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POSITIVE SOLUTIONS FOR LARGE RANDOM LINEAR SYSTEMS

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ABSTRACT. Consider a large linear system where A_n is an $n \times n$ matrix with independent real standard Gaussian entries, $\mathbf{1}_n$ is an $n \times 1$ vector of ones and with unknown the $n \times 1$ vector $\boldsymbol{x}_n$ satisfying

$$\boldsymbol{x}_n = \mathbf{1}_n + \frac{1}{\alpha_n \sqrt{n}} A_n \boldsymbol{x}_n \,.$$

We investigate the (componentwise) positivity of the solution $\boldsymbol{x}_n$ depending on the scaling factor α_n as the dimension n goes to infinity. We prove that there is a sharp phase transition at the threshold $\alpha_n^* = \sqrt{2\log n}$: below the threshold ($\alpha_n \ll \sqrt{2\log n}$), $\boldsymbol{x}_n$ has negative components with probability tending to 1 while above ($\alpha_n \gg \sqrt{2\log n}$), all the vector's components are eventually positive with probability tending to 1. At the critical scaling α_n^*, we provide a heuristics to evaluate the probability that $\boldsymbol{x}_n$ is positive.

Such linear systems arise as solutions at equilibrium of large Lotka-Volterra (LV) systems of differential equations, widely used to describe large biological communities with interactions. In the domain of positivity of $\boldsymbol{x}_n$ (a property known as *feasibility* in theoretical ecology), our results provide a stability criterion for such LV systems for which $\boldsymbol{x}_n$ is the solution at equilibrium.

1. Introduction

Denote by A_n an $n \times n$ matrix with independent Gaussian $\mathcal{N}(0,1)$ entries and by α_n a positive sequence. We are interested in the componentwise positivity of the $n \times 1$ vector $\boldsymbol{x}_n$, solution of the linear system

$$\boldsymbol{x}_n = \mathbf{1}_n + \frac{1}{\alpha_n \sqrt{n}} A_n \boldsymbol{x}_n \ , \tag{1.1}$$

where $\mathbf{1}_n$ is the $n \times 1$ vector with components 1.

It is well-known since Geman [7] that the limsup of the spectral radius of $\frac{A_n}{\sqrt{n}}$ is almost surely (a.s.) ≤ 1, so that matrix $\left(I_n - \frac{A_n}{\alpha_n\sqrt{n}}\right)$ is eventually invertible as long as $\alpha_n \gg 1$. In this case, vector $\boldsymbol{x}_n = (x_k)_{k\in[n]}$, where we denote by $[n] = \{1, \cdots, n\}$, is

$$\boldsymbol{x}_n = \left(I_n - \frac{A_n}{\alpha_n\sqrt{n}}\right)^{-1} \mathbf{1}_n \qquad \text{with} \qquad x_k = \boldsymbol{e}_k^* \left(I_n - \frac{A_n}{\alpha_n\sqrt{n}}\right)^{-1} \mathbf{1}_n \ ,$$

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where $\boldsymbol{e}_k$ is the $n \times 1$ canonical vector and B^* is the transconjugate of B (or simply its transpose if B is real).

The positivity of the x_k's is a key issue in the study of large Lotka-Volterra (LV) systems, widely used in mathematical biology and ecology to model populations with interactions.

Consider for instance a given foodweb and denote by $\boldsymbol{x}_n(t) = (x_k(t))_{k\in[n]}$ the vector of abundances of the various species within the foodweb at time t. A standard way to connect the various abundances is via a LV system of equations

$$\frac{dx_k(t)}{dt} = x_k(t)\left(1 - x_k(t) + \frac{1}{\alpha_n\sqrt{n}}\sum_{\ell\in[n]} A_{k\ell}\, x_\ell(t)\right) \qquad \text{for} \quad k \in [n]\,, \tag{1.2}$$

where the interactions $(A_{k\ell})$ can be modeled as random in the absence of any prior information. Here, the $A_{k\ell}$'s are assumed to be i.i.d. $\mathcal{N}(0,1)$. At the equilibrium $\frac{d\boldsymbol{x}_n}{dt} = 0$, the abundance vector $\boldsymbol{x}_n$ is a solution of (1.1) and a key issue is the existence of a *feasible* solution, that is a solution $\boldsymbol{x}_n$ where all the x_k's are positive. Dougoud et al. [5] based on Geman et al. [8] proved that a feasible solution is very unlikely to exist if $\alpha_n \equiv \alpha$ is a constant. In fact, the CLT proved in [8] asserts that for any fixed number M of components

$$(x_k - 1)_{k\in[M]} \quad \xrightarrow[n\to\infty]{\mathcal{D}} \quad Z \ \sim\ \mathcal{N}(0, \sigma_\alpha^2\, I_M)\ ,$$

where $\xrightarrow{\mathcal{D}}$ (resp. $\xrightarrow{\mathcal{P}}$) stands for the convergence in distribution (resp. in probability) and where $\sigma_\alpha^2 = \mathcal{O}(1)$. As an important consequence, vectors $\boldsymbol{x}_n$ with positive components will become extremely rare since

$$\mathbb{P}\{x_k > 0,\, k\in[M]\} \xrightarrow[n\to\infty]{} \left(\int_{-\frac{1}{\sigma_\alpha}}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx\right)^M \quad\Rightarrow\quad \mathbb{P}\{x_k > 0,\, k\in[n]\} \xrightarrow[n\to\infty]{} 0\,.$$

In this article, we consider a growing scaling factor $\alpha_n \to \infty$ and study the positivity of $\boldsymbol{x}_n$'s components in relation with α_n.

We find that there exists a critical threshold $\alpha_n^* = \sqrt{2\log n}$ below which feasible solutions hardly exist and above which feasible solutions are more and more likely to exist. More precisely, we prove the following:

Theorem 1.1 (Feasibility)**.** *Let* $\alpha_n \xrightarrow[n\to\infty]{} \infty$ *and denote by* $\alpha_n^* = \sqrt{2\log n}$. *Let* $\boldsymbol{x}_n = (x_k)_{k\in[n]}$ *be the solution of* (1.1).

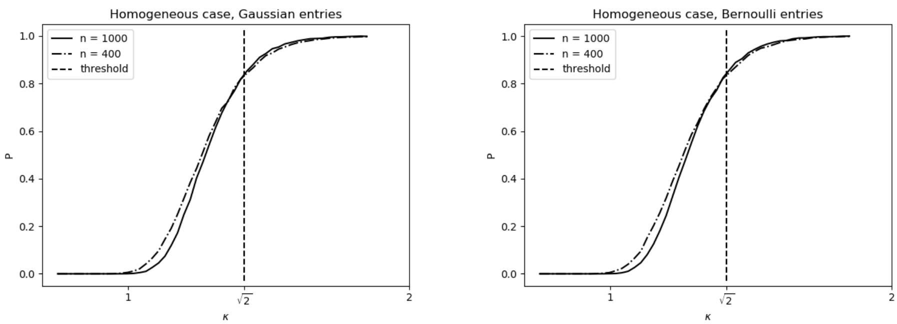
(1) *If* $\exists\, \varepsilon > 0$ *such that* $\alpha_n \le (1-\varepsilon)\alpha_n^*$ *then* $\mathbb{P}\left\{\min_{k\in[n]} x_k > 0\right\} \xrightarrow[n\to\infty]{} 0\,,$

(2) *If* $\exists\, \varepsilon > 0$ *such that* $\alpha_n \ge (1+\varepsilon)\alpha_n^*$ *then* $\mathbb{P}\left\{\min_{k\in[n]} x_k > 0\right\} \xrightarrow[n\to\infty]{} 1\,.$

We illustrate the transition toward feasibility in Figure 1.

