Random Matrices for Cooperative Spectrum Sensing: Some recent results

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joint work with

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Cooperative spectrum sensing

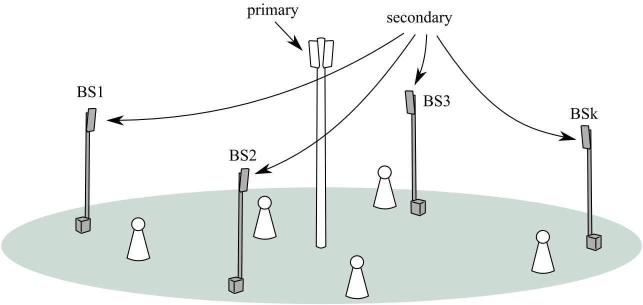

Figure: Considered scenario for cooperative spectrum sensing

Communication features

- A wireless network **sparsely** uses a certain bandwidth.
- A **secondary** wireless system wants to use this bandwidth whenever it is **available**.
- The base stations of the secondary system **share information** between them.
- The test does **not** require **any prior knowledge** on the signal structure.
- The test needs to be performed on a **real time** basis.

Modelisation

Hypothesis (H0): No signal. Every secondary sensor $k = 1 : N$ receives a signal $y_k(\ell)$ at the sampling time units $\ell = 1 : n$ with

$$y_k(\ell) = \sigma w_k(\ell) ,$$

where $w_k(\ell)$ is a white gaussian noise, and σ is its variance.

Hypothesis (H1): Presence of a signal. The signal has now the form

$$y_k(\ell) = h_k s(\ell) + \sigma w_k(\ell) ,$$

where $s(\ell)$ is a gaussian primary signal at time ℓ and h_k is the fading coefficient associated to the secondary station k.

Features

Constraints:

- The number of secondary sensors N and the dimension of the received signal n are of the **same order**.
- The noise variance σ and the fading coefficients h_k are **unknown**.
- **Real-time processing**.

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Collaboration between secondary sensors:

- Data is stored in a matrix

$$\mathbf{Y} = \left(\frac{y_k(\ell)}{\sqrt{n}} \right)_{k=1:N,\ \ell=1:n} ,$$

as all the signals are shared between the secondary sensors.

A few remarks

- Quantities σ and h_k are unknown. Therefore, **Neyman-Pearson** test **cannot** be implemented.
- There might not exist a uniformly most powerful test: **Type II errors** must be studied to **compare** statistical tests.

Presentation of the results

In the following, we shall:

- present the **Generalized Maximum Likelihood** statistical test (GML Test),
- study its **Type I and II** errors,
- compare them to the **reference test** based on the extreme eigenvalue ratio (EER test),
- establish that the GML test is **uniformly most powerful** than the EER test.

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These results are mainly based on:

- The **asymptotic study** of **various regimes** of extreme eigenvalues of large random matrices.

A few facts about large random matrices

Consider a $N \times n$ matrix $\mathbf{Z}_n$ with independent entries:

$$\mathbf{Z}_n = \frac{\sigma}{\sqrt{n}} (Z_{ij})$$

where

$$\begin{pmatrix} Z_{1j} \\ \vdots \\ Z_{Nj} \end{pmatrix} \sim \mathcal{C}N(0, \mathbf{\Sigma_n}) \quad \text{with} \quad \mathbf{\Sigma}_n = \begin{pmatrix} \rho & 0 & \cdots & 0 \\ 0 & 1 & 0 & \vdots \\ & & \ddots & \\ 0 & \cdots & & 1 \end{pmatrix}$$

Of major interest is the **spectrum** $(\lambda_1, \cdots, \lambda_N)$ of $\mathbf{Z}_n \mathbf{Z}_n^*$ under the asymptotic regime:

$$n, N \to \infty, \quad \frac{N}{n} \to c \in (0, 1) \ .$$

Global regime of the spectrum

Whatever the value of ρ, the **spectral measure**

$$\mathbf{L}_n([a,b]) = \frac{\#\{\lambda_i \in [a,b]\}}{N}$$

converges towards **Marčenko-Pastur** distribution:

$$\mathbf{L}_n([a,b]) \to \mathbb{P}_{\check{M}P}([a,b]) \ a.s.$$

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where

$$\mathbb{P}_{\check{M}P}([a,b]) = \int_a^b 1_{(\lambda^-,\lambda^+)}(x) \frac{\sqrt{(\lambda^+ - x)(x-\lambda^-)}}{2\pi c x} \, dx \ .$$

with

$$\begin{cases} \lambda^- = \sigma^2(1-\sqrt{c})^2 \\ \lambda^+ = \sigma^2(1+\sqrt{c})^2 \end{cases}$$

