ON BILINEAR FORMS BASED ON THE RESOLVENT OF LARGE RANDOM MATRICES

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ABSTRACT. Consider a $N \times n$ non-centered matrix Σ_n with a separable variance profile:

$$\Sigma_n = \frac{D_n^{1/2} X_n \tilde{D}_n^{1/2}}{\sqrt{n}} + A_n \ .$$

Matrices D_n and $\tilde{D}_n$ are non-negative deterministic diagonal, while matrix A_n is deterministic, and X_n is a random matrix with complex independent and identically distributed random variables, each with mean zero and variance one. Denote by $Q_n(z)$ the resolvent associated to $\Sigma_n \Sigma_n^*$, i.e.

$$Q_n(z) = \left(\Sigma_n \Sigma_n^* - zI_N\right)^{-1} \ .$$

Given two sequences of deterministic vectors (u_n) and (v_n) with bounded Euclidean norms, we study the limiting behavior of the random bilinear form:

$$u_n^* Q_n(z) v_n \ , \qquad \forall z \in \mathbb{C} - \mathbb{R}^+ \ ,$$

as the dimensions of matrix Σ_n go to infinity at the same pace. Such quantities arise in the study of functionals of $\Sigma_n \Sigma_n^*$ which do not only depend on the eigenvalues of $\Sigma_n \Sigma_n^*$, and are pivotal in the study of problems related to non-centered Gram matrices such as central limit theorems, individual entries of the resolvent, and eigenvalue separation.

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1. Introduction

The model. Consider a $N \times n$ random matrix $\Sigma_n = (\xi_{ij}^n)$ given by:

$$\Sigma_n = \frac{D_n^{\frac{1}{2}} X_n \tilde{D}_n^{\frac{1}{2}}}{\sqrt{n}} + A_n \stackrel{\triangle}{=} Y_n + A_n \ , \qquad (1.1)$$

where D_n and $\tilde{D}_n$ are respectively $N \times N$ and $n \times n$ non-negative deterministic diagonal matrices. The entries of matrices (X_n), $(X_{ij}^n\,; i, j, n)$ are complex, independent and identically distributed (i.i.d.) with mean 0 and variance 1, and $A_n = (a_{ij}^n)$ is a deterministic $N \times n$ matrix whose spectral norm is bounded in n.

The purpose of this article is to study bilinear forms based on the resolvent $Q_n(z)$ of matrix $\Sigma_n \Sigma_n^*$, where Σ_n^* stands for the hermitian adjoint of Σ_n:

$$Q_n(z) = \left(\Sigma_n \Sigma_n^* - zI_N\right)^{-1} \ ,$$

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as the dimensions N and n grow to infinity at the same pace, that is:

$$0 \quad < \quad \liminf \frac{N}{n} \quad \leq \quad \limsup \frac{N}{n} \quad < \quad \infty \ , \tag{1.2}$$

a condition that will be referred to as $N, n \to \infty$ in the sequel.

A lot of attention has been devoted to the study of quadratic forms $y^* A y$, where $y = n^{-1/2}(X_1, \cdots X_n)^T$, the X_i's being i.i.d., and A is a matrix independent from y. It is well-known, at least since Marcenko and Pastur's seminal paper [15, Lemma 1] (see also [4, Lemma 2.7]) that under fairly general conditions, $y^* A y \sim_\infty n^{-1} \mathrm{Tr}\, A$.

Such a result is of constant use in the study of centered random matrices, as it allows to describe the behavior of the Stieltjes transform associated to the spectral measure (empirical distribution of the eigenvalues) of the matrix under investigation, see for instance [19], [20], [11, 12], etc. Indeed, the Stieltjes transform of the spectral measure writes:

$$f_n(z) = \frac{1}{N} \mathrm{Tr}\, Q_n(z) = \frac{1}{N} \sum_{i=1}^{N} [Q_n(z)]_{ii}(z) \ ,$$

where the $[Q_n(z)]_{ii}$'s denote the diagonal elements of the resolvent. Denote by $\tilde{\eta}_i$ the ith row of Σ_n and by $\Sigma_{n,i}$ matrix Σ_n when row $\tilde{\eta}_i$ has been removed, then the matrix inversion lemma yields the following expression:

$$[Q_n(z)]_{ii} = -\frac{1}{z\left(1 + \tilde{\eta}_i (\Sigma_{n,i}^* \Sigma_{n,i} - zI)^{-1} \tilde{\eta}_i^*\right)} \ .$$

In the case where $\Sigma_n = n^{-1/2} X_n$, the quadratic form that appears in the previous expression can be handled by the aforementioned results. However, if Σ_n is non-centered and given by (1.1), then the quadratic form writes:

$$\tilde{\eta}_i \tilde{Q}_i(z) \tilde{\eta}_i^* = \tilde{y}_i \tilde{Q}_i(z) \tilde{y}_i^* + \tilde{a}_i \tilde{Q}_i(z) \tilde{y}_i^* + \tilde{y}_i \tilde{Q}_i(z) \tilde{a}_i^* + \tilde{a}_i \tilde{Q}_i(z) \tilde{a}_i^* \ ,$$

where $\tilde{Q}_i(z) = (\Sigma_{n,i}^* \Sigma_{n,i} - zI)^{-1}$, and $\tilde{y}_i$ and $\tilde{a}_i$ are the ith rows of matrices Y_n and A_n. The first term can be handled as in the centered case, the second and third terms go to zero; however, the fourth term involves a quadratic form $\tilde{a}_i \tilde{Q}_i(z) \tilde{a}_i^*$ based on deterministic vectors.

It is of interest to notice that, due to some fortunate cancellation, the particular study of bilinear forms of the type $u_n^* Q_n(z) v_n$ or their analogues of the type $\tilde{u}_n \tilde{Q}_n(z) \tilde{v}_n^*$ can be circumvented to establish first order results for non-centered random matrices (see for instance [8], [12]). However, such a study has to be addressed for finer questions such as: Asymptotic behavior of individual entries of the resolvent, Central Limit Theorems [14], behavior of the extreme eigenvalues of $\Sigma_n \Sigma_n^*$, behavior of the eigenvalues and eigenvectors associated with finite rank perturbations of $\Sigma_n \Sigma_n^*$ [6], behavior of eigenvectors or projectors on eigenspaces of $Q(z)$ (see for instance [3] in the context of sample covariance (centered) model), etc.

In a more applied setting, functionals based on individual entries of the resolvent [1, 2] naturally arise in the field of wireless communication. Moreover, the asymptotic study of the quadratic forms $u_n^* Q_n(z) u_n$ is important in statistical inference problems. In the non-correlated case (where $D_n = I_N$ and $\tilde{D}_n = I_n$), it is proved in [21] how such quadratic forms yield consistent estimates of projectors on the subspace orthogonal to the column space of A_n. Such projectors form the basis of MUSIC algorithm, very popular in the field of antenna array processing. A similar approach has been developed in [16], [17] for sample covariance matrix models.

It is the purpose of this article to provide a quantitative description of the limiting behavior of the bilinear form $u_n^* Q_n(z) v_n$, where u_n and v_n are deterministic, as the dimensions of Σ_n go to infinity as indicated in (1.2).

Assumptions, fundamental equations, deterministic equivalents. Formal assumptions for the model are stated below, where $\|\cdot\|$ either denotes the Euclidean norm of a vector or the spectral norm of a matrix.

Assumption A-1. *The random variables* $(X_{ij}^n\ ;\ 1\le i\le N,\, 1\le j\le n\,,\, n\ge 1)$ *are complex, independent and identically distributed. They satisfy* $\mathbb{E}X_{ij}^n = 0$ *and* $\mathbb{E}|X_{ij}^n|^2 = 1$.

Assumption A-2. *The family of deterministic* $N\times n$ *matrices* $(A_n, n\ge 1)$ *is bounded for the spectral norm as* $N, n\to\infty$*:*

$$\boldsymbol{a}_{\max} = \sup_{n\ge 1} \|A_n\| < \infty\ .$$

Notice that this assumption implies in particular that the Euclidean norm of any row or column of $\|A_n\|$ is uniformly bounded in N, n.

Assumption A-3. *The families of real deterministic* $N\times N$ *and* $n\times n$ *matrices* (D_n) *and* $(\tilde D_n)$ *are diagonal with non-negative diagonal elements, and are bounded for the spectral norm as* $N, n\to\infty$*:*

$$\boldsymbol{d}_{\max} = \sup_{n\ge 1}\|D_n\| < \infty \quad \textit{and} \quad \tilde{\boldsymbol{d}}_{\max} = \sup_{n\ge 1}\|\tilde D_n\| < \infty\ .$$

Moreover,

$$\boldsymbol{d}_{\min} = \inf_N \frac{1}{N}\mathrm{Tr}\, D_n > 0 \quad \textit{and} \quad \tilde{\boldsymbol{d}}_{\min} = \inf_n \frac{1}{n}\mathrm{Tr}\, \tilde D_n > 0\ .$$

We collect here results from [12].

The following system of equations:

$$\left\{\begin{array}{lcl} \delta(z) &=& \frac{1}{n}\mathrm{Tr}\, D_n \left(-z(I_N + \tilde\delta(z) D_n) I_N + A_n \left(I_n + \delta(z)\tilde D_n\right)^{-1} A_n^*\right)^{-1} \\ \tilde\delta(z) &=& \frac{1}{n}\mathrm{Tr}\, \tilde D_n \left(-z(I_n + \delta(z)\tilde D_n) + A_n^* \left(I_N + \tilde\delta(z) D_n\right)^{-1} A_n\right)^{-1} \end{array}\right. ,\quad z\in\mathbb{C}-\mathbb{R}^+ \tag{1.3}$$

admits a unique solution $(\delta,\tilde\delta)$ in the class of Stieltjes transforms of nonnegative measures[1] with support in $\mathbb{R}^+$. Matrices $T_n(z)$ and $\tilde T_n(z)$ defined by

$$\left\{\begin{array}{lcl} T_n(z) &=& \left(-z(I_N + \tilde\delta(z) D_n) + A_n \left(I_n + \delta(z)\tilde D_n\right)^{-1} A_n^*\right)^{-1} \\ \tilde T_n(z) &=& \left(-z(I_n + \delta(z)\tilde D_n) + A_n^* \left(I_N + \tilde\delta(z) D_n\right)^{-1} A_n\right)^{-1} \end{array}\right. \tag{1.4}$$

are approximations of the resolvent $Q_n(z)$ and the co-resolvent $\tilde Q_n(z) = (\Sigma_n^*\Sigma_n - zI_N)^{-1}$ in the sense that ($\xrightarrow{a.s.}$ stands for the almost sure convergence):

$$\frac{1}{N}\mathrm{Tr}\,\left(Q_n(z) - T_n(z)\right) \xrightarrow[N,n\to\infty]{a.s.} 0\ ,$$

[1] In fact, δ and $\tilde\delta$ are the Stieltjes transforms of measures with respective total mass $n^{-1}\mathrm{Tr}\, D_n$ and $n^{-1}\mathrm{Tr}\,\tilde D_n$.

which readily gives a deterministic approximation of the Stieltjes transform $N^{-1}\mathrm{Tr}\, Q_n(z)$ of the spectral measure of $\Sigma_n\Sigma_n^*$ in terms of T_n (and similarly for $\tilde{Q}_n$ and $\tilde{T}_n$). Matrices T_n and $\tilde{T}_n$ will play a fundamental role in the sequel.

Nice constants and nice polynomials. By nice constants, we mean positive constants which depend upon the limiting quantities $\boldsymbol{d_{\min}}$, $\tilde{\boldsymbol{d}}_{\boldsymbol{\min}}$, $\boldsymbol{d_{\max}}$, $\tilde{\boldsymbol{d}}_{\boldsymbol{\max}}$, $\boldsymbol{a_{\max}}$, $\liminf \frac{N}{n}$ and $\limsup \frac{N}{n}$ but are independent from n and N. Nice polynomials are polynomials with fixed degree (which is a nice constant) and with non-negative coefficients, each of them being a nice constant. Further dependencies are indicated if needed.

Statement of the main result. Let $\boldsymbol{\delta}_z$ be the distance between the point $z \in \mathbb{C}$ and the real nonnegative axis $\mathbb{R}^+$:

$$\boldsymbol{\delta}_z \;=\; \mathrm{dist}(z, \mathbb{R}^+)\ . \tag{1.5}$$

Here is the main result of the paper:

Theorem 1.1. *Assume that $N, n \to \infty$ and that assumptions A-1, A-2 and A-3 hold true. Assume moreover that there exists an integer $p \geq 1$ such that $\sup_n \mathbb{E}|X_{ij}^n|^{8p} < \infty$ and let (u_n) and (v_n) be sequences of $N \times 1$ deterministic vectors. Then, for every $z \in \mathbb{C} - \mathbb{R}^+$,*

$$\mathbb{E}\left|u_n^*\left(Q_n(z) - T_n(z)\right)v_n\right|^{2p} \leq \frac{1}{n^p}\Phi_p(|z|)\Psi_p\left(\frac{1}{\boldsymbol{\delta}_z}\right)\|u_n\|^{2p}\|v_n\|^{2p}, \tag{1.6}$$

where Φ_p and Ψ_p are nice polynomials depending on p but not on (u_n) neither on (v_n).

Remark 1.1. Apart from providing the convergence speed $\mathcal{O}(n^{-p})$, inequality (1.6) provides a fine control of the behavior of $\mathbb{E}|u^*(Q-T)v|^{2p}$ when z is near the real axis. Such a control should be helpful for studying the behavior of the extreme eigenvalues of $\Sigma_n\Sigma_n^*$ along the lines of [4] and [5].

Remark 1.2. *Influence of the eigenvectors of AA^* on the limiting behavior of u^*Qu.* Consider a matrix Σ with no variance profile ($D = I_N$, $\tilde{D} = I_n$) and let T be given by (1.4). Matrix T writes in this case:

$$T = \left(-z(1+\tilde{\delta})I + \frac{AA^*}{1+\delta}\right)^{-1}\ .$$

Denote by $V\Delta V^*$ the spectral decomposition of AA^*, and by T_Δ:

$$T_\Delta = \left(-z(1+\tilde{\delta})I + \frac{\Delta}{1+\delta}\right)^{-1}\ .$$

Obviously, $T = VT_\Delta V^*$ and by Theorem 1.1, $u^*Qu - u^*VT_\Delta V^*u \to 0$. Clearly, the limiting behavior of u^*Qu not only depends on the spectrum (matrix Δ) of AA^* but also on its eigenvectors (matrix V).

Contents. In Section 2, we set up the notations, state intermediate results among which Lemma 2.6, which is the cornerstone of the paper. Loosely speaking, this lemma whose idea can be found in the work of Girko [9] states that quantities such as

$$\sum_{i=1}^n u^*Q_ia_ia_i^*Q_iu$$

are bounded. This control turns out to be central to take into account Assumption A-2. An intermediate deterministic matrix R_n is introduced and the proof of Theorem 1.1 is outlined. Basically, the quantity of interest $u^*(Q-T)v$ is split into three parts:

$$u^*(Q-T)v = u^*(Q-\mathbb{E}Q)v + u^*(\mathbb{E}Q-R)v + u^*(R-T)v \ ,$$

each being studied separately.

In Section 3, the proof of estimate of $u^*(Q-\mathbb{E}Q)v$ is established, based on a decomposition of $Q-\mathbb{E}Q$ as a sum of martingale increments. Section 4 is devoted to the proof of estimate of $u^*(\mathbb{E}Q-R)v$; and Section 5, to the proof of estimate of $u^*(R-T)v$.

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2. Notations, preliminary results and sketch of proof

2.1. **Notations.** The indicator function of the set $\mathcal{A}$ will be denoted by $\mathbf{1}_{\mathcal{A}}(x)$, its cardinality by $\#\mathcal{A}$. Denote by $a \wedge b = \inf(a,b)$ and by $a \vee b = \sup(a,b)$. As usual, $\mathbb{R}^+ = \{x \in \mathbb{R} \ : \ x \geq 0\}$ and $\mathbb{C}^+ = \{z \in \mathbb{C} \ : \ \mathrm{Im}(z) > 0\}$; similarly $\mathbb{C}^- = \{z \in \mathbb{C} \ : \ \mathrm{Im}(z) < 0\}$; $\mathbf{i} = \sqrt{-1}$; if $z \in \mathbb{C}$, then $\bar{z}$ stands for its complex conjugate. Denote by $\xrightarrow{\mathcal{P}}$ the convergence in probability of random variables and by $\xrightarrow{\mathcal{D}}$ the convergence in distribution of probability measures. Denote by $\mathrm{diag}(a_i;\ 1 \leq i \leq k)$ the $k \times k$ diagonal matrix whose diagonal entries are the a_i's. Element (i,j) of matrix M will be either denoted m_{ij} or $[M]_{ij}$ depending on the notational context. if M is a $n \times n$ square matrix, $\mathrm{diag}(M) = \mathrm{diag}(m_{ii}; 1 \leq i \leq n)$. Denote by M^T the matrix transpose of M, by M^* its Hermitian adjoint, by $\mathrm{Tr}\,(M)$ its trace and $\det(M)$ its determinant (if M is square). We shall use Landau's notation: By $a_n = \mathcal{O}(b_n)$, it is meant that there exists a nice constant K such that $|a_n| \leq K|b_n|$ as $N, n \to \infty$.

