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Equilibria of large random Lotka–Volterra systems with vanishing species: a mathematical approach

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Abstract

Ecosystems with a large number of species are often modelled as Lotka–Volterra dynamical systems built around a large interaction matrix with random part. Under some known conditions, a global equilibrium exists and is unique. In this article, we rigorously study its statistical properties in the large dimensional regime. Such an equilibrium vector is known to be the solution of a so-called Linear Complementarity Problem. We describe its statistical properties by designing an Approximate Message Passing (AMP) algorithm, a technique that has recently aroused an intense research effort in the fields of statistical physics, machine learning, or communication theory. Interaction matrices based on the Gaussian Orthogonal Ensemble, or following a Wishart distribution are considered. Beyond these models, the AMP approach developed in this article has the potential to describe the statistical properties of equilibria associated to more involved interaction matrix models.

Mathematics Subject Classification Primary 15B52 · 37H30; Secondary 60B20 · 92D40

1 Introduction

Equilibrium of a large Lotka–Volterra system

In the field of mathematical ecology, Lotka–Volterra (LV) systems of coupled differential equations are widely used to model the time evolution of the abundances of N interacting species within an ecosystem Takeuchi (1996). Such systems take the form

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$$\frac{d\boldsymbol{x}_N}{dt}(t) = \boldsymbol{x}_N(t) \odot (\boldsymbol{r}_N - (I_N - \Gamma_N)\,\boldsymbol{x}_N(t))\,, \quad \boldsymbol{x}_N(0) \in (0,\infty)^N\,, \tag{1}$$

where the vector function $\boldsymbol{x}_N : [0,\infty) \to \mathbb{R}_+^N = [0,\infty)^N$ represents the abundances of the N species, $\odot$ is the componentwise product, $\boldsymbol{r}_N \in \mathbb{R}_+^N$ is the so-called vector of intrinsic growth rates of the species, and $-I_N + \Gamma_N = (-1_{(i=j)} + \Gamma_{ij}) \in \mathbb{R}^{N\times N}$ represents the interaction matrix. More precisely Γ_{ij} represents the effect of species j on the growth of species i for $i \neq j$ and $-1 + \Gamma_{ii}$ represents the intraspecific interaction. Equivalently, (1) can be written as a series of coupled ordinary differential equations:

$$\frac{dx_i}{dt}(t) = x_i(t)\left(r_i - x_i(t) + \sum_k \Gamma_{ik}\,x_k(t)\right)\,, \quad x_i(0) > 0\,, \quad 1 \le i \le N\,,$$

where $\Gamma_N = (\Gamma_{ij})$, $\boldsymbol{x}_N = (x_i)$ and $\boldsymbol{r}_N = (r_i)$.

In theoretical ecology, the matrix Γ_N and the vector $\boldsymbol{r}_N$ are often modelled as random when the number N of species is large, turning the ecological system into a large disordered system. Such systems have aroused an important amount of research in the fields of mathematical ecology, borrowing tools from statistical physics, high dimensional probability, or random matrix theory Akjouj et al. (2024).

In this paper, we shall be interested in the situation where the LV dynamical system is well-defined for all $t \in \mathbb{R}_+$ and possesses an unique globally stable equilibrium vector:

$$\boldsymbol{x}_N^\star = \left(x_i^\star\right)_{i=1}^N \qquad \text{with} \qquad \boldsymbol{x}_N(t) \xrightarrow[t\to\infty]{} \boldsymbol{x}_N^\star$$

for all initial conditions $\boldsymbol{x}_N(0) = (x_i(0))_i$ lying in the interior of the first orthant, that is $x_i(0) > 0$ for all $i \in \{1,\ldots,N\}$.

In general there does not exist a globally stable equilibrium. Even a single equilibrium might not exist. There are however various conditions ensuring the existence of such an equilibrium, see Hofbauer and Sigmund Hofbauer and Sigmund (1998), Takeuchi Takeuchi (1996), etc. In the present work, we will rely on Takeuchi's condition (cf. Proposition 2) and will assume the existence of an equilibrium $\boldsymbol{x}_N^\star$ for large N.

It is well-known that the property $\boldsymbol{x}_N(0) \in (0,\infty)^N$ is maintained for all $t > 0$ and $\boldsymbol{x}_N(t) \in (0,\infty)^N$. However, in general, the equilibrium vector $\boldsymbol{x}_N^\star$ may lie at the boundary of $\mathbb{R}_+^N$, i.e. may have vanishing components. Moreover, assuming that Γ_N and $\boldsymbol{r}_N$ are random, the vector $\boldsymbol{x}_N^\star$ is random as well.

When N becomes large, it is of interest to understand the statistical properties of $\boldsymbol{x}_N^\star$ such as for example its proportion of non-zero components, or the distribution of $\boldsymbol{x}_N^\star$'s components, etc. Many interesting features and properties of the equilibrium are encoded in the empirical measure of the abundances

$$\mu^{\boldsymbol{x}_N^\star} = \frac{1}{N}\sum_{i=1}^N \delta_{x_i^\star}\,,$$

where δ_a stands for the Dirac measure at a. For instance, the proportion of surviving species at equilibrium is given by

$$\frac{\#\,\text{surviving species}}{N} = \int 1_{(0,\infty)}(t)d\,\mu^{\boldsymbol{x}_N^\star}(dt)\,.$$

Mathematically speaking, the measure $\mu^{\boldsymbol{x}_N^\star}$ is a random probability measure on $\mathbb{R}$, defined on the same probability space Ω as $\boldsymbol{r}_N$ and Γ_N.

Note that, if $\boldsymbol{r}_N$ is exchangeable, the distribution of the first (and in fact any) component $[\boldsymbol{x}_N^\star]_1$ of the equilibrium vector should resemble[1] $\mu^{\boldsymbol{x}_N^\star}$.

In the literature devoted to large LV systems, standard choices for the matrix Γ_N are classical random matrix models such as the Gaussian Orthogonal Ensemble (GOE) model, the real Ginibre model (i.i.d. centered Gaussian entries for Γ_N), or the so-called elliptical model, that can be seen as an interpolation between the GOE and the real Ginibre models Allesina and Tang (2012). For these models, feasible equilibria where $x_i^\star > 0$ for $1 \leq i \leq N$ are studied in Bizeul and Najim (2021); Clenet et al. (2022); Akjouj and Najim (2022); Clenet et al. (2023).

The large-N properties of $\boldsymbol{x}_N^\star$ were recently considered in the theoretical ecology literature. In Bunin (Apr 2017), Bunin considered a non-centered elliptical model with the help of the dynamical cavity method. A similar result was obtained by Galla in Galla (2018) by means of generating functionals techniques, see also Opper and Diederich (1992); Tokita (Oct 2004). Many insights are provided by these techniques from a physicist point of view. However, up to our knowledge, no rigorous method to describe the asymptotic properties of $\boldsymbol{x}_N^\star$ can be found in the literature so far.

The purpose of this paper is to address this question in the case where matrix Γ_N is either taken from the GOE or follows a Wishart distribution. Our results on the asymptotics of $\mu^{\boldsymbol{x}_N^\star}$ mathematically confirm Bunin and Galla's works.

Linear complementarity problem

When it exists, the globally stable equilibrium $\boldsymbol{x}_N^\star = (x_i^\star)$ of the LV equation above is known to be the solution of a so-called Linear Complementarity Problem (LCP), see for instance (Takeuchi 1996, Chap. 3), which consists in finding a vector with real entries that satisfies a system of inequalities involving matrix Γ_N and vector $\boldsymbol{r}_N$:

$$\begin{cases} x_i^\star & \geq 0, \\ x_i^\star\left(r_i - \left[(I_N - \Gamma_N)\boldsymbol{x}_N^\star\right]_i\right) & = 0, \\ r_i - \left[(I_N - \Gamma_N)\boldsymbol{x}_N^\star\right]_i & \leq 0, \end{cases} \qquad \text{for all } i \in \{1, \ldots, N\}\,. \tag{2}$$

In the context of theoretical ecology, a vector satisfying (2) is often referred to as a saturated equilibrium or saturated rest point, see for instance Hofbauer and Sigmund (Hofbauer and Sigmund 1988, Section 19.4) and (Hofbauer and Sigmund 1998,Section 13.4).

[1] A thorough and rigorous study of this fact has been recently done in (Gueddari et al. 2024, Section 3.4).

The two first conditions are natural for an equilibrium to system (1): the abundances are necessarily non-negative and the equilibrium should be a critical point of the dynamics. The third one is more subtle, it is called *uninvadability* and its ecological interpretation is the following: the quantity $r_i - \left[(I_N - \Gamma_N)\boldsymbol{x}_N^\star\right]_i$ is the net growth rate (aka invasion fitness), that is the rate of exponential growth or decay of a small population $x_i \approx 0$ in an environment where the other species are at equilibrium $\boldsymbol{x}_N^\star$; these rates being all nonpositive is a stability requirement. Sufficient conditions on Γ_N to ensure existence and uniqueness of the solution $\boldsymbol{x}_N^\star$ are known. The problem boils down to the following question: how can we asymptotically extract statistical information on $\boldsymbol{x}_N^\star$, solution to the highly non-linear problem (2), given that Γ_N and $\boldsymbol{r}_N$ are random?

The reader is referred to Sect. 4.2 below for a quick overview of the LCP theory, and to Cottle et al. (2009); Murty and Yu (1988) for complete and comprehensive expositions.

Approximate message passing

The idea we develop in this paper is that the distribution $\mu^{\boldsymbol{x}_N^\star}$ can be estimated for large N by designing a proper Approximate Message Passing (AMP) algorithm.

Approximate Message Passing (AMP) is a technique that has recently aroused an intense research effort in the fields of statistical physics, machine learning, high-dimensional statistics and communication theory. Among the many landmark articles, we can cite Donoho et al. (2009), Bayati and Montanari (2011), Bolthausen (2014). More references can be found in the recent tutorial Feng et al. (2022).

An AMP algorithm produces a sequence of $\mathbb{R}^N$–valued random vectors, say $\boldsymbol{\xi}^k = (\xi_i^k)$, which are iteratively built around a $N \times N$ random matrix, sometimes called the measurement matrix. This algorithm is conceived in such a way that for any finite collection $\boldsymbol{\xi}^1, \dots, \boldsymbol{\xi}^k$ of these vectors, the following joint empirical distribution:

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{(\xi_i^1,\dots,\xi_i^k)}$$

converges as $N \to \infty$ to a Gaussian distribution on $\mathbb{R}^k$ whose parameters can be fully characterized by the so-called Density Evolution (DE) equations. In the context of our LV equilibrium problem, it turns out that an AMP algorithm can be designed in such a way that the AMP iterates approximate our LCP solution after an adequate transformation. Thanks to this approximation, the asymptotic properties of $\mu^{\boldsymbol{x}_N^\star}$ can be deduced from the DE equations.

Random matrix models and perspectives

Regarding the statistical model for Γ_N, we shall consider in this paper the GOE model Allesina and Tang (2012), and the Wishart model. The latter has been introduced to ecology in the context of resource-competition, see for instance the influential articles by MacArthur MacArthur (1970). Wishart models are also particular cases of a kernel matrix, which is considered when the interaction between two species depends on a distance between the values of some functional traits attached to these species, see (Akjouj et al. 2024, §4.6) and the references therein, or the recent paper Rozas et al. (2023). Both models are first studied under a Gaussianity assumption for the entries, see Assumptions 2-4. This assumption which might not seem biologically relevant is relaxed later and we provide similar results without the Gaussian requirement, see Assumptions 8-9.

We believe that this LCP/AMP approach for studying $\mu^{\boldsymbol{x}_N^\star}$ can be generalized and applied to more complex models for matrix Γ_N, see for instance Hachem (2024) (symmetric matrix, sparse variance profile) and Gueddari et al. (2024) (non-symmetric matrix, elliptical models). The recent results of Fan Fan (2022) might be used to cover the general rotationally invariant case; more general models are also considered in Bayati et al. (2015); Wang et al. (2022).

Outline of the article

The problem statement, the main results and simulations are presented in Sect. 2. In Sect. 2.2 (resp. Section 2.3) Theorem 1 (resp. Theorem 2) describes the statistical properties of the equilibrium for a matrix Γ_N drawn from the GOE (resp. from the Wishart ensemble). In Sect. 2.4, we extend these results to matrix ensembles based on non-Gaussian entries. Section 4 is devoted to the proof of Theorem 1, starting with an outline of the proof in Sect. 4.1, while elements of proof of Theorem 2 are provided in Sect. 5.

Main notations

For $x \in \mathbb{R}$, let $x_+ = \max(x, 0)$, $x_- = \max(-x, 0)$ and $[N] = \{1, \dots, N\}$. For a given set $\mathcal{S}$ denote by $|\mathcal{S}|$ its cardinality. Vectors will be denoted by lowercase bold letters $\boldsymbol{a} = (a_i), \boldsymbol{b} = (b_i)$, etc. If $f : \mathbb{R} \to \mathbb{R}$ is a real function, vector $f(\boldsymbol{a})$ is defined componentwise by $f(\boldsymbol{a}) = (f(a_i))_{i\in[N]}$. For vectors of same dimensions, $\boldsymbol{a} \odot \boldsymbol{b} = (a_i b_i)$ denotes the componentwise (Hadamard) product. Vector $\mathbf{1}_N$ is the $N \times 1$ vector of ones and $x \mapsto 1_{\mathcal{S}}(x)$ is the indicator function of set $\mathcal{S}$. Transpose of matrix A is $A^\top$ and its eigenvalues are $\lambda_i(A)$.

For $\boldsymbol{a} = (a_i)$, $\boldsymbol{a} \succcurlyeq 0$ (resp. $\boldsymbol{a} \succ 0$) refers to the componentwise inequalities $a_i \geq 0$ (resp. $a_i > 0$) for all $i \in [N]$. A positive (resp. negative) definite matrix A is denoted by $A > 0$ (resp. $A < 0$).