Remark 1.2 (Beyond the Gaussian case). Proof of Theorem 1.1 is based on an analysis of the order of magnitude of the extreme values of the x_k's, which heavily relies on sub-Gaussiannity of Lipschitz functionals with Gaussian entries. This property remains true if the A_{ij}'s satisfies a logarithmic sobolev inequality - details are provided in Section 4.3. The case of discrete entries remains open although simulations (see Figure 1B) indicate that a similar phase transition occurs.

Remark 1.3. Notice that $\frac{1}{\alpha_n^*}$ goes to zero extremely slowly, as shown in Table 1. For modeling purposes, the threshold $\sigma_n^* := \frac{1}{\alpha_n^*}$ acts as an n-dependent upper bound of the standard deviation of the entries of $(\alpha_n^{-1} A_n)$, under which feasibility occurs.



(A) Gaussian entries (B) Bernoulli ±1 entries

FIGURE 1. Transition toward feasibility. We consider different values of n, respectively 1000 (dashed line), 4000 (solid line). For each n and each κ on the x-axis, we simulate 10000 $n \times n$ matrices A_n and compute the solution $\boldsymbol{x}_n$ of (1.1)at the scaling $\alpha_n(\kappa) = \kappa\sqrt{\log(n)}$. Each curve represents the proportion of feasible solutions $\boldsymbol{x}_n$ obtained for 10000 simulations. The dotted vertical line corresponds to the critical scaling $\alpha_n^* = \sqrt{2\log(n)}$ for $\kappa = \sqrt{2}$.

To complement the picture, we provide the following heuristics at the critical scaling $\alpha_n^* = \sqrt{2\log n}$:

$$\mathbb{P}\left\{\min_{k\in[n]} x_k > 0\right\} \quad \approx \quad 1 - \sqrt{\frac{e}{4\pi \log n}} + \frac{e}{8\pi \log n} \qquad \text{as} \quad n \to \infty\,. \tag{1.3}$$

TABLE 1. The quantity $\frac{1}{\alpha_n^*} = \frac{1}{\sqrt{2\log n}}$ vanishes extremely slowly.

n	10^2	10^3	10^4	10^5	10^6
$\frac{1}{\alpha_n^*}$	0.33	0.27	0.23	0.21	0.19

Aside from the question of feasibility arises the question of *stability* : for a large complex system, that is a system of coupled differential equations describing the time evolution of the abundances of the various species of a given foodweb, how likely a perturbation of the solution $\boldsymbol{x}$ will return to the equilibrium? Gardner and Ashby [6] considered stability issues of complex systems connected at random. Based on the circular law for large matrices with i.i.d. entries, May [14] provided a complexity/stability criterion and motivated the systematic use of large random matrix theory in the study of foodwebs, see for instance Allesina et al. [1]. Recently, Stone [15] and Gibbs et al. [9] revisited the relation between feasibility and stability.

We complement the result of Theorem 1.1 by addressing the question of stability in the context of a LV system (1.2) and prove that under the second condition of the theorem feasibility and stability occur simultaneously.

Recall that the solution at equilibrium $\boldsymbol{x}_n$ is stable if the Jacobian matrix $\mathcal{J}$ of the Lotka-Volterra system evaluated at $\boldsymbol{x}_n$, that is

$$\mathcal{J}(\boldsymbol{x}_n) = \mathrm{diag}(\boldsymbol{x}_n)\left(-I_n + \frac{A_n}{\alpha_n\sqrt{n}}\right) \tag{1.4}$$

has all its eigenvalues with negative real part.

Corollary 1.4 (Stability). *Let $\boldsymbol{x}_n = (x_k)_{k\in[n]}$ be the solution of* (1.1)*. Assume that $\boldsymbol{\ell}^+ := \limsup_{n\to\infty} \frac{\sqrt{2\log n}}{\alpha_n} < 1$. Denote by $\mathcal{S}_n$ the spectrum of $\mathcal{J}(\boldsymbol{x}_n)$. Then*

$$\max_{\lambda\in\mathcal{S}_n} \min_{k\in[n]} |\lambda + x_k| \xrightarrow[n\to\infty]{\mathcal{P}} 0 \quad \textit{and} \quad \max_{\lambda\in\mathcal{S}_n} \mathrm{Re}(\lambda) \ \le\ -(1-\boldsymbol{\ell}^+) + o_P(1)\,. \tag{1.5}$$

Proof of Corollary 1.4 relies on standard perturbation results from linear algebra and on Theorem 1.1.

Organization of the paper. Theorem 1.1 is proved in Section 2, Corollary 1.4 in Section 3. In Section 4, elements supporting heuristics (1.3) are provided, together with extensions to nonhomogeneous systems (where vector $\mathbf{1}_n$ in (1.1) is replaced by a generic deterministic vector $\boldsymbol{r}_n$) and non-Gaussian entries.

2. Positive solutions: Proof of Theorem 1.1

We will use the following notations for the various norms at stake: if $\boldsymbol{v}$ is a vector then $\|\boldsymbol{v}\|$ stands for its euclidian norm; if A is a matrix then $\|A\|$ stands for its spectral norm and $\|A\|_F = \sqrt{\sum_{ij}|A_{ij}|^2}$ for its Frobenius norm. Let φ be a function from $\Sigma = \mathbb{R}$ or $\mathbb{C}$ to $\mathbb{C}$ then $\|\varphi\|_\infty = \sup_{x\in\Sigma}|\varphi(x)|$.

2.1. **Some preparation and strategy of the proof.** Denote by $Q_n = \left(I_n - \frac{A_n}{\alpha_n\sqrt{n}}\right)^{-1}$ the resolvent and by $s(B)$ the largest singular value of a given matrix B. Then it is well known that almost surely $s_n := s(n^{-1/2}A_n) \xrightarrow[n\to\infty]{} 2$ (see for instance [3, Chapter 5]) hence $s\left(\frac{1}{\alpha_n\sqrt{n}}A_n\right) \xrightarrow[n\to\infty]{} 0$. In particular, the solution

$$\boldsymbol{x}_n = (x_k)_{k\in[n]} = \left(I_n - \frac{A_n}{\alpha_n\sqrt{n}}\right)^{-1}\mathbf{1}_n = Q_n\,\mathbf{1}_n\ ,$$

with I_n the $n\times n$ identity, is uniquely defined almost surely for all n large. In order to study the minimum of $\boldsymbol{x}_n$'s components, we partially unfold the above resolvent (in the sequel, we will simply denote $A, \alpha, \mathbf{1}, Q$ instead of $A_n, \alpha_n, \mathbf{1}_n, Q_n$) and write:

$$\begin{aligned} x_k \quad &= \quad \boldsymbol{e}_k^*\boldsymbol{x} \quad = \quad \boldsymbol{e}_k^* Q\,\mathbf{1} \quad = \quad \sum_{\ell=0}^{\infty} \boldsymbol{e}_k^*\left(\frac{A}{\alpha\sqrt{n}}\right)^{\ell}\mathbf{1}\,, \\ &= \quad 1 + \frac{1}{\alpha}\boldsymbol{e}_k^*\left(n^{-1/2}A\right)\mathbf{1} + \frac{1}{\alpha^2}\boldsymbol{e}_k^*\left(n^{-1/2}A\right)^2 Q\,\mathbf{1} \quad = \quad 1 + \frac{1}{\alpha}Z_k + \frac{1}{\alpha^2}R_k\ , \end{aligned} \tag{2.1}$$

where

$$Z_k = \boldsymbol{e}_k^*\left(n^{-1/2}A\right)\mathbf{1} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} A_{ki} \qquad \text{and} \qquad R_k = \boldsymbol{e}_k^*\left(n^{-1/2}A\right)^2 Q\,\mathbf{1}\,. \tag{2.2}$$

Notice that the Z_k's are i.i.d. $\mathcal{N}(0,1)$.