Recall that when dealing with vectors, $\|\cdot\|$ will refer to the Euclidean norm; in the case of matrices, $\|\cdot\|$ will refer to the spectral norm.

Due to condition (1.2), we can assume (without loss of generality) that there exist $0 < \boldsymbol{\ell}^- \leq \boldsymbol{\ell}^+ < \infty$ such that

$$\forall N, n \in \mathbb{N}^*, \qquad \boldsymbol{\ell}^- \quad \leq \quad \frac{N}{n} \quad \leq \quad \boldsymbol{\ell}^+ \ .$$

We may drop subscripts and superscripts n for readability.

Denote by Y the $N \times n$ matrix $n^{-1/2} D^{1/2} X \tilde{D}^{1/2}$; by (η_j), (a_j), (x_j) and (y_j) the columns of matrices Σ, A, X and Y. Denote by Σ_j, A_j and Y_j, the matrices Σ, A and Y where column j has been removed. The associated resolvent is $Q_j(z) = (\Sigma_j \Sigma_j^* - zI_N)^{-1}$.

Denote by $\mathbb{E}_j$ the conditional expectation with respect to the σ-field $\mathcal{F}_j$ generated by the vectors $(y_\ell,\ 1 \leq \ell \leq j)$. By convention, $\mathbb{E}_0 = \mathbb{E}$.

Denote by $\mathbb{E}_{y_j}$ the conditional expectation with respect to the σ-field generated by the vectors $(y_\ell,\ \ell \neq j)$.

2.2. **Classical and useful results.** We remind here classical identities of constant use in the sequel. The first one expresses the diagonal elements of the co-resolvent; the other ones are based on low-rank perturbations of inverses (see for instance [13, Sec. 0.7.4]).

Diagonal elements of the co-resolvent; rank-one perturbation of the resolvent.

$$\tilde{q}_{jj}(z) = -\frac{1}{z(1+\eta_j^* Q_j(z)\eta_j)} , \quad (2.1)$$

$$Q(z) = Q_j(z) - \frac{Q_j(z)\eta_j\eta_j^* Q_j(z)}{1+\eta_j^* Q_j\eta_j} , \quad (2.2)$$

$$Q_j(z) = Q(z) + \frac{Q(z)\eta_j\eta_j^* Q(z)}{1-\eta_j^* Q\eta_j} , \quad (2.3)$$

$$1+\eta_j^* Q_j\eta_j = \frac{1}{1-\eta_j^* Q\eta_j} . \quad (2.4)$$

A useful consequence of (2.2) is:

$$\eta_j^* Q(z) = \frac{\eta_j^* Q_j(z)}{1+\eta_j^* Q_j(z)\eta_j} = -z\tilde{q}_{jj}(z)\eta_j^* Q_j(z) . \quad (2.5)$$

Recall that $\boldsymbol{\delta}_z = \mathrm{dist}(z, \mathbb{R}^+)$. Considering the eigenvalues of $Q(z)$ immediately yields:

$$\|Q(z)\| \le \frac{1}{\boldsymbol{\delta}_z} .$$

Taking into account the fact that

$$-\frac{1}{z(1+n^{-1}\tilde{d}_j \operatorname{Tr} Q_j + a_j^* Q_j a_j)} \quad \text{and} \quad -\frac{1}{z(1+\eta_j^* Q_j\eta_j)}$$

are Stieltjes transforms of probability measures over $\mathbb{R}^+$, and based on standard properties of Stieltjes transforms (see for instance [12, Proposition 2.2]), we readily obtain the following estimates:

$$\frac{1}{\left|1+\frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j + a_j^* Q_j a_j\right|} \le \frac{|z|}{\boldsymbol{\delta}_z} \quad \text{and} \quad \frac{1}{|1+\eta_j^* Q_j\eta_j|} \le \frac{|z|}{\boldsymbol{\delta}_z} , \quad \forall z \in \mathbb{C}-\mathbb{R}^+ . \quad (2.6)$$

The following lemma describes the behavior of quadratic forms based on random vectors (see for instance [4, Lemma 2.7]).

Lemma 2.1. *Let $\boldsymbol{x} = (x_1, \cdots, x_n)$ be a $n\times 1$ vector where the x_i's are centered i.i.d. complex random variables with unit variance; consider $p \ge 2$ and assume that $\mathbb{E}|x_1|^{2p} < \infty$. Let $M = (m_{ij})$ be a $n \times n$ complex matrix independent of $\boldsymbol{x}$. Then there exists a constant K_p such that*

$$\mathbb{E}\left|\boldsymbol{x}^* M\boldsymbol{x} - \operatorname{Tr} M\right|^p \le K_p \left(\operatorname{Tr} MM^*\right)^{p/2} .$$

Let $\boldsymbol{u} \in \mathbb{C}^n$ be a deterministic vector, then:

$$\mathbb{E}|\boldsymbol{x}^*\boldsymbol{u}|^p = \mathcal{O}(\|\boldsymbol{u}\|^p) .$$

Moreover,

$$\mathbb{E}\|\boldsymbol{x}\|^p = \mathcal{O}(n^{p/2}) .$$

Note by $D = \mathrm{diag}(d_i\,;\ 1 \le i \le N)$ and $\tilde{D} = \mathrm{diag}(\tilde{d}_i\,;\ 1 \le i \le n)$. Gathering the previous estimates yields the following useful corollary:

Corollary 2.2. *Let $z \in \mathbb{C}-\mathbb{R}^+$, and let $p \ge 2$. Denote by Δ_j the quantity:*

$$\Delta_j = \eta_j^* Q_j\eta_j - \frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j - a_j^* Q_j a_j .$$

Then

$$\mathbb{E}_{y_j} |\Delta_j|^p = \mathcal{O}\left(\frac{1}{n^{p/2}\,\boldsymbol{\delta}_z^p}\right) .$$

Theorem 2.3 (Burkholder inequality)**.** *Let (X_k) be a complex martingale difference sequence with respect to the filtration $(\mathcal{F}_k)$. For every $p \geq 1$, there exists K_p such that:*

$$\mathbb{E}\left|\sum_{k=1}^n X_k\right|^{2p} \leq K_p\left(\mathbb{E}\left(\sum_{k=1}^n \mathbb{E}\left(|X_k|^2 \mid \mathcal{F}_{k-1}\right)\right)^p + \sum_{k=1}^n \mathbb{E}|X_k|^{2p}\right) .$$

A result on holomorphic functions:

Lemma 2.4 (Part of Schwarz's lemma, Th.12.2 in [18])**.** *Let f be an holomorphic function on the open unit disc U such that $f(0)=0$ and $\sup_{z\in U}|f(z)| \leq 1$. Then $|f(z)| \leq |z|$ for every $z \in U$.*

Rules about nice polynomials and nice constants. Some very simple rules of calculus related to nice polynomials will be particularly helpful in the sequel:

If $(\Phi_k, 1 \leq k \leq K)$ and $(\Psi_k, 1 \leq k \leq K)$ are nice polynomials, then there exist nice polynomials Φ and Ψ such that:

$$\sum_{k=1}^K \Phi_k(x)\Psi_k(y) \leq \Phi(x)\Psi(y) \quad \text{for} \quad x, y > 0. \tag{2.7}$$

Take for instance $\Phi(x) = \sum_{k=1}^K \Phi_k(x)$ and $\Psi(x) = \sum_{k=1}^K \Psi_k(x)$.

If Φ_1 and Ψ_1 are nice polynomials, then there exist nice polynomials Φ and Ψ such that:

$$\sqrt{\Phi_1(x)\Psi_1(y)} \leq \Phi(x)\Psi(y) \quad \text{for} \quad x, y > 0. \tag{2.8}$$

Take for instance $\Phi = 2^{-1}(1+\Phi_1)$ and $\Psi = (1+\Psi_1)$ and note that:

$$\sqrt{\Phi_1(x)\Psi_1(y)} \leq \frac{1}{2}(1+\Phi_1(x)\Psi_1(y)) \leq \frac{(1+\Phi_1(x))}{2}(1+\Psi_1(y)) .$$

The values of nice constants or nice polynomials may change from line to line within the proofs, the constant or the polynomial remaining nice.

2.3. **Important estimates.**

Lemma 2.5. *Assume that the setting of Theorem 1.1 holds true. Let u be a deterministic complex $N \times 1$ vector. Then, for every $z \in \mathbb{C} - \mathbb{R}^+$, the following estimates hold true:*

$$\mathbb{E}\left(\sum_{j=1}^n \mathbb{E}_{j-1}\left(u^* Q a_j a_j^* Q^* u\right)\right)^p \leq K_p \frac{\|u\|^{2p}}{\boldsymbol{\delta}_z^{2p}} , \tag{2.9}$$

$$\mathbb{E}\left(\sum_{j=1}^n \mathbb{E}_{j-1}\left(u^* Q \eta_j \eta_j^* Q^* u\right)\right)^p \leq \tilde{K}_p \frac{|z|^p\,\|u\|^{2p}}{\boldsymbol{\delta}_z^{2p}} , \tag{2.10}$$

where K_p and $\tilde{K}_p$ are nice constants depending on p but not on $\|u\|$.

Proof of Lemma 2.5 is postponed to Appendix A.

Lemma 2.6. *Assume that the setting of Theorem 1.1 holds true. Let u be a deterministic complex $N \times 1$ vector. Then, for every $z \in \mathbb{C} - \mathbb{R}^+$, the following estimates hold true:*

$$\sum_{j=1}^{n} \mathbb{E}\left(u^* Q_j a_j a_j^* Q_j^* u\right)^2 \quad \leq \quad \Phi(|z|)\Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right) \|u\|^4 \ , \tag{2.11}$$

$$\mathbb{E}\left(\sum_{j=1}^{n} \mathbb{E}_{j-1}\left(u^* Q_j a_j a_j^* Q_j^* u\right)\right)^p \quad \leq \quad \tilde{\Phi}(|z|)\tilde{\Psi}\left(\frac{1}{\boldsymbol{\delta}_z}\right) \|u\|^{2p}, \tag{2.12}$$

where $\Phi, \Psi, \tilde{\Phi}$ and $\tilde{\Psi}$ are nice polynomials not depending on $\|u\|$.

Proof of Lemma 2.6 is postponed to Appendix A.

In order to proceed, it is convenient to introduce the following intermediate quantities ($z \in \mathbb{C} - \mathbb{R}^+$):

$$\alpha_n(z) \quad = \quad \frac{1}{n}\mathrm{Tr}\, D_n \mathbb{E}Q_n(z), \qquad \tilde{\alpha}_n(z) \quad = \quad \frac{1}{n}\mathrm{Tr}\, \tilde{D}_n \mathbb{E}\tilde{Q}_n(z), \tag{2.13}$$

$$R_n(z) \quad = \quad \left(-z(I_N + \tilde{\alpha}(z) D_n) I_N + A_n \left(I_n + \alpha(z)\tilde{D}_n\right)^{-1} A_n^*\right)^{-1} \ , \tag{2.14}$$

$$\tilde{R}_n(z) \quad = \quad \left(-z(I_n + \alpha(z)\tilde{D}_n) + A_n^* \left(I_N + \tilde{\alpha}(z) D_n\right)^{-1} A_n\right)^{-1} \ . \tag{2.15}$$

A slight modification of the proof of [12, Proposition 5.1-(3)] yields the following estimates:

$$\|R_n(z)\| \leq \frac{1}{\boldsymbol{\delta}_z} \ , \quad \|\tilde{R}_n(z)\| \leq \frac{1}{\boldsymbol{\delta}_z} \quad \text{for } z \in \mathbb{C} - \mathbb{R}^+ \ ,$$

and

$$\|T_n(z)\| \leq \frac{1}{\boldsymbol{\delta}_z} \ , \quad \|\tilde{T}_n(z)\| \leq \frac{1}{\boldsymbol{\delta}_z} \quad \text{for } z \in \mathbb{C} - \mathbb{R}^+ \ .$$

2.4. **Main steps of the proof.** In order to prove Theorem 1.1, we split the quantity of interest $u^*(Q - T)u$ into three parts:

$$u^*(Q - T)v = u^*(Q - \mathbb{E}Q)v + u^*(\mathbb{E}Q - R)v + u^*(R - T)v \ ,$$

and handle each term separately. Precise results are stated in the following three propositions.

Proposition 2.7. *Assume that the setting of Theorem 1.1 holds true. Let (u_n) and (v_n) be sequences of $N \times 1$ deterministic vectors. Then, for every $z \in \mathbb{C} - \mathbb{R}^+$,*

$$\mathbb{E}\left|u_n^* \left(Q_n(z) - \mathbb{E}Q_n(z)\right) v_n\right|^{2p} \leq \frac{1}{n^p}\Phi_p(|z|)\Psi_p\left(\frac{1}{\boldsymbol{\delta}_z}\right) \|u_n\|^{2p}\|v_n\|^{2p},$$

where Φ_p and Ψ_p are nice polynomials depending on p but not on (u_n) nor on (v_n).

Proposition 2.7 is proved in Section 3.

Proposition 2.8. *Assume that the setting of Theorem 1.1 holds true.*

(i) *Let (u_n) and (v_n) be sequences of $N \times 1$ deterministic vectors. Then, for every $z \in \mathbb{C} - \mathbb{R}^+$,*

$$\left|u_n^* \left(\mathbb{E}Q_n(z) - R_n(z)\right) v_n\right| \leq \frac{1}{\sqrt{n}}\Phi(|z|)\Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right) \|u_n\| \, \|v_n\|,$$

where Φ and Ψ are nice polynomials, not depending on (u_n) nor on (v_n).

(ii) *Let M_n be a $N \times N$ deterministic matrix. Then, for every $z \in \mathbb{C} - \mathbb{R}^+$,*

$$\left| \frac{1}{n} \mathrm{Tr}\, M_n \mathbb{E} Q_n(z) - \frac{1}{n} \mathrm{Tr}\, M_n R_n(z) \right| \le \frac{1}{n} \Phi(|z|) \Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right) \|M_n\|,$$

where Φ and Ψ are nice polynomials, not depending on M_n.

Proposition 2.8-(i) is proved in Section 4; proof of Proposition 2.8-(ii) is very similar and thus omitted.

Proposition 2.9. *Assume that the setting of Theorem 1.1 holds true. Let (u_n) and (v_n) be sequences of $N \times 1$ deterministic vectors.*

Then, for every $z \in \mathbb{C} - \mathbb{R}^+$,

$$|u_n^* \left(R_n(z) - T_n(z)\right) v_n| \le \frac{1}{n} \Phi(|z|) \Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right) \|u_n\| \, \|v_n\|,$$

where Φ and Ψ are nice polynomials, not depending on (u_n) nor on (v_n).

Proposition 2.9 is proved in Section 5.

Theorem 1.1 is then easily proved using these three propositions together with inequality $|x+y+z|^{2p} \le K_p(|x|^{2p} + |y|^{2p} + |z|^{2p})$ and (2.7).

3. Proof of Proposition 2.7

Recall the decomposition:

$$u^*(Q-T)v = u^*(Q - \mathbb{E}Q)v + u^*(\mathbb{E}Q - R)v + u^*(R-T)v \ .$$

In this section, we establish the estimate:

$$\mathbb{E} \left| u^* \left(Q(z) - \mathbb{E}Q(z)\right) v \right|^{2p} \le \frac{1}{n^p} \Phi_p(|z|) \Psi_p\left(\frac{1}{\boldsymbol{\delta}_z}\right) \|u\|^{2p} \, \|v\|^{2p} \ , \tag{3.1}$$

for $z \in \mathbb{C} - \mathbb{R}^+$.

3.1. Reduction to unit vectors and quadratic forms.

Assume first that estimate (3.1) holds true for deterministic vectors of norm one, then it holds true for any deterministic vector. Indeed, just consider

$$\tilde{u}_n = \frac{u_n}{\|u_n\|} \quad \text{and} \quad \tilde{v}_n = \frac{v_n}{\|v_n\|}$$

and use the bilinear property. It is therefore sufficient to establish (3.1) for unit vectors u_n and v_n.

Assume now that (3.1) holds true for quadratic forms, that is:

$$\mathbb{E} \left| u^* \left(Q(z) - \mathbb{E}Q(z)\right) u \right|^{2p} \le \frac{1}{n^p} \Phi_p(|z|) \Psi_p\left(\frac{1}{\boldsymbol{\delta}_z}\right) \|u\|^{4p} \ . \tag{3.2}$$

Let u_n and v_n be unit real vectors and write:

$$\begin{aligned} 2u^t(Q-T)v &= (u+v)^t(Q-T)(u+v) - \mathbf{i}(u+\mathbf{i}v)^*(Q-T)(u+\mathbf{i}v) \\ &\quad -(1-\mathbf{i})\left(u^t(Q-T)u + v^t(Q-T)v\right) \ . \end{aligned}$$

All the terms of the right hand side can be estimated with the help of (3.2); hence, applying (2.7) yields (3.1) for unit and real vectors. Generalization to complex unit vectors u_n and v_n is straightforward.

In order to establish estimate (3.1) for the bilinear form $u^*(Q - \mathbb{E}Q)v$, it is therefore sufficient to establish estimate (3.2) for the quadratic form $u^*(Q-\mathbb{E}Q)u$ and for unit vectors $\|u\|$ (just consider $u/\|u\|$ if necessary).