Extreme eigenvalues

In the case where $1 \leq \rho \leq \sqrt{c}$:
Convergence of λ_{max} and λ_{min} toward the **endpoints** of Marčenko-Pastur distribution:

$$\lambda_{\text{max}} \to \lambda^{+} = \sigma^2(1+\sqrt{c})^2, \qquad \lambda_{\text{min}} \to \lambda^{-} = \sigma^2(1-\sqrt{c})^2 .$$

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In the case where $\rho > \sqrt{c}$:
The eigenvalue $\lambda_{\max}$ converges **outside** the bulk of MP distribution!

$$\lambda_{\max} \to \sigma^2(1+\rho)\left(1+\frac{c}{\rho}\right) \quad > \quad \sigma^2(1+\sqrt{c})^2 \ ,$$

while $\lambda_{\min} \to \lambda^-$.

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while $\lambda_{\min} \to \lambda^-$.

In this case, we speak of a **spiked model**.

Back to our problem

Aim: To build a test statistics based on matrix:

$$\mathbf{R} = \mathbf{Y}\mathbf{Y}^* \quad \text{where} \quad \mathbf{Y} = \left(\frac{y_k(\ell)}{\sqrt{n}} \right)_{k=1:N,\ \ell=1:n} .$$

in the case where:

- the number N of secondary sensors is of **the same order** as the number n of observations:

$$N \to \infty, \quad n \to \infty, \quad c_n = \frac{N}{n} \to c \in (0,1) .$$

When there is no signal: Model with i.i.d. entries

The matrix model is: $\mathbf{Y}$ with i.i.d. entries $\mathcal{C}N(0,1)$

$$\mathbf{Y} = \frac{\sigma}{\sqrt{n}} \begin{pmatrix} w_1(1) & \cdots & w_1(n) \\ \vdots & & \vdots \\ w_N(1) & \cdots & w_N(n) \end{pmatrix} .$$

and we are interested in $\hat{\mathbf{R}} = \mathbf{Y}\mathbf{Y}^*$.

Asymptotics of the spectrum

- Convergence of the extreme eigenvalues:

$$\lambda_{\max}(\hat{\mathbf{R}}) \xrightarrow[n\to\infty]{} \lambda^{+} \stackrel{\triangle}{=} \sigma^2(1+\sqrt{c})^2,$$
$$\lambda_{\min}(\hat{\mathbf{R}}) \xrightarrow[n\to\infty]{} \lambda^{-} \stackrel{\triangle}{=} \sigma^2(1-\sqrt{c})^2,$$

- Convergence of the normalised trace:

$$\frac{1}{N}\mathrm{Trace}(\hat{\mathbf{R}}) \to \sigma^2.$$

When there is some signal

Matrix model:

$$\check{\mathbf{Y}} = \frac{1}{\sqrt{n}} \begin{pmatrix} h_1 & \sigma & & 0 \\ \vdots & & \ddots & \\ h_N & 0 & & \sigma \end{pmatrix} \begin{pmatrix} s(1) & \cdots & s(n) \\ w_1(1) & \cdots & w_1(n) \\ \vdots & & \vdots \\ w_N(1) & \cdots & w_N(n) \end{pmatrix}$$

Let: $\check{\mathbf{R}} = \check{\mathbf{Y}}\check{\mathbf{Y}}^*$

Equivalence with a spiked model

Performing a SVD yields:

$$\begin{pmatrix} h_1 & \sigma & & 0 \\ \vdots & & \ddots & \\ h_N & 0 & & \sigma \end{pmatrix} = \mathbf{U} \begin{pmatrix} \sqrt{|\mathbf{h}|^2 + \sigma^2} & 0 & \cdots & 0 \\ 0 & \sigma & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma \end{pmatrix} \mathbf{V}$$

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Thus $\check{\mathbf{R}} = \check{\mathbf{Y}}\check{\mathbf{Y}}^*$ has the **same spectrum** as $\tilde{\mathbf{Y}}\tilde{\mathbf{Y}}^*$ with:

$$\tilde{\mathbf{Y}} = \frac{1}{\sqrt{n}} \begin{pmatrix} \sqrt{|\mathbf{h}|^2 + \sigma^2} & 0 & \cdots & 0 \\ 0 & \sigma & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma \end{pmatrix} \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{N1} & \cdots & X_{Nn} \end{pmatrix},$$

the X_{ij}'s being i.i.d. $\mathcal{C}N(0,1)$.