Extreme values of Gaussian random variables. Consider the sequence (Z_k) of standard Gaussian i.i.d. random variables, recall that $\alpha_n^* = \sqrt{2\log n}$ and let

$$M_n = \max_{k\in[n]} Z_k\,,\quad \check{M}_n = \min_{k\in[n]} Z_k \quad \text{and} \quad \beta_n^* = \alpha_n^* - \frac{1}{2\alpha_n^*}\log(4\pi\log n)\,. \tag{2.3}$$

Denote by $G(x) = e^{-e^{-x}}$ the Gumbel cumulative distribution. Then the following results are standard, (i.e. [12, Theorem 1.5.3]): for all $x \in \mathbb{R}$

$$\mathbb{P}\left\{\alpha_n^*(M_n - \beta_n^*) \le x\right\} \xrightarrow[n\to\infty]{} G(x)\,,\quad \mathbb{P}\left\{\alpha_n^*(\check{M}_n + \beta_n^*) \ge -x\right\} \xrightarrow[n\to\infty]{} G(x)\,. \tag{2.4}$$

Strategy of the proof. Equation (2.1) immediately yields

$$\begin{cases} \min_{k\in[n]} x_k & \ge & 1 + \frac{1}{\alpha}\check{M} + \frac{1}{\alpha^2}\min_{k\in[n]} R_k\ , \\ \min_{k\in[n]} x_k & \le & 1 + \frac{1}{\alpha}\check{M} + \frac{1}{\alpha^2}\max_{k\in[n]} R_k\,. \end{cases}$$

We rewrite the first equation as

$$\begin{aligned} \min_{k\in[n]} x_k \ &\ge\ 1 + \frac{\alpha_n^*}{\alpha_n}\left(\frac{\check{M}+\beta_n^*}{\alpha_n^*} - \frac{\beta_n^*}{\alpha_n^*} + \frac{\min_{k\in[n]} R_k}{\alpha_n^*\alpha_n}\right) \\ &=\ 1 + \frac{\alpha_n^*}{\alpha_n}\left(-1 + o_P(1) + \frac{\min_{k\in[n]} R_k}{\alpha_n^*\alpha_n}\right)\,, \end{aligned} \tag{2.5}$$

where we have used the fact that $(\alpha_n^*)^{-1}(\check{M} + \beta_n^*) = o_P(1)$. Similarly,

$$\min_{k\in[n]} x_k \ \le\ 1 + \frac{\alpha_n^*}{\alpha_n}\left(-1 + o_P(1) + \frac{\max_{k\in[n]} R_k}{\alpha_n^*\alpha_n}\right)\,.$$

The theorem will then follow from the following lemma.

Lemma 2.1. *The following convergence holds*

$$\frac{\max_{k\in[n]} R_k}{\alpha_n\sqrt{2\log n}} \xrightarrow[n\to\infty]{\mathcal{P}} 0 \qquad \textit{and} \qquad \frac{\min_{k\in[n]} R_k}{\alpha_n\sqrt{2\log n}} \xrightarrow[n\to\infty]{\mathcal{P}} 0\,.$$

Proof of Lemma 2.1 requires a careful analysis of the order of magnitude of the extreme values of the remaining term $(R_k)_{k\in[n]}$. It is postponed to Section 2.3.

2.2. **Lipschitz property and tightness of $R_k(A)$.** Let $\eta \in (0,1)$ and $\varphi : \mathbb{R}^+ \to [0,1]$ be a smooth (infinitely differentiable) function with values

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in [0, 2+\eta] \\ 0 & \text{if } x \ge 3 \end{cases}\ ,$$

and strictly decreasing from 1 to zero as x goes from $2+\eta$ to 3. Notice in particular that $\|\varphi'\|_\infty$ is finite. Recall that $s_n = s(n^{-1/2}A)$ is the largest singular value of the normalized matrix $n^{-1/2}A$ and denote by

$$\varphi_n := \varphi(s_n) = \varphi\left(s(n^{-1/2}A)\right)\,.$$

Notice that $\mathbb{P}\{\varphi_n < 1\} = \mathbb{P}\{s_n > 2+\eta\} \xrightarrow[n\to\infty]{} 0$ (by the a.s. convergence of s_n to 2). Instead of working with R_k we introduce the truncated quantity:

$$\widetilde{R}_k = \varphi_n R_k\ . \tag{2.6}$$

For a given $n \times n$ matrix A, we may consider its $2n \times 2n$ hermitized matrix $\mathcal{H}(A)$ defined as $\mathcal{H}(A) = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$. Notice that the singular values of A together with their negatives are the eigenvalues of $\mathcal{H}(A)$.

Lemma 2.2. *Let $\widetilde{R}_k$ be given by* (2.6)*, then the function $A \mapsto \widetilde{R}_k(A)$ is Lipschitz, i.e.*

$$\left|\widetilde{R}_k(A) - \widetilde{R}_k(B)\right| \le K\|A - B\|_F \ , \tag{2.7}$$

where $\|A\|_F$ is the Frobenius norm and K is a constant independent from k and n.

Proof. Notice that $\varphi(s_n) = 0$ and $\varphi'(s_n) = 0$ for $s_n \ge 3$, which implies that one may consider the bound $s_n \le 3$ in the following computations, for $\widetilde{R}_k$ or its derivatives would be zero otherwise. Recall the definition of the resolvent $Q = \left(I - \frac{A}{\alpha\sqrt{n}}\right)^{-1}$ then $Q^{-1}Q = I$ which yields $Q = I + \frac{A}{\alpha\sqrt{n}}Q$ from which we deduce that

$$\varphi_n \|Q\| \quad\le\quad \varphi_n \left(1 - \frac{1}{\alpha}\left\|n^{-\frac{1}{2}}A\right\|\right)^{-1} \quad\le\quad \frac{1}{1 - 3\alpha^{-1}} \quad\le\quad 3 \tag{2.8}$$

for n large enough.

We first consider a matrix A such that $\mathcal{H}(A)$ has simple spectrum (i.e. with $2n$ distinct eigenvalues, each with multiplicity 1). We denote by $\partial_{ij} = \frac{\partial}{\partial A_{ij}}$ and prove that the vector $\nabla\widetilde{R}_k(A) = \left(\partial_{ij}\widetilde{R}(A),\ i,j \in [n]\right)$ satisfies

$$\|\nabla\widetilde{R}_k(A)\| = \sqrt{\sum_{ij}\left|\partial_{ij}\widetilde{R}_k(A)\right|^2} \le K\,. \tag{2.9}$$

We may occasionally drop the dependence of $\widetilde{R}_k$ in A. We begin by computing

$$\begin{aligned}
\partial_{ij}\widetilde{R}_k \quad &= \quad \lim_{h\to 0}\frac{\widetilde{R}_k(A + h\boldsymbol{e}_i\boldsymbol{e}_j^*) - \widetilde{R}_k(A)}{h}\ , \\
&= \quad (\partial_{ij}\varphi_n)R_k + \varphi_n\,\boldsymbol{e}_k^*\left(\partial_{ij}\left(n^{-\frac{1}{2}}A\right)^2\right)Q\,\mathbf{1} + \varphi_n\,\boldsymbol{e}_k^*\left(n^{-\frac{1}{2}}A\right)^2(\partial_{ij}Q)\,\mathbf{1} \\
&=: \quad T_{1,ij} + T_{2,ij} + T_{3,ij}\,.
\end{aligned}$$

Straightforward computations yield

$$\partial_{ij}\left(n^{-\frac{1}{2}}A\right)^2 = \frac{1}{n}\left(A\boldsymbol{e}_i\boldsymbol{e}_j^* + \boldsymbol{e}_i\boldsymbol{e}_j^*A\right) \quad\text{and}\quad \partial_{ij}Q = \frac{1}{\alpha\sqrt{n}}Q\boldsymbol{e}_i\boldsymbol{e}_j^*Q\,. \tag{2.10}$$