3.2. **Martingale difference sequence and Burkholder inequality.** We first express the difference $u^*(Q - \mathbb{E}Q)u$ as the sum of martingale difference sequences:

$$\begin{aligned} u^*(Q-\mathbb{E}Q)u &= \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1})(u^* Q u) \\ &= \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1})(u^*(Q - Q_j)u) \\ &= -\sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1})\left(\frac{u^* Q_j \eta_j^* \eta_j Q_j u}{1+\eta_j^* Q_j \eta_j}\right) \quad \triangleq \quad -\sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1})\Gamma_j \ . \end{aligned}$$

One can easily check that $((\mathbb{E}_j - \mathbb{E}_{j-1})\Gamma_j)$ is the sum of a martingale difference sequence with respect to the filtration $(\mathcal{F}_j, j \le n)$; hence Burkholder's inequality yields:

$$\begin{aligned} &\mathbb{E}\left|\sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1})\Gamma_j\right|^{2p} \\ &\qquad \le K\left(\mathbb{E}\left(\sum_{j=1}^n \mathbb{E}_{j-1}\left|(\mathbb{E}_j - \mathbb{E}_{j-1})\Gamma_j\right|^2\right)^p + \sum_{j=1}^n \mathbb{E}\left|(\mathbb{E}_j - \mathbb{E}_{j-1})\Gamma_j\right|^{2p}\right) \ . \qquad (3.3) \end{aligned}$$

Recall the definition of $\Delta_j = \eta_j^* Q_j \eta_j - n^{-1}\tilde{d}_j \operatorname{Tr} DQ_j - a_j^* Q_j a_j$. In order to control the right-hand side of Burkholder's inequality, we write Γ_j as:

$$\begin{aligned} \Gamma_j &= \frac{u^* Q_j \eta_j^* \eta_j Q_j u}{1+\eta_j^* Q_j \eta_j} = \frac{u^* Q_j \eta_j^* \eta_j Q_j u}{1+\eta_j^* Q_j \eta_j} \times \frac{1+\frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j + a_j^* Q_j a_j}{1+\frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j + a_j^* Q_j a_j} \\ &= \frac{u^* Q_j \eta_j^* \eta_j Q_j u}{1+\eta_j^* Q_j \eta_j} \times \frac{1+\eta_j^* Q_j \eta_j - \Delta_j}{1+\frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j + a_j^* Q_j a_j} \\ &\triangleq \Gamma_{1j} - \Gamma_{2j} \ , \end{aligned}$$

where

$$\Gamma_{1j} = \frac{u^* Q_j \eta_j \eta_j^* Q_j u}{1+\frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j + a_j^* Q_j a_j} \quad \text{and} \quad \Gamma_{2j} = \frac{\Gamma_j \Delta_j}{1+\frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j + a_j^* Q_j a_j} \ . \qquad (3.4)$$

In the following proposition, we establish relevant estimates.

Proposition 3.1. *Assume that the setting of Theorem 1.1 holds true. There exist nice polynomials* $(\Phi_i, 1\le i\le 4)$ *and* $(\Psi_i, 1\le i\le 4)$ *such that the following estimates hold true:*

$$\mathbb{E}\bigg(\sum_{j=1}^n \mathbb{E}_{j-1}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})\Gamma_{1j}\right|^2\bigg)^p \le \frac{1}{n^p}\Phi_1(|z|)\Psi_1\left(\frac{1}{\boldsymbol{\delta}_z}\right)\ , \tag{3.5}$$

$$\sum_{j=1}^n \mathbb{E}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})\Gamma_{1j}\right|^{2p} \le \frac{1}{n^p}\Phi_2(|z|)\Psi_2\left(\frac{1}{\boldsymbol{\delta}_z}\right)\ , \tag{3.6}$$

$$\mathbb{E}\bigg(\sum_{j=1}^n \mathbb{E}_{j-1}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})\Gamma_{2j}\right|^2\bigg)^p \le \frac{1}{n^p}\Phi_3(|z|)\Psi_3\left(\frac{1}{\boldsymbol{\delta}_z}\right)\ , \tag{3.7}$$

$$\sum_{j=1}^n \mathbb{E}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})\Gamma_{2j}\right|^{2p} = \frac{1}{n^p}\Phi_4(|z|)\Psi_4\left(\frac{1}{\boldsymbol{\delta}_z}\right)\ . \tag{3.8}$$

It is now clear that the proof of Proposition 2.7 directly follows from Burkholder's inequality together with the estimates of Proposition 3.1. The rest of the section is devoted to the proof of Proposition 3.1.

3.3. **Proof of Proposition 3.1: Estimates** (3.5) **and** (3.6)**.** We split Γ_{1j} as $\Gamma_{1j}=\chi_{1j}+\chi_{2j}+\chi_{3j}$, where:

$$\begin{aligned}
\chi_{1j} &= \frac{u^*Q_jy_jy_j^*Q_ju}{1+\frac{\tilde d_j}{n}\operatorname{Tr} DQ_j + a_j^*Q_ja_j}\ ,\\
\chi_{2j} &= \frac{y_j^*Q_juu^*Q_ja_j}{1+\frac{\tilde d_j}{n}\operatorname{Tr} DQ_j + a_j^*Q_ja_j} + \frac{a_j^*Q_juu^*Q_jy_j}{1+\frac{\tilde d_j}{n}\operatorname{Tr} DQ_j + a_j^*Q_ja_j}\ ,\\
\chi_{3j} &= \frac{u^*Q_ja_ja_j^*Q_ju}{1+\frac{\tilde d_j}{n}\operatorname{Tr} DQ_j + a_j^*Q_ja_j}\ .
\end{aligned}$$

Notice that $(\mathbb{E}_j-\mathbb{E}_{j-1})(\chi_{3j})=0$, hence χ_{3j} will play no further role in the sequel. As Q_j is independent from column y_j, we have:

$$(\mathbb{E}_j-\mathbb{E}_{j-1})(\chi_{1j}) = \frac{\tilde d_j}{n}\mathbb{E}_j\left(\frac{x_j^*D^{1/2}Q_juu^*Q_jD^{1/2}x_j - \operatorname{Tr} DQ_juu^*Q_j}{1+\frac{\tilde d_j}{n}\operatorname{Tr} DQ_j + a_j^*Q_ja_j}\right)\ , \tag{3.9}$$

and

$$\begin{aligned}
\mathbb{E}_{j-1}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})(\chi_{1j})\right|^2 &\overset{(a)}{\le} \frac{\tilde{\boldsymbol{d}}_{\max}^2}{n^2}\times\mathbb{E}_{j-1}\left|\frac{x_j^*D^{1/2}Q_juu^*Q_jD^{1/2}x_j - \operatorname{Tr} DQ_juu^*Q_j}{1+\frac{\tilde d_j}{n}\operatorname{Tr} DQ_j + a_j^*Q_ja_j}\right|^2\ ,\\
&\overset{(b)}{\le} \frac{\tilde{\boldsymbol{d}}_{\max}^2}{n^2}\frac{|z|^2}{\boldsymbol{\delta}_z^2}\times\mathbb{E}_{j-1}\left[\mathbb{E}_{y_j}\left|x_j^*D^{1/2}Q_juu^*Q_jD^{1/2}x_j - \operatorname{Tr} DQ_juu^*Q_j\right|^2\right]\\
&\overset{(c)}{\le} K\frac{\tilde{\boldsymbol{d}}_{\max}^2}{n^2}\frac{|z|^2}{\boldsymbol{\delta}_z^2}\times\mathbb{E}_{j-1}\left(\operatorname{Tr} D^{1/2}Q_juu^*Q_jD^{1/2}D^{1/2}Q_j^*uu^*Q_j^*D^{1/2}\right)\\
&= \mathcal{O}\left(\frac{|z|^2}{n^2\,\boldsymbol{\delta}_z^6}\right)\ ,
\end{aligned} \tag{3.10}$$

where (a) follows from Jensen's inequality, (b) from estimate (2.6), and (c) from Lemma 2.1. Thus

$$\mathbb{E}\Big(\sum_{j=1}^{n}\mathbb{E}_{j-1}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})(\chi_{1j})\right|^2\Big)^p \quad = \quad \mathcal{O}\left(\frac{|z|^{2p}}{n^p\,\boldsymbol{\delta}_z^{6p}}\right)\ . \tag{3.11}$$

We now turn to the contribution of χ_{2j}. Arguments similar as previously yield:

$$\begin{aligned}
\mathbb{E}_{j-1}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})(\chi_{2j})\right|^2 = \quad & \mathbb{E}_{j-1}\left|\mathbb{E}_j\chi_{2j}\right|^2 \le \mathbb{E}_{j-1}\left|\chi_{2j}\right|^2 \\
\le \quad & \frac{2}{n}\mathbb{E}_{j-1}\left(\left|\frac{x_j^*D^{1/2}Q_juu^*Q_ja_j}{1+\frac{\tilde{d}_j}{n}\operatorname{Tr}DQ_j+a_j^*Q_ja_j}\right|^2+\left|\frac{a_j^*Q_juu^*Q_jD^{1/2}x_j}{1+\frac{\tilde{d}_j}{n}\operatorname{Tr}DQ_j+a_j^*Q_ja_j}\right|^2\right)\ , \\
\le \quad & \frac{2}{n}\frac{|z|^2}{\boldsymbol{\delta}_z^2}\mathbb{E}_{j-1}\Big(\mathbb{E}_{y_j}(x_j^*D^{1/2}Q_juu^*Q_j^*D^{1/2}x_j)\times u^*Q_ja_ja_j^*Q_j^*u \\
& \qquad +\mathbb{E}_{y_j}(x_j^*D^{1/2}Q_j^*uu^*Q_jD^{1/2}x_j)\times u^*Q_j^*a_ja_j^*Q_ju\Big)\ , \\
\le \quad & \frac{K}{n}\frac{|z|^2}{\boldsymbol{\delta}_z^4}\left(\mathbb{E}_{j-1}\left(u^*Q_j^*a_ja_j^*Q_ju\right)+\mathbb{E}_{j-1}\left(u^*Q_ja_ja_j^*Q_j^*u\right)\right)\ .
\end{aligned} \tag{3.12}$$

Now, using Eq. (2.12) in Lemma 2.6 yields:

$$\mathbb{E}\Big(\sum_{j=1}^{n}\mathbb{E}_{j-1}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})(\chi_{2j})\right|^2\Big)^p \quad \le \quad \frac{1}{n^p}\Phi(|z|)\Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right). \tag{3.13}$$

Hence, gathering (3.11) and (3.13) yields estimate (3.5).

We now establish estimate (3.6). As previously, consider identity (3.9); take it this time to the power p. Using the same arguments as for (3.10), we obtain:

$$\mathbb{E}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})(\chi_{1j})\right|^{2p} \quad = \quad \mathcal{O}\left(\frac{|z|^{2p}}{n^{2p}\,\boldsymbol{\delta}_z^{6p}}\right)\ ,$$

hence:

$$\mathbb{E}\sum_{j=1}^{n}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})(\chi_{1j})\right|^{2p} \quad = \quad \mathcal{O}\left(\frac{|z|^{2p}}{n^{2p-1}\,\boldsymbol{\delta}_z^{6p}}\right)\ . \tag{3.14}$$

Similarly, using the same arguments as in (3.12), together with elementary manipulations, we obtain:

$$\mathbb{E}_{j-1}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})(\chi_{2j})\right|^{2p} \le \frac{K}{n^p}\frac{|z|^{2p}}{\boldsymbol{\delta}_z^{4p}}\left(\mathbb{E}_{j-1}\left(u^*Q_j^*a_ja_j^*Q_ju\right)^p+\mathbb{E}_{j-1}\left(u^*Q_ja_ja_j^*Q_j^*u\right)^p\right)\ .$$

Due to the rough estimate (A.1), we obtain

$$\mathbb{E}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})(\chi_{2j})\right|^{2p} \quad \le \quad \frac{K}{n^p}\frac{|z|^{2p}}{\boldsymbol{\delta}_z^{6p-4}}\left(\mathbb{E}\left(u^*Q_j^*a_ja_j^*Q_ju\right)^2+\mathbb{E}\left(u^*Q_ja_ja_j^*Q_j^*u\right)^2\right)\ ,$$

which after summation, and the estimate obtained in Lemma 2.6, yields:

$$\mathbb{E}\sum_{j=1}^{n}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})(\chi_{2j})\right|^{2p} \quad \le \quad \frac{1}{n^p}\Phi'(|z|)\Psi'\left(\frac{1}{\boldsymbol{\delta}_z}\right)\ , \tag{3.15}$$

where Φ' and Ψ' are nice polynomials. Gathering (3.14) and (3.15) yields estimate (3.6).

3.4. **Proof of Proposition 3.1: Estimates** (3.7) **and** (3.8)**.** We split Γ_{2j} as $\Gamma_{2j} = \chi_{1j} + \chi_{2j} + \chi_{3j}$, where:

$$\begin{aligned}
\chi_{1j} &= \Delta_j \times \frac{u^* Q_j a_j a_j^* Q_j u}{(1+\eta_j^* Q_j \eta_j)(1+\frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j + a_j^* Q_j a_j)}\ , \\
\chi_{2j} &= \Delta_j \times \frac{u^* Q_j y_j y_j^* Q_j u}{(1+\eta_j^* Q_j \eta_j)(1+\frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j + a_j^* Q_j a_j)}\ , \\
\chi_{3j} &= \Delta_j \times \frac{u^* Q_j y_j a_j^* Q_j u + u^* Q_j a_j y_j^* Q_j u}{(1+\eta_j^* Q_j \eta_j)(1+\frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j + a_j^* Q_j a_j)}\ .
\end{aligned}$$

Consider first:

$$\begin{aligned}
\mathbb{E}_{j-1}\left|(\mathbb{E}_j - \mathbb{E}_{j-1})(\chi_{1j})\right|^2 &\le 2\mathbb{E}_{j-1}|\chi_{1j}|^2 \\
&\overset{(a)}{\le} \frac{K|z|^4}{\boldsymbol{\delta}_z^4}\mathbb{E}_{j-1}\left|u^* Q_j a_j a_j^* Q_j u\left(y_j^* Q_j y_j - n^{-1}\tilde{d}_j \operatorname{Tr} DQ_j\right)\right|^2 \\
&\quad + \frac{K|z|^4}{\boldsymbol{\delta}_z^4}\mathbb{E}_{j-1}\left|u^* Q_j a_j a_j^* Q_j u\left(y_j^* Q_j a_j + a_j^* Q_j y_j\right)\right|^2\ , \\
&\overset{(b)}{\le} \frac{K|z|^4}{n^2\,\boldsymbol{\delta}_z^6}\mathbb{E}_{j-1}\left[u^* Q_j a_j a_j^* Q_j^* u\ \mathbb{E}_{y_j}\left|x_j^* D^{1/2} Q_j D^{1/2} x_j - \operatorname{Tr} DQ_j\right|^2\right] \\
&\quad + \frac{K|z|^4}{n\,\boldsymbol{\delta}_z^6}\mathbb{E}_{j-1}\left[u^* Q_j a_j a_j^* Q_j^* u\ \mathbb{E}_{y_j}(x_j^* D^{1/2} Q_j a_j a_j^* Q_j^* D^{1/2} x_j)\right] \\
&\quad\quad + \frac{K|z|^4}{n\,\boldsymbol{\delta}_z^6}\mathbb{E}_{j-1}\left[u^* Q_j a_j a_j^* Q_j^* u\ \mathbb{E}_{y_j}(x_j^* D^{1/2} Q_j^* a_j a_j^* Q_j D^{1/2} x_j)\right] \\
&\overset{(c)}{\le} \frac{K|z|^4}{n\,\boldsymbol{\delta}_z^8}\mathbb{E}_{j-1}(u^* Q_j a_j a_j^* Q_j^* u)\ ,
\end{aligned}$$

where (a) follows from (2.6), (b) from the fact that $|u^* Q_j a_j a_j^* Q_j u| \le K\boldsymbol{\delta}_z^{-2}$ and $|u^* Q_j a_j a_j^* Q_j^* u| \le K\boldsymbol{\delta}_z^{-2}$, and (c) from Lemma 2.1. From this and Lemma 2.6, we deduce that:

$$\mathbb{E}\left(\sum_{j=1}^n \mathbb{E}_{j-1}\left|(\mathbb{E}_j - \mathbb{E}_{j-1})(\chi_{1j})\right|^2\right)^p \le \frac{1}{n^p}\Phi(|z|)\Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right)\ . \tag{3.16}$$

Consider now:

$$\mathbb{E}_{j-1}\left|(\mathbb{E}_j - \mathbb{E}_{j-1})(\chi_{2j})\right|^2 \overset{(a)}{\le} 2\mathbb{E}_{j-1}|\chi_{2j}|^2 \overset{(b)}{\le} \frac{K|z|^4}{\boldsymbol{\delta}_z^4}\mathbb{E}_{j-1}\left|y_j^* Q_j u\right|^4|\Delta_j|^2 \overset{(c)}{\le} \frac{K|z|^4}{n^3\,\boldsymbol{\delta}_z^{10}}\ ,$$

where (a) follows from the triangle and Jensen's inequality, (b) from (2.6) and (c) from Cauchy-Schwarz inequality, Lemma 2.1 and Corollary 2.2.
Hence,

$$\mathbb{E}\left(\sum_{j=1}^n \mathbb{E}_{j-1}\left|(\mathbb{E}_j - \mathbb{E}_{j-1})(\chi_{2j})\right|^2\right)^p = \mathcal{O}\left(\frac{|z|^{4p}}{n^{2p}\,\boldsymbol{\delta}_z^{10p}}\right)\ .$$