Given a vector $\boldsymbol{a}$ and a matrix A, $\|\boldsymbol{a}\|$ denotes the Euclidian norm of $\boldsymbol{a}$ and $\|A\|$ the spectral norm of A. For a vector $\boldsymbol{a}$, $\|\boldsymbol{a}\|_0 = |\{i; a_i \neq 0\}|$ is the number of its non-zero elements and $\mathrm{supp}(\boldsymbol{a})$ is its support, that is the set of indices of non-zero elements.

Given vectors $\boldsymbol{a} = (a_i)$, $\boldsymbol{a}^1 = (a_i^1), \dots, \boldsymbol{a}^k = (a_i^k)$ of the same size N, we denote as $\mu^{\boldsymbol{a}}$ and $\mu^{\boldsymbol{a}^1,\dots,\boldsymbol{a}^k}$ the probability measures

$$\mu^{\boldsymbol{a}} = \frac{1}{N} \sum_{i \in [N]} \delta_{a_i} \qquad \text{and} \qquad \mu^{\boldsymbol{a}^1,\dots,\boldsymbol{a}^k} = \frac{1}{N} \sum_{i \in [N]} \delta_{(a_i^1,\dots,a_i^k)} \,.$$

We call $\mu^{\boldsymbol{a}}$ the *empirical distribution* of the components of $\boldsymbol{a}$ and $\mu^{\boldsymbol{a}^1,\dots,\boldsymbol{a}^k}$ the *joint empirical distribution* of the components of $\boldsymbol{a}^1, \dots, \boldsymbol{a}^k$.

If μ_N, μ are probability measures over $\mathbb{R}^d$ then $\mu_N \xrightarrow[N\to\infty]{w} \mu$ stands for the weak convergence of probability measures. The distribution of a random variable X is denoted by $\mathcal{L}(X)$ and we express that two random variables X, Y have the same distribution by $X \stackrel{\mathcal{L}}{=} Y$. As usual, abbreviation a.s. stands for almost sure/surely.

2 Problem statement, assumptions, and main results

2.1 Equilibria, Wasserstein space and pseudo-Lipschitz functions

Independently of the structure of Γ_N, it is known that if $\|\Gamma_N\| < 1$, then the ODE (1) admits a unique solution $(\boldsymbol{x}_N(t), t \geq 0)$ with a bounded trajectory, for any arbitrary initial value $\boldsymbol{x}_N(0) \succ 0$, see Li et al. (2009). Moreover the same condition $\|\Gamma_N\| < 1$ guarantees, as we shall recall in more detail in Sect. 4, the existence of a globally stable equilibrium point $\boldsymbol{x}_N^\star$ in the classical sense of the Lyapounov theory (Takeuchi 1996, Chapter 3).

Given $k \geq 1$, the *Wasserstein space* $\mathcal{P}_k(\mathbb{R}^d)$ is defined as the set of probability measures μ over $\mathbb{R}^d$ with finite k^{th} moment: $\int_{\mathbb{R}^d} \|\boldsymbol{x}\|^k \mu(d\boldsymbol{x}) < \infty$. Given $\mu, \nu \in \mathcal{P}_k(\mathbb{R}^d)$, we denote by $\mathcal{M}_k(\mu, \nu)$ the set of probability measures in $\mathcal{P}_k(\mathbb{R}^d \times \mathbb{R}^d)$ with marginals μ and ν, i.e.

$$\eta \in \mathcal{M}_k(\mu, \nu) \qquad \Rightarrow \qquad \begin{cases} \eta(A \times \mathbb{R}^d) &= \mu(A)\,, \\ \eta(\mathbb{R}^d \times B) &= \nu(B)\,, \end{cases}$$

for all A, B Borel sets in $\mathbb{R}^d$. We can endow the space $\mathcal{P}_k(\mathbb{R}^d)$ with the distance:

$$d_k(\mu, \nu) = \inf_{\eta \in \mathcal{M}_k(\mu,\nu)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \|\boldsymbol{x} - \boldsymbol{y}\|^k \eta(d\boldsymbol{x} d\boldsymbol{y}) \right\}^{1/k} .$$

A function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is *pseudo-Lipschitz* with constant L and degree $k \geq 2$ if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, the following inequality holds:

$$|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})| \quad \leq \quad L\|\boldsymbol{x} - \boldsymbol{y}\| \left(1 + \|\boldsymbol{x}\|^{k-1} + \|\boldsymbol{y}\|^{k-1}\right) .$$

We denote by $PL_k(\mathbb{R}^d)$ this set of functions. We will rely later on the following classical lemma, see for instance (Feng et al. 2022, Sects. 1.1 and 7.4) and Villani (2009).

Lemma 1 *Let $\mu_N, \mu \in \mathcal{P}_k(\mathbb{R}^d)$ for $k \geq 2$. The following conditions are equivalent:*

(i) $d_k(\mu_N, \mu) \xrightarrow[N\to\infty]{} 0$,
(ii) For all $\varphi \in PL_k(\mathbb{R}^d)$, $\int \varphi d\mu_N \xrightarrow[N\to\infty]{} \int \varphi d\mu$,
(iii) $\mu_N \xrightarrow[N\to\infty]{w} \mu$ *and* $\int_{\mathbb{R}^d} \|\boldsymbol{x}\|^k \mu_N(d\boldsymbol{x}) \xrightarrow[N\to\infty]{} \int_{\mathbb{R}^d} \|\boldsymbol{x}\|^k \mu(d\boldsymbol{x})$.

If one of the equivalent conditions of Lemma 1 is satisfied, we say that the sequence (μ_N) converges in $\mathcal{P}_k(\mathbb{R}^d)$ to μ and denote it by

$$\mu_N \xrightarrow[N\to\infty]{\mathcal{P}_k(\mathbb{R}^d)} \mu \,.$$

If not misleading, we will occasionally drop $\mathbb{R}^d$ and simply write $\mathcal{P}_k$, PL_k.

Let $\boldsymbol{r}_N$ be a random vector of dimension $N \times 1$ that satisfies the following assumption.

Assumption 1 The following hold true.

(i) For all $N \geq 1$, $\boldsymbol{r}_N \succcurlyeq 0$ is defined on the same probability space as matrix Γ_N and is independent from Γ_N.
(ii) There exists a probability measure $\bar{\mu} \in \mathcal{P}_2(\mathbb{R}^+)$ such that $\bar{\mu} \neq \boldsymbol{\delta}_0$ and

$$(a.s.) \qquad \mu^{\boldsymbol{r}_N} \xrightarrow[N\to\infty]{\mathcal{P}_2(\mathbb{R})} \bar{\mu} \,.$$

2.2 The GOE case

We first define rigorously the symmetric matrix Γ_N and express sufficient conditions for the existence of a unique global equilibrium $\boldsymbol{x}_N^\star$ to (1).

Assumption 2 Let A_N be a $N \times N$ matrix from the Gaussian Orthogonal Ensemble. Namely, considering that X_N is a real $N \times N$ matrix with independent $\mathcal{N}(0, 1)$ elements,

$$A_N \stackrel{\mathcal{L}}{=} \frac{X_N + X_N^\top}{\sqrt{2}} \,.$$

Let κ be a positive real number. Then,

$$\Gamma_N = \frac{A_N}{\kappa\sqrt{N}} \,. \tag{3}$$

Remark 1 (biological interpretation of the interactions) The symmetric interactions correspond to competitive interactions when negative and mutualistic interactions when positive. Predator-prey interactions are not accounted for in this model. Let $\mathcal{O}$ be the standard "big O" notation then interspecific interactions Γ_{ij} $(i \neq j)$ are of order $\mathcal{O}\left(1/\sqrt{N}\right)$ while intraspecific interactions $-1 + \Gamma_{ii}$ are of order $\mathcal{O}(1)$. The scaling $1/\sqrt{N}$ ensures that asymptotically in N the interaction matrix Γ_N has a "macroscopic" effect in the sense that $\|\Gamma_N\| = \mathcal{O}(1)$ (see also the remark below for mathematical details).

Remark 2 Denote by $A_{ij}^{(N)}$ the element (i, j) of A_N, then $A_{ij}^{(N)} = A_{ji}^{(N)}$ and $\mathcal{L}(A_{ij}^{(N)}) = \mathcal{N}(0, 1+\delta_{ij})$ where δ_{ij} is the Kronecker symbol with value 1 if $i = j$, zero else. Much is known about this model, in particular the asymptotic behaviour of the spectral measure of $A_N/\sqrt{N}$ (Wigner's theorem) and its spectral norm, see for instance Bai and Silverstein (2010); Pastur and Shcherbina (2011) and the references therein:

$$
(a.s.) \qquad \frac{1}{N}\sum_{i\in[N]} \delta_{\lambda_i\left(A_N/\sqrt{N}\right)} \xrightarrow[N\to\infty]{w} \frac{\sqrt{(4-x^2)_+}}{2\pi}\,dx \qquad \text{and}
$$
$$
\left\|\frac{A_N}{\sqrt{N}}\right\| \xrightarrow[N\to\infty]{} 2\,. \tag{4}
$$

We shall consider the following assumption:

Assumption 3 The normalizing factor κ in (3) satisfies $\kappa > 2$.

Note that non-optimality of this assumption is discussed at length in Remark 4 and Sect. 3.2. Before stating the main theorem, we recall its direct mathematical consequences.

Combining Assumption 3 and the a.s. convergence of $\|A_N/\sqrt{N}\|$ toward 2, we get that with probability one, eventually

$$
\|\Gamma_N\| < 1\,.
$$

Formally, this property means that there exists a set $\widetilde{\Omega}$ with probability one such that

$$
\forall \omega \in \widetilde{\Omega}\,, \quad \exists N^\star(\omega)\,, \quad \forall N \geq N^\star(\omega)\,, \qquad \|\Gamma_N\| < 1\,.
$$

As a consequence, for every $\omega \in \widetilde{\Omega}$, the existence and uniqueness of $\boldsymbol{x}_N^\star$ is granted for N large enough.

We can now state the main result of this section : after justifying the existence of a globally stable equilibrium $\boldsymbol{x}_N^\star$, one can describe the asymptotic distribution of the abundances at equilibrium, expressed mathematically through the convergence of the empirical measure $\mu^{\boldsymbol{x}_N^\star}$ as $N \to \infty$. The limiting distribution is expressed in terms of a random variable $\bar{r}$, with law $\bar{\mu}$, the limiting distribution of the intrinsic growth rates and three auxiliary parameters γ, σ and δ that will be defined as the solutions of a fixed point equation. Biologically, γ represents the proportion of surviving species (more details are given below the theorem), σ measures the diversity at equilibrium and δ is a bit more subtle to interpret but can be linked to the sensitivity to the introduction of a new species when the system is near equilibrium. We now give the precise statement in the GOE case:

Theorem 1 *(i) (existence of equilibrium) Let $\boldsymbol{r}_N \succcurlyeq 0$ and let Assumptions 2 and 3 hold true. Then, $\|\Gamma_N\| < 1$ eventually with probability one. For such N's, the ODE (1) is defined for all $t \in \mathbb{R}_+$ and has a globally stable equilibrium $\boldsymbol{x}_N^\star$. For the other N's, let $\boldsymbol{x}_N^\star = 0$.*

(ii) (asymptotic distribution of the abundances)

(a) Let $\bar{r} \geq 0$ be a real valued random variable with finite second moment and $\mathcal{L}(\bar{r}) \neq \delta_0$. Let $\bar{Z}$ be a $\mathcal{N}(0,1)$ random variable independent of $\bar{r}$. Then, for any $\kappa > \sqrt{2}$, the system of equations

$$\kappa = \delta + \frac{\gamma}{\delta}, \tag{5a}$$

$$\sigma^2 = \frac{1}{\delta^2}\mathbb{E}\left(\sigma\bar{Z} + \bar{r}\right)_+^2, \tag{5b}$$

$$\gamma = \mathbb{P}\Big[\sigma\bar{Z} + \bar{r} > 0\Big], \tag{5c}$$

admits an unique solution (δ, σ, γ) in $(1/\sqrt{2}, \infty) \times (0, \infty) \times (0, 1)$.

(b) Let Assumptions 1, 2 and 3 hold. Define $\boldsymbol{x}_N^\star$ as previously. The distribution $\mu^{\boldsymbol{x}_N^\star}$ is a $\mathcal{P}_2(\mathbb{R})$–valued random variable on the probability space where A_N and $\boldsymbol{r}_N$ are defined. Assume that $\bar{r}$ is a r.v. with $\mathcal{L}(\bar{r}) = \bar{\mu}$, independent of $\bar{Z} \sim \mathcal{N}(0,1)$. Then, the convergence

$$(a.s.) \qquad \mu^{\boldsymbol{x}_N^\star} \xrightarrow[N\to\infty]{\mathcal{P}_2(\mathbb{R})} \mathscr{L}\left(\left(1 + \gamma/\delta^2\right)\left(\sigma\bar{Z} + \bar{r}\right)_+\right) \tag{6}$$

holds true, where δ, σ, γ are defined as solutions of system (5).

This theorem, which proof is postponed to Sect. 4, calls for some remarks.

Remark 3 Equations (5a)-(5c) have already been obtained[2] at a physical level of rigor by Bunin Bunin (Apr 2017) and Galla Galla (2018). Up to our knowledge, Theorem 1 is the first rigorous statement to describe the asymptotic properties of the distribution of the elements of $\boldsymbol{x}_N^\star$.