It remains to compute $\partial_{ij}\varphi_n = \varphi'(s_n)\partial_{ij}s_n$. Recall that $\mathcal{H}(A)$ has a simple spectrum and notice that $A \mapsto s_n(A)$ is differentiable. In fact, since s_n is simple, it is a simple root of the characteristic polynomial. In particular, it is not a root of its derivative and one can use the implicit function theorem to conclude its differentiability. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be respectively the left and right normalized singular vectors associated to $s(A)$. Then

$$\mathcal{H}(A)\boldsymbol{w} = s(A)\boldsymbol{w} \quad\text{with}\quad \boldsymbol{w} = \begin{pmatrix}\boldsymbol{u} \\ \boldsymbol{v}\end{pmatrix} \quad\text{and}\quad \|\boldsymbol{w}\|^2 = 2\ ,$$

moreover $\boldsymbol{w}$ is (up to scaling) the unique eigenvector of $s(A)$ since $s(A)$ is simple by assumption. We now apply [10, Theorem 6.3.12] to compute s_n's derivative:
(2.11)

$$\partial_{ij}s(A) = \frac{1}{\|\boldsymbol{w}\|^2}\left(\boldsymbol{u}^*\boldsymbol{e}_i\boldsymbol{e}_j^*\boldsymbol{v} + \boldsymbol{v}^*\boldsymbol{e}_j\boldsymbol{e}_i^*\boldsymbol{u}\right) = \boldsymbol{u}^*\boldsymbol{e}_i\boldsymbol{e}_j^*\boldsymbol{v} \quad \text{hence} \quad \partial_{ij}s_n = \frac{1}{\sqrt{n}}\boldsymbol{u}^*\boldsymbol{e}_i\boldsymbol{e}_j^*\boldsymbol{v}$$

(recall that all the considered vectors are real). We first handle the term $T_{1,ij}$.

$$\begin{aligned}\sum_{ij}|T_{1,ij}|^2 &= \sum_{ij}\left|\boldsymbol{u}^*\boldsymbol{e}_i\boldsymbol{e}_j^*\boldsymbol{v}\varphi'(s_n)\boldsymbol{e}_k^*\left(n^{-1/2}A\right)^2 Q\frac{\mathbf{1}}{\sqrt{n}}\right|^2 ,\\ &\le 3^6\|\varphi'\|_\infty^2\sum_i|\boldsymbol{u}^*\boldsymbol{e}_i|^2\sum_j\left|\boldsymbol{e}_j^*\boldsymbol{v}\right|^2 \quad\le\quad 3^6\|\varphi'\|_\infty^2 .\end{aligned}$$

We now handle the term $T_{2,ij}$.

$$\begin{aligned}\sum_{ij}|T_{2,ij}|^2 &= \sum_{ij}\left|\varphi_n\boldsymbol{e}_k^*\left(\frac{A}{\sqrt{n}}\boldsymbol{e}_i\boldsymbol{e}_j^* + \boldsymbol{e}_i\boldsymbol{e}_j^*\frac{A}{\sqrt{n}}\right)Q\frac{\mathbf{1}}{\sqrt{n}}\right|^2\\ &\le 2\varphi_n^2\sum_i\left|\boldsymbol{e}_k^*\frac{A}{\sqrt{n}}\boldsymbol{e}_i\right|^2\sum_j\left|\boldsymbol{e}_j^*Q\frac{\mathbf{1}}{\sqrt{n}}\right|^2 + 2\varphi_n^2\sum_i|\boldsymbol{e}_k^*\boldsymbol{e}_i|^2\sum_j\left|\boldsymbol{e}_j^*\frac{A}{\sqrt{n}}Q\frac{\mathbf{1}}{\sqrt{n}}\right|^2 ,\\ &= 2\varphi_n^2\left(\boldsymbol{e}_k^*\frac{A}{\sqrt{n}}\frac{A^*}{\sqrt{n}}\boldsymbol{e}_k\right)\left(\frac{\mathbf{1}^*}{\sqrt{n}}Q^*Q\frac{\mathbf{1}}{\sqrt{n}}\right) + 2\varphi_n^2\left(\frac{\mathbf{1}^*}{\sqrt{n}}Q^*\frac{A^*A}{n}Q\frac{\mathbf{1}}{\sqrt{n}}\right) \le \quad 2^2\times 3^4 .\end{aligned}$$

The term $T_{3,ij}$ can be handled similarly and one can prove $\sum_{ij}|T_{3,ij}|^2 \le 3^8$. Gathering all these estimates, we finally obtain the desired bound:

$$\sqrt{\sum_{ij}\left|\partial_{ij}\widetilde{R}_k\right|^2} \le \sqrt{3\sum_{ij}|T_{1,ij}|^2 + 3\sum_{ij}|T_{2,ij}|^2 + 3\sum_{ij}|T_{3,ij}|^2} \le K ,$$

where K neither depends on k nor on n.

Having proved a local estimate over $\|\nabla\widetilde{R}_k(A)\|$ for each matrix A such that $\mathcal{H}(A)$ has simple spectrum, we now establish the Lipschitz estimate (2.7) for two such matrices A, B.

Let A, B such that $\mathcal{H}(A)$ and $\mathcal{H}(B)$ have simple spectrum and consider $A_t = (1-t)A + tB$ for $t \in [0,1]$. Notice first that the continuity of the eigenvalues implies that there exists $\delta > 0$ sufficiently small such that $\mathcal{H}(A_t)$ has a simple spectrum for $t \le \delta$ and $t \ge 1-\delta$. To go beyond $[0,\delta)\cup(1-\delta,1]$ and prove that $\mathcal{H}(A_t)$ has simple spectrum for the entire interval $[0,1]$ except maybe for a finite number of points, we rely on the argument in Kato [11, Chapter 2.1] which states that apart from a finite number of t_ℓ's: $t_0 = 0 < t_1 < \cdots < t_L < t_{L+1} = 1$, the number of eigenvalues of $\mathcal{H}(A_t)$ remains constant for $t \in [0,1]$ and $t \neq t_\ell, \ell \in [L]$. Since $\mathcal{H}(A_t)$ has simple spectrum for $t \in [0,\delta)\cup(1-\delta,1]$, it has simple spectrum for all $t \notin \{t_\ell, \ell \in [L]\}$.

We can now proceed:

$$\begin{aligned}\left|\widetilde{R}_k(A_{t_1}) - \widetilde{R}_k(A)\right| &= \left|\lim_{\tau\nearrow t_1}\int_0^\tau \frac{d}{dt}\widetilde{R}_k(A_t)\,dt\right| = \left|\lim_{\tau\nearrow t_1}\int_0^\tau \nabla\widetilde{R}_k(A_t)\circ\frac{d}{dt}A_t\,dt\right| ,\\ &\le \lim_{\tau\nearrow t_1}\int_0^\tau \|\nabla\widetilde{R}_k(A_t)\|\times\|B-A\|_F\,dt \le K\,t_1\,\|B-A\|_F .\end{aligned}$$

By iterating this process, we obtain

$$\begin{aligned}\left|\widetilde{R}_k(B) - \widetilde{R}_k(A)\right| &\leq \sum_{\ell=1}^{L+1} \left|\widetilde{R}_k(A_{t_\ell}) - \widetilde{R}_k(A_{t_{\ell-1}})\right| \\ &\leq \sum_{\ell=1}^{L+1} K(t_\ell - t_{\ell-1})\|B - A\|_F = K\|B - A\|_F\,,\end{aligned}$$

hence the Lipschitz property along the segment $[A, B]$.