Similarly, one can prove that:

$$\mathbb{E}\left(\sum_{j=1}^{n}\mathbb{E}_{j-1}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})(\chi_{3j})\right|^2\right)^p \quad = \quad \mathcal{O}\left(\frac{|z|^{4p}}{n^p\,\boldsymbol{\delta}_z^{10p}}\right) .$$

Gathering the previous results yields the bound:

$$\mathbb{E}\left(\sum_{j=1}^{n}\mathbb{E}_{j-1}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})(\Gamma_{2j})\right|^2\right)^p \quad \le \quad \frac{1}{n^p}\Phi'(|z|)\Psi'\left(\frac{1}{\boldsymbol{\delta}_z}\right) .$$

We now evaluate the second part of Burkholder's inequality (and may re-use notations Φ and Ψ for different polynomials).

$$\begin{aligned}
\sum_{j=1}^{n}\mathbb{E}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})(\chi_{1j})\right|^{2p} \quad &\le \quad \frac{K|z|^{4p}}{\boldsymbol{\delta}_z^{4p}}\sum_{j=1}^{n}\mathbb{E}\left(u^*Q_ja_ja_j^*Q_j^*u\right)^{2p}\mathbb{E}_{y_j}\left|\Delta_j\right|^{2p} \\
&\overset{(a)}{\le} \quad \frac{K|z|^{4p}}{n^p\,\boldsymbol{\delta}_z^{6p}}\sum_{j=1}^{n}\mathbb{E}\left(u^*Q_ja_ja_j^*Q_j^*u\right)^{2}\left(u^*Q_ja_ja_j^*Q_j^*u\right)^{2p-2} \\
&\le \quad \frac{K|z|^{4p}}{n^p\,\boldsymbol{\delta}_z^{10p-4}}\sum_{j=1}^{n}\mathbb{E}\left(u^*Q_ja_ja_j^*Q_j^*u\right)^{2} \\
&\le \quad \frac{1}{n^p}\Phi(|z|)\Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right) ,
\end{aligned}$$

where (a) follows from Corollary 2.2 and the last estimate, from Lemma 2.6. Similar computations yield:

$$\begin{aligned}
\sum_{j=1}^{n}\mathbb{E}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})(\chi_{2j})\right|^{2p} \quad &\le \quad \frac{1}{n^{3p-1}}\Phi'(|z|)\Psi'\left(\frac{1}{\boldsymbol{\delta}_z}\right) , \\
\sum_{j=1}^{n}\mathbb{E}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})(\chi_{3j})\right|^{2p} \quad &\le \quad \frac{1}{n^{2p-1}}\Phi''(|z|)\Psi''\left(\frac{1}{\boldsymbol{\delta}_z}\right) ,
\end{aligned}$$

the first of these inequalities requiring the assumption $\sup_n \mathbb{E}|X_{ij}^n|^{8p} < \infty$ in the statement of Theorem 1.1. Gathering these three results yields:

$$\sum_{j=1}^{n}\mathbb{E}\left|(\mathbb{E}_j-\mathbb{E}_{j-1})(\Gamma_{2j})\right|^{2p} \quad \le \quad \frac{1}{n^p}\tilde{\Phi}(|z|)\tilde{\Psi}\left(\frac{1}{\boldsymbol{\delta}_z}\right) ,$$

and Proposition 3.1 is proved.

4. Proof of Proposition 2.8

Recall the decomposition:

$$u^*(Q-T)v = u^*(Q-\mathbb{E}Q)v + u^*(\mathbb{E}Q-R)v + u^*(R-T)v .$$

In this section, we establish the estimate:

$$\left|u^*\left(\mathbb{E}Q(z)-R(z)\right)v\right| \le \frac{1}{\sqrt{n}}\Phi(|z|)\Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right)\|u\|\,\|v\| ,$$

The argument referred to in Section (3.1) still holds true here; therefore it is sufficient to establish:

$$|u^* (\mathbb{E}Q(z) - R(z))\, u| \le \frac{1}{\sqrt{n}} \Phi(|z|) \Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right) \ , \tag{4.1}$$

for $z \in \mathbb{C} - \mathbb{R}^+$ and for a unit vector u.
Recalling that

$$R = \left[-z(I + \tilde{\alpha} D) + A(I + \alpha \tilde{D})^{-1} A^* \right]^{-1} \ ,$$

the resolvent identity yields:

$$\begin{aligned}
u^*(R-Q)u &= u^* R (Q^{-1} - R^{-1}) Q u \ , \\
&= u^* R \left(\Sigma\Sigma^* - A(I + \alpha \tilde{D})^{-1} A^* \right) Q u + z \tilde{\alpha} u^* R D Q u \ , \\
&= u^* R \left(\sum_{j=1}^n \eta_j \eta_j^* - \sum_{j=1}^n \frac{a_j a_j^*}{1 + \alpha \tilde{d}_j} \right) Q u + z \tilde{\alpha} u^* R D Q u \ , \\
&\overset{(a)}{=} \sum_{j=1}^n \frac{u^* R \eta_j \eta_j^* Q_j u}{1 + \eta_j^* Q_j \eta_j} - \sum_{j=1}^n \frac{u^* R a_j a_j^* Q_j u}{1 + \alpha \tilde{d}_j} \\
&\qquad + \sum_{j=1}^n \frac{u^* R a_j a_j^* Q_j \eta_j \eta_j^* Q_j u}{(1 + \eta_j^* Q_j \eta_j)(1 + \alpha \tilde{d}_j)} - \sum_{j=1}^n \frac{\tilde{d}_j}{n} \mathbb{E}\left(\frac{1}{1 + \eta_j^* Q_j \eta_j} \right) u^* R D Q u \ , \\
&\triangleq \sum_{j=1}^n Z_j \ .
\end{aligned}$$

where (a) follows from (2.2) and (2.5), together with the mere definition of $\tilde{\alpha}$.

As usual, we now write $\eta_j = y_j + a_j$, group the terms that compensate one another and split Z_j accordingly:

$$Z_j = Z_{1j} + Z_{2j} + Z_{3j} + Z_{4j} \ ,$$

where

$$\begin{aligned}
Z_{1j} &= \frac{y_j^* Q_j u u^* R y_j}{1 + \eta_j^* Q_j \eta_j} - \frac{\tilde{d}_j}{n} \mathbb{E}\left(\frac{1}{1 + \eta_j^* Q_j \eta_j} \right) u^* R D Q u \ , \\
Z_{2j} &= \frac{(\alpha \tilde{d}_j - y_j^* Q_j y_j) u^* R a_j a_j^* Q_j u}{(1 + \eta_j^* Q_j \eta_j)(1 + \alpha \tilde{d}_j)} \ , \\
Z_{3j} &= \frac{y_j^* Q_j u a_j^* Q_j y_j \times u^* R a_j}{(1 + \eta_j^* Q_j \eta_j)(1 + \alpha \tilde{d}_j)} \ , \\
Z_{4j} &= \frac{u^* R y_j a_j^* Q_j u + u^* R a_j y_j^* Q_j u}{1 + \eta_j^* Q_j \eta_j} \\
&\quad - \frac{y_j^* Q_j a_j u^* R a_j a_j^* Q_j u + a_j^* Q_j y_j u^* R a_j a_j^* Q_j u}{(1 + \eta_j^* Q_j \eta_j)(1 + \alpha \tilde{d}_j)} \\
&\quad + \frac{u^* R a_j a_j^* Q_j a_j y_j^* Q_j u + u^* R a_j a_j^* Q_j y_j a_j^* Q_j u}{(1 + \eta_j^* Q_j \eta_j)(1 + \alpha \tilde{d}_j)}
\end{aligned}$$

The rest of the section is devoted to establish the convergence to zero of the terms $\mathbb{E} \sum_{j=1}^n Z_{\ell j}$ for $1 \le \ell \le 4$.

4.1. **Convergence to zero of $\sum_j \mathbb{E} Z_{1j}$.** We have

$$
\begin{aligned}
\mathbb{E} Z_{1j} &= \mathbb{E}\left(\frac{y_j^* Q_j u u^* R y_j}{1+\eta_j^* Q_j \eta_j}\right) - \frac{\tilde{d}_j}{n}\mathbb{E}\left(\frac{1}{1+\eta_j^* Q_j \eta_j}\right)\mathbb{E}(u^* R D Q u)\\
&= \mathbb{E}\left[\left(\frac{y_j^* Q_j u u^* R y_j}{1+\eta_j^* Q_j \eta_j}\right) - \frac{\tilde{d}_j}{n}\left(\frac{u^* R D Q_j u}{1+\eta_j^* Q_j \eta_j}\right)\right]\\
&\quad + \frac{\tilde{d}_j}{n}\left[\mathbb{E}\left(\frac{u^* R D Q_j u}{1+\eta_j^* Q_j \eta_j}\right) - \mathbb{E}\left(\frac{1}{1+\eta_j^* Q_j \eta_j}\right)\mathbb{E}(u^* R D Q_j u)\right]\\
&\quad + \frac{\tilde{d}_j}{n}\mathbb{E}\left(\frac{1}{1+\eta_j^* Q_j \eta_j}\right)\mathbb{E}(u^* R D (Q_j - Q) u)\\
&\triangleq \chi_{1j} + \chi_{2j} + \chi_{3j} \ .
\end{aligned}
$$

We first handle χ_{ij}. Recall that $\Delta_j = \eta_j^* Q_j \eta_j - n^{-1}\tilde{d}_j \operatorname{Tr} D Q_j - a_j^* Q_j a_j$. Since $\mathbb{E}_{y_j}(y_j^* Q_j u u^* R y_j) = \tilde{d}_j n^{-1} u^* R D Q_j u$, we get:

$$
\begin{aligned}
\chi_{1j} &= \mathbb{E}\left[\left(\frac{y_j^* Q_j u u^* R y_j}{1+\eta_j^* Q_j \eta_j}\right) - \frac{\tilde{d}_j}{n}\left(\frac{u^* R D Q_j u}{1+\eta_j^* Q_j \eta_j}\right)\right] \ ,\\
&= \mathbb{E}\left[\left(\frac{1}{1+\eta_j^* Q_j \eta_j} - \frac{1}{1+\frac{\tilde{d}_j}{n}\operatorname{Tr} D Q_j + a_j^* Q_j a_j}\right)\left(y_j^* Q_j u u^* R y_j - \frac{\tilde{d}_j}{n}(u^* R D Q_j u)\right)\right] \ ,\\
&= \mathbb{E}\left[\Delta_j \frac{y_j^* Q_j u u^* R y_j - \frac{\tilde{d}_j}{n}(u^* R D Q_j u)}{(1+\eta_j^* Q_j \eta_j)(1+\frac{\tilde{d}_j}{n}\operatorname{Tr} D Q_j + a_j^* Q_j a_j)}\right] \ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|\chi_{1j}| &\le \frac{|z|^2}{\boldsymbol{\delta}_z^2}\sqrt{\mathbb{E}|\Delta_j|^2}\left[\mathbb{E}\left|y_j^* Q_j u u^* R y_j - \frac{\tilde{d}_j}{n}(u^* R D Q_j u)\right|^2\right]^{1/2} \ ,\\
&\le \frac{|z|^2}{\boldsymbol{\delta}_z^2} \times \frac{1}{\sqrt{n}\boldsymbol{\delta}_z} \times \frac{1}{n\boldsymbol{\delta}_z^2} \quad = \quad \mathcal{O}\left(\frac{|z|^2}{n^{3/2}\boldsymbol{\delta}_z^5}\right) \ .
\end{aligned}
$$

Summing over j yields the estimate $\sum_j |\chi_{1j}| = \mathcal{O}\left(|z|^2 n^{-1/2}\boldsymbol{\delta}_z^{-5}\right)$.

We now handle χ_{2j}. Using the inequality $\operatorname{cov}(XY) \le \sqrt{\operatorname{var}(X)\operatorname{var}(Y)}$, we get:

$$
|\chi_{2j}| \quad \le \quad \frac{K}{n}\frac{|z|}{\boldsymbol{\delta}_z}\sqrt{\mathbb{E}\left|u^* R D (Q_j - \mathbb{E} Q_j) u\right|^2}
$$

Hence, applying Proposition 2.7 to $|u^* R D (Q_j - \mathbb{E} Q_j) u|^2$ and summing over j yields the estimate $\sum_j |\chi_{2j}| = n^{-1/2}\Phi(|z|)\Psi(\boldsymbol{\delta}_z^{-1})$.

Let us now handle the term χ_{3j}.

$$\begin{aligned}
|\chi_{3j}| &= \left| \frac{\tilde{d}_j}{n} \mathbb{E}\left(\frac{1}{1+\eta_j^* Q_j \eta_j} \right) \mathbb{E}(u^* RD(Q_j - Q)u) \right| , \\
&\le \frac{K}{n} \frac{|z|}{\boldsymbol{\delta}_z} \mathbb{E} \left| \frac{u^* RDQ_j \eta_j \eta_j^* Q_j u}{1+\eta_j^* Q_j \eta_j} \right| , \\
&\le \frac{K}{n} \frac{|z|^2}{\boldsymbol{\delta}_z^2} \sqrt{\mathbb{E}|u^* RDQ_j\eta_j|^2} \sqrt{\mathbb{E}|\eta_j^* Q_j u|^2} , \\
&\le \frac{K}{n} \frac{|z|^2}{\boldsymbol{\delta}_z^2} \left(\mathbb{E}|u^* RDQ_j\eta_j|^2 + \mathbb{E}|\eta_j^* Q_j u|^2 \right) . \qquad (4.2)
\end{aligned}$$

Now, as:

$$\begin{aligned}
\mathbb{E}|u^* RDQ_j \eta_j|^2 &= \mathbb{E} u^* RDQ_j y_j y_j^* Q_j^* DR^* u + \mathbb{E} u^* RDQ_j a_j a_j^* Q_j^* DR^* u , \\
\mathbb{E}|\eta_j^* Q_j u|^2 &= \mathbb{E} u^* Q_j^* y_j y_j^* Q_j u + \mathbb{E} u^* Q_j^* a_j a_j^* Q_j u ,
\end{aligned}$$

it remains to sum over j and to apply Lemma 2.6 to get the estimate $\sum_j |\chi_{3j}| = n^{-1}\Phi(|z|)\Psi(\boldsymbol{\delta}_z^{-1})$. Gathering the partial estimates yields:

$$\left| \mathbb{E} \sum_j Z_{1j} \right| \le \frac{\Phi(|z|)\Psi(\boldsymbol{\delta}_z^{-1})}{\sqrt{n}} . \qquad (4.3)$$

4.2. **Convergence to zero of $\sum_j \mathbb{E} Z_{2j}$.** Recall that

$$Z_{2j} = \frac{(\alpha \tilde{d}_j - y_j^* Q_j y_j) u^* R a_j a_j^* Q_j u}{(1+\eta_j^* Q_j \eta_j)(1+\alpha \tilde{d}_j)} .$$

We have:

$$\begin{aligned}
|\mathbb{E} Z_{2j}| &\overset{(a)}{\le} \frac{|z|^2}{\boldsymbol{\delta}_z^2} |u^* R a_j| \mathbb{E} \left| (\alpha \tilde{d}_j - y_j^* Q_j y_j) a_j^* Q_j u \right| \\
&\le \frac{|z|^2}{\boldsymbol{\delta}_z^2} |u^* R a_j| \sqrt{\mathbb{E}|a_j^* Q_j u|^2} \sqrt{\mathbb{E}\left| \alpha \tilde{d}_j - y_j^* Q_j y_j \right|^2} \\
&\le \frac{|z|^2}{\boldsymbol{\delta}_z^2} \left(\frac{u^* R a_j a_j^* R u + \mathbb{E} u^* Q_j a_j a_j^* Q_j u}{2} \right) \sqrt{\mathbb{E}\left| \alpha \tilde{d}_j - y_j^* Q_j y_j \right|^2} , \qquad (4.4)
\end{aligned}$$

where (a) follows from (2.6). In order to estimate the remaining square root, we decompose the difference as:

$$\alpha \tilde{d}_j - y_j^* Q_j y_j = \frac{\tilde{d}_j}{n} \operatorname{Tr} D(\mathbb{E}Q - Q) + \frac{\tilde{d}_j}{n} \operatorname{Tr} D(Q - Q_j) + \frac{\tilde{d}_j}{n} \operatorname{Tr} DQ_j - y_j^* Q_j y_j .$$

Hence,

$$\begin{aligned}
&\mathbb{E}|\alpha \tilde{d}_j - y_j^* Q_j y_j|^2 \\
&\quad \le K \left(\frac{1}{n^2} \mathbb{E} \left| \operatorname{Tr} D(\mathbb{E}Q - Q) \right|^2 + \frac{1}{n^2} \mathbb{E} |\operatorname{Tr} D(Q - Q_j)|^2 + \mathbb{E} \left| \frac{\tilde{d}_j}{n} \operatorname{Tr} DQ_j - y_j^* Q_j y_j \right|^2 \right) .
\end{aligned}$$