Remark 4 Notice that system (5) admits an unique solution for $\kappa > \sqrt{2}$ while Convergence (6) is only established for $\kappa > 2$. The range of solutions ($\kappa > \sqrt{2}$) to equations (5a)–(5c) supports the fact that the true threshold should be $\kappa > \sqrt{2}$ instead of $\kappa > 2$ (as in Assumption 3), a fact already noticed in the theoretical ecology literature Bunin (Apr 2017), see also Sect. 3.2.

[2] Notice that in Bunin (Apr 2017); Galla (2018), the authors consider more general models such as the elliptical model, which encompasses the Wigner model as a particular case.

Ecological interpretations

Theorem 1 brings valuable ecological information on the equilibrium for large N. Some important features, detailed hereafter, are illustrated in Fig. 1.

- Proportion of surviving species at equilibrium.

This is a key property of the equilibrium and Theorem 1 sheds some light on this proportion for large N: by inspecting (5c) and (6), the parameter γ can be interpreted as an approximation of the proportion of surviving species $\|x_N^\star\|_0/N$. Simulations in Fig. 1a confirm this fact.

One can see from Eq. (5c) that $\gamma > 1/2$, which means that in this model, more than half the species survive.

Furthermore, an easy calculation involving Eq. (5b) and (5c) shows that γ does not change if we replace $\bar{r}$ with $K\bar{r}$ where $K > 0$ is an arbitrary constant.

We should note however that rigorously speaking, Theorem 1 does not assert that γ is the limit of $\|x_N^\star\|_0/N$. Indeed, one can only deduce from this theorem that

$$\sup_{\varphi} \left\{ (a.s) \lim_{N\to\infty} \frac{1}{N} \sum_{i\in[N]} \varphi(x_i^\star) \right\} = \gamma,$$

where $\sup_\varphi$ is taken on the set of functions $\{\varphi : \mathbb{R} \to [0,1] \text{ continuous},\ \varphi(0) = 0\}$. Since the function $1_{\{x>0\}}$ is not continuous at zero, the convergence (6) does not imply that $\|\boldsymbol{x}_N^\star\|_0/N$ converges to γ, for any type of convergence. Up to our knowledge, the study of the asymptotic behavior of $\|\boldsymbol{x}_N^\star\|_0/N$ is an open question.

- Distribution of surviving species at equilibrium.

Denote by $\boldsymbol{s}(\boldsymbol{x}^\star)$ the subvector of $\boldsymbol{x}^\star$ with the positive components of $\boldsymbol{x}^\star$. Its dimension $|\boldsymbol{s}(\boldsymbol{x}^\star)|$ is random and the distribution of the surviving species is given by:

$$\mu^{\boldsymbol{s}(\boldsymbol{x}^\star)} = \frac{1}{|\boldsymbol{s}(\boldsymbol{x}^\star)|} \sum_{i\in[|\boldsymbol{s}(\boldsymbol{x}^\star)|]} \delta_{[\boldsymbol{s}(\boldsymbol{x}^\star)]_i} .$$

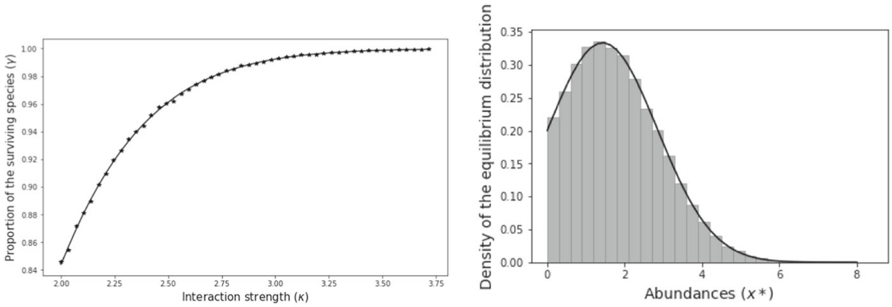
For a similar reason as previously the convergence of $\mu^{\boldsymbol{s}(\boldsymbol{x}_n^\star)}$ is out of reach but a good proxy for the limiting law should be:

$$\mathcal{L}\left(\left(1+\gamma/\delta^2\right)\left(\sigma\bar{Z}+\bar{r}\right)_+ \,\middle|\, \sigma\bar{Z}+\bar{r} > 0 \right),$$

the density of which is explicit and given by

$$f_{\text{surv}}(y) = \frac{\delta}{\kappa} f_{\sigma\bar{Z}+\bar{r}}\left(\frac{\delta\, y}{\kappa}\right) \frac{\mathbf{1}_{(y>0)}}{\gamma} \quad \text{where} \quad f_{\sigma\bar{Z}+\bar{r}}(y) = \int_{\mathbb{R}} \frac{e^{-\frac{(y-r)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma} \bar{\mu}(dr) \tag{7}$$

(recall that $1+\frac{\gamma}{\delta^2} = \frac{\kappa}{\delta}$ by Eq.(5a)). The matching between $\mu^{\boldsymbol{s}(\boldsymbol{x}^\star)}$ and f_{surv} is illustrated in Fig. 1b.



(A) Proportion of surviving species (GOE) (B) Density of a surviving species

Fig. 1 Subplot 1a represents the proportion of surviving species, that is the proportion of positive components of the equilibrium $\boldsymbol{x}^\star$ (star), versus the theoretical value of γ (solid line), given the parameter κ which varies from 2 to 3.75. In the plot, $N = 1000$ and each point (star) is the mean of proportions obtained out of 100 Monte-Carlo simulations. Subplot 1b represents the distribution of a surviving species ($N = 1000$ and 100 Monte-Carlo simulations). The solid line represents the theoretical value of the density f_{surv}, see (7)

Notice that if the r.v. $\bar{r}$ is constant then f_{surv} is the density of a truncated Gaussian distribution.

2.3 The Wishart case

As pointed out in the introduction, Wishart matrices are also relevant in theoretical ecology. They were introduced to ecology in the context of resource-competition, see for instance the influential articles by MacArthur MacArthur (1970). Wishart matrices model interactions between two species which depend on the distance between values of some given functional traits, see for instance (Akjouj et al. 2024, § 4.6) or Rozas et al. (2023).

Assumption 4 Let B_N be a $P \times N$ matrix with i.i.d. Gaussian $\mathcal{N}(0, 1)$ entries. Let κ be a real positive number and define the $N \times N$ matrix Γ_N as:

$$\Gamma_N = \frac{B_N^\top B_N}{\kappa P} \,. \tag{8}$$

For this model, the ith column of matrix B_N is a vector modelling the traits of species i.

We will be interested in the specific regime where N, P go to infinity at the same pace:

Assumption 5 Let $N = N(P)$ and assume that

$$\frac{N}{P} \xrightarrow[P \to \infty]{} c \in (0, \infty) \,.$$

This regime will be denoted by $N, P \to \infty$ in the sequel.

Model (8) has been thoroughly studied under Assumption 5. Marchenko-Pastur's theorem describes the asymptotic behaviour of the spectral limit of $B_N^\top B_N/P$. The limiting spectral norm has been studied by Bai and Yin, see for instance Bai and Silverstein (2010); Pastur and Shcherbina (2011) and the references therein:

$$(a.s.) \qquad \left\| \frac{B_N^\top B_N}{P} \right\| \xrightarrow[N,P\to\infty]{} (1+\sqrt{c})^2 \,.$$

Assumption 6 The normalizing factor in (8) satisfies $\kappa > (1+\sqrt{c})^2$.

For this model, a similar result as Theorem 1 can be stated, giving the existence of a globally stable equilibrium and characterising the limiting behavior of the distribution of the abundances. Again, three auxiliary parameters are necessary to describe the limiting law, they obey a system of equations which slightly differes from (5a)-(5c). The respective interpretation of the three parameters is the same as in the GOE case.

Theorem 2 *(i) (existence of equilibrium) Let* $\boldsymbol{r}_N \succcurlyeq 0$ *and let Assumptions 4, 5 and 6 hold. Then,* $\|\Gamma_N\| < 1$ *eventually with probability one. For such N's, the LV ODE solution is defined for all* $t \in \mathbb{R}_+$ *and has a globally stable equilibrium* $\boldsymbol{x}_N^\star$. *For the other N, set* $\boldsymbol{x}_N^\star = 0$.

(ii) (asymptotic distribution of the abundances)

(a) Let $\bar{r} \geq 0$ *be a real valued r.v. with* $\mathcal{L}(\bar{r}) \neq \delta_0$. *Let* $\bar{Z}$ *be a* $\mathcal{N}(0,1)$ *r.v. independent from* $\bar{r}$. *Then, for every* $\kappa > \left(1+\sqrt{\frac{c}{2}}\right)^2$, *the system of equations*

$$\kappa = (\delta + c\gamma)\left(1+\frac{1}{\delta}\right), \tag{9a}$$

$$\tau^2 = \frac{c}{\delta^2}\mathbb{E}\left[\left(\tau\bar{Z}+\bar{r}\right)_+^2\right], \tag{9b}$$

$$\gamma = \mathbb{P}\left[\tau\bar{Z}+\bar{r} > 0\right], \tag{9c}$$

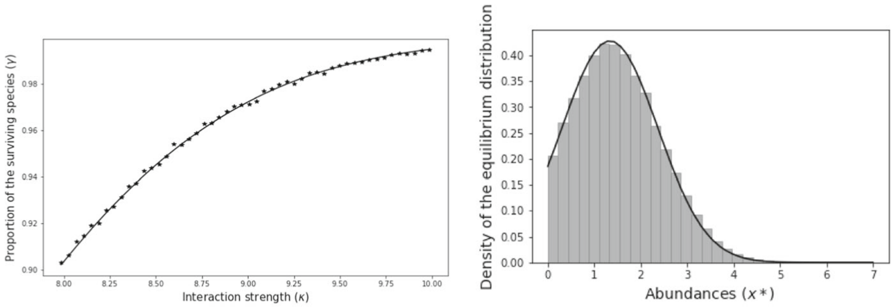
admits an unique solution (δ,τ,γ) *in* $(\sqrt{c/2},\infty)\times(0,\infty)\times(0,1)$.

(b) Let Assumptions 1, 4, 5 and 6 hold. Define $\boldsymbol{x}_N^\star$ *as previously. The distribution* $\mu^{\boldsymbol{x}_N^\star}$ *is a* $\mathcal{P}_2(\mathbb{R})$*–valued random variable on the probability space where* A_N *and* $\boldsymbol{r}_N$ *are defined. Assume that* $\bar{r}$ *is a r.v. with* $\mathcal{L}(\bar{r}) = \bar{\mu}$, *independent of* $\bar{Z} \sim \mathcal{N}(0,1)$. *The following convergence holds true:*

$$(a.s.) \qquad \mu^{\boldsymbol{x}_N^\star} \xrightarrow[N,P\to\infty]{\mathcal{P}_2(\mathbb{R})} \mathcal{L}\left((1+1/\delta)\left(\tau\bar{Z}+\bar{r}\right)_+\right), \tag{10}$$

where δ, τ *and* γ *are defined as solutions of system (9).*

There is a strong matching between the parameters obtained by solving system (9) and their empirical counterparts obtained by Monte-Carlo simulations, as illustrated in Fig. 2.



(A) Proportion of surviving species (GOE) (B) Density of a surviving species

Fig. 2 Subplot 2a represents the proportion of surviving species, that is the proportion of positive components of the equilibrium $\boldsymbol{x}^{\star}$ (star), versus the theoretical value of γ (solid line), given the parameter κ which varies from 8 to 10 (in this case, $c = 1000/300$ and the threshold is $(1+\sqrt{c})^2 \simeq 7.98$). In the plot, $N = 1000$, $P = 300$ and each point (star) is the mean of proportions obtained out of 100 Monte-Carlo simulations. Subplot 2b represents the distribution of a surviving species ($N = 1000$, $P = 300$ and 100 Monte-Carlo simulations). The solid line represents the theoretical value of the density $f_{Z|Z>0}$ where Z is the random variable with limiting distribution of $\mu^{\boldsymbol{x}^{\star}_N}$ given in (10) - cf. Theorem 2

Remark 5 There is again a gap between the range of values of the parameter κ for which the system (9a)-(9c) has a unique solution, that is $\kappa > \left(1+\sqrt{\frac{c}{2}}\right)^2$, and the range of values for which we can prove the convergence (10).

The proof of this theorem relies on an asymmetric version of the AMP algorithm and is otherwise very close to the proof of Theorem 1. We provide some details in Sect. 5.

2.4 Toward universality

From the ecological point of view, there is no obvious reason why the interactions between species should be Gaussian. It is therefore natural to wonder to what extent one should get rid of this Gaussianity assumption. This is the question we mathematically address in this section. We mentioned in the introduction that AMP techniques have been generalized to matrices with non-necessarily Gaussian entries, see Bayati et al. (2015); Chen and Lam (2021); Dudeja et al. (2023); Wang et al. (2022). It is possible, at low cost, to relax the Gaussiannity assumption of the entries in Assumptions 2 and 5.

We first strenghten Assumption 1 and replace it by the following stronger assumption:

Assumption 7 The following holds true:

(i) For all $N \geq 1, \boldsymbol{r}_N \succcurlyeq 0$ is defined on the same space as matrix Γ_N and is independent of Γ_N.
(ii) There exists a probability measure $\bar{\mu} \in \mathcal{P}(\mathbb{R}^+)$ such that $\bar{\mu} \neq \boldsymbol{\delta}_0$, the moment generating function of $\bar{\mu}$ is analytical near zero (which implies that $\bar{\mu}$ has all its

moments finite), and

$$(a.s.) \qquad \mu^{r_N} \xrightarrow[N\to\infty]{\mathcal{P}_k(\mathbb{R})} \bar{\mu} \qquad \text{for all} \quad k \geq 1\,.$$

We now relax the GOE assumption (Assumption 2).