$\Rightarrow$ **spiked model!**

Asymptotics of the spectrum

Limit of the largest eigenvalue. If the **Signal to noise ratio** is above the threshold:

$$\rho = \frac{\sum_{k=1}^{N} |h_k|^2}{\sigma^2} = \frac{\text{signal power}}{\text{noise variance}} > \sqrt{c} \ ,$$

then the limit of the largest eigenvalue is **no longer the same**:

$$\lambda_{\max}(\check{\mathbf{R}}) \xrightarrow[n\to\infty]{} \sigma^2(1+\rho)\left(1+\frac{c}{\rho}\right) \quad > \quad \sigma^2(1+\sqrt{c})^2 \ !$$

References:

- Baik, Ben Arous, Péché - Annals of Probab. (2005)
- Baik, Silverstein - J. Mult. Analysis (2006)

Other limits are not modified:

- smallest eigenvalue:

$$\lambda_{\min}(\check{\mathbf{R}}) \to \lambda^{-} = \sigma^2(1-\sqrt{c})^2;$$

- normalized trace:

$$\frac{1}{N}\mathrm{Trace}(\check{\mathbf{R}}) \to \sigma^2.$$

Hypothesis Test

We shall thus test the hypotheses

- (H_0) No primary signal, i.e.

$$\mathbf{R} = \hat{\mathbf{R}} \text{ (with i.i.d. entries)},$$

versus

- (H_1) Presence of a noticeable primary signal, i.e.

$$\mathbf{R} = \check{\mathbf{R}} \text{ (spiked model)} \quad \text{with } \rho > \sqrt{c} \ .$$

The likelihood ratio

Consider the likelihood functions of the observation matrix **Y** under hypotheses **H0** and **H1**:

$$p_0(\mathbf{Y};\sigma^2) = \frac{1}{(\pi\sigma^2)^{NK}} \exp\left(-\frac{N}{\sigma^2}\mathrm{Tr}\,\mathbf{R}\right)$$

$$p_1(\mathbf{Y};\sigma^2,\mathbf{h}) = \frac{1}{\pi^K \det(\mathbf{h}\mathbf{h}^* + \sigma^2\mathbf{I}_K)} \exp\left(-N\mathrm{Tr}\,\mathbf{R}(\mathbf{h}\mathbf{h}^* + \sigma^2\mathbf{I}_K)^{-1}\right)$$

Neyman-Pearson Lemma: If σ^2 and **h** are known, then the test

$$L_N = \frac{p_0(\mathbf{Y};\sigma^2)}{p_1(\mathbf{Y};\sigma^2,\mathbf{h})}$$

is uniformly most powerful: For a given **level of significance**, its error of second kind is minimum.

The Generalized maximum likelihood ratio test

Unfortunately, σ^2 and $\mathbf{h}$ are unknown. A **suboptimal** but classical approach consists in considering the test:

$$L_N = \frac{\sup_{\sigma^2} p_0(\mathbf{Y}; \sigma^2)}{\sup_{\sigma^2, \mathbf{h}} p_1(\mathbf{Y}; \sigma^2, \mathbf{h})}$$

which yields, in our case, to the test:

$$T_1 \quad = \quad \frac{\lambda_{\max}}{\frac{1}{N}\mathrm{Tr}\mathbf{R}}$$

Limits of T_1 depending on the hypotheses:

$$T_1 \xrightarrow[n\to\infty]{(under \ \ H_0)} (1+\sqrt{c})^2 \quad \text{and} \quad T_1 \xrightarrow[n\to\infty]{(under \ \ H_1)} (1+\rho)\left(1+\frac{c}{\rho}\right)$$

Type I error

- Type I error represents the probability of choosing H_1 while the true hypothesis is H_0.
- Describing the **fluctuations** of T_1 enables us to compute the threshold t_α associated to an a priori fixed type I erro α.

Computation of the threshold

Fluctuations. Under (H_0), $\hat{\mathbf{R}} = \mathbf{Y}\mathbf{Y}^*$ with $\mathbf{Y}$ with i.i.d. gaussian entries

- Fluctuations of $\lambda_{\max}$ are of order $N^{-2/3}$,
- Fluctuations of $\frac{1}{N}\mathrm{Trace}\,\hat{\mathbf{R}}$ are of order N^{-2}.