Writing $\mathbb{E}|n^{-1}\mathrm{Tr}\, D(Q-\mathbb{E}Q)|^2 \le \boldsymbol{\ell}^+ \sup_{j\le n} \mathbb{E}|e_j^* D(Q-\mathbb{E}Q)e_j|^2$ where e_j represents canonical vector number j and using the result of Section 3, the first term of the right hand side is of order $n^{-1}\Phi(|z|)\Psi(\boldsymbol{\delta}_z^{-1})$. The second term is of order $(n\boldsymbol{\delta}_z)^{-2}$ (minor modification of [20, Lemma 2.6] to encompass the case $\mathrm{Re}(z) < 0$). Finally, the third term is of order $n^{-1}\boldsymbol{\delta}_z^{-2}$ by Lemma 2.1. Collecting these results, we obtain:

$$\begin{aligned} \sqrt{\mathbb{E}|\alpha\tilde{d}_j - y_j^* Q_j y_j|^2} &\le \frac{K}{\sqrt{n}}\left(\Phi_1\Psi_1 + \frac{\Phi_2\Psi_2}{n} + \Phi_3\Psi_3\right)^{1/2} \\ &\le \frac{K}{\sqrt{n}}\left(\Phi_1\Psi_1 + \Phi_2\Psi_2 + \Phi_3\Psi_3\right)^{1/2} \\ &\overset{(a)}{\le} \frac{K}{\sqrt{n}}\sqrt{\tilde{\Phi}\tilde{\Psi}} \quad \overset{(b)}{\le} \frac{K}{\sqrt{n}}\Phi\Psi \ , \end{aligned}$$

where the Φ's are nice polynomials with argument $|z|$ and the Ψ's are nice polynomials with argument $|\boldsymbol{\delta}_z^{-1}|$, and where (a) follows from (2.7) and (b) from (2.8). It remains to plug this estimate into (4.4), to sum over j and to use Assumption 2 together with Lemma 2.6 to obtain:

$$\begin{aligned} \left|\mathbb{E}\sum_{j=1}^n Z_{2j}\right| &\le \frac{K|z|^2}{\sqrt{n}\boldsymbol{\delta}_z^2}\left(u^* RAA^* Ru + \sum_{j=1}^n \mathbb{E}u^* Q_j a_j a_j^* Q_j u\right)\Phi(|z|)\Psi(\boldsymbol{\delta}_z^{-1}) \ , \\ &\le \frac{1}{\sqrt{n}}\Phi'(|z|)\Psi'(\boldsymbol{\delta}_z^{-1}) \ . \end{aligned} \tag{4.5}$$

4.3. **Convergence to zero of** $\sum_j \mathbb{E}Z_{3j}$**.** Recall that

$$Z_{3j} = \frac{y_j^* Q_j u a_j^* Q_j y_j \times u^* R a_j}{(1+\eta_j^* Q_j \eta_j)(1+\alpha\tilde{d}_j)} \ .$$

We have:

$$\begin{aligned} \mathbb{E}|Z_{3j}| &\overset{(a)}{\le} \frac{|z|^2}{\boldsymbol{\delta}_z^2}|u^* R a_j| \times \mathbb{E}|y_j^* Q_j u a_j^* Q_j y_j| \quad \le \quad \frac{|z|^2}{\boldsymbol{\delta}_z^2}|u^* R a_j|\sqrt{\mathbb{E}|y_j^* Q_j u|^2 \mathbb{E}|a_j^* Q_j y_j|^2} \\ &\overset{(b)}{\le} \frac{K}{n}\frac{|z|^2}{\boldsymbol{\delta}_z^4}|u^* R a_j| \ , \end{aligned}$$

where (a) follows from (2.6), and (b) from Lemma 2.1. Hence,

$$\begin{aligned} \left|\sum_{j=1}^n \mathbb{E}Z_{3j}\right| &\le \frac{K}{n}\frac{|z|^2}{\boldsymbol{\delta}_z^4}\sum_{j=1}^n |u^* R a_j| \\ &\le \frac{K}{n}\frac{|z|^2}{\boldsymbol{\delta}_z^4}\sqrt{n} \times \sqrt{\sum_{j=1}^n u^* R a_j a_j^* R^* u} \quad = \quad \mathcal{O}\left(\frac{|z|^2}{\sqrt{n}\,\boldsymbol{\delta}_z^5}\right) \ . \end{aligned} \tag{4.6}$$

4.4. **Convergence to zero of** $\sum_j \mathbb{E}Z_{4j}$**.** Write Z_{4j} as

$$Z_{4j} = \frac{W_{4j}}{(1+\eta_j^* Q_j \eta_j)(1+\alpha\tilde{d}_j)}$$

with

$$W_{4j} = (1+\alpha\tilde{d}_j)(u^* R y_j a_j^* Q_j u + u^* R a_j y_j^* Q_j u) \\ - y_j^* Q_j a_j u^* R a_j a_j^* Q_j u - a_j^* Q_j y_j u^* R a_j a_j^* Q_j u \\ + u^* R a_j a_j^* Q_j a_j y_j^* Q_j u + u^* R a_j a_j^* Q_j y_j a_j^* Q_j u$$

Write

$$\frac{1}{1+\eta_j^* Q_j \eta_j} = \frac{1}{1+\frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j + a_j^* Q_j a_j} - \frac{\Delta_j}{(1+\eta_j^* Q_j \eta_j)(1+\frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j + a_j^* Q_j a_j)} .$$

Plugging this identity into Z_{4j} and taking into account the fact that $\mathbb{E}_{y_j} W_{4j} = 0$, we obtain:

$$\begin{aligned} |\mathbb{E} Z_{4j}| &= \left| \mathbb{E}\left(\frac{\Delta_j W_{4j}}{(1+\alpha\tilde{d}_j)(1+\eta_j^* Q_j \eta_j)(1+\frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j + a_j^* Q_j a_j)} \right) \right| \\ &\le \frac{|z|^3}{\boldsymbol{\delta}_z^3} \sqrt{\mathbb{E}|\Delta_j|^2} \sqrt{\mathbb{E}|W_{4j}|^2} \quad \le \quad \frac{K}{\sqrt{n}} \frac{|z|^3}{\boldsymbol{\delta}_z^4} \sqrt{\mathbb{E}|W_{4j}|^2} . \end{aligned}$$

Hence,

$$\left| \mathbb{E} \sum_j Z_{4j} \right| \le \frac{K}{\sqrt{n}} \frac{|z|^3}{\boldsymbol{\delta}_z^4} \sum_j \sqrt{\mathbb{E}|W_{4j}|^2} \le \frac{K|z|^3}{\boldsymbol{\delta}_z^4} \sqrt{\sum_j \mathbb{E}|W_{4j}|^2} . \tag{4.7}$$

We therefore estimate $\sum_j \mathbb{E}|W_{4j}|^2$. First, write:

$$\begin{aligned} \mathbb{E}|W_{4j}|^2 &\le \frac{K}{n}\left(1+\frac{1}{\boldsymbol{\delta}_z}\right)^2 \left(\mathbb{E}|a_j^* Q_j u|^2 u^* R D R^* u + |u^* R a_j|^2 \mathbb{E}(u^* Q_j^* D Q_j u) \right) \\ &\quad + \frac{K}{n} |u^* R a_j|^2 \mathbb{E}\left[|a_j^* Q_j u|^2 \left(a_j^* Q_j^* D Q_j a_j + a_j^* Q_j D Q_j^* a_j \right) \right] \\ &\quad + \frac{K}{n} |u^* R a_j|^2 \mathbb{E}\left(|a_j^* Q_j a_j|^2 u^* Q_j^* D Q_j u + |a_j^* Q_j u|^2 a_j^* Q_j D Q_j a_j \ . \right) \end{aligned}$$

Now, summing over j yields:

$$\begin{aligned} \sum_{j=1}^n \mathbb{E}|W_{4j}|^2 &\le \frac{K}{n}\left(\sum_{j=1}^n \mathbb{E}(u^* Q_j^* a_j a_j^* Q_j u) \right) \left(1+\frac{1}{\boldsymbol{\delta}_z}\right) \frac{1}{\boldsymbol{\delta}_z^2} \\ &\quad + \frac{K}{n}\left(\sum_{j=1}^n \mathbb{E}(u^* R a_j a_j^* R^* u) \right) \left(\frac{1}{\boldsymbol{\delta}_z^4} + \frac{1}{\boldsymbol{\delta}_z^2}\left(1+\frac{1}{\boldsymbol{\delta}_z}\right) \right) \\ &\le \frac{1}{n} \Phi(|z|) \Psi(\boldsymbol{\delta}_z^{-1}) . \end{aligned}$$

Plugging this into (4.7) yields the estimate

$$\left| \mathbb{E} \sum_j Z_{4j} \right| \le \frac{1}{\sqrt{n}} \Phi'(|z|) \Psi'(\boldsymbol{\delta}_z^{-1}) . \tag{4.8}$$

4.5. **End of proof.** Recall that:

$$|u^*(R-\mathbb{E}Q)u| \quad \leq \quad \left|\mathbb{E}\sum_{j=1}^n Z_{1j}\right| + \left|\mathbb{E}\sum_{j=1}^n Z_{2j}\right| + \left|\mathbb{E}\sum_{j=1}^n Z_{3j}\right| + \left|\mathbb{E}\sum_{j=1}^n Z_{4j}\right| .$$

It remains to gather estimates (4.3), (4.5), (4.6) and (4.8) to get the desired estimate:

$$|u^*(R-\mathbb{E}Q)u| \quad \leq \quad \frac{1}{\sqrt{n}}\Phi(|z|)\Psi(\boldsymbol{\delta}_z^{-1}) .$$

5. Proof of Proposition 2.9

Recall the decomposition:

$$u^*(Q-T)v = u^*(Q-\mathbb{E}Q)v + u^*(\mathbb{E}Q-R)v + u^*(R-T)v .$$

As mentioned in Section 3.1, it is sufficient to establish the estimate:

$$|u^*\left(R(z)-T(z)\right)u| \leq \frac{1}{n}\Phi(|z|)\Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right) , \tag{5.1}$$

for $z \in \mathbb{C}-\mathbb{R}^+$ in the case where u has norm one.

5.1. **The estimate for $u^*(R-T)u$.** Recall the definitions of $\delta, \tilde{\delta}$ (1.3), $\alpha, \tilde{\alpha}$ (2.13) and $R, \tilde{R}$ (2.14-2.15). Using twice the resolvent identity yields:

$$u^*(R-T)u = (\tilde{\alpha}-\tilde{\delta})\kappa_1 + (\alpha-\delta)\kappa_2 , \tag{5.2}$$

where

$$\begin{cases} \kappa_1 = zu^*RDTu \\ \kappa_2 = u^*RA(I+\alpha\tilde{D})^{-1}\tilde{D}(I+\delta\tilde{D})^{-1}A^*Tu \end{cases} .$$

The following bounds are straightforward:

$$|\kappa_1| \leq \frac{|z|\tilde{\boldsymbol{d}}_{\mathbf{max}}}{\boldsymbol{\delta}_z^2} \quad \text{and} \quad |\kappa_2| \leq \frac{\|A\|^2\tilde{\boldsymbol{d}}_{\mathbf{max}}}{\boldsymbol{\delta}_z^2} \times \|(I+\alpha\tilde{D})^{-1}\| \times \|(I+\delta\tilde{D})^{-1}\| .$$

It remains to control the spectral norms of $(I+\alpha\tilde{D})^{-1}$ and $(I+\delta\tilde{D})^{-1}$. Recall that α is the Stieltjes transform of a positive measure with support included in $\mathbb{R}^+$. This in particular implies that $\mathrm{Im}(z\alpha) > 0$ for $z \in \mathbb{C}^+$. One can check that

$$\Upsilon_j(z) = \frac{1}{-z(1+\alpha\tilde{d}_j)}$$

is analytic and satisfies $\mathrm{Im}(\Upsilon_j) > 0$ and $\mathrm{Im}(z\Upsilon_j) > 0$ on $\mathbb{C}^+$ and that $\lim_{y\to\infty}(-\mathbf{i}y\Upsilon_j(\mathbf{i}y)) = 1$. As a consequence, Υ_j is the Stieltjes transform of a probability measure with support included in $\mathbb{R}^+$ (see *e.g.* [12, Prop. 2.2(2)]). In particular,

$$|\Upsilon_j(z)| \leq \frac{1}{\boldsymbol{\delta}_z} \quad \text{for} \quad 1 \leq j \leq n ,$$

which readily implies that $\|(I+\alpha\tilde{D})^{-1}\| \leq |z|\boldsymbol{\delta}_z^{-1}$. The same argument applies for $\|(I+\delta\tilde{D})^{-1}\|$. Finally,

$$|\kappa_2| \leq \frac{|z|^2\|A\|^2\tilde{\boldsymbol{d}}_{\mathbf{max}}}{\boldsymbol{\delta}_z^4} .$$

In view of the estimates obtained for κ_1 and κ_2, it is sufficient, in order to establish (5.1), to obtain estimates for $\alpha - \delta$ and $\tilde{\alpha} - \tilde{\delta}$. Assume that the following estimate holds true:

$$\forall z \in \mathbb{C} - \mathbb{R}^+ , \quad \max\left(|\alpha - \delta|, |\tilde{\alpha} - \tilde{\delta}|\right) \le \frac{1}{n}\Phi(|z|)\Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right) , \tag{5.3}$$

where Φ and Ψ are nice polynomials. Then, plugging (5.3) into (5.2) immediately yields the desired result (5.1).

The rest of the section is devoted to establish (5.3).

5.2. **Auxiliary estimates over** $(\alpha - \delta)$ **and** $(\tilde{\alpha} - \tilde{\delta})$**.** Writing $\alpha = n^{-1}\mathrm{Tr}\, DR + n^{-1}\mathrm{Tr}\, D(\mathbb{E}Q - R)$ and $\delta = n^{-1}\mathrm{Tr}\, DT$, the difference $\alpha - \delta$ expresses as $n^{-1}\mathrm{Tr}\, D(R-T) + n^{-1}\mathrm{Tr}\, D(\mathbb{E}Q - R)$. Now using the resolvent identity $R - T = -R(R^{-1} - T^{-1})T$ and performing the same computation for the tilded quantities yields the following system of equations:

$$\begin{pmatrix} \alpha - \delta \\ \tilde{\alpha} - \tilde{\delta} \end{pmatrix} = C_0 \begin{pmatrix} \alpha - \delta \\ \tilde{\alpha} - \tilde{\delta} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon} \\ \tilde{\boldsymbol{\varepsilon}} \end{pmatrix} \quad \text{where} \quad C_0 = \begin{pmatrix} u_0 & z v_0 \\ z\tilde{v}_0 & \tilde{u}_0 \end{pmatrix} , \tag{5.4}$$

the coefficients being defined as:

$$\left\{ \begin{array}{lcl} u_0 & = & \frac{1}{n}\mathrm{Tr}\, D^{1/2} R A (I + \alpha\tilde{D})^{-1} \tilde{D} (I + \delta\tilde{D})^{-1} A^* T D^{1/2} \\ \tilde{u}_0 & = & \frac{1}{n}\mathrm{Tr}\, \tilde{D}^{1/2} \tilde{R} A^* (I + \tilde{\alpha} D)^{-1} D (I + \tilde{\delta} D)^{-1} A \tilde{T} \tilde{D}^{1/2} \\ v_0 & = & \frac{1}{n}\mathrm{Tr}\, DRDT \\ \tilde{v}_0 & = & \frac{1}{n}\mathrm{Tr}\, \tilde{D}\tilde{R}\tilde{D}\tilde{T} \end{array} \right. , \tag{5.5}$$

and the quantities $\boldsymbol{\varepsilon}$ and $\tilde{\boldsymbol{\varepsilon}}$ being given by:

$$\boldsymbol{\varepsilon} \; = \; \frac{1}{n}\mathrm{Tr}\, D(\mathbb{E}Q - R) \quad \text{and} \quad \tilde{\boldsymbol{\varepsilon}} \; = \; \frac{1}{n}\mathrm{Tr}\, \tilde{D}(\mathbb{E}\tilde{Q} - \tilde{R}) \; . \tag{5.6}$$

The general idea, in order to transfer the estimates over $\boldsymbol{\varepsilon}$ and $\tilde{\boldsymbol{\varepsilon}}$ (as provided in Proposition 2.8-(ii)), to $\alpha - \delta$ and $\tilde{\alpha} - \tilde{\delta}$, is to obtain an estimate over $1/\det(I - C_0)$, and then to solve the system (5.4).