The general property follows by density of such matrices in the set of $n \times n$ matrices and by continuity of $A \mapsto \widetilde{R}_k(A)$. Let A, B be given and $A_\varepsilon \to A$ and $B_\varepsilon \to B$ be such that $\mathcal{H}(A_\varepsilon)$ and $\mathcal{H}(B_\varepsilon)$ have simple spectrum then:

$$\begin{aligned}\left|\widetilde{R}_k(B) - \widetilde{R}_k(A)\right| \leq \left|\widetilde{R}_k(B_\varepsilon) - \widetilde{R}_k(B)\right| + K\|B_\varepsilon - A_\varepsilon\| + \left|\widetilde{R}_k(A_\varepsilon) - \widetilde{R}_k(A)\right| \\ \xrightarrow[\varepsilon\to 0]{} K\|B - A\|_F\,.\end{aligned}$$

Proof of Lemma 2.2 is completed. □

We now use concentration arguments to bound $\mathbb{E}\max_{k\in[n]}(\widetilde{R}_k - \mathbb{E}\widetilde{R}_k)$.

Proposition 2.3. *Let K be as in Lemma 2.2, then $\mathbb{E}\max_{k\in[n]}(\widetilde{R}_k - \mathbb{E}\widetilde{R}_k) \leq K\sqrt{2\log n}$.*

Proof. By applying Tsirelson-Ibragimov-Sudakov inequality [4, Theorem 5.5] to $\widetilde{R}_k(A)$, we obtain the following exponential estimate:

$$\mathbb{E}e^{\lambda(\widetilde{R}_k(A) - \mathbb{E}\widetilde{R}_k(A))} \leq e^{\frac{\lambda^2 K^2}{2}} \quad \forall \lambda \in \mathbb{R}\,.$$

We now estimate the expectation of the maximum (we drop the dependence in A).

$$\begin{aligned}\exp\left(\lambda\mathbb{E}\max_{k\in[n]}(\widetilde{R}_k - \mathbb{E}\widetilde{R}_k)\right) &\leq \mathbb{E}\exp\left(\lambda\max_{k\in[n]}(\widetilde{R}_k - \mathbb{E}\widetilde{R}_k)\right) \\ &\leq \sum_{k=1}^{n} \mathbb{E}e^{\lambda(\widetilde{R}_k - \mathbb{E}\widetilde{R}_k)} \leq ne^{\frac{\lambda^2 K^2}{2}}\,.\end{aligned}$$

Hence for $\lambda > 0$,

$$\mathbb{E}\max_{k\in[n]}(\widetilde{R}_k - \mathbb{E}\widetilde{R}_k) \leq \frac{\log n}{\lambda} + \frac{\lambda K^2}{2} =: \Phi(\lambda)\,.$$

Optimizing in λ, we get $\lambda^* = \frac{\sqrt{2\log n}}{K}$ and $\Phi(\lambda^*) = K\sqrt{2\log n}$, the desired estimate. □

Proposition 2.4. *We have $\mathbb{E}\widetilde{R}_k(A_n) = \mathcal{O}(1)$ uniformly in $k \in [n]$.*

Proof. By exchangeability, we have $\mathbb{E}\widetilde{R}_k = \frac{1}{n}\sum_{i=1}^n \mathbb{E}\widetilde{R}_i$ and

$$\left|\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\widetilde{R}_i\right| = \left|\frac{1}{n}\mathbb{E}\,\varphi_n \mathbf{1}^*\left(\frac{A}{\sqrt{n}}\right)^2 Q\,\mathbf{1}\right| \leq \left\|\frac{\mathbf{1}}{\sqrt{n}}\right\|^2 \mathbb{E}\left(\varphi_n\left\|\frac{A}{\sqrt{n}}\right\|^2\|Q\|\right) = \mathcal{O}(1)\,.$$

by (2.8). Proof of Proposition 2.4 is completed. □

We are now in position to prove Lemma 2.1.

2.3. **Proof of Lemma 2.1.** Since the $\widetilde{R}_i(A)$'s are exchangeable, $\mathbb{E}\widetilde{R}_k(A) = \mathbb{E}\widetilde{R}_1(A)$. Notice that $\max_{k\in[n]} \widetilde{R}_k(A) - \widetilde{R}_1(A)$ is nonnegative hence by Markov inequality,

$$\mathbb{P}\left\{\frac{\max_{k\in[n]} \widetilde{R}_k(A) - \widetilde{R}_1(A)}{\alpha\sqrt{2\log n}} \geq \varepsilon\right\} \leq \frac{\mathbb{E}\left(\max_{k\in[n]} \widetilde{R}_k(A) - \widetilde{R}_1(A)\right)}{\varepsilon\alpha\sqrt{2\log n}}$$

$$= \frac{\mathbb{E}\left(\max_{k\in[n]}\left(\widetilde{R}_k(A) - \mathbb{E}\widetilde{R}_k(A) + \mathbb{E}\widetilde{R}_1(A)\right) - \widetilde{R}_1(A)\right)}{\varepsilon\alpha\sqrt{2\log n}},$$

$$= \frac{\mathbb{E}\left(\max_{k\in[n]}\left(\widetilde{R}_k(A) - \mathbb{E}\widetilde{R}_k(A)\right)\right)}{\varepsilon\alpha\sqrt{2\log n}} \leq \frac{K}{\varepsilon\alpha}$$

by Proposition 2.3. This implies that

$$\frac{\max_{k\in[n]} \widetilde{R}_k(A) - \widetilde{R}_1(A)}{\alpha\sqrt{2\log n}} \xrightarrow[n\to\infty]{\mathcal{P}} 0\,. \tag{2.12}$$

We now prove that

$$\widetilde{R}_1(A)\,/\,\left(\alpha\sqrt{2\log n}\right) \xrightarrow[n\to\infty]{\mathcal{P}} 0\,. \tag{2.13}$$

By Proposition 2.4, $\mathbb{E}\widetilde{R}_1(A) = \mathcal{O}(1)$ hence $\mathbb{E}\widetilde{R}_1(A)/(\alpha\sqrt{2\log(n)}) \to 0$. Applying Poincaré's inequality (cf. [4, Theorem 3.20] and its extension to Lipschitz functionals on p. 73) to the Lipschitz functional $A \mapsto \widetilde{R}_1(A)$ (cf. Lemma 2.2), we can bound $\widetilde{R}_1(A)$'s variance by K^2 and obtain

$$\mathbb{P}\left(\left|\frac{\widetilde{R}_1(A) - \mathbb{E}\widetilde{R}_1(A)}{\alpha\sqrt{2\log n}}\right| > \delta\right) \leq \frac{\mathrm{var}(\widetilde{R}_1(A))}{2\delta^2\alpha^2\log n} \leq \frac{K^2}{2\delta^2\alpha^2\log n} \xrightarrow[n\to\infty]{} 0\,.$$

This and Proposition 2.4 yield (2.13). Combining (2.12) and (2.13) finally yields:

$$\frac{\max_{k\in[n]} \widetilde{R}_k(A)}{\alpha\sqrt{2\log n}} \xrightarrow[n\to\infty]{\mathcal{P}} 0\,.$$

In order to obtain the result for the untilded quantities, we write

$$\mathbb{P}\left\{\left|\frac{\max_k R_k(A)}{\alpha\sqrt{2\log n}}\right| > \varepsilon\right\} \leq \mathbb{P}\left\{\max_k R_k(A) \neq \max_k \widetilde{R}_k(A)\right\} + \mathbb{P}\left\{\left|\frac{\max_k \widetilde{R}_k(A)}{\alpha\sqrt{2\log n}}\right| > \frac{\varepsilon}{2}\right\},$$

$$= \mathbb{P}\{\varphi_n < 1\} + \mathbb{P}\left\{\left|\frac{\max_k \widetilde{R}_k(A)}{\alpha\sqrt{2\log n}}\right| > \varepsilon/2\right\} \xrightarrow[n\to\infty]{} 0\,.$$

One proves the second assertion similarly. This concludes the proof of Lemma 2.1.