Assumption 8 Let $A_N = \left(A_{ij}^{(N)}\right)$ be a $N \times N$ symmetric matrix where the $A_{ij}^{(N)}$'s are centered independent random variables satisfying

$$\mathbb{E}(A_{ij}^{(N)})^2 = 1 \quad (i < j)\,, \quad \sup_N \max_i \mathbb{E}(A_{ii}^{(N)})^2 < C\,,$$

and

$$\max_{i,j} N^{1-k/2}\mathbb{E}\left|A_{ij}^{(N)}\right|^k \xrightarrow[N\to\infty]{} 0 \quad (k \geq 3)\,.$$

Moreover, assume that the following holds true:

$$\left\| \frac{A_N}{\sqrt{N}} \right\| \xrightarrow[N\to\infty]{\text{a.s.}} 2. \tag{11}$$

Denote by $\Gamma_N = A_N/(\kappa\sqrt{N})$.

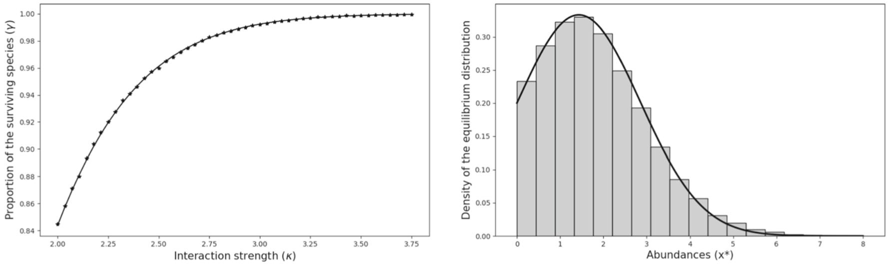
Example 6 (Wigner matrices) The standard example of a matrix A_N that generalizes the GOE model and that complies with Assumption 8 corresponds to the case where $A_{ij}^{(N)} \stackrel{\mathcal{L}}{=} \chi$ for $i \neq j$ and $A_{ii}^{(N)} \stackrel{\mathcal{L}}{=} \chi'$, where the centered random variables χ and χ' do not depend on N, $\mathbb{E}\chi^2 = 1$, and χ and χ' have all their moments finite. Note that in this case, the convergence (11) is a standard result in Random Matrix theory Bai and Silverstein (2010); Pastur and Shcherbina (2011).

Beyond the model described in Example 6, some sparse models can also be covered by Assumption 8, as the following example shows.

Example 7 (Sparse models) Sparsity of the food interactions is often justified from an ecological point of view, see Busiello et al. (2017). Let $p_N \in (0, 1)$, and

$$A_{ij}^{(N)} = \begin{cases} 1/\sqrt{p_N} & \text{with probability } p_N/2 \\ -1/\sqrt{p_N} & \text{with probability } p_N/2 \\ 0 & \text{with probability } 1 - p_N. \end{cases}$$

Since $\mathbb{E}\big|A_{ij}^{(N)}\big|^k = p_N^{1-k/2}$, the moment condition in Assumption 8 is satisfied as soon as $Np_N \xrightarrow[N\to\infty]{} \infty$. Furthermore, the spectral norm convergence condition (11) is satisfied when $\frac{Np_N}{\log N} \xrightarrow[N\to\infty]{} \infty$, as shown in Benaych-Georges et al. (2020), see also Benaych-Georges et al. (2019). Therefore, according to this model, a species within our LV system can interact with an average number of species much smaller than N but of an order $\gg \log N$.



(A) Proportion of surviving species (uniform) (B) Density of a surviving species (sparse matrix)

Fig. 3 In Subplot 3a, we consider a Wigner matrix whose entries are uniform on $[-\sqrt{3}, \sqrt{3}]$ (hence centered with variance one) as in Example 6. The plot represents the proportion of surviving species in the equilibrium $\boldsymbol{x}^\star$ (star), versus the theoretical value of γ (solid line), given the parameter κ which varies from 2 to 3.75. In Subplot 3b, we consider entries as described in Example 7 with $p_N = \sqrt{N}$. The plot represents the distribution of a surviving species. The solid line represents the theoretical value of the density $f_{Z|Z>0}$ where Z is the random variable with limiting distribution f_{surv} of $\mu^{\boldsymbol{x}^\star}{}_N$, see Eq. (7). In both simulations, we consider $N = 1000$ and 100 Monte-Carlo simulations

We are now in position to state a non-Gaussian version of Theorem 1:

Theorem 3 (Non-Gaussian symmetric matrix) *All the conclusions of Theorem 1 remain true if Assumptions 1 and 2 in the statement of this theorem are replaced with Assumptions 7 and 8 respectively.*

Elements of proof are provided in Appendix 1. In Fig. 3, simulations illustrate the matching between theoretical curves and simulated equilibria for Wigner matrices with uniform entries and sparse matrices (cf. Example 7).

We now provide the proper assumption to state a non-Gaussian version of Theorem 2.

Assumption 9 • We have $N = N(P)$, and there exists $c > 0$ such that

$$\frac{N(P)}{N} \xrightarrow[P\to\infty]{} c.$$

- The $P \times N$ random matrix $B_N = \left(B_{ij}^{(N)}\right)_{i,j=1}^{P,N}$ is such that the random variables $B_{ij}^{(N)}$ for $i \in [P]$ and $j \in [N]$ are centered, independent, with variance one and satisfy

$$\max_{i,j} P^{1-k/2}\mathbb{E}\left|B_{ij}^{(N)}\right|^k \xrightarrow[N\to\infty]{} 0\,, \qquad (k \geq 3)\,.$$

We denote by

$$\Gamma_N = \frac{B_N^\top B_N}{\kappa P}\,.$$

- Finally, in this asymptotic regime, the convergence

$$\left\| \frac{B_N^\top B_N}{P} \right\| \xrightarrow[P\to\infty]{\text{a.s.}} (1+\sqrt{c})^2 \tag{12}$$

  holds true.

Example 8 The standard model for a matrix B_N satisfying this assumption is the model for which $B_{ij}^{(N)} \stackrel{\mathcal{L}}{=} \chi$, where χ is a centered random variable with unit variance having all its moments finite. In this case, the convergence (12) is a standard random matrix theory result Bai and Silverstein (2010); Pastur and Shcherbina (2011).

With this assumption at hand, we are in position to provide a counterpart to Theorem 2.

Theorem 4 (Non-Gaussian Wishart matrices) *All the conclusions of Theorem 2 remain true if Assumption 1 is replaced with Assumption 7 and Assumptions 4 and 5 are replaced with Assumption 9 in the statement of this theorem.*

Elements of proof are provided in Appendix B.

3 Discussion

We summarize hereafter our contributions, discuss its limitations and the open problems raised by the present work.

3.1 Main contribution of the present work

In this article we are interested in large Lotka–Volterra dynamical systems, popular in theoretical ecology to model large foodwebs. In this context, the interaction matrix writes $-I + \Gamma$ where Γ is a large random matrix. We focus on symmetric models for Γ, either based on Gaussian Orthogonal Ensemble or on Wishart matrices under normalizations which yield the existence of a stable equilibrium. Symmetric matrices account for competitive or mutualistic interactions but cannot model predator-prey interactions.

We develop a new mathematical method to describe the statistical properties of the equilibrium. We are able in particular to estimate the number of surviving species at equilibrium.

We show that the distribution at equilibrium is completely characterized by a few parameters of the model, in particular the limiting law of the intrinsic growth rates, and three auxiliary parameters, the proportion of surviving species, the diversity at equilibrium of the sensitivity to perturbations, that are shown to obey a simple 3×3 system of equations.

Our work is based on the mathematical technique known as Approximate Message Passing and developed this last decade by Montanari and many others Donoho et al.

(2009); Bayati and Montanari (2011), etc. This rigorous method complements numerous works on the subject Bunin (Apr 2017); Galla (2018); Clenet et al. (2023), etc. based on replica methods and other non-rigorous heuristics. Our theoretical results are illustrated by simulations which show a strong matching between the (theoretically) predicted quantities and simulated quantities.

Up to our knowledge, the application of AMP to theoretical ecology is new and we believe that this method is robust and could pave the way to a rigorous and systematic study of large Lotka–Volterra systems beyond the specific random matrix chosen here.

3.2 Further developments and open questions

Elliptic models. A natural question is to extend the present approach to non-symmetric random matrices, such as real Ginibre random matrices (all the A_{ij} are i.i.d.) or elliptic random matrices (there is a fixed correlation ρ between A_{ij} and A_{ji} for $i < j$). Contrary to the Wigner case, no AMP results were readily available to cope with these models. Some time after the release of the present work, an AMP algorithm has been developed in Gueddari et al. (2024) for elliptic random matrix and extends the present strategy to the elliptic Gaussian context.

Optimal threshold. In Theorem 1, we assume that $\kappa > 2$ (see Assumption 3). Extensive simulations and theoretical physicists' results Bunin (Apr 2017); Galla (2018) suggest that the right condition should be[3] $\kappa > \sqrt{2}$. This is a very interesting open problem, see also Remarks 3 and 4. In our proof, we need the condition $\kappa > 2$ to apply Takeuchi's result (see Prop. 2) which asserts the existence and uniqueness of the LV equilibrium. A similar gap occurs in the Wishart model, see Remark 5.

Consistent estimation of the number of surviving species. Given a LV system fulfilling the assumptions of Theorem 1 there is a strong matching between the empirical quantity $\frac{1}{N}\sum_{i=1}^{N} 1_{(x_i^\star>0)}$ and parameter γ defined in (5a)-(5c) as illustrated in Fig.1. A rigorous proof of the convergence is currently out of reach, see the comments at the end of Sect. 2.2.

4 Proof of theorem 1

4.1 Outline of the proof

There are four steps in the proof.

Step 1 In Sect. 4.2, we characterize the stable equilibrium $\boldsymbol{x}_N^\star$ of (1) as the solution of a Linear Complementarity Problem (LCP). We give an equivalent formulation of the solution of a LCP as the solution of a fixed-point equation, see Proposition 3.

Step 2 In Sect. 4.3, we establish the uniqueness and existence of parameters δ, σ and γ, solutions to system (5). These parameters will play a crucial role to design an AMP

[3] See for instance Bunin Bunin (Apr 2017), Fig. 5(a), yellow curve corresponding to $\gamma = 1$ and $\mu = 0$ (GOE). In Bunin's figure, $\sigma = 1/\kappa$. Bunin asserts that the system is stable below the yellow curve, corresponding to $\sigma = 1/\sqrt{2}$ at $\mu = 0$, which reads $\kappa > \sqrt{2}$.

algorithm fitted for our purpose. Equations (5a)-(5c) will progressively appear during the proof.

Step 3 In Sect. 4.4, we first recall some general facts about Approximate Message Passing (AMP) algorithms and present a specific algorithm (20) whose output $(\boldsymbol{\xi}_N^k)_+$ will converge toward $\boldsymbol{x}_N^\star$, characterized as the solution of the fixed-point equation associated to the corresponding LCP. The approximate fixed-point equation satisfied by $\boldsymbol{\xi}_N^k$ is given in (23), see also (25).

Step 4 The strength of the AMP procedure is that we can track down via the Density Evolution (DE) equations the asymptotic distribution of $(\boldsymbol{\xi}_N^k)_+$'s empirical measure for any k. We can then transfer it to $\boldsymbol{x}_N^\star$ by using a perturbation result by Chen and Xiang in Chen and Xiang (2007), see (33). A central argument borrowed from Montanari and Richard Montanari and Richard (2016) is that vectors $\boldsymbol{\xi}_N^k$ tend to be aligned for large k.

4.2 Characterization of $\boldsymbol{x}_N^\star$ through a LCP

In this section, we recall the connection between the possible stable equilibrium of the ODE (1) and the solution of an underlying LCP in the theory of mathematical programming. We mainly rely on chapter 3 of Takeuchi's book Takeuchi (1996).

Given a matrix $M \in \mathbb{R}^{N\times N}$ and a vector $\boldsymbol{c} \in \mathbb{R}^N$, the LCP problem, denoted as LCP$(M, \boldsymbol{c})$, consists in finding couples of vectors $(\boldsymbol{y}, \boldsymbol{w}) \in \mathbb{R}^N \times \mathbb{R}^N$ satisfying

$$\begin{cases} \boldsymbol{w} &= M\boldsymbol{y} + \boldsymbol{c} \succcurlyeq 0\,, \\ \boldsymbol{y} &\succcurlyeq 0\,, \\ \boldsymbol{w}^\top \boldsymbol{y} &= 0\,. \end{cases} \tag{13}$$

Notice that the last condition can be written equivalently either $w_i y_i = 0$ for all $i \in [N]$ or $\mathrm{supp}(\boldsymbol{w}) \cap \mathrm{supp}(\boldsymbol{y}) = \emptyset$. When a solution $(\boldsymbol{y}, \boldsymbol{w})$ exists we write $\boldsymbol{y} \in \mathrm{LCP}(M, \boldsymbol{c})$. If a solution exists and is unique, we write

$$\boldsymbol{y} = \mathrm{LCP}(M, \boldsymbol{c})\,.$$

A necessary and sufficient condition for the existence of a unique solution to the LCP problem has been given by Murty Murty (1972), see also Cottle et al. (2009). For a symmetric matrix, this condition is simply to be positive definite.

The following proposition establishes a connection between the solution of an LCP problem and globally stable equilibrium for a LV system .

