estimate $\hat{\eta}_N(\theta)$ of the pseudo-spectrum $\eta(\theta)$.

Theorem 2. *Assume that Assumption 1 holds and that the separation condition*

$$\lambda_K > \sigma^2\sqrt{c} \qquad (11)$$

holds. Then, the pseudo-spectrum estimate $\hat{\eta}_N(\theta)$ defined by

$$\hat{\eta}_N(\theta) = \mathbf{a}_{M-L+1}^*(\theta) \left(\mathbf{I} - \sum_{k=1}^{K} \frac{1}{h\left(\hat{\lambda}_{k,N}\right)} \hat{\mathbf{u}}_{k,N}\hat{\mathbf{u}}_{k,N}^* \right) \mathbf{a}_{M-L+1}(\theta) \qquad (12)$$

verifies

$$\sup_{\theta\in[-\pi,\pi]} |\hat{\eta}_N(\theta) - \eta_N(\theta)| \xrightarrow[N\to\infty]{a.s.} 0, \qquad (13)$$

Finally, the corresponding DoA estimates $(\hat{\theta}_{k,N})_{k=1,\ldots,K}$ are consistent, and verify

$$M(\hat{\theta}_{k,N} - \theta_k) \to 0 \ a.s. \qquad (14)$$

This result can be proved as Theorem 3 in [7]. We remark that in [7], it is assumed that the angles $(\theta_k)_{k=1,\ldots,K}$ do not scale with M, N, L. However, it is possible to extend the results of [7], and thus Theorem 2 to the case where certain DoAs are spaced of the order of the beamwidth, i.e. for some k, $\theta_{k+1} - \theta_k = \mathcal{O}(\frac{1}{M})$.

Under the separation condition (11), it is thus possible to derive consistent subspace DoA estimators in the context of spatial smoothing schemes. Roughly speaking, (11) means that the smallest non zero eigenvalue of the signal matrix $\mathbf{B}_N\mathbf{B}_N^*$ should be large enough in order to ensure the separation of the noise and signal subspaces of the empirical covariance matrix $\mathbf{X}_N\mathbf{X}_N^*$. In the case where parameter L does not converge towards ∞, it is interesting to get some insights on the separation condition, and to evaluate how it behaves when L increases. If L does not converge towards ∞, β is reduced to 1, and $\frac{M}{N} \to d$, where $d = cL$. If $\mathbf{A}_{M-L+1}$ is the matrix $\mathbf{A}_{M-L+1} = (\mathbf{a}_{M-L+1}(\theta_1), \ldots, \mathbf{a}_{M-L+1}(\theta_K))$, it is easy to check that

$$\mathbf{B}_N\mathbf{B}_N^* = \mathbf{A}_{M-L+1} \left(\frac{\mathbf{S}_N\mathbf{S}_N^*}{N} \bullet \mathbf{A}_L^T\overline{\mathbf{A}}_L \right) \mathbf{A}_{M-L+1}^* \qquad (15)$$

where $\bullet$ represents the Hadamard (i.e. element wise) product of matrices, and where $\overline{\mathbf{B}}$ stands for the complex conjugation operator of the elements of matrix $\mathbf{B}$. In order to simplify, we assume that $\frac{\mathbf{S}_N\mathbf{S}_N^*}{N}$ converges towards a diagonal matrix $\mathbf{D}$ when N increases. Therefore, $\frac{\mathbf{S}\mathbf{S}^*}{N} \bullet (\mathbf{A}_L^T\overline{\mathbf{A}}_L) \to \mathbf{D}$. Therefore, the separation condition is equivalent to

$$\lim_{N\to+\infty} \lambda_K \left(\mathbf{A}_{M-L+1}\mathbf{D}\ \mathbf{A}_{M-L+1}^* \right) > \frac{\sigma^2\sqrt{d}}{\sqrt{L}}$$

If the DoAs $(\theta_k)_{k=1,\ldots,K}$ do not scale with M, N, matrix $\mathbf{A}_{M-L+1}^*\mathbf{A}_{M-L+1}$ converges towards $\mathbf{I}_K$ when $N \to +\infty$, and the separation condition reduces to

$$\min_{k=1,\ldots,K} \mathbf{D}_{k,k} > \frac{\sigma^2\sqrt{d}}{\sqrt{L}}$$

This analysis means that, provided that M and N are large enough and that L is much lower than M, spatial smoothing allows to reduce the threshold $\sigma^2\sqrt{d}$ corresponding to G-MUSIC methods without spatial smoothing by the factor $\sqrt{L}$. Therefore, if M and N are of the same order of magnitude, our asymptotic analysis allows to predict an improvement of the performance of the spatial smoothing subspace methods when L increases provided $L << M$. If L is however too large, the above analysis is no more justified, and the impact of the diminution of the number of antennas becomes dominant, and the performance tends to decrease.

4. NUMERICAL EXPERIMENTS.

In this section, we provide numerical simulations illustrating the results given in the previous sections. In the following experiments, we consider 2 closely spaced sources with DoAs $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2M}$, and we assume that $M = 160$ and $N = 20$. The $2 \times N$ signal matrix is the realization of a random matrix with $\mathcal{N}_{\mathbb{C}}(0, 1)$ i.i.d. entries. The 2 source signals are normalized in order to force the sources to have power 1, and so that the signal to noise ratio is defined by $\text{SNR} = 1/\sigma^2$. Table 1 provides the minimum value of SNR for which the separation condition (in its finite length version) holds, i.e.

$$(\sigma^2)^{-1} = \frac{1}{\lambda_{K,N}} \sqrt{(M - L + 1)/NL}$$

When L increases, $\sqrt{(M - L + 1)/NL}$ decreases. However, when L increases, $M - L + 1$ decreases and $\lambda_{K,N}$ also decreases because

the smallest eigenvalue of matrix $\mathbf{A}^*_{M-L+1}\mathbf{A}_{M-L+1}$ decreases. This explains why the minimal SNR first decreases, and then increases.