Lower bound for $\det(I - C_0)$. The mere definition of $I - C_0$ yields

$$\begin{array}{rcl} |\det(I - C_0)| & = & \left|(1 - u_0)(1 - \tilde{u}_0) - z^2 v_0 \tilde{v}_0\right| \\ & \ge & (1 - |u_0|) \times (1 - |\tilde{u}_0|) - |z|^2 |v_0| \times |\tilde{v}_0| \end{array}$$

In order to control the quantities $u_0, \tilde{u}_0, v_0$ and $\tilde{v}_0$, we shall use the following inequality:

$$|\mathrm{Tr}\, AB^*| \le (\mathrm{Tr}\, AA^*)^{1/2} \times (\mathrm{Tr}\, BB^*)^{1/2} \;\; , \tag{5.7}$$

together with the following quantities:

$$\left\{ \begin{array}{lcl} u_1 & = & \frac{1}{n}\mathrm{Tr}\, DTA(I + \delta^* \tilde{D})^{-1} \tilde{D} (I + \delta\tilde{D})^{-1} A^* T^* \\ \tilde{u}_1 & = & \frac{1}{n}\mathrm{Tr}\, \tilde{D}\tilde{T} A^* (I + \tilde{\delta} D)^{-1} D (I + \tilde{\delta}^* D)^{-1} A \tilde{T}^* \\ v_1 & = & \frac{1}{n}\mathrm{Tr}\, DTDT^* \\ \tilde{v}_1 & = & \frac{1}{n}\mathrm{Tr}\, \tilde{D}\tilde{T}\tilde{D}\tilde{T}^* \end{array} \right.$$

and

$$\left\{ \begin{array}{lcl} u_2 & = & \frac{1}{n}\mathrm{Tr}\, DRA(I + \alpha^* \tilde{D})^{-1} \tilde{D} (I + \alpha\tilde{D})^{-1} A^* R^* \\ \tilde{u}_2 & = & \frac{1}{n}\mathrm{Tr}\, \tilde{D}\tilde{R} A^* (I + \tilde{\alpha} D)^{-1} D (I + \tilde{\alpha}^* D)^{-1} A \tilde{R}^* \\ v_2 & = & \frac{1}{n}\mathrm{Tr}\, DRDR^* \\ \tilde{v}_2 & = & \frac{1}{n}\mathrm{Tr}\, \tilde{D}\tilde{R}\tilde{D}\tilde{R}^* \end{array} \right. \tag{5.8}$$

Using (5.7) together with identity $(I+\delta\tilde{D})^{-1}A^*T = \tilde{T}A^*(I+\tilde{\delta}D)^{-1}$ (and similar ones for related quantities), we obtain:

$$|u_0| \le (\tilde{u}_1 u_2)^{1/2}\ , \quad |\tilde{u}_0| \le (u_1\tilde{u}_2)^{1/2}\ , \quad |v_0| \le (v_1 v_2)^{1/2}\ , \quad |\tilde{v}_0| \le (\tilde{v}_1\tilde{v}_2)^{1/2}\ ,$$

hence the lower bound:

$$|\det(I-C_0)| \ge (1-(\tilde{u}_1 u_2)^{1/2})(1-(u_1\tilde{u}_2)^{1/2}) - |z|^2(v_1 v_2 \tilde{v}_1\tilde{v}_2)^{1/2}\ . \tag{5.9}$$

Notice that it is not proved yet that the right hand side of the previous inequality is non-negative.

In order to handle estimate (5.9), we shall rely on the following proposition.

Proposition 5.1. *Consider the nonnegative real numbers x_i, y_i, s_i, t_i $(i=1,2)$. Assume that:*

$$x_i \le 1,\ y_i \le 1 \qquad \textit{and} \qquad (1-x_i)(1-y_i) - s_i t_i \ge 0 \quad \textit{for} \quad i=1,2.$$

Then:

$$(1-\sqrt{x_1x_2})\,(1-\sqrt{y_1y_2}) - \sqrt{s_1s_2t_1t_2} \ge \sqrt{(1-x_1)(1-y_1)-s_1t_1}\sqrt{(1-x_2)(1-y_2)-s_2t_2}\ .$$

Proof. If $a \ge c$ (≥ 0) and $b \ge d$ (≥ 0), then:

$$\sqrt{ab} - \sqrt{cd} \ge \sqrt{a-c}\sqrt{b-d}\ .$$

To prove this, simply take the difference of the squares. Applying once this inequality yields $1-\sqrt{x_1x_2} \ge \sqrt{(1-x_1)(1-x_2)}$, hence:

$$(1-\sqrt{x_1x_2})\,(1-\sqrt{y_1y_2}) - \sqrt{s_1s_2t_1t_2} \ge \sqrt{(1-x_1)(1-x_2)(1-y_1)(1-y_2)} - \sqrt{s_1s_2t_1t_2}$$

Applying again the first inequality yields then the desired result. □

Our goal is to apply Proposition 5.1 to (5.9). The main idea, in order to fulfill assumptions of Proposition 5.1 (at least on some portions of $\mathbb{C}-\mathbb{R}^+$), is to consider the quantities of interest, i.e. $u_i, \tilde{u}_i, v_i, \tilde{v}_i$ $(i=1,2)$ as coefficients of linear systems whose determinants are the desired quantities $(1-u_i)(1-\tilde{u}_i) - |z|^2 v_i\tilde{v}_i$.

Consider the following matrices:

$$C_i(z) = \begin{pmatrix} u_i & v_i \\ |z|^2\tilde{v}_i & \tilde{u}_i \end{pmatrix}, \quad i=1,2\ .$$

The following proposition holds true:

Proposition 5.2. *Assume that $z \in \mathbb{C}-\mathbb{R}^+$. Then:*

(i) *The following holds true: $1-u_1(z) \ge 0$ and $1-\tilde{u}_1(z) \ge 0$. Moreover, there exists positive constants K, η such that:*

$$\det(I-C_1(z)) \ \ge\ K\frac{\boldsymbol{\delta}_z^8}{(\eta^2+|z|^2)^4}\ .$$

(ii) *There exist nice polynomials* Φ *and* Ψ *and a set*

$$\mathcal{E}_n = \left\{ z \in \mathbb{C}^+, \quad \frac{1}{n}\Phi(|z|)\Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right) \le 1/2 \right\} ,$$

such that for every $z \in \mathcal{E}_n$, $1 - u_2(z) \ge 0$, $1 - \tilde{u}_2(z) \ge 0$, *and*

$$\det(I - C_2) \ge K \frac{\boldsymbol{\delta}_z^8}{(\eta^2 + |z|^2)^4} ,$$

where K, η *are positive constants.*

Proof of Proposition 5.2 is postponed to Appendix B.

We are now in position to establish the following estimate:

$$\forall z \in \mathcal{E}_n, \quad \max\left(|\alpha - \delta|, |\tilde{\alpha} - \tilde{\delta}|\right) \le \frac{1}{n}\Phi(|z|)\Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right) . \tag{5.10}$$

Assume $z \in \mathcal{E}_n$. Thanks to Proposition 5.2, assumptions of Proposition 5.1 are fulfilled by $u_i, \tilde{u}_i, v_i$ and $\tilde{v}_i$, and (5.9) yields:

$$\det(I - C_0) \ \ge \ \sqrt{\det(I - C_1)}\sqrt{\det(I - C_2)} \ \ge \ K \frac{\boldsymbol{\delta}_z^8}{(\eta^2 + |z|^2)^4} , \tag{5.11}$$

where K, η are nice constants.
Solving now the system (5.4), we obtain:

$$\left\{ \begin{array}{lcl} \alpha - \delta & = & (\det(I - C_0))^{-1}\left((1 - \tilde{u}_0)\boldsymbol{\varepsilon} + z v_0 \tilde{\boldsymbol{\varepsilon}}\right) \\ \tilde{\alpha} - \tilde{\delta} & = & (\det(I - C_0))^{-1}\left((1 - u_0)\tilde{\boldsymbol{\varepsilon}} + z \tilde{v}_0 \boldsymbol{\varepsilon}\right) \end{array} \right.$$

It remains to use (5.11), Proposition 2.8-(ii), and obvious bounds over $u_0, \tilde{u}_0, v_0$ and $\tilde{v}_0$ to conclude and obtain (5.10).

We turn out to the case where $z \in \mathbb{C} - \mathbb{R}^+ - \mathcal{E}_n$, and rely on the same argument as in Haagerup and Thorbjornsen [10] (see also [7]). In this case,

$$\frac{1}{n}\Phi(|z|)\Psi(\boldsymbol{\delta}_z^{-1}) \ge \frac{1}{2} .$$

As $|\alpha - \delta| = |n^{-1}\mathrm{Tr}\, D(\mathbb{E}Q - T)| \le 2\boldsymbol{\ell}^+ \boldsymbol{d}_{\mathbf{max}} \boldsymbol{\delta}_z^{-1}$, we obtain:

$$\forall z \in \mathbb{C} - \mathbb{R}^+ - \mathcal{E}_n, \quad |\alpha - \delta| \le \frac{2\boldsymbol{\ell}^+ \boldsymbol{d}_{\mathbf{max}}}{\boldsymbol{\delta}_z} \times \frac{2\Phi(|z|)\Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right)}{n} ;$$

a similar estimate holds for $\tilde{\alpha} - \tilde{\delta}$ for $z \notin \mathcal{E}_n$. Gathering the cases where $z \in \mathcal{E}_n$ and $z \notin \mathcal{E}_n$ yields (5.3).

Appendix A. Remaining proofs for Section 2

Proof of Lemma 2.5. Note that it is sufficient to establish the result for a vector u with norm one (which is assumed in the sequel). The general result follows by considering $u/\|u\|$.

We proceed by induction over p. Let $p = 1$ and consider:

$$0 \ \le \ \mathbb{E}\sum_{j=1}^{n} \mathbb{E}_{j-1} u^* Q a_j a_j^* Q^* u \ = \ \mathbb{E} u^* Q A A^* Q^* u \ \le \ {\boldsymbol{a}_{\mathbf{max}}}^2 \mathbb{E}\|Q\|^2 .$$

As $\|Q\| \le \boldsymbol{\delta}_z^{-1}$, we obtain the desired bound.

Now, write

$$
\begin{aligned}
\mathbb{E}\left|\sum_{j=1}^n \mathbb{E}_{j-1}(u^*Qa_ja_j^*Q^*u)\right|^p &= \sum_{j_1,\cdots,j_p} \mathbb{E}\left[\mathbb{E}_{j_1-1}(u^*Qa_{j_1}a_{j_1}^*Q^*u)\cdots\mathbb{E}_{j_p-1}(u^*Qa_{j_p}a_{j_p}^*Q^*u)\right] \\
&\le p!\sum_{j_1\le\cdots\le j_p} \mathbb{E}\left[\mathbb{E}_{j_1-1}(u^*Qa_{j_1}a_{j_1}^*Q^*u)\cdots\mathbb{E}_{j_p-1}(u^*Qa_{j_p}a_{j_p}^*Q^*u)\right] \\
&= p!\sum_{j_1\le\cdots\le j_p} \mathbb{E}\Big[\mathbb{E}_{j_p-1}(u^*Qa_{j_p}a_{j_p}^*Q^*u)\underbrace{\prod_{k=1}^{p-1}\mathbb{E}_{j_k-1}(u^*Qa_{j_k}a_{j_k}^*Q^*u)}_{\mathcal{F}_{j_p-1}\ \text{measurable}}\Big] \\
&= p!\sum_{j_1\le\cdots\le j_{p-1}} \mathbb{E}\Big[\sum_{j_p=j_{p-1}}^n (u^*Qa_{j_p}a_{j_p}^*Q^*u)\prod_{k=1}^{p-1}\mathbb{E}_{j_k-1}(u^*Qa_{j_k}a_{j_k}^*Q^*u)\Big] \\
&\stackrel{(a)}{\le} p!\,\frac{\boldsymbol{a}_{\max}{}^2}{\boldsymbol{\delta}_z^2}\,\mathbb{E}\left|\sum_{j=1}^n \mathbb{E}_{j-1}(u^*Qa_ja_j^*Q^*u)\right|^{p-1} ,
\end{aligned}
$$

where (a) follows from the fact that

$$
\sum_{j_p=j_{p-1}}^n (u^*Qa_{j_p}a_{j_p}^*Q^*u) \le \sum_{j_p=1}^n (u^*Qa_{j_p}a_{j_p}^*Q^*u) \le \frac{\boldsymbol{a}_{\max}{}^2}{\boldsymbol{\delta}_z^2} \ .
$$

It remains to plug the induction assumption to conclude. Hence (2.9) is established.

In order to establish (2.10), one may use the same arguments as previously together with the identity $Q\Sigma\Sigma^* = I + zQ$, which yields the factor $|z|^p$ in estimate (2.10).

Proof of Lemma 2.6. We prove the lemma in the case where $\|u\| = 1$, the general result readily follows by considering $u/\|u\|$.

Write $u^*Q_ja_ja_j^*Q_j^*u = \chi_{1j} + \chi_{2j} + \chi_{3j} + \chi_{4j}$ with:

$$
\begin{aligned}
\chi_{1j} &= u^*(Q_j - Q)a_ja_j^*(Q_j - Q)^*u \\
\chi_{2j} &= u^*Qa_ja_j^*Q^*u \\
\chi_{3j} &= u^*(Q_j - Q)a_ja_j^*Q^*u \\
\chi_{4j} &= u^*Qa_ja_j^*(Q_j - Q)^*u
\end{aligned}
$$

Hence,

$$
\sum_{j=1}^n \mathbb{E}\left(u^*Q_ja_ja_j^*Q_j^*u\right)^2 \le \sum_{j=1}^n \mathbb{E}\chi_{1j}^2 + \sum_{j=1}^n \mathbb{E}\chi_{2j}^2 + \sum_{j=1}^n \mathbb{E}|\chi_{3j}|^2 + \sum_{j=1}^n \mathbb{E}|\chi_{4j}|^2 \ .
$$

Notice that:

$$
\mathbb{E}|\chi_{3j}|^2 \le \frac{1}{2}\left(\mathbb{E}\chi_{1j}^2 + \mathbb{E}\chi_{2j}^2\right) \quad \text{and} \quad \mathbb{E}|\chi_{4j}|^2 \le \frac{1}{2}\left(\mathbb{E}\chi_{1j}^2 + \mathbb{E}\chi_{2j}^2\right) \ .
$$

Note that using the facts that $a_ja_j^* \le AA^*$ and $\eta_j\eta_j^* \le \Sigma\Sigma^*$ together with the identity $Q\Sigma\Sigma^* = I + zQ$ yield the rough but useful estimates:

$$
u^*Qa_ja_j^*Q^*u = \mathcal{O}\left(\boldsymbol{\delta}_z^{-2}\right) \quad \text{and} \quad u^*Q\eta_j\eta_j^*Q^*u = \mathcal{O}\left(\frac{|z|}{\boldsymbol{\delta}_z^2}\right) \ . \tag{A.1}
$$

We first begin by the contribution of $\sum_j \mathbb{E}\chi_{2j}^2$:

$$\begin{aligned}
\sum_{j=1}^n \chi_{2j}^2 &= \sum_{j=1}^n u^*Qa_ja_j^*Q^*u \times u^*Qa_ja_j^*Q^*u \ , \\
&\leq \sum_{j=1}^n u^*Qa_ja_j^*Q^*u \times u^*QAA^*Q^*u \ , \\
&\leq (u^*QAA^*Q^*u)^2 \quad = \quad \mathcal{O}\left(\boldsymbol{\delta}_z^{-4}\right) \\
&\leq \Phi_2(|z|)\Psi_2\left(\frac{1}{\boldsymbol{\delta}_z}\right) \ . \qquad \text{(A.2)}
\end{aligned}$$

Similarly,

$$\sum_{j=1}^n \left(u^*Q\eta_j\eta_j^*Q^*u\right)^2 \quad = \quad \mathcal{O}\left(\frac{|z|^2}{\boldsymbol{\delta}_z^4}\right) \ . \qquad \text{(A.3)}$$

We now turn to the contribution of $\sum_j \mathbb{E}\chi_{1j}^2$. Using the decompositions (2.2) and (2.3), χ_{1j} writes:

$$\begin{aligned}
\chi_{1j} &= \left|\frac{1+\eta_j^*Q_j\eta_j}{1-\eta_j^*Q\eta_j}\right| \times \left|u^*Q\eta_j\eta_j^*Qa_ja_j^*Q^*\eta_j\eta_j^*Q^*u\right| \\
&= \left|1+\eta_j^*Q_j\eta_j\right| \times \left|u^*Q\eta_j\eta_j^*Q^*u\right| \times \left|\frac{a_j^*Q^*\eta_j\eta_j^*Qa_j}{1-\eta_j^*Q\eta_j}\right| \ . \qquad \text{(A.4)}
\end{aligned}$$

We first prove that

$$\frac{a_j^*Q^*\eta_j\eta_j^*Qa_j}{1-\eta_j^*Q\eta_j} = \mathcal{O}\left(\frac{|z|}{\boldsymbol{\delta}_z^2}\right) \ . \qquad \text{(A.5)}$$

In fact:

$$\begin{aligned}
\left|\frac{a_j^*Q^*\eta_j\eta_j^*Qa_j}{1-\eta_j^*Q\eta_j}\right| &\leq \left|\frac{a_j^*Q^*\eta_j\eta_j^*Q^*a_j}{1-\eta_j^*Q\eta_j}\right| + \left|\frac{a_j^*Q^*\eta_j\eta_j^*(Q-Q^*)a_j}{1-\eta_j^*Q\eta_j}\right| \\
&\overset{(a)}{\leq} \left|a_j^*(Q_j-Q)^*a_j\right| + 2|\mathrm{Im}(z)||a_j^*(Q_j-Q)Qa_j| \\
&= \mathcal{O}\left(\frac{1}{\boldsymbol{\delta}_z}\right) + \mathcal{O}\left(\frac{|z|}{\boldsymbol{\delta}_z^2}\right) \quad = \quad \mathcal{O}\left(\frac{|z|}{\boldsymbol{\delta}_z^2}\right) \ ,
\end{aligned}$$

where we use the fact that $Q - Q^* = 2\mathbf{i}\mathrm{Im}(z)Q^*Q$ to obtain (a). Now,

$$\left|1+\eta_j^*Q_j\eta_j\right| \leq 1 + |\Delta_j| + \left|\frac{\tilde{d}_j}{n}\mathrm{Tr}\, DQ_j + a_j^*Q_ja_j\right| \ . \qquad \text{(A.6)}$$

Since $|n^{-1}\tilde{d}_j \operatorname{Tr} DQ_j + a_j^* Q_j a_j| = \mathcal{O}(\boldsymbol{\delta}_z^{-1})$, we obtain:

$$\begin{aligned}
\sum_{j=1}^n \mathbb{E}\chi_{1j}^2 &= \left(\mathcal{O}\left(\frac{|z|^2}{\boldsymbol{\delta}_z^4}\right) + \mathcal{O}\left(\frac{|z|^2}{\boldsymbol{\delta}_z^6}\right)\right) \times \sum_{j=1}^n \mathbb{E}\left(u^* Q \eta_j \eta_j^* Q^* u\right)^2 \\
&\quad + \mathcal{O}\left(\frac{|z|^2}{\boldsymbol{\delta}_z^4}\right) \times \sum_{j=1}^n \mathbb{E}\left(u^* Q \eta_j \eta_j^* Q^* u\right)^2 \times |\Delta_j|^2 \\
&\overset{(a)}{=} \mathcal{O}\left(\frac{|z|^4}{\boldsymbol{\delta}_z^8}\right) + \mathcal{O}\left(\frac{|z|^4}{\boldsymbol{\delta}_z^{10}}\right) + \mathcal{O}\left(\frac{|z|^4}{\boldsymbol{\delta}_z^8}\right) \times \sum_{j=1}^n \mathbb{E}\,|\Delta_j|^2 \\
&\overset{(b)}{=} \mathcal{O}\left(\frac{|z|^4}{\boldsymbol{\delta}_z^8}\right) + \mathcal{O}\left(\frac{|z|^4}{\boldsymbol{\delta}_z^{10}}\right) \\
&\leq \Phi_1(|z|)\Psi_1\left(\frac{1}{\boldsymbol{\delta}_z}\right),
\end{aligned}$$

where (a) follows from (A.3) and (A.1) and (b), from Corollary 2.2.