Therefore,

$$T_1 = \frac{\lambda_{\max}}{\frac{1}{N}\mathrm{Trace}\,\hat{\mathbf{R}}} \approx \frac{\lambda_{\max}}{\sigma^2}$$

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Limit. When correctly centered and rescaled, T_1 converges to a **Tracy-Widom** distribution:

$$\widetilde{T}_1 \;\stackrel{\triangle}{=}\; N^{2/3} \frac{T_1 - (1+\sqrt{c_n})^2}{(1+\sqrt{c_n})\left(\frac{1}{\sqrt{c_n}}+1\right)^{1/3}} \quad \xrightarrow[N\to\infty]{\mathcal{L}} \quad TW\ .$$

Computation of the threshold (followed)

- Knowing the quantiles of **Tracy-Widom** distribution enables us to compute the threshold for a given level of significance α:

$$\mathbb{P}_{TW}\{\widetilde{T}_1 > t_{\boldsymbol{\alpha}}\} = \boldsymbol{\alpha} \ .$$

- For the level $\boldsymbol{\alpha}$, decision will be:

$$\begin{array}{lll} \text{choose } (H_0) & \text{if} & \widetilde{T}_1 \leq t_{\boldsymbol{\alpha}}, \\ \text{choose } (H_1) & \text{if} & \widetilde{T}_1 > t_{\boldsymbol{\alpha}}, \end{array}$$

Type II Error

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- This probability goes to zero. Indeed:

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$$\mathbf{s}^n \xrightarrow{(H_1)} (1+\sqrt{c})^2 \quad \text{while} \quad T_1 \xrightarrow{(H_1)} (1+\rho)\left(1+\frac{c}{\rho}\right)$$

- Therefore $\mathbb{P}_{H_1}\left(T_1 \leq \mathbf{s}^n\right)$ is a **large deviation**: It goes **exponentially fast** to zero

$$\mathbb{P}_{H_1}\left(T_1 \leq \mathbf{s}_1^n\right) \asymp_\infty e^{-N\mathcal{E}} \ .$$

The Error Exponent

It is defined by:

$$\boldsymbol{\mathcal{E}} = -\lim_{n \to \infty} \frac{1}{N} \log \mathbb{P}_{H_1}\left(T_1 \leq \mathbf{s}_1^n\right)$$

$$\left(\Leftrightarrow \quad \mathbb{P}_{H_1}\left(T_1 \leq \mathbf{s}_1^n\right) \asymp_\infty e^{-N\boldsymbol{\mathcal{E}}}\right)$$

- As $T_1 = \frac{\lambda_{\max}}{\frac{1}{N}\mathrm{Tr}\mathbf{R}}$, the deviations can either come from λ_{max} or from $\frac{1}{N}\mathrm{Tr}\mathbf{R}$.

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- As $T_1 = \frac{\lambda_{\max}}{\frac{1}{N}\mathrm{Tr}\mathbf{R}}$, the deviations can either come from λ_{max} or from $\frac{1}{N}\mathrm{Tr}\mathbf{R}$.
- It turns out that $\lambda_{\max}$ drives the large deviations because the deviations of $\frac{1}{N}\mathrm{Tr}\mathbf{R}$ are far smaller:

$$\mathbb{P}_{H_1}\left(\frac{1}{N}\mathrm{Tr}\mathbf{R} \text{ away from } \sigma^2 \right) \asymp_\infty e^{-N^2 \boldsymbol{\kappa}}$$

Large deviations of $\lambda_{\max}$

- By the previous discussion, the computation of the error exponent relies on:

  The **study of the large deviations of** $\lambda_{\max}$ under $\mathbf{H_1}$.

- There exists a **rate function** I_ρ which describes the large deviations of $\lambda_{\max}$:

$$\mathbb{P}\left(\lambda_{\max} \in A\right) \asymp_\infty \exp\left(-inf_{x \in A}\, I_\rho(x)\right) \ .$$

Computation of the Error Exponent

- The error exponent $\mathcal{E}$ is given by the **rate function** associated to the large deviations of $\lambda_{\max}$ under (H_1):