L	2	4	8	16	32	64	96	128
SNR	33.46	30.30	27.46	25.31	24.70	28.25	36.11	51.52

Table 1. Minimum value of SNR for separation condition

In figure 1, we represent the mean-square errors of the G-MUSIC estimator $\hat{\theta}_1$ for $L = 2, 4, 8, 16$ versus SNR. The corresponding Cramer-Rao bounds is also represented. As expected, it is seen that the performance tends to increase with L until $L = 16$. In figure 2, L is equal to 16, 32, 64, 96, 128.

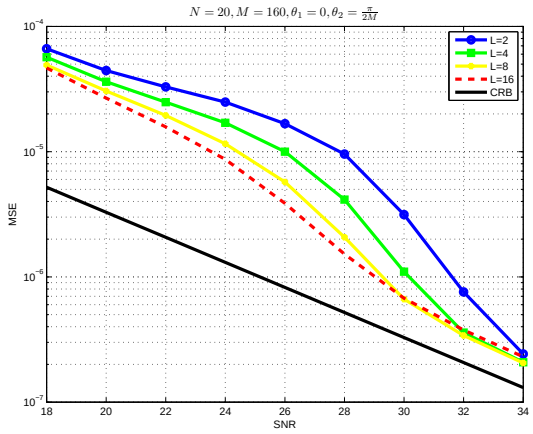

Fig. 1. Empirical MSE of G-MUSIC SS estimator $\hat{\theta}_1$ versus SNR

For $L = 32$, it is seen that the MSE tends to degrade at high SNR w.r.t. $L = 16$, while the performance severely degrades for larger values of L.

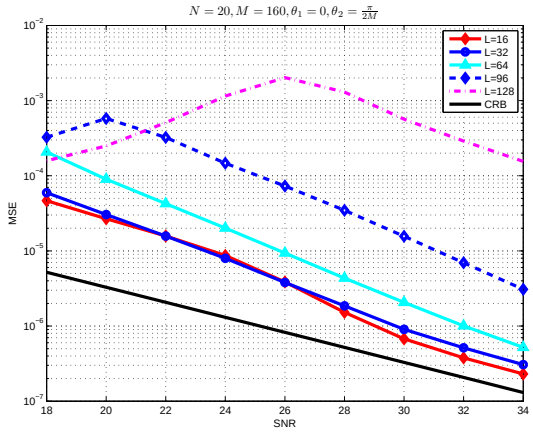

Fig. 2. Empirical MSE of G-MUSIC SS estimator $\hat{\theta}_1$ versus SNR

In Figure 3, parameter L is equal to 16. We compare the performance of G-MUSIC SS with the standard MUSIC method with spatial smoothing. We also represent the MSE provided by G-MUSIC and MUSIC for $L = 1$. The standard unsmoothed MUSIC method of course completely fails, while the use of the G-MUSIC SS provides a clear improvement of the performance w.r.t. MUSIC SS and unsmoothed G-MUSIC.

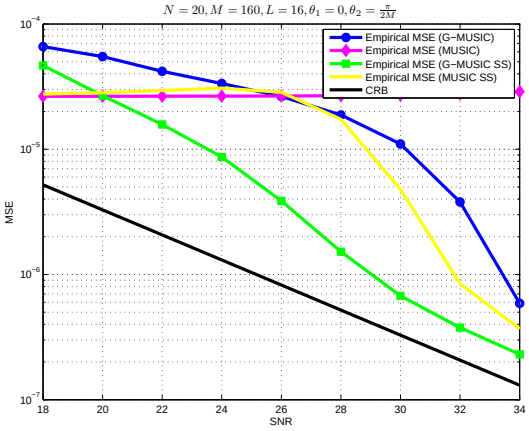

Fig. 3. Empirical MSE of different estimators of θ_1 when L=16

We finally consider the case $L = 128$, and compare as above G-MUSIC SS, MUSIC SS, unsmoothed G-MUSIC and unsmoothed MUSIC. G-MUSIC completely fails because L and M are of the same order of magnitude. Theorem 2 is thus no more valid, and the pseudo-spectrum estimate is not consistent.

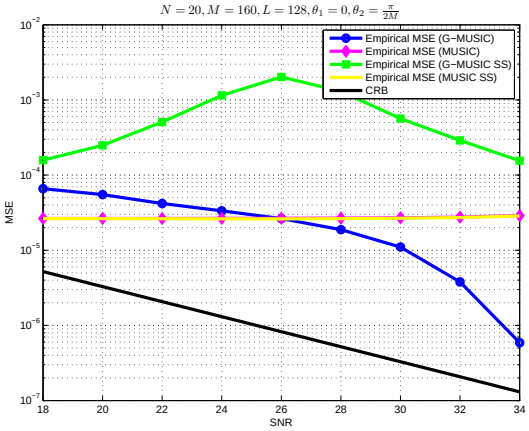

Fig. 4. Empirical MSE of different estimators of θ_1 when L=128

5. CONCLUSION

In this paper, we have addressed the behaviour of subspace DoA estimators in the case where the number of observations may be much lower than the number of sensors. In this context, we have studied the statistical performance of subspace estimators based on spatial smoothing schemes. For this, we have evaluated the behaviour of the largest singular values and corresponding singular vectors of large random matrices defined as additive low rank perturbations of certain random block-Hankel matrices, and established that they behave as if the entries of the block-Hankel matrices were i.i.d. Starting from this result, we have shown that it is possible to generalize the G-estimators introduced in [10] and [14], and have proved their consistency.

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