3. Stability: Proof of Corollary 1.4

In order to study the stability of large Lotka-Volterra systems, we are led to study the matrix

$$\mathcal{J}(\boldsymbol{x}_n) = \mathrm{diag}(\boldsymbol{x}_n)\left(-I_n + \frac{A_n}{\alpha_n\sqrt{n}}\right)\,.$$

We first establish the following estimates

$$\begin{cases} \min_{k\in[n]} x_k \geq 1 - \boldsymbol{\ell}^+ - o_P(1)\,, \\ \max_{k\in[n]} x_k \leq 1 + \boldsymbol{\ell}^+ + o_P(1)\,. \end{cases} \tag{3.1}$$

The first estimate immediately follows from (2.5) together with Lemma 2.1. From x_k's decomposition (2.1) we have

$$\begin{aligned}\max_{k\in[n]} x_k &\le 1+\frac{M_n}{\alpha_n}+\frac{\max_{k\in[n]} R_k}{\alpha_n^2} = 1+\frac{\alpha_n^*}{\alpha_n}\left(\frac{M_n-\beta_n^*}{\alpha_n^*}+\frac{\beta_n^*}{\alpha_n^*}+\frac{\max_{k\in[n]} R_k}{\alpha_n^*\alpha_n}\right)\\ &\le 1+\boldsymbol{\ell}^+ + o_P(1)\,,\end{aligned}$$

where the last inequality follows from Lemma 2.1 and the fact that $(\alpha_n^*)^{-1}(M_n - \beta_n^*) \xrightarrow{\mathcal{P}} 0$.

We now compare the spectra of matrices $\mathcal{D}(\boldsymbol{x}_n) = -\mathrm{diag}(\boldsymbol{x}_n)$ and $\mathcal{J}(\boldsymbol{x}_n)$ by relying on Bauer and Fike's theorem [10, Theorem 6.3.2]: for every $\lambda \in \mathcal{S}_n$, there exists a component x_k of vector $\boldsymbol{x}_n$ such that

$$\begin{aligned}|\lambda + x_k| &\le \left\|\mathrm{diag}(\boldsymbol{x}_n)\frac{A_n}{\alpha_n\sqrt{n}}\right\| \le \frac{1}{\alpha_n}\|\mathrm{diag}(\boldsymbol{x}_n)\|\left\|\frac{A_n}{\sqrt{n}}\right\|\\ &\overset{(a)}{\le} \frac{1}{\alpha_n}\left(1+\boldsymbol{\ell}^+ + o_P(1)\right)\left(2+o_P(1)\right) = o_P(1),\end{aligned}$$

where (a) follows from the second estimate in (3.1) and from the spectral norm estimate. Notice that the majorization above is uniform for $\lambda \in \mathcal{S}_n$. The first part of the corrolary is proved. Finally,

$$\mathrm{Re}(\lambda) + x_k \le |\lambda + x_k| = o_P(1) \qquad \Rightarrow \qquad \mathrm{Re}(\lambda) \le -\min_{k\in[n]} x_k + o_P(1)\,.$$

The estimate (1.5) finally follows from the first estimate in (3.1).

4. Heuristics at critical scaling, nonhomogeneous systems and non-Gaussian entries

4.1. **A heuristics at the critical scaling.**

We provide here a heuristics to compute the probability that a solution $\boldsymbol{x}_n$ is feasible at critical scaling $\alpha_n^* = \sqrt{2\log n}$.

Heuristics 4.1. *The probability that a solution is feasible at the critical scaling α_n^* is asymptotically given by*

$$\mathbb{P}(x_k > 0,\ k\in[n]) \approx 1 - \sqrt{\frac{e}{4\pi\log n}} + \frac{e}{8\pi\log n} =: H_1(n)\,. \tag{4.1}$$

In Figure 2, we compare the heuristics with results from simulations.

Arguments. Consider

$$x_k = 1 + \boldsymbol{e}_k^*\frac{A_n}{\alpha_n^*\sqrt{n}}\mathbf{1}_n + \frac{R_k}{(\alpha_n^*)^2} = 1+\frac{Z_k}{\alpha_n^*}+\frac{R_k}{(\alpha_n^*)^2} = 1+\frac{1}{\alpha_n^*}\left(Z_k + \frac{R_k}{\alpha_n^*}\right)\,.$$

Following Geman and Hwang [8, Lemma A.1], one could prove that Z_k and R_k are asymptotically independent centered Gaussian random variables, each with variance one. We thus approximate the quantity $Z_k + \frac{R_k}{\alpha_n^*}$ by a Gaussian random variable with distribution $\mathcal{N}\left(0, 1+\frac{1}{(\alpha_n^*)^2}\right)$ and set $x_k \approx 1 + \left(\frac{1}{\alpha_n^*}\sqrt{1+\frac{1}{(\alpha_n^*)^2}}\right)U_k$ where the U_k's are i.i.d. $\mathcal{N}(0,1)$. Denote by $\check{M}_n^U = \min_{k\in[n]} U_k$ then

$$\mathbb{P}(x_k > 0\,,\ k\in[n]) \approx \mathbb{P}\left(1+\left(\frac{1}{\alpha_n^*}\sqrt{1+\frac{1}{(\alpha_n^*)^2}}\right)\check{M}_n^U > 0\right)\,.$$

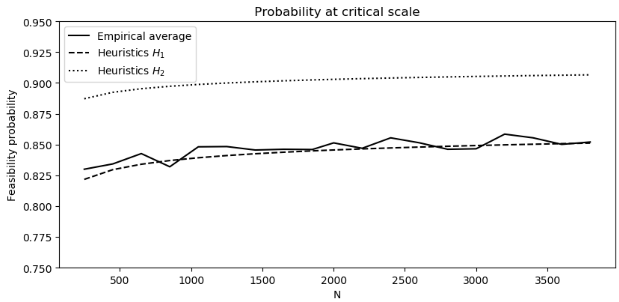


FIGURE 2. Probability at critical scaling. The solid curve corresponds to the proportion of feasible solutions at critical scaling α_n^* obtained for 10000 simulations (for n ranging from 50 to 3750 with a 200-increment) - notice the strong standard deviation. The dashed curve represents the heuristics H_1 defined in (4.1). The dotted curve represents the heuristics H_2 introduced in Remark 4.2. Notice the substantial discrepancy between H_1 and H_2.

Recall that standard Gaussian extreme value convergence results yield
(4.2)
$$\mathbb{P}\left\{\alpha_n^*\left(-\check{M}_n^U-\beta_n^*\right)<x\right\}=\mathbb{P}\left\{\alpha_n^*(\check{M}_n^U+\beta_n^*)>-x\right\}\quad\xrightarrow[n\to\infty]{}\quad G(x)=e^{-e^{-x}}\,,$$
where β_n^* is defined in (2.3). Denote by $\Theta(\alpha)=\sqrt{1+\alpha^{-2}}$ then
$$\mathbb{P}\left(1+\Theta(\alpha_n^*)\frac{\check{M}_n^U}{\alpha_n^*}>0\right)=\mathbb{P}\left(\alpha_n^*(\check{M}_n^U+\beta_n^*)>-\frac{(\alpha_n^*)^2}{\Theta(\alpha_n^*)}+\alpha_n^*\beta_n^*\right)\,.$$
Notice that
$$-\frac{(\alpha_n^*)^2}{\Theta(\alpha_n^*)}+\alpha_n^*\beta_n^*=\frac{1}{2}-\frac{1}{2}\log(4\pi\log n)+\mathcal{O}\left(\frac{1}{(\alpha_n^*)^2}\right)=\frac{1}{2}+\log\frac{1}{\sqrt{2\pi}\alpha_n^*}+\mathcal{O}\left(\frac{1}{(\alpha_n^*)^2}\right)\,.$$
Hence
$$\begin{aligned}\mathbb{P}\left(1+\Theta(\alpha_n^*)\frac{\check{M}_n^U}{\alpha_n^*}>0\right)&=\mathbb{P}\left(\alpha_n^*(\check{M}_n^U+\beta_n^*)>\frac{1}{2}+\log\frac{1}{\sqrt{2\pi}\alpha_n^*}+\mathcal{O}\left(\frac{1}{(\alpha_n^*)^2}\right)\right)\\&\overset{(a)}{\approx}e^{-\exp\left(\frac{1}{2}+\log\frac{1}{\sqrt{2\pi}\alpha_n^*}+\mathcal{O}\left(\frac{1}{(\alpha_n^*)^2}\right)\right)}=e^{-\sqrt{\frac{e}{2\pi}}\frac{1}{\alpha_n^*}\left(1+\mathcal{O}((\alpha_n^*)^{-2})\right)}\\&=1-\sqrt{\frac{e}{2\pi}}\frac{1}{\alpha_n^*}+\frac{1}{2}\frac{e}{2\pi}\frac{1}{(\alpha_n^*)^2}+\mathcal{O}\left(\frac{1}{(\alpha_n^*)^3}\right).\end{aligned}\tag{4.3}$$
We finally end up with the announced approximation
$$\mathbb{P}(x_k>0\,,\ k\in[n])\quad\approx\quad H_1(n):=1-\sqrt{\frac{e}{4\pi\log n}}+\frac{e}{8\pi\log n}\,.$$