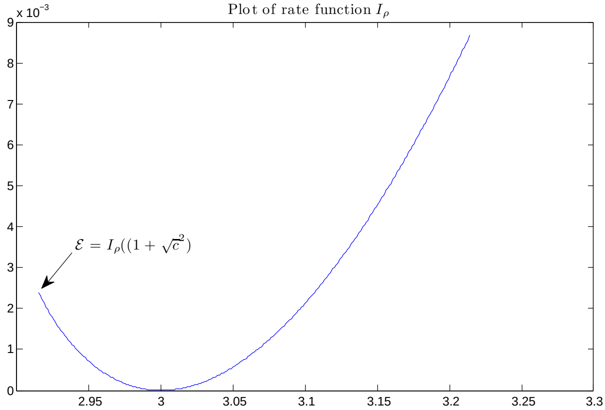

Figure: Rate function I_ρ for $c = 0.5$, $\rho = 1$

Error Exponent - Neyman-Pearson bound

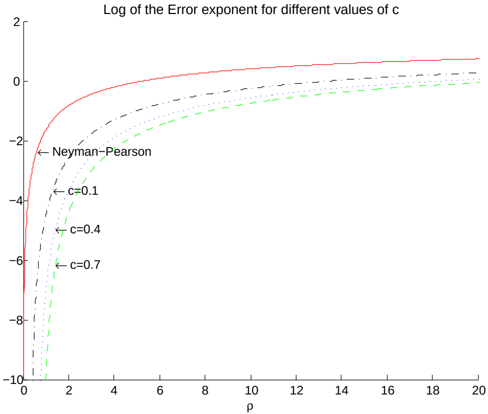

Figure: Computation of the logarithm of the Error Exponent for different values of c ($\rho \geq \sqrt{c}$), and comparison with the optimal bound (Neyman-Pearson) obtained in the case where all the parameters are perfectly known.

Beyond the Error Exponent: The Error Exponent curve

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- We are interested in the regime where both the level of significance **and** the type II error decrease to zero and consider the pairs $(\mathcal{E}_1, \mathcal{E}_2)$ which **jointly** satisfy:

$$\begin{aligned} \mathbb{P}_{H_0}\left(T_1 \geq \mathbf{s}_1^n\right) &\asymp_\infty e^{-N\mathcal{E}_1} \\ \mathbb{P}_{H_1}\left(T_1 \leq \mathbf{s}_1^n\right) &\asymp_\infty e^{-N\mathcal{E}_2} \end{aligned}$$

Beyond the Error Exponent: The Error Exponent curve

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- The set of these pairs $(\mathcal{E}_1, \mathcal{E}_2)$ is the **Error Exponent Curve**. It relies on:
  - Large deviations of $\lambda_{\max}$ under $(H_0) \Rightarrow \mathcal{E}_1$,
  - Large deviations of $\lambda_{\max}$ under $(H_1) \Rightarrow \mathcal{E}_2$.

The Error Exponent curve for T_1

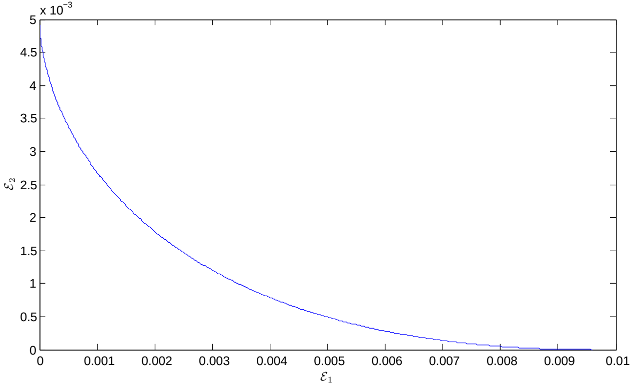

Figure: The Error Exponent curve: $\mathcal{E}_2$ versus $\mathcal{E}_1$.

The Extreme Eigenvalue Ratio test

Recall the following asymptotic results:

Under $\mathbf{H_0}$: Convergence of the extreme eigenvalues

$$\left.\begin{array}{l}\lambda_{\max} \rightarrow \sigma^2(1+\sqrt{c})^2 \\ \lambda_{\min} \rightarrow \sigma^2(1-\sqrt{c})^2\end{array}\right\} \quad \Rightarrow \quad \frac{\lambda_{\max}}{\lambda_{\min}} \rightarrow \frac{(1+\sqrt{c})^2}{(1-\sqrt{c})^2}$$

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Under $\mathbf{H_1}$: Convergence of the extreme eigenvalues

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The EER statistics: Based on the previous remarks, the EER statistics writes:

$$T_2 = \frac{\lambda_{\max}}{\lambda_{\min}}$$

.