Proposition 2 (Lemma 3.2.2 and Theorem 3.2.1 of Takeuchi (1996)) *Given a symmetric matrix $B \in \mathbb{R}^{N\times N}$ and a vector $\boldsymbol{c} \in \mathbb{R}^N$, consider the following LV system of ODE:*

$$\frac{d\boldsymbol{y}}{dt}(t) = \boldsymbol{y}(t) \odot (\boldsymbol{c} + B\boldsymbol{y}(t))\ , \quad \boldsymbol{y}(0) \succ 0\,. \tag{14}$$

for all $t \geq 0$. *Then, the LCP problem* $\mathrm{LCP}(-B, -\boldsymbol{c})$ *has an unique solution for each* $\boldsymbol{c} \in \mathbb{R}^N$ *if and only if* $B < 0$, *i.e.* B *is negative definite. On the domain where* $B < 0$, $c \in \mathbb{R}^N$, *the function* $x = \mathrm{LCP}(-B, -c)$ *is measurable. Moreover, if* $B < 0$, *then for every* $\boldsymbol{c} \in \mathbb{R}^N$, *the ODE* (14) *has a globally stable equilibrium* $\boldsymbol{y}^\star$ *given by* $\boldsymbol{y}^\star = \mathrm{LCP}(-B, -\boldsymbol{c})$.

Indeed, the equilibrium is characterized by the conditions $\boldsymbol{y}^\star \succcurlyeq 0$ and for all $i \in [N]$, $y_i^\star(c_i + (B\boldsymbol{y}^\star)_i) = 0$ whereas the condition $-\boldsymbol{c} - B\boldsymbol{y}^\star \preccurlyeq 0$ (with the obvious meaning of $\preccurlyeq$) turns out to be a necessary condition for the equilibrium $\boldsymbol{y}^\star$ to be stable in the classical sense of Lyapounov theory (see (Takeuchi 1996, Chapter 3) to recall the different notions of stability, and (Takeuchi 1996, Theorem 3.2.5) for this result).

Going back to system (1), a potential equilibrium $\boldsymbol{x}_N^\star$ should satisfy

$$\boldsymbol{x}_N^\star \succcurlyeq 0 \qquad \text{and} \qquad x_i^\star\left(r_i - \left[(I_N - \Gamma_N)\boldsymbol{x}_N^\star\right]_i\right) = 0 \quad \text{for all} \quad i \in [N]$$

and

$$\boldsymbol{r}_N + (\Gamma_N - I_N)\,\boldsymbol{x}_N^\star \preccurlyeq 0,$$

which means that the couple $(\boldsymbol{x}_N^\star, \boldsymbol{w}_N^\star)$ solves the problem $\mathrm{LCP}(I_N - \Gamma_N, -\boldsymbol{r}_N)$.

Applying the reminder (4) and Assumption 3, matrix $I_N - \Gamma_N$ is eventually positive definite with probability one. Define now the vector $\boldsymbol{x}_N^\star$ by

$$\boldsymbol{x}_N^\star = \begin{cases} \mathrm{LCP}(I_N - \Gamma_N, -\boldsymbol{r}_N) & \text{if } \|\Gamma_N\| < 1, \\ 0 & \text{otherwise }. \end{cases} \tag{15}$$

Then, from Proposition 2, we get that vector $\boldsymbol{x}_N^\star$ satisfies the statement of Theorem 1-(ii).

We end this section by providing an alternative expression of the LCP problem as the solution of a fixed point equation.

Alternative expression for the LCP solution

This fact will be useful in Sect. 4.4.

Proposition 3 *Let* $z = (z_i) \in \mathbb{R}^N$ *and consider the fixed-point equation:*

$$z = \Upsilon_N z_+ + \boldsymbol{\rho}_N \tag{16}$$

where $z_+ = ((z_i)_+)$. *Then* z *is a solution of* (16) *iff* $z_+ \in \mathrm{LCP}(I_N - \Upsilon_N, -\boldsymbol{\rho}_N)$.

Proof Suppose that z is a solution of (16) and write $z = z_+ - z_-$. Then

$$z_+, z_- \succcurlyeq 0\,, \quad (z_+)^\top z_- = 0 \quad \text{and} \quad z_- = (I_N - \Upsilon_N)z_+ - \boldsymbol{\rho}_N\,.$$

Hence $z_+ \in \mathrm{LCP}(I_N - \Upsilon_N, -\boldsymbol{\rho}_N)$.

To establish the converse, let $(\boldsymbol{y}, \boldsymbol{w})$ a solution of LCP$(I_N - \Upsilon_N, -\boldsymbol{\rho}_N)$. Define $\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{w}$ then

$$\begin{cases} \boldsymbol{z}_+ = \boldsymbol{y} \\ \boldsymbol{z}_- = \boldsymbol{w} \end{cases} \qquad \text{and} \qquad \boldsymbol{w} = (I_N - \Upsilon_N)\boldsymbol{y} - \boldsymbol{\rho}_N \quad \Rightarrow \quad \boldsymbol{z} = \Upsilon_N \boldsymbol{z}_+ + \boldsymbol{\rho}_N \,.$$

□

4.3 Existence and uniqueness of the solution of system (5)

We begin with the following technical lemma, the third part of which will be used in Sect. 4.4. To avoid any ambiguity, we shall always refer to σ as the unique positive root of $\sigma^2 > 0$.

Lemma 4 *Let $\bar{r}$ be a non negative r.v. with $\mathcal{L}(\bar{r}) \neq \delta_0$.*

(i) For a given $\delta > 0$, Eq. (5b) admits a solution σ^2 if and only if $\delta > 1/\sqrt{2}$. In this case, this solution is unique, and is denoted by $\sigma^2(\delta)$.

(ii) Let $\delta > 1/\sqrt{2}$ then

$$\mathbb{P}\{\sigma(\delta)\bar{Z} + \bar{r} \geq 0\} < \delta^2 \,.$$

(ii) Assume $\delta > 1/\sqrt{2}$. Starting with an arbitrary $\sigma_0 \geq 0$, consider the iterative scheme:

$$\sigma_{t+1}^2 = \frac{1}{\delta^2}\mathbb{E}\left(\sigma_t \bar{Z} + \bar{r}\right)_+^2 \,, \qquad \textit{then} \quad \sigma_t^2 \xrightarrow[t\to\infty]{} \sigma^2(\delta)\,.$$

Proof of Lemma 4 is postponed to Appendix 1.

We now establish that system (5) has a unique solution

$$(\delta, \sigma, \gamma) \in (1/\sqrt{2}, \infty) \times (0, \infty) \times (0, 1)\,.$$

Let $\delta > 1/\sqrt{2}$, $\sigma^2(\delta)$ be defined by (5b), and $\gamma(\delta)$ by (5c). Setting $f(\sigma^2) = \mathbb{E}(\sigma\bar{Z} + \bar{r})_+^2$, we have established in the proof of Lemma 4-(i) that

$$\gamma(\delta) = \left.\frac{df}{d\sigma^2}\right|_{\sigma^2 = \sigma^2(\delta)} .$$

Moreover $\gamma(\delta) < \delta^2$ by Lemma 4-(ii). All what remains to show is that the equation

$$\kappa = \delta + \frac{\gamma(\delta)}{\delta} \tag{17}$$

has a unique solution $\delta > 1/\sqrt{2}$. We thus need to study the behavior of $\gamma(\delta)$. In all the remainder, differentiability issues can be easily checked and are skipped.

Recall that $df(\sigma^2)/d\sigma^2$ decreases asymptotically to $1/2$ as σ^2 increases from 0 to ∞, from which we can deduce that $\sigma^2(\delta) \to \infty$ as $\delta \downarrow 1/\sqrt{2}$ by Lemma 4-(ii). Using the fact that

$$\sigma^2(\delta) = \frac{f(\sigma^2(\delta))}{\delta^2}$$

and taking the derivatives with respect to δ, we get that

$$\frac{d\sigma^2(\delta)}{d\delta}\Big(1-\frac{1}{\delta^2}\left.\frac{df(\sigma^2)}{d\sigma^2}\right|_{\sigma^2=\sigma^2(\delta)}\Big)=-\frac{2f(\sigma^2(\delta))}{\delta^3},$$

which shows that $\sigma^2(\delta)$ is a decreasing function. Hence $\gamma(\delta)$ is increasing since $\sigma\mapsto\mathbb{P}\{\sigma\bar{Z}+\bar{r}\geq 0)$ is decreasing (cf. proof of Lemma 4).

We can now conclude. For $\delta\downarrow 1/\sqrt{2}$, $\sigma^2(\delta)\to\infty$ by what precedes, thus, $\gamma(\delta)\downarrow 1/2$, and $\delta+\gamma(\delta)/\delta\to\sqrt{2}<\kappa$. Near infinity, $\delta+\gamma(\delta)/\delta\sim\delta>\kappa$. Consequently, Eq. (17) has a solution by continuity. To establish uniqueness, we prove that the function $\delta\mapsto\delta+\gamma(\delta)/\delta$ is increasing. Indeed,

$$\frac{d}{d\delta}\left(\delta+\frac{\gamma(\delta)}{\delta}\right)=1+\frac{\gamma'(\delta)}{\delta}-\frac{\gamma(\delta)}{\delta^2}\geq 1-\frac{\gamma(\delta)}{\delta^2}>0$$

as shown by Lemma 4-(ii), and we are done. Proof of Theorem 1-1 is completed.

4.4 Design of an AMP algorithm to approximate the LCP solution

The AMP principles in a nutshell

We begin with some of the fundamental results of the AMP theory. The now classical form of an AMP iterative algorithm, as formalized in the article Bayati and Montanari (2011) of Bayati and Montanari based in part on a result of Bolthausen Bolthausen (2014), can be presented as follows. Let $(h^k)_{k\geq 0}$ be a sequence of Lipschitz $\mathbb{R}^2\to\mathbb{R}$ functions. By the Lipschitz assumption, the derivative

$$\frac{\partial h^k(u,a)}{\partial u}$$

is defined almost everywhere and the function $\partial_1 h^k(u,a)$ is any function that coincides with this derivative where it is defined. For $\boldsymbol{x}=(x_i)_{i\in[N]}$, define by $\langle\boldsymbol{x}\rangle_N$ the scalar quantity:

$$\langle\boldsymbol{x}\rangle_N := \frac{1}{N}\sum_{i\in[N]} x_i\,.$$

Let $\boldsymbol{a}_N\in\mathbb{R}^N$ be a random vector of so-called auxiliary information. Recall that A_N is the GOE matrix introduced in Assumption 2. Starting with a vector $\boldsymbol{u}_N^0\in\mathbb{R}^N$, the AMP recursion is written

$$\boldsymbol{u}_N^{k+1}=\frac{A_N}{\sqrt{N}}h^k(\boldsymbol{u}_N^k,\boldsymbol{a}_N)\;-\;\langle\partial_1 h^k(\boldsymbol{u}_N^k,\boldsymbol{a}_N)\rangle_N\;h^{k-1}(\boldsymbol{u}_N^{k-1},\boldsymbol{a}_N)\,, \tag{18}$$

where $h^k(\boldsymbol{u},\boldsymbol{a})=\big(h^k(u_i,a_i)\big)_{i\in[N]}$.

From this recursion, it is possible to precisely evaluate the asymptotic behavior of the empirical measures

$$\mu^{\boldsymbol{a}_N, \boldsymbol{u}_N^1, \ldots, \boldsymbol{u}_N^k}$$

as $N \to \infty$ for any k, and to prove that $\mu^{\boldsymbol{a}_N, \boldsymbol{u}_N^1, \ldots, \boldsymbol{u}_N^k}$ converges toward a centered vector $(\bar{a}, Z^1, \ldots, Z^k)$ whose covariance structure is defined by the so-called Density Evolution (DE). In particular $\bar{a} \perp\!\!\!\perp (Z^1, \ldots, Z^k)$ and $(Z^1, \ldots, Z^k)$ is a Gaussian vector. The term

$$\langle \partial_1 h^k(\boldsymbol{u}_N^k, \boldsymbol{a}_N) \rangle_N \; h^{k-1}(\boldsymbol{u}_N^{k-1}, \boldsymbol{a}_N)$$

(equal to zero for $k = 0$) is referred to as the Onsager term and plays a crucial role in making possible this convergence. For a detailed exposition of the AMP theory, along with the description of many of its applications, the reader is referred to the recent tutorial Feng et al. (2022).