Remark 4.2. A rougher approximation would have been to set $x_k\approx 1+\frac{Z_k}{\alpha_n^*}$ with $Z_k\sim\mathcal{N}(0,1)$ and to drop the next term $\frac{R_k}{(\alpha_n^*)^2}$ in the heuristics but this would have resulted in the following approximation
$$\mathbb{P}(x_k>0,\ k\in[n])\quad\approx\quad 1-(4\pi\log(n))^{-1/2}+(8\pi\log(n))^{-1}=:H_2(n)\,,$$
which is worse than $H_1(n)$, as illustrated in Figure 2. $\square$

4.2. **Positivity for a nonhomogeneous linear system.** By homogeneous, we refer to a LV system where the intrinsic growth rate of species i is equal to 1. If not, the system is nonhomogeneous (NH). The results developed so far extend to a NH linear system where $\mathbf{1}_n$ is replaced by a vector $\boldsymbol{r}_n$ with slight modifications. In particular, we identify a regime where feasibility and stability occur simultaneously. Denote by $\boldsymbol{r}_n = (r_k)$ a $n \times 1$ deterministic vector with positive components and consider the linear system

$$\boldsymbol{x}_n = \boldsymbol{r}_n + \frac{1}{\alpha_n \sqrt{n}} A_n \boldsymbol{x}_n \,. \tag{4.4}$$

Introduce the notations

$$r_{\min}(n) = \min_{k\in[n]} r_k\,,\ \ r_{\max}(n) = \max_{k\in[n]} r_k \text{ and } \sigma_{\boldsymbol{r}}(n) = \|\boldsymbol{r}/\sqrt{n}\| = \sqrt{n^{-1} \sum_{k\in[n]} r_k^2}\,.$$

Assume that there exist $\rho_{\min}, \rho_{\max}$ independent from n such that eventually

$$0 < \rho_{\min} \leq r_{\min}(n) \leq \sigma_{\boldsymbol{r}}(n) \leq r_{\max}(n) \leq \rho_{\max} < \infty\,.$$

Then

Theorem 4.3 (Feasibility - NH case)**.** *Let $\alpha_n \xrightarrow[n\to\infty]{} \infty$ and denote by $\alpha_n^* = \sqrt{2\log n}$. Let $\boldsymbol{x}_n = (x_k)_{k\in[n]}$ be the solution of* (4.4)*.*

(1) *If $\exists\, \varepsilon > 0$ such that $\alpha_n \leq (1-\varepsilon)\frac{\alpha_n^* \sigma_{\boldsymbol{r}}(n)}{r_{\max}(n)}$ then $\mathbb{P}\left\{\min_{k\in[n]} x_k > 0\right\} \xrightarrow[n\to\infty]{} 0$.*
(2) *If $\exists\, \varepsilon > 0$ such that $\alpha_n \geq (1+\varepsilon)\frac{\alpha_n^* \sigma_{\boldsymbol{r}}(n)}{r_{\min}(n)}$ then $\mathbb{P}\left\{\min_{k\in[n]} x_k > 0\right\} \xrightarrow[n\to\infty]{} 1$.*

Remark 4.4. Contrary to the homogeneous system where there is a sharp transition at $\alpha_n^* = \sqrt{2\log(n)}$, the situation is not as clean-cut here and there is a buffer zone

$$\alpha_n \in \left[\frac{\sigma_{\boldsymbol{r}}(n)}{r_{\max}(n)}\sqrt{2\log(n)}\,,\ \frac{\sigma_{\boldsymbol{r}}(n)}{r_{\min}(n)}\sqrt{2\log(n)}\right]$$

in which the study of the feasibility is not clear.

This buffer zone is illustrated in Figure 3 where we simulate the transition toward feasibility for a nonhomogeneous system (4.4) in the case where deterministic vector $\boldsymbol{r}_n$ is equally distributed over $[1, 3]$, i.e.

$$\boldsymbol{r}_n(i) = 1 + \frac{2i}{n}\,, \qquad \sigma_{\boldsymbol{r}}(n) = \sqrt{\frac{1}{n}\sum_{i\in[n]} \boldsymbol{r}_n^2(i)} \xrightarrow[n\to\infty]{} \sqrt{\int_0^1 (1+2x)^2\, dx} \tag{4.5}$$

We introduce the quantities

$$t_1 = \lim_N \frac{\sqrt{2}\sigma_{\boldsymbol{r}}(N)}{\boldsymbol{r}_{\max}} \simeq 0.98 \quad \text{and} \quad t_2 = \lim_N \frac{\sqrt{2}\sigma_{\boldsymbol{r}}(N)}{\boldsymbol{r}_{\min}} \simeq 2.94\,. \tag{4.6}$$

As one may notice, the transition region is wider than in the homogeneous case.

Elements of proof. We have

$$x_k = \boldsymbol{e}_k^*\, Q\, \boldsymbol{r}_n = r_k + \frac{1}{\alpha}\frac{\sum_{i=1}^n r_i A_{ki}}{\sqrt{n}} + \frac{1}{\alpha^2}\boldsymbol{e}_k^* \left(\frac{A}{\sqrt{n}}\right) Q\, \boldsymbol{r}_n = r_k + \frac{\sigma_{\boldsymbol{r}}(n)}{\alpha} U_k + \frac{1}{\alpha^2} R_k^{(\boldsymbol{r})}$$

where the U_k's are i.i.d. $\mathcal{N}(0,1)$. One can check by carefully reading the proof of Lemma 2.1 that the conclusions of the lemma apply to $R_k^{(\boldsymbol{r})}$. In particular, one may

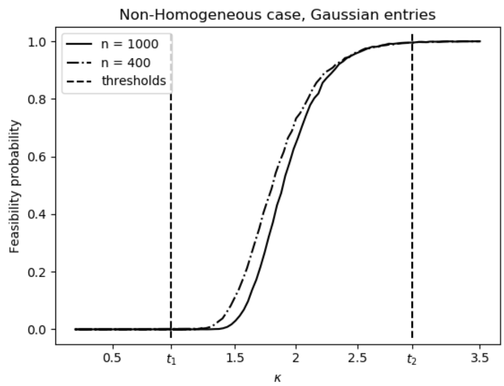


FIGURE 3. Transition toward feasibility for a NH system. The curves are obtained as for Figure 1 for $\boldsymbol{r}_N$ defined in (4.5). The thresholds t_1 and t_2 are computed in (4.6).