It remains to gather the contributions of $\chi_{1j}, \chi_{2j}, \chi_{3j}$ and χ_{4j} to get:

$$\sum_{j=1}^n \mathbb{E}\left(u^* Q_j a_j a_j^* Q_j^* u\right)^2 \leq 2\Phi_1(|z|)\Psi_1\left(\frac{1}{\boldsymbol{\delta}_z}\right) + 2\Phi_2(|z|)\Psi_2\left(\frac{1}{\boldsymbol{\delta}_z}\right) \overset{(a)}{\leq} \Phi(|z|)\,\Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right),$$

where (a) follows from (2.7). Eq. (2.11) is proved.

In order to prove (2.12), first note that:

$$\begin{aligned}
&\mathbb{E}\left(\sum_{j=1}^n \mathbb{E}_{j-1}\left(u^* Q_j a_j a_j^* Q_j^* u\right)\right)^p \\
&\leq K\left(\mathbb{E}\left|\sum_{j=1}^n \mathbb{E}_{j-1}\chi_{1j}\right|^p + \mathbb{E}\left|\sum_{j=1}^n \mathbb{E}_{j-1}\chi_{2j}\right|^p + \mathbb{E}\left|\sum_{j=1}^n \mathbb{E}_{j-1}\chi_{3j}\right|^p + \mathbb{E}\left|\sum_{j=1}^n \mathbb{E}_{j-1}\chi_{4j}\right|^p\right).
\end{aligned}$$

Hence, it remains to evaluate the contributions of each term. Using decomposition (A.4) together with the estimate (A.5), we obtain:

$$\mathbb{E}\left|\sum_{j=1}^n \mathbb{E}_{j-1}\chi_{1j}\right|^p = \mathcal{O}\left(\frac{|z|^p}{\boldsymbol{\delta}_z^{2p}}\right) \times \mathbb{E}\left(\sum_{j=1}^n \mathbb{E}_{j-1}|1 + \eta_j^* Q_j \eta_j| \times u^* Q \eta_j \eta_j^* Q^* u\right)^p.$$

Using (A.6) together with (2.10) yields:

$$\mathbb{E}\left|\sum_{j=1}^n \mathbb{E}_{j-1}\chi_{1j}\right|^p = \mathcal{O}\left(\frac{|z|^{2p}}{\boldsymbol{\delta}_z^{4p}}\right) + \mathcal{O}\left(\frac{|z|^{2p}}{\boldsymbol{\delta}_z^{5p}}\right) + \mathcal{O}\left(\frac{|z|^p}{\boldsymbol{\delta}_z^{2p}}\right) \times \mathbb{E}\left|\sum_{j=1}^n \mathbb{E}_{j-1}\left(|\Delta_j| \times u^* Q \eta_j \eta_j^* Q^* u\right)\right|^p.$$

Combining standard inequalities (Cauchy-Schwarz, $|\sum_j a_j b_j| \leq (\sum_j a_j^2)^{1/2}(\sum_j b_j^2)^{1/2}$, and Cauchy-Schwarz again), we obtain:

$$\begin{aligned}
&\mathbb{E}\left(\sum_{j=1}^n \mathbb{E}_{j-1}\left(|\Delta_j| \times u^* Q \eta_j \eta_j^* Q^* u\right)\right)^p \\
&\leq \left[\mathbb{E}\left(\sum_{j=1}^n \mathbb{E}_{j-1}(u^* Q \eta_j \eta_j^* Q^* u)^2\right)^p \times \mathbb{E}\left(\sum_{j=1}^n \mathbb{E}_{j-1}|\Delta_j|^2\right)^p\right]^{1/2} \overset{(a)}{=} \mathcal{O}\left(\frac{|z|^p}{\boldsymbol{\delta}_z^{3p}}\right),
\end{aligned}$$

where (a) follows from (A.1), Corollary 2.2 and (2.10). Finally,

$$\mathbb{E}\left|\sum_{j=1}^{n}\mathbb{E}_{j-1}\chi_{1j}\right|^{p} \;=\; \mathcal{O}\left(\frac{|z|^{2p}}{\boldsymbol{\delta}_z^{4p}}\right)+\mathcal{O}\left(\frac{|z|^{2p}}{\boldsymbol{\delta}_z^{5p}}\right)+\mathcal{O}\left(\frac{|z|^{2p}}{\boldsymbol{\delta}_z^{5p}}\right) \;\le\; \Phi_1(|z|)\Psi_1(\boldsymbol{\delta}_z^{-1})\ . \tag{A.7}$$

Eq. (2.9) directly yields the estimate:

$$\mathbb{E}\left|\sum_{j=1}^{n}\mathbb{E}_{j-1}\chi_{2j}\right|^{p}=\mathcal{O}\left(\frac{1}{\boldsymbol{\delta}_z^{2p}}\right)\le\Phi_2(|z|)\Psi_2(\boldsymbol{\delta}_z^{-1})\ . \tag{A.8}$$

Finally,

$$\mathbb{E}\left|\sum_{j=1}^{n}\mathbb{E}_{j-1}\chi_{3j}\right|^{p} \;\le\; \left(\mathbb{E}\left|\sum_{j=1}^{n}\mathbb{E}_{j-1}\chi_{1j}\right|^{p}\mathbb{E}\left|\sum_{j=1}^{n}\mathbb{E}_{j-1}\chi_{2j}\right|^{p}\right)^{1/2} \;\le\; \Phi_3(|z|)\Psi_3\left(\frac{1}{\boldsymbol{\delta}_z}\right)\ . \tag{A.9}$$

A corresponding inequality exists for $\mathbb{E}|\sum\mathbb{E}_{j-1}\chi_{4j}|^{p}$: obtain:

$$\mathbb{E}\left|\sum_{j=1}^{n}\mathbb{E}_{j-1}\chi_{4j}\right|^{p} \;\le\; \Phi_4(|z|)\Psi_4\left(\frac{1}{\boldsymbol{\delta}_z}\right)\ . \tag{A.10}$$

Gathering (A.7), (A.8), (A.9) and (A.10), we end up with (2.12), and Lemma 2.6 is proved.

Appendix B. Remaining proofs for Section 5

Proof of Proposition 5.2-(i). Recall that $\delta=\frac{1}{n}\operatorname{Tr}DT$ and $\tilde\delta=\frac{1}{n}\operatorname{Tr}\tilde D\tilde T$. We consider first the case where $z\in\mathbb{C}^+\cup\mathbb{C}^-$. We have

$$\operatorname{Im}(\delta)=\frac{1}{2\mathbf{i}n}\operatorname{Tr}DT(T^{-*}-T^{-1})T^{*}\quad\text{and}\quad\operatorname{Im}(z\tilde\delta)=\frac{1}{2\mathbf{i}n}\operatorname{Tr}\tilde D(z\tilde T)\left[(z\tilde T)^{-*}-(z\tilde T)^{-1}\right](z\tilde T)^{*}\ .$$

Developing the previous identities, we end up with the system:

$$(I-C_1)\begin{pmatrix}\operatorname{Im}(\delta)\\ \operatorname{Im}(z\tilde\delta)\end{pmatrix}=\operatorname{Im}(z)\ \begin{pmatrix}w_1(z)\\ \tilde x_1(z)\end{pmatrix} \tag{B.1}$$

where

$$\left\{\begin{array}{lcll} w_1(z) &=& \frac{1}{n}\operatorname{Tr}DTT^{*} & (>0)\\ \tilde x_1(z) &=& \frac{1}{n}\operatorname{Tr}\tilde D\tilde T A^{*}(I+\tilde\delta D)^{-1}(I+\tilde\delta^{*}D)^{-1}A\tilde T^{*} & (>0)\end{array}\right.\ .$$

By developing the first equation of this system, and by recalling that $\delta(z)$ is the Stieltjes transform of a positive measure μ_n with support included in $\mathbb{R}^+$, we obtain

$$1-u_1=w_1\frac{\operatorname{Im}(z)}{\operatorname{Im}(\delta)}+v_1\frac{\operatorname{Im}(z\tilde\delta)}{\operatorname{Im}(\delta)}\ge w_1\frac{\operatorname{Im}(z)}{\operatorname{Im}(\delta)}\ge 0\ .$$

Replacing $(\operatorname{Im}(\delta),\operatorname{Im}(z\tilde\delta))$ with $(\operatorname{Im}(\tilde\delta),\operatorname{Im}(z\delta))$ and repeating the same argument, we obtain

$$1-\tilde u_1=\tilde w_1\frac{\operatorname{Im}(z)}{\operatorname{Im}(\tilde\delta)}+\tilde v_1\frac{\operatorname{Im}(z\delta)}{\operatorname{Im}(\tilde\delta)}\ge\tilde w_1\frac{\operatorname{Im}(z)}{\operatorname{Im}(\tilde\delta)}\ge 0\ .$$

By continuity of $u_1(z)$ and $\tilde u_1(z)$ at any point of the open real negative axis, we have $1-u_1\ge 0$ and $1-\tilde u_1\ge 0$ for any $z\in\mathbb{C}-\mathbb{R}^+$. The first two inequalities in the statement of Proposition 5.2-(i) are proven.

By applying Cramer's rule ([13, Sec. 0.8.3]) where the first column of $I - C_1$ is replaced with the right hand member of (B.1), we obtain

$$\det(I-C_1) = (1-\tilde{u}_1)w_1\frac{\mathrm{Im}(z)}{\mathrm{Im}(\delta)} + v_1\tilde{x}_1\frac{\mathrm{Im}(z)}{\mathrm{Im}(\delta)} \geq (1-\tilde{u}_1)w_1\frac{\mathrm{Im}(z)}{\mathrm{Im}(\delta)} \geq w_1\tilde{w}_1\frac{\mathrm{Im}(z)}{\mathrm{Im}(\delta)}\frac{\mathrm{Im}(z)}{\mathrm{Im}(\tilde{\delta})}. \quad \text{(B.2)}$$

Using the fact that the positive measure μ_n is supported by $\mathbb{R}^+$ and has a total mass $n^{-1}\mathrm{Tr}\, D$, we have

$$0 \leq \frac{\mathrm{Im}(\delta)}{\mathrm{Im}(z)} = \int \frac{1}{|t-z|^2}\mu_n(dt) \leq \frac{1}{\boldsymbol{\delta}_z^2}\frac{1}{n}\mathrm{Tr}\, D \leq \frac{\boldsymbol{\ell}^+\boldsymbol{d}_{\mathbf{max}}}{\boldsymbol{\delta}_z^2}, \quad \text{and} \quad 0 \leq \frac{\mathrm{Im}(\tilde{\delta})}{\mathrm{Im}(z)} \leq \frac{\tilde{\boldsymbol{d}}_{\mathbf{max}}}{\boldsymbol{\delta}_z^2}\ . \quad \text{(B.3)}$$

In order to find a lower bound on w_1 and $\tilde{w}_1$, we begin by finding a lower bound on $|\delta|$. A computation similar to [12, Lemma C.1] shows that the sequence of measures (μ_n) is tight. Hence there exists $\eta > 0$ such that:

$$\mu_n[0,\eta] \geq \frac{1}{2}\ \frac{1}{n}\mathrm{Tr}\, D \geq \frac{\boldsymbol{\ell}^-\boldsymbol{d}_{\mathbf{min}}}{2}\ .$$

We have

$$|\delta| \geq |\,\mathrm{Im}(\delta)| \ = \ |\,\mathrm{Im}(z)|\int\frac{\mu_n(dt)}{|t-z|^2} \ \geq \ |\,\mathrm{Im}(z)|\int_0^\eta \frac{\mu_n(dt)}{2(t^2+|z|^2)} \geq |\,\mathrm{Im}(z)|\frac{\boldsymbol{\ell}^-\boldsymbol{d}_{\mathbf{min}}}{4(\eta^2+|z|^2)}\ . \quad \text{(B.4)}$$

Furthermore, when $\mathrm{Re}(z) < 0$, we have

$$|\delta| \geq \mathrm{Re}(\delta) \ = \ \int\frac{t-\mathrm{Re}(z)}{|t-z|^2}\mu_n(dt) \geq -\mathrm{Re}(z)\int\frac{\mu_n(dt)}{|t-z|^2} \geq -\mathrm{Re}(z)\frac{\boldsymbol{\ell}^-\boldsymbol{d}_{\mathbf{min}}}{4(\eta^2+|z|^2)}\ .$$

which results in

$$|\delta| \geq \boldsymbol{\delta}_z\frac{\boldsymbol{\ell}^-\boldsymbol{d}_{\mathbf{min}}}{4(\eta^2+|z|^2)}\ .$$

We can now find a lower bound to w_1:

$$\begin{aligned}
w_1 \ &= \ \frac{1}{n}\mathrm{Tr}\, DTT^* \ = \ \frac{1}{n}\sum_{i=1}^N d_i\sum_{j=1}^N |T_{ij}|^2 \ = \ \frac{1}{n}\mathrm{Tr}\, D\sum_{i=1}^N \kappa_i\sum_{j=1}^N |T_{ij}|^2 \qquad \text{with} \quad \kappa_i = \frac{d_i}{\mathrm{Tr}\, D} \\
&\overset{(a)}{\geq} \ \frac{1}{n}\mathrm{Tr}\, D\left(\sum_{i=1}^N\kappa_i\left(\sum_{j=1}^N|T_{ij}|^2\right)^{1/2}\right)^2 \ \geq \ \frac{1}{n}\mathrm{Tr}\, D\left(\sum_{i=1}^N\kappa_i|T_{ii}|\right)^2 \ \geq \ \frac{1}{n}\mathrm{Tr}\, D\left|\sum_{i=1}^N\kappa_i T_{ii}\right|^2 \\
&= \ \frac{|\delta|^2}{\frac{1}{n}\mathrm{Tr}\, D} \ \geq \ \frac{(\boldsymbol{\delta}_z\boldsymbol{\ell}^-\boldsymbol{d}_{\mathbf{min}})^2}{16\ \boldsymbol{\ell}^+\boldsymbol{d}_{\mathbf{max}}(\eta^2+|z|^2)^2}
\end{aligned}$$

where (a) follows by convexity. A similar computation yields $\tilde{w}_1 \geq (\boldsymbol{\delta}_z\tilde{\boldsymbol{d}}_{\mathbf{min}})^2/(16\ \tilde{\boldsymbol{d}}_{\mathbf{max}}(\tilde{\eta}^2 + |z|^2)^2)$ where $\tilde{\eta}$ is a positive constant. Grouping these estimates with those in (B.3) and plugging them into (B.2), we obtain

$$\begin{aligned}
\det(I-C_1) &\geq \frac{\boldsymbol{\delta}_z^8\ (\boldsymbol{\ell}^-\tilde{\boldsymbol{d}}_{\mathbf{min}}\tilde{\boldsymbol{d}}_{\mathbf{min}})^2}{256\ (\boldsymbol{\ell}^+\boldsymbol{d}_{\mathbf{max}}\tilde{\boldsymbol{d}}_{\mathbf{max}})^2(\eta^2+|z|^2)^2(\tilde{\eta}^2+|z|^2)^2} \\
&\geq K\frac{\boldsymbol{\delta}_z^8}{(\max(\eta,\tilde{\eta})^2+|z|^2)^4}
\end{aligned}$$

where K is a nice constant.
The same bound holds for $z \in (-\infty, 0)$ by continuity of $\det(I - C_1(z))$ at any point of the open real negative axis.