A specific AMP algorithm for the LCP

To establish Theorem 1, we design the following AMP algorithm and study its properties. For each N, let $(\boldsymbol{u}_N^0, \boldsymbol{a}_N) \in \mathbb{R}^N \times \mathbb{R}^N$ be a couple of random vectors independent of A_N, with $\boldsymbol{a}_N \succcurlyeq 0$. Assume that there exists a couple of L^2 random variables $(\bar{u}, \bar{a})$ such that

$$(a.s.) \qquad \mu^{\boldsymbol{u}_N^0, \boldsymbol{a}_N} \xrightarrow[N\to\infty]{\mathcal{P}_2(\mathbb{R}^2)} \mathscr{L}((\bar{u}, \bar{a}))\,, \quad \bar{a} \neq 0\,. \tag{19}$$

Vectors $\boldsymbol{u}_N^0$ and $\boldsymbol{a}_N$ will be specified later, see (24). Notice that $\bar{a} \geq 0$. By Assumption 3, κ is larger than $\sqrt{2}$ hence (5) admits an unique solution $(\delta, \sigma^2, \gamma)$ by the first part of the theorem. Let $h^t \equiv h$ for all $k \geq 0$, where

$$h(u, a) = \frac{(u + a)_+}{\delta} \qquad \text{and} \qquad \partial_1 h(u, a) = \frac{\mathbb{1}_{\{u+a>0\}}}{\delta}\,.$$

The AMP iteration 18 now reads

$$\boldsymbol{u}_N^{k+1} = \frac{A_N}{\delta\sqrt{N}} \left(\boldsymbol{u}_N^k + \boldsymbol{a}_N\right)_+ - \frac{\langle \mathbb{1}_{\{\boldsymbol{u}_N^k + \boldsymbol{a}_N > 0\}} \rangle_N \left(\boldsymbol{u}_N^{k-1} + \boldsymbol{a}_N\right)_+}{\delta^2}. \tag{20}$$

The DE equations for this algorithm are provided by the following proposition, which is a direct application of (Feng et al. 2022, Theorem 2.3) (see also (Bayati and Montanari (2011), Theorem 4)):

Proposition 5 *For $N \geq 1$, Let A_N be a GOE matrix and let $(\boldsymbol{u}_N^0, \boldsymbol{a}_N) \in \mathbb{R}^N \times \mathbb{R}^N$ be a couple of random vectors independent of A_N, with $\boldsymbol{a}_N \succcurlyeq 0$. Assume (19) and consider the recursion (20). Then, for every $k \geq 1$,*

$$(a.s.) \qquad \mu^{\boldsymbol{a}_N, \boldsymbol{u}_N^1, \ldots, \boldsymbol{u}_N^k} \xrightarrow[N\to\infty]{\mathcal{P}_2(\mathbb{R}^{k+1})} \mathscr{L}((\bar{a}, Z^1, \ldots, Z^k))\,,$$

where $(Z^1, \dots, Z^k)$ *is a centered Gaussian vector, independent of* $(\bar{u}, \bar{a})$*. The* $k \times k$ *covariance matrix* R^k *of the random vector* $(Z^1, \dots, Z^k)$ *is defined recursively in* k *as follows:*

$$R^1 = \mathbb{E}(Z^1)^2 = \frac{1}{\delta^2}\mathbb{E}(\bar{u}+\bar{a})_+^2 ,$$

and given R^k*, matrix* R^{k+1}*'s first principal submatrix is* R^k*,*

$$\left[R^{k+1}\right]_{ij} = \left[R^k\right]_{ij} \qquad \text{for} \quad i, j \in [k] ,$$

whereas the last row and column of R^{k+1} *are defined via the equations:*

$$\left[R^{k+1}\right]_{k+1,\ell} = \mathbb{E}Z^{k+1}Z^{\ell} = \frac{1}{\delta^2}\begin{cases} \mathbb{E}(Z^k+\bar{a})_+(Z^{\ell-1}+\bar{a})_+ & \text{if } \ell \in \{2, \dots, k+1\} , \\ \mathbb{E}(Z^k+\bar{a})_+(\bar{u}+\bar{a})_+ & \text{if } \ell = 1 . \end{cases}$$

Notice that by writing $\boldsymbol{\alpha}^{k+1} = \left((\bar{u}+\bar{a})_+, (Z^1+\bar{a})_+, \dots, (Z^k+\bar{a})_+\right)^{\top}$, we see that $R^{k+1} = \mathbb{E}\boldsymbol{\alpha}^{k+1}(\boldsymbol{\alpha}^{k+1})^{\top}$, which immediately shows that R^{k+1} is a positive semidefinite matrix (actually, one can prove that it is definite, see Feng et al. (2022)).

Denote by

$$\boldsymbol{\xi}_N^k = \boldsymbol{u}_N^k + \boldsymbol{a}_N .$$

What is going to drive the following computations is the fact that the vectors $\boldsymbol{\xi}_N^k$ and $\boldsymbol{\xi}_N^{k+1}$ will tend to be aligned as $N \to \infty$ then $k \to \infty$. This will be formalized and proved in Lemma 6. Denote by $\gamma_N^k = \left\langle \mathbb{1}_{\left\{\xi_N^k > 0\right\}} \right\rangle_N$ and recall the expression of γ given in (5c). With these notations at hand, the AMP recursion (20) reads:

$$\begin{aligned}
\boldsymbol{\xi}_N^{k+1} &= \frac{A_N}{\delta\sqrt{N}}\left(\boldsymbol{\xi}_N^k\right)_+ - \frac{\gamma_N^k}{\delta^2}\left(\boldsymbol{\xi}_N^{k-1}\right)_+ + \boldsymbol{a}_N , \\
&= \frac{A_N}{\delta\sqrt{N}}\left(\boldsymbol{\xi}_N^k\right)_+ - \frac{\gamma}{\delta^2}\left(\boldsymbol{\xi}_N^{k-1}\right)_+ + \boldsymbol{a}_N + \frac{\gamma-\gamma_N^k}{\delta^2}\left(\boldsymbol{\xi}_N^{k-1}\right)_+ , \\
&= \frac{A_N}{\delta\sqrt{N}}\left(\boldsymbol{\xi}_N^k\right)_+ - \frac{\gamma}{\delta^2}\left(\boldsymbol{\xi}_N^k\right)_+ + \boldsymbol{a}_N + \frac{\gamma-\gamma_N^k}{\delta^2}\left(\boldsymbol{\xi}_N^{k-1}\right)_+ \\
&\quad + \frac{\gamma}{\delta^2}\left(\left(\boldsymbol{\xi}_N^k\right)_+ - \left(\boldsymbol{\xi}_N^{k-1}\right)_+\right) .
\end{aligned}$$

Replacing now $\boldsymbol{\xi}_N^{k+1}$ by $\boldsymbol{\xi}_N^k$, we end up with:

$$\boldsymbol{\xi}_N^k = \frac{A_N}{\delta\sqrt{N}}\left(\boldsymbol{\xi}_N^k\right)_+ - \frac{\gamma}{\delta^2}\left(\boldsymbol{\xi}_N^k\right)_+ + \boldsymbol{a}_N + \boldsymbol{\varepsilon}_N^k, \tag{21}$$

where

$$\boldsymbol{\varepsilon}_N^k = \frac{\gamma-\gamma_N^k}{\delta^2}\left(\boldsymbol{\xi}_N^{k-1}\right)_+ + \boldsymbol{\xi}_N^k - \boldsymbol{\xi}_N^{k+1} + \frac{\gamma}{\delta^2}\left(\left(\boldsymbol{\xi}_N^k\right)_+ - \left(\boldsymbol{\xi}_N^{k-1}\right)_+\right). \tag{22}$$

Massaging (21) and relying on (5a) we obtain:

$$\left(\boldsymbol{\xi}_N^k\right)_+ - \frac{\left(\boldsymbol{\xi}_N^k\right)_-}{1+\gamma/\delta^2} = \frac{A_N}{\kappa\sqrt{N}}\left(\boldsymbol{\xi}_N^k\right)_+ + \frac{\delta(\boldsymbol{a}_N + \boldsymbol{\varepsilon}_N^k)}{\kappa}\,. \tag{23}$$

Denote by

$$\boldsymbol{z} = \left(\boldsymbol{\xi}_N^k\right)_+ - \frac{\left(\boldsymbol{\xi}_N^k\right)_-}{1+\gamma/\delta^2}\,.$$

Notice that $\boldsymbol{z}_+ = \left(\boldsymbol{\xi}_N^k\right)_+$ and set finally

$$\boldsymbol{u}_N^0 = \mathbf{1}_N \qquad \text{and} \qquad \boldsymbol{a}_N = \frac{\kappa}{\delta}\boldsymbol{r}_N\,. \tag{24}$$

With these notations, (23) is rewritten

$$\boldsymbol{z} = \Gamma_N \boldsymbol{z}_+ + \boldsymbol{r}_N + \frac{\delta}{\kappa}\boldsymbol{\varepsilon}_N^k\,. \tag{25}$$

Relying on Proposition 3 and on the fact that $\|\Gamma_N\| < 1$ eventually, we conclude that $\boldsymbol{z}_+ = \left(\boldsymbol{\xi}_N^k\right)_+$ is the unique solution of

$$\mathrm{LCP}\left(I_N - \Gamma_N, -\boldsymbol{r}_N - \frac{\delta}{\kappa}\boldsymbol{\varepsilon}_N^k\right)$$

for N large enough, which is almost what is aimed, up to the term $\frac{\delta}{\kappa}\boldsymbol{\varepsilon}_N^k$ - see Eq. (15).

Remark 9 Retrospectively, notice that with the choice (24), assumptions of Proposition 5 are satisfied: $(\boldsymbol{u}_N^0, \boldsymbol{a}_N)$ is independent of A_N and (19) holds thanks to Assumption 1 with $\bar{a} = \frac{\kappa}{\delta}\bar{r}$.

Before bounding $\boldsymbol{\varepsilon}_N^k$, let us first study the behavior of $\mu^{\left(\boldsymbol{\xi}_N^k\right)_+}$. Applying Proposition 5, we get that for all $k \geq 2$:

$$\mu^{\boldsymbol{u}_N^k} \xrightarrow[N\to\infty]{\mathcal{P}_2(\mathbb{R})} \mathscr{L}(Z^k)\,,$$

where $Z^k \stackrel{\mathcal{L}}{=} \theta_k \bar{Z}$ with $\bar{Z} \stackrel{\mathcal{L}}{=} \mathcal{N}(0,1)$ and θ_k satisfying the following DE equation:

$$\theta_{k+1}^2 = \frac{1}{\delta^2}\mathbb{E}(\theta_k \bar{Z} + \bar{a})_+^2\,. \tag{26}$$

Since function $\varphi(u,a) = (u+a)_+$ is Lipschitz, it is clear that

$$\mu^{(\boldsymbol{\xi}_N^k)_+} \quad \xrightarrow[N\to\infty]{\mathcal{P}_2(\mathbb{R})} \quad \mathscr{L}\left((\theta_k\bar{Z} + \bar{a})_+\right)\,. \tag{27}$$

Furthermore, since the distribution function of $\theta_k \bar{Z} + \bar{a}$ has no discontinuity, the following convergence holds:

$$(a.s.) \qquad \gamma_N^k \xrightarrow[N\to\infty]{} \mathbb{P}\left(\theta_k \bar{Z} + \bar{a} > 0\right) \qquad \text{where} \qquad \gamma_N^k = \left\langle \mathbb{1}_{\left\{\xi_N^k > 0\right\}} \right\rangle_N .$$

Introduce the quantity:

$$\sigma_k = \frac{\delta}{\kappa} \theta_k . \tag{28}$$

Following (26), the recursive equation satisfied by σ_k is

$$\sigma_{k+1}^2 = \frac{1}{\delta^2} \mathbb{E}\left(\sigma_k \bar{Z} + \bar{r}\right)_+^2$$

which is precisely the equation appearing in Lemma 4-(ii). As a conclusion, $\sigma_k \xrightarrow[k\to\infty]{} \sigma$, where σ satisfies (5b). This convergence has two interesting consequences:

$$\mathbb{P}\left(\theta_k \bar{Z} + \bar{a} > 0\right) = \mathbb{P}\left(\sigma_k \bar{Z} + \bar{r} > 0\right) \quad \xrightarrow[k\to\infty]{} \quad \mathbb{P}\left(\sigma \bar{Z} + \bar{r} > 0\right) = \gamma,$$

where γ satisfies (5c), and

$$\mathcal{L}\left((\theta_k \bar{Z} + \bar{a})_+\right) = \mathcal{L}\left(\left(1 + \gamma/\delta^2\right)(\sigma_k \bar{Z} + \bar{r})_+\right) \quad \xrightarrow[k\to\infty]{\mathcal{P}_2(\mathbb{R})} \mathcal{L}\left(\left(1 + \gamma/\delta^2\right)(\sigma \bar{Z} + \bar{r})_+\right) ,$$

the latter being the distribution appearing in Theorem 1-(iii).

Control of the error term $\boldsymbol{\varepsilon}_N^k$

Recall the expression of $\boldsymbol{\varepsilon}_N^k$ given in (22):

$$\boldsymbol{\varepsilon}_N^k = \frac{\gamma - \gamma_N^k}{\delta^2} \left(\boldsymbol{\xi}_N^{k-1}\right)_+ + \boldsymbol{\xi}_N^k - \boldsymbol{\xi}_N^{k+1} + \frac{\gamma}{\delta^2}\left(\left(\boldsymbol{\xi}_N^k\right)_+ - \left(\boldsymbol{\xi}_N^{k-1}\right)_+\right) .$$

A direct consequence of (27) yields that

$$\frac{\|(\boldsymbol{\xi}_N^{k-1})_+\|^2}{N} \xrightarrow[N\to\infty]{a.s.} \mathbb{E}\left(\theta_{k-1} \bar{Z} + \bar{a}\right)_+^2 = \theta_k^2 \delta^2 .$$

In particular, the sequence $\left(\frac{\|(\boldsymbol{\xi}_N^{k-1})_+\|^2}{N}\right)_N$ is bounded. Furthermore, $\lim_k (a.s.) \lim_N (\gamma - \gamma_N^k) = 0$. We thus have

$$\lim_{k\to\infty} (a.s.) \lim_{N\to\infty} \frac{(\gamma - \gamma_N^k)^2}{\delta^4} \frac{\|(\boldsymbol{\xi}_N^{k-1})_+\|^2}{N} = 0 . \tag{29}$$

The main idea to control the two remaining terms $\boldsymbol{\xi}_N^k - \boldsymbol{\xi}_N^{k+1}$ and $\left(\boldsymbol{\xi}_N^k\right)_+ - \left(\boldsymbol{\xi}_N^{k-1}\right)_+$ is to establish that the correlation coefficient

$$Q_k := \frac{\mathbb{E} Z^{k-1} Z^k}{\theta_{k-1}\theta_k} \tag{30}$$

converges to 1 as $k \to \infty$. This can be interpreted as an alignement of vectors $\boldsymbol{\xi}_N^k$ and $\boldsymbol{\xi}_N^{k-1}$. This argument was developed in a similar context in Montanari and Richard (2016), see also Donoho and Montanari (2016). For self-containedness, we state and prove the following lemma:

Lemma 6 *The sequence $(Q_k)_{k\geq 2}$ defined in (30) satisfies $Q_k \xrightarrow[k\to\infty]{} 1$.*

Proof of Lemma 6 is postponed to Appendix 1.