check that Proposition 2.4 holds uniformly in $k \in [n]$ in the nonhomogeneous case. Denote by $\check{M} = \min_{k\in[n]} U_k$, then

$$\begin{aligned}\min_{k\in[n]} x_k &\leq r_{\max}(n) + \frac{\sigma_{\boldsymbol{r}}(n)}{\alpha}\check{M} + \frac{\max_{k\in[n]} R_k^{(\boldsymbol{r})}}{\alpha^2}\,,\\ &\leq r_{\max}(n) + \frac{\sigma_{\boldsymbol{r}}(n)\alpha^*}{\alpha}\left(\frac{\check{M}+\beta^*}{\alpha^*} - \frac{\beta^*}{\alpha^*} + \frac{\max_{k\in[n]} R_k^{(\boldsymbol{r})}}{\sigma_{\boldsymbol{r}}(n)\alpha^*\alpha}\right)\,,\\ &= r_{\max}(n) + \frac{\sigma_{\boldsymbol{r}}(n)\alpha^*}{\alpha}\left(-1 + o_P(1)\right)\,.\end{aligned}$$

The first statement of the theorem follows. The second statement follows similarly, noticing that $\min_{k\in[n]} x_k \geq r_{\min}(n) + \alpha^{-1}\sigma_{\boldsymbol{r}}(n)\check{M} + \alpha^{-2}\min_{k\in[n]} R_k^{(\boldsymbol{r})}$. Proof of Theorem 4.3 is completed. □

A nonhomogeneous system (4.4) is associated to the following Lotka-Volterra system

$$\frac{dx_k(t)}{dt} = x_k(t)\left(r_k - x_k(t) + \frac{1}{\alpha_n\sqrt{n}}\sum_{\ell\in[n]} A_{k\ell}x_\ell(t)\right)$$

for $k \in [n]$ whose jacobian at equilibrium is still given by (1.4).

Theorem 4.5 (Stability - NH case). *Let $\boldsymbol{x}_n = (x_k)_{k\in[n]}$ be the solution of (4.4) and assume that*

$$\boldsymbol{\ell}^+ := \limsup_{n\to\infty} \frac{\alpha_n^*\,\sigma_{\boldsymbol{r}}(n)}{\alpha_n\, r_{\min}(n)} < 1\,.$$

Denote by $\mathcal{S}_n$ the spectrum of $\mathcal{J}(\boldsymbol{x}_n)$. Then for every $\lambda \in \mathcal{S}_n$,

$$\max_{\lambda\in\mathcal{S}_n}\min_{k\in[n]} |\lambda + x_k| \xrightarrow[n\to\infty]{\mathcal{P}} 0 \qquad \textit{and} \qquad \max_{\lambda\in\mathcal{S}_n} \operatorname{Re}\lambda \leq -(1-\boldsymbol{\ell}^+) + o_P(1)\,.$$

4.3. **Beyond the Gaussian case.** In this section, we extend the result to the class of random variables satisfying a logarithmic sobolev inequality. We first recall standard facts that can be found in [13].

A random variable X on $\mathbb{R}^n$ satisfies a logarithmic sobolev inequality with constant ρ (denoted by $X \in LSI(\rho)$) if for every function $g : \mathbb{R}^n \to \mathbb{R}$ regular enough (for instance g^2 is of finite entropy [13, Section 5.1] and g locally Lipschitz[1]):

$$\mathrm{Ent}(g^2(X)) \;:=\; \mathbb{E}\left(g^2(X)\log g^2(X)\right) - \mathbb{E}g^2(X)\log \mathbb{E}g^2(X) \;\leq\; \frac{2}{\rho}\mathbb{E}[\|\nabla g(X)\|^2] \;.$$

We say that X satisfies a Poincaré inequality with constant c (denoted by $X \in Poinc(c)$) if for every $f : \mathbb{R}^n \to \mathbb{R}$ locally Lipschitz:

$$\mathrm{Var}(f(X)) \;\leq\; c\,\mathbb{E}[\|\nabla f(X)\|^2] \;.$$

A well known fact is that logarithmic sobolev inequalities are stronger than Poincaré's :

Proposition 4.6. *If X satisfies $LSI(\rho)$ then it satisfies $Poinc\left(\rho^{-1}\right)$.*

Another standard result is that $LSI(\rho)$ implies Gaussian type concentration for Lipschitz functions:

Theorem 4.7. *If $X \in LSI(\rho)$, then for every function $f : \mathbb{R}^n \to \mathbb{R}$ L-Lipschitz:*

$$\mathbb{E}\left[e^{\lambda(f(X)-\mathbb{E}f(X))}\right] \;\leq\; e^{\lambda^2 L^2/2\rho} \;.$$

Important examples of random variables satisfying $LSI(\rho)$ are strongly log-concave variables and couples of independent variables satisfying LSI:

Proposition 4.8 ([13, Theorem 5.2])**.** *If X in $\mathbb{R}^n$ is strongly log-concave, that is has a density $e^{-V(x)}dx$ where $V : \mathbb{R}^n \mapsto \mathbb{R}$ is C^2 with $Hess(V) \geq \rho I_n$ ($\rho > 0$), then X satisfies $LSI(\rho)$.*

Proposition 4.9 ([13, Corollary 5.7])**.** *If $X \in LSI(\rho_X)$ and $Y \in LSI(\rho_Y)$ are independent, then $Z = (X, Y)$ satisfies $LSI(\rho_Z)$, where $\rho_Z = \min(\rho_X, \rho_Y)$.*

We finally mention that the uniform measure on an interval also satisfies $LSI(\rho)$ for some ρ depending on the interval. Indeed it is a Lipschitz push-forward of the Gaussian distribution. In fact, let $Z \sim \mathcal{N}(0,1)$ and $\Phi : \mathbb{R} \to (0,1)$ its cumulative distribution function, then Φ is $\frac{1}{\sqrt{2\pi}}$-Lipschitz and $U = \Phi(Z) \sim \mathcal{U}(0,1)$, the uniform distribution satisfies $LSI(\frac{1}{2\pi})$ by direct computations. In our case, the random variable $A = (A_{ij})_{i,j\in[n]}$ is a random vector of $\mathbb{R}^{n^2}$ whose entries are i.i.d. By virtue of Proposition 4.9, vector A satisfies $LSI(\rho)$ as long as the r.v. A_{11} does.

We now go back to the extension of Theorem 1.1 for non-Gaussian entries. The Gaussianity of the entries is used at three crucial steps, and in each case, a milder $LSI(\rho)$ assumption, for some $\rho > 0$ is enough :

(1) Gaussian entries immediately imply that the Z_k's are independent standard Gaussian random variables, for which the study of the extrema is standard.

[1] A function $g : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz if for all $x \in \mathbb{R}^n$, there exist $\xi_x, L_x > 0$ such that $y \in B(x, \xi_x) \Rightarrow |g(x) - g(y) \leq L_x\|x - y\|$.

In the case where the entries are not Gaussian, the Z_k's are no longer Gaussian but this issue can easily be circumvented since by the CLT the Z_k's converge in distribution to a standard Gaussian. The extreme value study of such families of Z_k's has been carried out in [2, Propositions 2 & 3]. In our case, it is easy to check that the $LSI(\rho)$ condition ensures that we can apply [2, Proposition 3].

(2) Gaussian concentration has been used to prove sub-Gaussiannity of $\widetilde{R}_k(A)$. The value of the constant being irrelevant to us, Theorem 4.7 yields the same conclusion.

(3) In the proof of Lemma 2.1 the Gaussian Poincaré inequality is used to prove that $\widetilde{R}_1(A)/(\alpha\sqrt{2\log(n)})$ goes to zero in probability. A Poincaré inequality remains available by Proposition 4.6 for $LSI(\rho)$ r.v.

Hence we can extend Theorem 1.1 as :

Theorem 4.10. *Assume that the entries A_{ij} are i.i.d. centered, with finite variance equal to 1 and satisfy $LSI(\rho)$ for some $\rho > 0$, then the conclusions of Theorem* 1.1 *hold.*

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