The EER statistics

In the context of cooperative sensing, people have devoted a lot of attention to the statistics $T_2 = \frac{\lambda_{\max}}{\lambda_{\min}}$.

References:

- Zeng and Liang. *Maximum-Minimum Eigenvalue Detection for Cognitive Radio*. IEEE PIMRC 2007.
- L. Cardoso, M. Debbah, P. Bianchi, J. Najim. *Cooperative Spectrum Sensing Using Random Matrix Theory*. Proceedings IEEE ISWPC 2008.
- T. Lim, R. Zhang, Y. Liang and Y. Zeng. *GLRT-Based Spectrum Sensing for Cognitive Radio*. IEEE Globecom 2008
- Zeng and Liang (2008). *Eigenvalue based Spectrum Sensing Algorithms for Cognitive Radio*, `arXiv:0804.2960`.
- Penna et al. (2009) *Cooperative Spectrum Sensing based on the Limiting Eigenvalue Ratio Distribution in Wishart Matrices*. `arXiv:0902.1947`.

Theoretical study of the EER statistics

Using the tools of **Large Random Matrix theory** (as developped previously for the GMLR test) , one can:

- Study the fluctuations of $\frac{\lambda_{\max}}{\lambda_{\min}}$ and compute the **threshold** for a given **level of significance** α,
- Compute the **Error Exponent** for a fixed **level of significance** α,
- Plot the **Error Exponent Curve** associated to the test T_2.

In particular, the latter allow us to compare performances of T_2 and T_1.

The Error Exponent curve

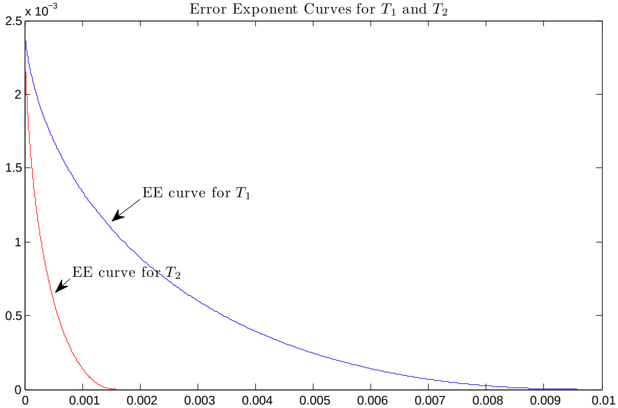

Figure: The Error Exponent curves for T_1 and T_2: The test statistics T_1 is uniformly more powerful than T_2.

Simulations: The type II error for a realistic scenario

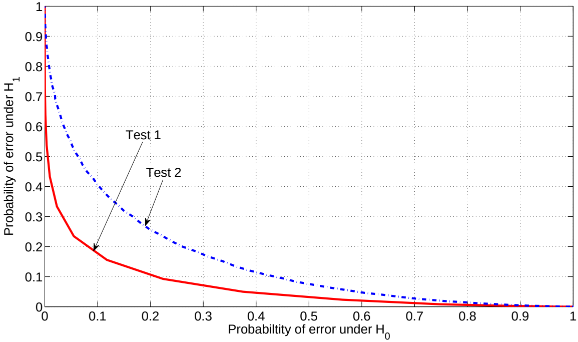

Figure: Type II Errors for T_1 and T_2 in the case where: $N = 10$, $n = 50$ et $\rho = 1$. Probabilities are computed via Monte-Carlo simulations (10^6 simulations).

Conclusion

In this presentation, we introduced recent and new results from Large Random Matrix Theory such as:

- Fluctuations of $\lambda_{\max}$ in a i.i.d. model,
- Large deviations for $\lambda_{\max}$ in a spiked model

and showed how to apply them in the context of Cooperative Spectrum Sensing.

Large deviations in particular allowed us to give a clean, theoretical study of the powers of the tests under investigation, and also to compare these tests.

As often with Random Matrices, we believe that the methods presented here will soon find other applications in wireless communication, beyond the context of cooperative sensing.

References: All the results presented here are developed in the forthcoming preprint:

- Bianchi, Debbah, Maïda, Najim. *Cooperative Sensing using the Sampled Covariance matrix*. soon to be posted.

As often with Random Matrices, we believe that the methods presented here will soon find other applications in wireless communication, beyond the context of cooperative sensing.

References: All the results presented here are developed in the forthcoming preprint:

- Bianchi, Debbah, Maïda, Najim. *Cooperative Sensing using the Sampled Covariance matrix*. soon to be posted.

Thank you for your attention!