We now conclude the proof of Theorem 1. Consider $\varphi(x_1, x_2) = (x_1 - x_2)^2 \in \mathrm{PL}_2(\mathbb{R}^2)$. By Proposition 5, we have

$$\begin{aligned}(a.s.) \quad \frac{\|\boldsymbol{\xi}_N^k - \boldsymbol{\xi}_N^{k+1}\|^2}{N} &= \frac{1}{N}\sum_{i=1}^N \varphi(u_i^k, u_i^{k+1}) \xrightarrow[N\to\infty]{} \mathbb{E}\left(Z^{k+1} - Z^k\right)^2 \\ &= \theta_{k+1}^2 + \theta_k^2 - 2\theta_{k+1}\theta_k Q_{k+1}\,.\end{aligned}$$

Applying Lemma 6, we get that:

$$\lim_{k\to\infty} (a.s.) \lim_{N\to\infty} \frac{\|\boldsymbol{\xi}_N^k - \boldsymbol{\xi}_N^{k+1}\|^2}{N} = 0\,. \tag{31}$$

A similar argument applies to the last term.

$$\begin{aligned}&\frac{1}{N}\left\|\left(\boldsymbol{\xi}_N^k\right)_+ - \left(\boldsymbol{\xi}_N^{k-1}\right)_+\right\|^2 = \frac{1}{N}\left\|\left(\boldsymbol{u}_N^k + \boldsymbol{a}_N\right)_+ - \left(\boldsymbol{u}_N^{k-1} + \boldsymbol{a}_N\right)_+\right\|^2 \\ &\xrightarrow[N\to\infty]{a.s.} \quad \mathbb{E}\left((Z^k + \bar{a})_+ - (Z^{k-1} + \bar{a})_+\right)^2 = \mathbb{E}\left(Z^{k+1} - Z^k\right)^2 .\end{aligned}$$

Finally, using that

$$\begin{aligned}\frac{\|\boldsymbol{\varepsilon}_N^k\|^2}{N} \leq \frac{3}{N}\Bigg(&\frac{(\gamma - \gamma_N^k)^2}{\delta^4}\|(\boldsymbol{\xi}_N^{k-1})_+\|^2 + \|\boldsymbol{\xi}_N^k - \boldsymbol{\xi}_N^{k+1}\|^2 \\ &+ \frac{\gamma^2}{\delta^4}\left\|\left(\boldsymbol{\xi}_N^k\right)_+ - (\boldsymbol{\xi}_N^{k-1})_+\right\|^2\Bigg),\end{aligned}$$

we conclude that

$$\lim_{k\to\infty} (a.s.) \lim_{N\to\infty} \frac{\|\boldsymbol{\varepsilon}_N^k\|^2}{N} = 0\,. \tag{32}$$

Notice that the fact that the a.s. $\lim_N$ at the left hand side exists can be deduced again from Proposition 5.

From the approximated LCP to the genuine LCP

Recall that whenever $\|\Gamma_N\| < 1$, which happens eventually,

$$\boldsymbol{x}_N^\star = \mathrm{LCP}(I_N - \Gamma_N, -\boldsymbol{r}_N) \qquad \text{and} \qquad \left(\boldsymbol{\xi}_N^k\right)_+ = \mathrm{LCP}\left(I_N - \Gamma_N, -\boldsymbol{r}_N - \frac{\delta}{\kappa}\boldsymbol{\varepsilon}_N^k\right).$$

Statistical properties have been established for $\left(\boldsymbol{\xi}_N^k\right)_+$ via the AMP procedure, see for instance (27). Using LCP perturbation results, we shall identify the limiting empirical distribution of $\boldsymbol{x}_N^\star$. Let us introduce:

$$\mu^\star = \mathcal{L}\left((1+\gamma/\delta^2)(\sigma\bar{Z}+\bar{r})_+\right) = \mathcal{L}\left(\frac{\kappa}{\delta}(\sigma\bar{Z}+\bar{r})_+\right)$$

In (Chen and Xiang 2007, Th. 2.7, Th. 2.8), Chen and Xiang provide the following bound:

$$\begin{aligned} \|\boldsymbol{x}_N^\star - (\boldsymbol{\xi}_N^k)_+\| \ &\leq \ \left\|(I_N-\Gamma_N)^{-1}\right\| \times \frac{\kappa}{\delta}\left\|\boldsymbol{\varepsilon}_N^k\right\| \ = \ b_N\left\|\boldsymbol{\varepsilon}_N^k\right\| \\ \text{where} \quad b_N \ &:= \left\|(I_N-\Gamma_N)^{-1}\right\| \times \frac{\kappa}{\delta}. \end{aligned} \tag{33}$$

Let $\varphi:\mathbb{R}\to\mathbb{R}$ be an arbitrary function in $\mathrm{PL}(\mathbb{R}^2)$ with Lipschitz constant L_φ. For a given positive integer k, we have

$$\begin{aligned} \frac{1}{N}\sum_{i=1}^N \varphi(x_i^\star) - \int \varphi d\mu^\star &= \frac{1}{N}\sum_{i=1}^N\left(\varphi(x_i^\star)-\varphi((\xi_i^k)_+)\right) + \frac{1}{N}\sum_{i=1}^N \varphi((\xi_i^k)_+) - \int \varphi d\mu^\star \\ &:= \epsilon_N^1(k) + \epsilon_N^2(k). \end{aligned}$$

We first handle $\epsilon_N^2(k)$. By Proposition 5, we have:

$$\epsilon_N^2(k) \xrightarrow[N\to\infty]{\text{a.s.}} \mathbb{E}\varphi\left(\frac{\kappa}{\delta}(\sigma_k\bar{Z}+\bar{r})_+\right) - \mathbb{E}\varphi\left(\frac{\kappa}{\delta}(\sigma\bar{Z}+\bar{r})_+\right).$$

The r.h.s. is easily bounded by a constant $C(k)$ which converges to zero as $k\to\infty$, using the fact that $\lim_k \sigma_k = \sigma$.

We now turn to $\epsilon_N^1(k)$. By Cauchy-Schwarz inequality

$$
\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}\left|\varphi(x_i^\star)-\varphi((\xi_i^k)_+)\right| &\le \frac{L_\varphi}{N}\sum_{i\in[N]}\left|x_i^\star-(\xi_i^k)_+\right|\left(1+|x_i^\star|+|(\xi_i^k)_+|\right)\\
&\le \frac{L_\varphi}{N}\left\|\boldsymbol{x}_N^\star-(\boldsymbol{\xi}_N^k)_+\right\|\left(\sum_{i\in[N]}(1+|x_i^\star|+|\xi_i^k)_+|)^2\right)^{1/2}\\
&\le 3L_\varphi\frac{\left\|\boldsymbol{x}_N^\star-(\boldsymbol{\xi}_N^k)_+\right\|}{\sqrt{N}}\left(1+\frac{\|\boldsymbol{x}_N^\star\|}{\sqrt{N}}+\frac{\|(\boldsymbol{\xi}_N^k)_+\|}{\sqrt{N}}\right).
\end{aligned}
$$

Recall the bound (33) and the definition of b_N, then

$$
|\epsilon_N^1(k)|\le 3L_\varphi b_N\frac{\|\boldsymbol{\varepsilon}_N^k\|}{\sqrt{N}}\left(1+2\frac{\|(\boldsymbol{\xi}_N^k)_+\|}{\sqrt{N}}+b_N\frac{\|\boldsymbol{\varepsilon}_N^k\|}{\sqrt{N}}\right).
$$

By Assumption 3, b_N a.s. converges to a positive constant. By Proposition 5, we furthermore have

$$
\frac{\|(\boldsymbol{\xi}_N^k)_+\|}{\sqrt{N}}\xrightarrow[N\to\infty]{\text{a.s.}}\left(\mathbb{E}(\theta_k\bar{Z}+\bar{a})_+^2\right)^{1/2},
$$

which is bounded in k. Using (32), we obtain that $\limsup_N |\epsilon_N^1(k)|$ is bounded with probability one by a constant $C_1(k)$ which converges to zero as $k\to\infty$. Finally,

$$
(a.s.)\qquad \limsup_N\left|\frac{1}{N}\sum_{i\in[N]}\varphi(x_i^\star)-\int\varphi d\mu^\star\right|\le C(k)+C_1(k).
$$

Since $C(k)+C_1(k)$ can be made arbitrarily small, we have

$$
(a.s.)\qquad \frac{1}{N}\sum_{i=1}^{N}\varphi(x_i^\star)\xrightarrow[N\to\infty]{}\int\varphi d\mu^\star,
$$

which ends the proof of Theorem 1.

5 Elements of proof of Theorem 2

The strategy of proof is similar to that of Theorem 1. The Wishart model induces differences for the design of the AMP algorithm that we describe hereafter. The full mathematical proof is a matter of careful bookkeeping of Sect. 4. We provide the main steps of the proof but skip many mathematical details which can be found in Akjouj (2023).

5.1 Existence and uniqueness of the solution of system (9)

This can be established as in the case of the GOE model with minor modifications and is hence skipped.

5.2 Design of an AMP algorithm to approximate the LCP solution

We shall rely on the framework of asymmetric AMP as presented in (Feng et al. 2022, Sect. 2.2). Suppose that for a given κ satisfying Assumption 6, (δ, τ^2, γ) is the unique solution of (9). Consider the following recursive system:

$$\boldsymbol{u}_N^{k+1} = \frac{B_N^\top}{\sqrt{P}}\boldsymbol{v}_P^k - \frac{(\boldsymbol{u}_N^k + \boldsymbol{a}_N)_+}{\delta} \tag{34a}$$

$$\boldsymbol{v}_P^k = \frac{B_N}{\delta\sqrt{P}}(\boldsymbol{u}_N^k + \boldsymbol{a}_N)_+ - \frac{N}{P}\frac{\langle \mathbf{1}_{\{\boldsymbol{u}_N^k + \boldsymbol{a}_N > 0\}}\rangle_N}{\delta}\boldsymbol{v}_P^{k-1} \tag{34b}$$

where $\boldsymbol{u}_N^k, \boldsymbol{u}_N^{k+1}$ are $N \times 1$ vectors and and $\boldsymbol{v}_P^{k-1}, \boldsymbol{v}_P^k$, $P \times 1$ vectors with initial conditions

$$\boldsymbol{u}_N^0 = \mathbf{1}_N \qquad \text{and} \qquad \boldsymbol{v}_P^0 = \frac{B_N}{\delta\sqrt{P}}(\boldsymbol{u}_N^0 + \boldsymbol{a}_N)_+ \,.$$

The following proposition is the counterpart of Proposition 5 for asymmetric AMP.

Proposition 7 (consequence of Theorem 2.5 of Feng et al. (2022)) *For $N, P \geq 1$, let Assumptions 4, 5 and 6 hold true. Suppose that $\boldsymbol{a}_N \succcurlyeq 0$ is a random vector independent of A_N satisfying*

$$(a.s.) \qquad \mu^{\boldsymbol{a}_N} \xrightarrow[\mathcal{P}_2(\mathbb{R})]{N\to\infty} \mathcal{L}(\bar{a})$$

and consider the recursions (34). Then for every fixed $k \geq 1$,

$$(a.s.) \qquad \mu^{\boldsymbol{a}_N, \boldsymbol{u}_N^1, \dots, \boldsymbol{u}_N^k} \xrightarrow[N,P\to\infty]{\mathcal{P}_2(\mathbb{R}^{k+1})} \mathcal{L}\left((\bar{a}, U^1, \dots, U^k)\right) \,,$$

$$(a.s.) \qquad \mu^{\boldsymbol{v}_N^0, \dots, \boldsymbol{v}_N^{k-1}} \xrightarrow[N,P\to\infty]{\mathcal{P}_2(\mathbb{R}^{k})} \mathcal{L}\left((\bar{a}, V^0, \dots, V^{k-1})\right) \,,$$

where $(U^1, \dots, U^k)$ is a centered Gaussian random vector independent of $\bar{a}$ with covariance $T^{[k]}$, and $(V^0, \dots, V^{k-1})$ is a centered Gaussian random vector with covariance matrix $\Gamma^{[k]}$. More precisely the covariance matrices

$$T^{[k]} = (T_{ij})_{i,j\in[k]}\,; \qquad T_{ij} = \mathbb{E}U^iU^j \,,$$
$$\Sigma^{[k]} = (\Sigma_{i-1,j-1})_{i,j\in[k]}\,; \qquad \Sigma_{i-1,j-1} = \mathbb{E}V^{i-1}V^{j-1}$$

are defined inductively. First, let $\bar{Z} \sim \mathcal{N}(0,1)$ and introduce τ_k, θ_k such that

$$V^k \stackrel{\mathcal{L}}{=} \theta_k \bar{Z} \qquad \text{and} \qquad U^k \stackrel{\mathcal{L}}{=} \tau_k \bar{Z} \,,$$

so that $\theta_k^2 = \Sigma_{k,k}$ *and* $\tau_k^2 = T_{kk}$. *We define these quantities by induction:*

$$\theta_0^2 = \mathbb{E}(1+\bar{a})_+^2\,, \qquad \tau_{k+1}^2 = \mathbb{E}V_k^2 = \theta_k^2\,, \qquad \theta_{k+1}^2 = \frac{c}{\delta^2}\mathbb{E}(U_{k+1}+\bar{a})_+^2\,.$$

Now given $\Sigma^{[k]} = (\Sigma_{i-1,j-1})$, $\Sigma^{[k+1]}$ *is defined by*

$$\Sigma_{\ell,k} = \frac{c}{\delta^2}\mathbb{E}(U^\ell+\bar{a})_+(U^k+\bar{a})_+ \quad \textit{for } \ell \in [k]\,,$$
$$\Sigma_{0,k} = \frac{c}{\delta^2}\mathbb{E}(1+\bar{a})_+(U^k+\bar{a})_+\,.$$

Given $T^{[k]} = (T_{ij})$, $T^{[k+1]}$ *is defined by*

$$T_{\ell,k+1} = \mathbb{E}V^{\ell-1}V^k \;=\; \Sigma_{\ell-1,k} \quad \textit{for } \ell \in [k+1]\,.$$

From AMP recursions to an approximate LCP solution

We introduce the following notations:

$$\boldsymbol{\xi}_N^k = \boldsymbol{u}_N^k + \boldsymbol{a}_N\,, \qquad \gamma_N^k = \langle 1_{\{\xi_N^k>0\}}\rangle_N\,.$$

Recall the definition of γ solution to (9). Performing similar computations as in Sect. 4.4, we obtain:

$$\boldsymbol{\xi}_N^k + \frac{(\boldsymbol{\xi}_N^k)_+}{\delta} = \frac{B_N^\top B_N}{\left(1+\frac{c\gamma}{\delta}\right)\delta P}(\boldsymbol{\xi}_N^k)_+ + \boldsymbol{a}_N + \widetilde{\boldsymbol{\varepsilon}}_N^k \tag{35}$$

where

$$\widetilde{\boldsymbol{\varepsilon}}_N^k = \frac{B_N^\top}{\left(1+\frac{c\gamma}{\delta}\right)\sqrt{P}}\left(\frac{c\gamma - N/P\gamma_N^k}{\delta}\boldsymbol{v}_P^{k-1} + \frac{c\gamma}{\delta}\left(\boldsymbol{v}_P^k - \boldsymbol{v}_P^{k-1}\right)\right) + \boldsymbol{\xi}_N^k - \boldsymbol{\xi}_N^{k+1}\,.$$

We introduce the following notations:

$$z = (\boldsymbol{\xi}_N^k)_+ - \frac{(\boldsymbol{\xi}_N^k)_-}{1+1/\delta}\,, \quad \boldsymbol{r}_N = \frac{\boldsymbol{a}_N}{1+1/\delta}\,, \quad \boldsymbol{\varepsilon}_N^k = \frac{\widetilde{\boldsymbol{\varepsilon}}_N^k}{1+1/\delta}\,.$$

Then (35) can be rewritten as

$$z = \Gamma_N z_+ + \boldsymbol{r}_N + \boldsymbol{\varepsilon}_N^k\,,$$

where Γ_N is given by (8). Applying Proposition 3, we finally obtain that

$$z^+ = LCP\left(I_N - \Gamma_N, -\boldsymbol{r}_N - \boldsymbol{\varepsilon}_N^k\right)\,.$$

The rest of the proof closely follows the corresponding part in the proof of Theorem 1 and is omitted.

Appendix A: Theorem 1: remaining proofs

A.1 Proof of Lemma 4

Consider the function $f(\sigma^2) = \mathbb{E}(\sigma \bar{Z} + \bar{r})_+^2$. Then, Eq. (5b) is equivalent to the fixed-point equation:

$$\frac{f(\sigma^2)}{\delta^2} = \sigma^2. \tag{36}$$

We can prove by elementary means that

$$\frac{df}{d\sigma^2}(\sigma^2) = \frac{1}{2\sigma}\frac{df}{d\sigma}(\sigma^2) = \frac{1}{\sigma}\mathbb{E}\bar{Z}(\sigma\bar{Z}+\bar{r})_+ \,.$$

Moreover, conditioning on $\bar{r}$ and applying the integration by parts formula for the Gaussian r.v. $\bar{Z}$ we get

$$\frac{1}{\sigma}\mathbb{E}\left(\bar{Z}(\sigma\bar{Z}+\bar{r})_+ \mid \bar{r}\right) = \mathbb{E}\left(1_{\{\sigma\bar{Z}+\bar{r}\geq 0\}} \mid \bar{r}\right).$$

Hence

$$\frac{df}{d\sigma^2}(\sigma^2) = \mathbb{P}\{\sigma\bar{Z}+\bar{r}\geq 0\} = \mathbb{P}\{\bar{Z}+\bar{r}/\sigma \geq 0\}\,.$$

Notice that $\frac{df}{d\sigma^2}$ is a decreasing function since

$$\sigma < \sigma' \qquad \Rightarrow \qquad \{\bar{Z}+\bar{r}/\sigma' \geq 0\} \subset \{\bar{Z}+\bar{r}/\sigma \geq 0\}\,,$$

with

$$\lim_{\sigma^2\to\infty}\frac{df}{d\sigma^2}(\sigma^2) = \frac{1}{2}\,.$$

We now introduce function $g(\sigma^2) = \frac{f(\sigma^2)}{\delta^2} - \sigma^2$. Notice that $g(0) = \mathbb{E}\bar{r}^2/\delta^2 > 0$ and that

$$\frac{dg}{d\sigma^2}(\sigma^2) = \frac{\mathbb{P}\{\bar{Z}+\bar{r}/\sigma \geq 0\}}{\delta^2} - 1 > \frac{1}{2\delta^2} - 1\,. \tag{37}$$

If $\frac{1}{2\delta^2} - 1 \geq 0$ which is equivalent to the condition $\delta < (\sqrt{2})^{-1}$ then g's derivative is positive hence g is increasing with a positive starting point and never vanishes.

Suppose now that $\delta > 1/\sqrt{2}$. We shall prove that g vanishes at a unique point $\sigma^2(\delta)$:

$$g(\sigma^2(\delta)) = 0 \qquad \text{for} \qquad \sigma^2(\delta) > 0\,. \tag{38}$$

Notice that the derivative $dg/d\sigma^2$ is decreasing with a negative limit at infinity

$$\lim_{\sigma^2\to\infty} \frac{dg}{d\sigma^2}(\sigma^2) = \frac{1}{2\delta^2} - 1 < 0\,.$$

Depending on the sign of the value of $dg/d\sigma^2$ at zero, either g is constantly decreasing from the positive value $g(0)$ or g is first increasing then eventually decreasing. We now prove that

$$\lim_{\sigma^2\to\infty} g(\sigma^2) < 0\,. \tag{39}$$

This will yield (38).

$$\frac{g(\sigma^2)}{\sigma^2} = \frac{\mathbb{E}(\sigma\bar{Z}+\bar{r})_+^2}{\delta^2\sigma^2} - 1 = \frac{\mathbb{E}(\bar{Z}+\bar{r}/\sigma)_+^2}{\delta^2} - 1 \xrightarrow[\sigma^2\to\infty]{} \frac{1}{2\delta^2} - 1 < 0\,.$$

Hence g's limit is $-\infty$ at infinity. Eq. (39) is proved, so is (38). The first statement of the lemma is proved.

We now address the second point of the lemma. Let $\delta > 1/\sqrt{2}$ be fixed. From the previous analysis, we know that

$$\left.\frac{dg}{d\sigma^2}\right|_{\sigma^2=\sigma^2(\delta)} < 0.$$

From (37), one can compute

$$\left.\frac{dg}{d\sigma^2}\right|_{\sigma^2=\sigma^2(\delta)} = \frac{\mathbb{P}\{\sigma(\delta)\bar{Z}+\bar{r}\geq 0\}}{\delta^2} - 1,$$

and this gives the second point :

$$\mathbb{P}\{\sigma(\delta)\bar{Z}+\bar{r}\geq 0\} < \delta^2\,.$$

We now address the third point of the lemma. Consider a sequence (σ_t) such that

$$\sigma_0^2 > 0 \text{ and } \sigma_{p+1}^2 = \frac{1}{\delta^2} f(\sigma_p^2)\,.$$

One can easily prove that $\sigma_p^2 \uparrow_p \sigma^2(\delta)$ (resp. $\sigma_p^2 \downarrow \sigma^2(\delta)$) if $\sigma_0^2 < \sigma^2(\delta)$ (resp. $\sigma_0^2 > \sigma^2(\delta)$). The sequence remains constant if $\sigma_0^2 = \sigma^2(\delta)$. Lemma 4 is proved.

Proof of Lemma 6

Proof Let (X_1, X_2) be a centered Gaussian vector with covariance matrix $\Gamma(X_1, X_2)$ given by

$$\Gamma(X_1, X_2) = \begin{pmatrix} 1 & q \\ q & 1 \end{pmatrix} \qquad \text{with} \qquad q \in [0, 1] \, .$$

Let W be a (real) random variable independent of (X_1, X_2) with finite second moment $\mathbb{E}W^2 < \infty$. Consider the function $\mathcal{H} : [0, 1] \to [0, 1]$ defined as

$$q \longmapsto \mathcal{H}(q) = \frac{\mathbb{E}(X_1 + W)_+(X_2 + W)_+}{\mathbb{E}(X_1 + W)_+^2} \, .$$

It is shown in (Montanari and Richard 2016, Lemma 38 and proof of Lemma 37) that $\mathcal{H}$ is a continuous increasing function on $[0, 1]$ such that

$$\mathcal{H}(q) > q \quad \text{for all} \quad q < 1 \qquad \text{and} \qquad \mathcal{H}(1) = 1 \, .$$

Let Z^k be defined in Proposition 5, θ_k in (26) and Q_k in (30). Writing $Z^k = \theta_k \bar{Z}^k$ where $\mathcal{L}\left(\bar{Z}^k\right) = \mathcal{N}(0, 1)$, notice that

$$\operatorname{Cov}\left(\bar{Z}^k, \bar{Z}^{k-1}\right) = Q_k \, .$$

We have

$$\begin{aligned} Q_{k+1} = \frac{\mathbb{E}Z^k Z^{k+1}}{\theta_k \theta_{k+1}} &= \frac{\mathbb{E}(\theta_{k-1}\bar{Z}^{k-1} + \bar{a})_+(\theta_k \bar{Z}^k + \bar{a})_+}{\sqrt{\mathbb{E}(\theta_{k-1}\bar{Z}^{k-1} + \bar{a})_+^2 \mathbb{E}(\theta_k \bar{Z}^k + \bar{a})_+^2}} \, , \\ &= \frac{\mathbb{E}(\bar{Z}^{k-1} + \bar{a}/\theta_{k-1})_+(\bar{Z}^k + \bar{a}/\theta_k)_+}{\sqrt{\mathbb{E}(\bar{Z}^{k-1} + \bar{a}/\theta_{k-1})_+^2 \mathbb{E}(\bar{Z}^k + \bar{a}/\theta_k)_+^2}} \, . \end{aligned}$$

Notice that the last expression only depends on θ_{k-1}, θ_k and Q_k, the covariance between $\bar{Z}^k$ and $\bar{Z}^{k-1}$. We thus introduce the following function

$$H(Q_k, \theta_{k-1}, \theta_k) = \frac{\mathbb{E}(\bar{Z}^{k-1} + \bar{a}/\theta_{k-1})_+(\bar{Z}^k + \bar{a}/\theta_k)_+}{\sqrt{\mathbb{E}(\bar{Z}^{k-1} + \bar{a}/\theta_{k-1})_+^2 \mathbb{E}(\bar{Z}^k + \bar{a}/\theta_k)_+^2}} \, .$$

The function H is continuous. Combining Eq. (28) and the convergence of σ_k, denote by $\theta_\infty = \frac{\kappa}{\delta}\sigma$ where σ satisfies (5b). If we set $W = \bar{a}/\theta_\infty$ in the definition of $\mathcal{H}$ above, then

$$\mathcal{H}(q) = H(q, \theta_\infty, \theta_\infty) \, .$$

The lemma is established if we prove that $Q_\star := \liminf_k Q_k$ satisfies $Q_\star = 1$. Let us first show that $\liminf \mathcal{H}(Q_k) \geq \mathcal{H}(Q_\star)$. If $Q_\star = 0$, then $Q_k \geq Q_\star$ and since $\mathcal{H}$ is increasing we have $\liminf \mathcal{H}(Q_k) \geq \mathcal{H}(Q_\star)$. It is thus sufficient to assume that $Q_\star >$

0. For each $\varepsilon > 0$, $Q_k \geq Q_\star - \varepsilon$ for all k large enough. Thus, $\mathcal{H}(Q_k) \geq \mathcal{H}(Q_\star - \varepsilon)$ for all k large, which implies that $\liminf \mathcal{H}(Q_k) \geq \mathcal{H}(Q_\star - \varepsilon)$. Since $\varepsilon > 0$ is arbitrary, we have $\liminf \mathcal{H}(Q_k) \geq \mathcal{H}(Q_\star)$. With this, we have

$$Q_\star = \liminf_k H(Q_k, \theta_{k-1}, \theta_k) \stackrel{(a)}{=} \liminf_k H(Q_k, \theta_\infty, \theta_\infty) = \liminf_k \mathcal{H}(Q_k)\,,$$
$$\geq \mathcal{H}(Q_\star),$$

where (a) follows from the continuity of H. By $\mathcal{H}$'s properties, this implies that $Q_\star = 1$. □

Appendix B: Elements of proof for Theorems 3 and 4 (universality)

We provide hereafter arguments to complete proofs of Theorems 3 and 4 based on what has already been developed in the proofs of Theorems 1 and 2 and on various results available in the literature.

Proof of Theorem 3 We just need to prove that Proposition 5 above remains true when Assumptions 2 and 1 are replaced with Assumptions 8 and 7 respectively. This is a direct application of (Wang et al. 2022, Theorem 2.4). □

Proof of Theorem 2 We only need to prove that Proposition 7 remains true with the assumptions of Theorem 4. To that end, it is enough to notice that (Feng et al. 2022, Theorem 2.5), from which Proposition 7 follows directly, can in turn be cast in the framework of the AMP algorithm for GOE matrices (18), thanks to the embedding of Javanmard and Montanari described in Javanmard and Montanari (2013). Indeed, Assumptions 9 and 7 used in conjuction with this embedding provide a version of Algorithm (18) that enters the framework of (Wang et al. 2022, Theorem 2.4). This leads to Proposition 7. □

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