Proof of Proposition 5.2-(ii). Recall that

$$\boldsymbol{\varepsilon}_n = \frac{1}{n} \operatorname{Tr} D(\mathbb{E}Q - R) \ .$$

We first establish useful estimates.

Lemma B.1. *There exists nice polynomials Φ and Ψ such that:*

$$\left| \frac{\operatorname{Im}(\boldsymbol{\varepsilon}_n(z))}{\operatorname{Im}(z)} \right| \le \frac{1}{n} \Phi(|z|) \Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right) \ \text{ and } \ \left| \frac{\operatorname{Im}(z\boldsymbol{\varepsilon}_n(z))}{\operatorname{Im}(z)} \right| \le \frac{1}{n} \Phi(|z|) \Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right) \quad \text{for } z \in \mathbb{C} - \mathbb{R}^+ \ .$$

Proof. We prove the first inequality. By Proposition (2.8)-(ii), the sequence of functions $(\boldsymbol{\varepsilon}_n)$ satisfies over $\mathbb{C} - \mathbb{R}_+$

$$|\boldsymbol{\varepsilon}_n(z)| \le \frac{1}{n} \Phi(|z|) \Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right)$$

where Φ and Ψ are nice polynomials. Let $\mathcal{R}$ be the region of the complex plane defined as $\mathcal{R} = \{z \ : \ \operatorname{Re}(z) < 0, \ |\operatorname{Im}(z)| < -\operatorname{Re}(z)/2\}$. If $z \in \mathbb{C} - \mathbb{R}^+ - \mathcal{R}$, then $|\operatorname{Im}(z)| \ge \boldsymbol{\delta}_z/\sqrt{5}$, therefore $|\operatorname{Im} \boldsymbol{\varepsilon}(z)/\operatorname{Im} z| \le n^{-1}\sqrt{5}\boldsymbol{\delta}_z^{-1}\Phi(|z|)\Psi(\boldsymbol{\delta}_z^{-1})$ and the result is proven. Assume now that $z \in \mathcal{R}$. In this case, z belongs to the open disc $\mathcal{D}_z$ centered at $\operatorname{Re}(z)$ with radius $-\operatorname{Re}(z)/2$. For any $u \in \mathcal{D}_z$, we have $|\boldsymbol{\varepsilon}(u)| \le n^{-1}\Phi(|u|)\Psi(|u|^{-1})$. Moreover,

$$\forall u \in \mathcal{D}_z, \quad \frac{\boldsymbol{\delta}_z}{\sqrt{5}} \le -\frac{\operatorname{Re}(z)}{2} \le |u| \le -\frac{3\operatorname{Re}(z)}{2} \le \frac{3|z|}{2} \ .$$

As $\Phi(x)$ is increasing and $\Psi(1/x)$ is decreasing in $x > 0$, we obtain:

$$|\boldsymbol{\varepsilon}_n(u)| \le \frac{1}{n} \Phi\left(\frac{3|z|}{2}\right) \Psi\left(\frac{\sqrt{5}}{\boldsymbol{\delta}_z}\right) \quad \text{for} \quad u \in \mathcal{D}_z \ . \tag{B.5}$$

The function $\boldsymbol{\varepsilon}$ is holomorphic on $\mathcal{D}_z$. Consider the function: Applying Lemma 2.4 with

$$f(\zeta) = \frac{\boldsymbol{\varepsilon}\left(|\operatorname{Re}(z)/2|\zeta + \operatorname{Re}(z)\right) - \boldsymbol{\varepsilon}(\operatorname{Re}(z))}{\sup_{u \in \mathcal{D}_z} |\boldsymbol{\varepsilon}(u) - \boldsymbol{\varepsilon}(\operatorname{Re}(z))|} \ .$$

Let $\zeta = \mathbf{i}2\operatorname{Im}(z)/\operatorname{Re}(z)$, apply Lemma 2.4, and use (B.5). This yields:

$$|\boldsymbol{\varepsilon}(z) - \boldsymbol{\varepsilon}(\operatorname{Re}(z))| \le \frac{2|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|} \times \frac{1}{n} \Phi\left(|z|\right) \Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right) \le \frac{\sqrt{5}|\operatorname{Im}(z)|}{\boldsymbol{\delta}_z} \times \frac{1}{n} \Phi\left(|z|\right) \Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right) \ ,$$

where Φ and Ψ are nice polynomials. As $\operatorname{Im}(\boldsymbol{\varepsilon}(\operatorname{Re}(z))) = 0$, we obtain

$$\left| \frac{\operatorname{Im}(\boldsymbol{\varepsilon}_n(z))}{\operatorname{Im}(z)} \right| \le \left| \frac{\boldsymbol{\varepsilon}(z) - \boldsymbol{\varepsilon}(\operatorname{Re}(z))}{\operatorname{Im}(z)} \right| \le \frac{\sqrt{5}}{\boldsymbol{\delta}_z n} \Phi\left(|z|\right) \Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right) \ .$$

This proves the first inequality. The second one can be proved similarly. □

We now tackle the proof of Proposition 5.2-(ii), following closely the line of the proof of Proposition 5.2-(i). Recall that $\alpha = \frac{1}{n}\operatorname{Tr} D\mathbb{E}Q$, $\tilde{\alpha} = \frac{1}{n}\operatorname{Tr}\tilde{D}\mathbb{E}\tilde{Q}$, $\boldsymbol{\varepsilon} = \frac{1}{n}\operatorname{Tr} D(\mathbb{E}Q - R)$, and $\tilde{\boldsymbol{\varepsilon}} = \frac{1}{n}\operatorname{Tr}\tilde{D}(\mathbb{E}\tilde{Q} - \tilde{R})$. We begin by establishing the lower bound on $\det(I - C_2)$. Assume that $z \in \mathbb{C}^+ \cup \mathbb{C}^-$. Writing $\alpha = \frac{1}{n}\mathrm{Tr}\, DR + \boldsymbol{\varepsilon}$ and $\tilde{\alpha} = \frac{1}{n}\mathrm{Tr}\, \tilde{D}\tilde{R} + \tilde{\boldsymbol{\varepsilon}}$ and developing $\mathrm{Im}(\alpha)$ and $\mathrm{Im}(z\tilde{\alpha})$ with the help of the resolvent identity, we get the following system:

$$(I - C_2)\begin{pmatrix} \mathrm{Im}(\alpha) \\ \mathrm{Im}(z\tilde{\alpha}) \end{pmatrix} = \mathrm{Im}(z)\begin{pmatrix} w_2(z) \\ \tilde{x}_2(z) \end{pmatrix} + \begin{pmatrix} \mathrm{Im}(\boldsymbol{\varepsilon}) \\ \mathrm{Im}(z\tilde{\boldsymbol{\varepsilon}}) \end{pmatrix} ,$$

where $w_2(z) = \frac{1}{n}\mathrm{Tr}\, DRR^*$ and $\tilde{x}_2(z) > 0$. By developing the first equation of this system, and by recalling that $\alpha_n(z)$ is the Stieltjes transform of a positive measure μ_n with support included in $\mathbb{R}^+$, we obtain

$$1 - u_2 = w_2 \frac{\mathrm{Im}(z)}{\mathrm{Im}(\alpha)} + v_2 \frac{\mathrm{Im}(z\tilde{\alpha})}{\mathrm{Im}(\alpha)} + \frac{\mathrm{Im}(\boldsymbol{\varepsilon})}{\mathrm{Im}(\alpha)} \geq w_2 \frac{\mathrm{Im}(z)}{\mathrm{Im}(\alpha)} + \frac{\mathrm{Im}(\boldsymbol{\varepsilon})}{\mathrm{Im}(\alpha)} . \tag{B.6}$$

Similarly,

$$1 - \tilde{u}_2 = \tilde{w}_2 \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tilde{\alpha})} + \tilde{v}_2 \frac{\mathrm{Im}(z\alpha)}{\mathrm{Im}(\tilde{\alpha})} + \frac{\mathrm{Im}(\tilde{\boldsymbol{\varepsilon}})}{\mathrm{Im}(\tilde{\alpha})} \geq \tilde{w}_2 \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tilde{\alpha})} + \frac{\mathrm{Im}(\tilde{\boldsymbol{\varepsilon}})}{\mathrm{Im}(\tilde{\alpha})} . \tag{B.7}$$

where $\tilde{w}_2 = n^{-1}\mathrm{Tr}\, \tilde{D}\tilde{R}\tilde{R}^*$. By Cramer's rule,

$$\begin{aligned} \det(I - C_2) &= (1 - \tilde{u}_2) w_2 \frac{\mathrm{Im}(z)}{\mathrm{Im}(\alpha)} + v_2 \tilde{x}_2 \frac{\mathrm{Im}(z)}{\mathrm{Im}(\alpha)} + (1 - \tilde{u}_2)\frac{\mathrm{Im}(\boldsymbol{\varepsilon})}{\mathrm{Im}(\alpha)} + v_2 \frac{\mathrm{Im}(z\tilde{\boldsymbol{\varepsilon}})}{\mathrm{Im}(\alpha)} \\ &\geq w_2 \tilde{w}_2 \frac{\mathrm{Im}(z)}{\mathrm{Im}(\alpha)} \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tilde{\alpha})} + w_2 \frac{\mathrm{Im}(z)}{\mathrm{Im}(\alpha)} \frac{\mathrm{Im}(\tilde{\boldsymbol{\varepsilon}})}{\mathrm{Im}(\tilde{\alpha})} + (1 - \tilde{u}_2)\frac{\mathrm{Im}(\boldsymbol{\varepsilon})}{\mathrm{Im}(\alpha)} + v_2 \frac{\mathrm{Im}(z\tilde{\boldsymbol{\varepsilon}})}{\mathrm{Im}(\alpha)} \\ &\triangleq w_2 \tilde{w}_2 \frac{\mathrm{Im}(z)}{\mathrm{Im}(\alpha)} \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tilde{\alpha})} + e(z) . \end{aligned} \tag{B.8}$$

We now find an upper bound on the perturbation term $e(z)$. To this end, we have $0 \leq w_2 \leq \boldsymbol{\ell}^+ \boldsymbol{d}_{\mathbf{max}} / \boldsymbol{\delta}_z^2$ and $0 \leq v_2 \leq \boldsymbol{\ell}^+ \boldsymbol{d}_{\mathbf{max}}^2 / \boldsymbol{\delta}_z^2$. Recalling (5.8), we also have

$$|1 - \tilde{u}_2| \leq 1 + \frac{\boldsymbol{d}_{\mathbf{max}} \tilde{\boldsymbol{d}}_{\mathbf{max}} \boldsymbol{a}_{\mathbf{max}}^2 |z|^2}{\boldsymbol{\delta}_z^4} .$$

It has been proven in [12, Lemma C.1] that the sequence of positive measures (μ_n) with total mass $n^{-1}\mathrm{Tr}\,(D)$ is tight. In these conditions, a computation similar to (B.4) shows that for every $z \in \mathbb{C}^+ \cup \mathbb{C}^-$,

$$\frac{\mathrm{Im}(z)}{\mathrm{Im}(\alpha)} \leq \frac{4(\eta^2 + |z|^2)}{\boldsymbol{\ell}^- \boldsymbol{d}_{\mathbf{min}}} ,$$

where η is a positive constant. Combining these estimates with the result of Lemma B.1, we obtain that $|e(z)| \leq n^{-1}\Phi(|z|)\Psi(\boldsymbol{\delta}_z^{-1})$ where Φ and Ψ are nice polynomials. Considering now the first term at the right hand side of (B.8), an argument similar to the one made in the proof of Proposition 5.2-(i) (involving this time the tightness of the measures associated with the Stieltjes transforms $\frac{1}{n}\mathrm{Tr}\, DR$ and $\frac{1}{n}\mathrm{Tr}\, \tilde{D}\tilde{R}$) shows that

$$w_2 \tilde{w}_2 \frac{\mathrm{Im}(z)}{\mathrm{Im}(\alpha)} \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tilde{\alpha})} \geq K \frac{\boldsymbol{\delta}_z^8}{(\eta^2 + |z|^2)^4} ,$$

where K and η are nice constants. Finally, we can state that there exist nice polynomials Φ and Ψ such that:

$$\det(I - C_2) \ \geq K \frac{\boldsymbol{\delta}_z^8}{(\eta^2 + |z|^2)^4} \left(1 - \frac{1}{n}\Phi(|z|)\Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right)\right) .$$

By continuity of $\det(I - C_2(z))$ at any point of the open real negative axis, this inequality is true for any $z \in \mathbb{C} - \mathbb{R}^+$. Denote by $\mathcal{E}_n$ the set:

$$\mathcal{E}_n = \left\{ z \in \mathbb{C} - \mathbb{R}^+, \quad \frac{1}{n}\Phi(|z|)\Psi\left(\frac{1}{\boldsymbol{\delta}_z}\right) \leq 1/2 \right\} .$$

If $z \in \mathcal{E}_n$, then $\det(I - C_2)$ is readily lower-bounded by the quantity stated in Proposition 5.2-(ii).
By considering inequalities (B.6) and (B.7) and by possibly modifying the polynomials Φ and Ψ, we have $1 - u_2 \geq 0$ and $1 - \tilde{u}_2 \geq 0$ for $z \in \mathcal{E}_n$. The proof of proposition 5.2-(ii) is completed.

References

[1] C. Artigue and P. Loubaton. On the ergodic capacity and precoder design of flat fading MIMO systems equipped with MMSE receivers. In *Proceedings of the IEEE International Symposium on Information Theory*, pages 1095–1099, june 2009.

[2] C. Artigue and P. Loubaton. On the precoder design of flat fading MIMO systems equipped with MMSE receivers: a large system approach. *submitted to IEEE Inf. Th.*, 2009. available on `http://front.math.ucdavis.edu/0911.0351`.

[3] Z. D. Bai, B. Q. Miao, and G. M. Pan. On asymptotics of eigenvectors of large sample covariance matrix. *Ann. Probab.*, 35(4):1532–1572, 2007.

[4] Z. D. Bai and J. W. Silverstein. No eigenvalues outside the support of the limiting spectral distribution of large-dimensional sample covariance matrices. *Ann. Probab.*, 26(1):316–345, 1998.

[5] Z. D. Bai and J. W. Silverstein. Exact separation of eigenvalues of large-dimensional sample covariance matrices. *Ann. Probab.*, 27(3):1536–1555, 1999.

[6] F. Benaych-Georges and R.N. Rao. The eigenvalues and eigenvectors of finite, low rank perturbations of large random matrices (preprint). [online] `arXiv:math.PR/0910.2120`, 2009.

[7] M. Capitaine, C. Donati-Martin, and D. Féral. The largest eigenvalues of finite rank deformation of large Wigner matrices: convergence and nonuniversality of the fluctuations. *Ann. Probab.*, 37(1):1–47, 2009.

[8] R. Brent Dozier and J. W. Silverstein. On the empirical distribution of eigenvalues of large dimensional information-plus-noise-type matrices. *J. Multivariate Anal.*, 98(4):678–694, 2007.

[9] V.L. Girko. *An introduction to statistical analysis of random arrays*. VSP, 1998.

[10] U. Haagerup and S. Thorbjørnsen. A new application of random matrices: $\mathrm{Ext}(C^*_{\mathrm{red}}(F_2))$ is not a group. *Ann. of Math. (2)*, 162(2):711–775, 2005.

[11] W. Hachem, P. Loubaton, and J. Najim. The empirical distribution of the eigenvalues of a Gram matrix with a given variance profile. *Ann. Inst. H. Poincaré Probab. Statist.*, 42(6):649–670, 2006.

[12] W. Hachem, P. Loubaton, and J. Najim. Deterministic equivalents for certain functionals of large random matrices. *Ann. Appl. Probab.*, 17(3):875–930, 2007.

[13] R. A. Horn and C. R. Johnson. *Topics in matrix analysis*. Cambridge University Press, Cambridge, 1994.

[14] A. Kammoun, M. Kharouf, W. Hachem, J. Najim, and A. El Kharroubi. On the Fluctuations of the Mutual Information for Non Centered MIMO Channels: The Non Gaussian Case. In *the Proceedings of the IEEE International Workshop on Signal Processing Advances in Wireless Communications (SPAWC)*, Marrakech, Morocco, 2010.

[15] V. A. Marčenko and L. A. Pastur. Distribution of eigenvalues in certain sets of random matrices. *Mat. Sb. (N.S.)*, 72 (114):507–536, 1967.

[16] X. Mestre. Improved Estimation of Eigenvalues and Eigenvectors of Covariance Matrices using their Sample Estimates. *IEEE Trans. Inf. Theory*, 54(11):5113–5129, Nov. 2008.

[17] X. Mestre. On the asymptotic behavior of the sample estimates of eigenvalues and eigenvectors of covariance matrices. *IEEE Trans. Signal Process.*, 56(11):5353–5368, 2008.

[18] W. Rudin. *Real and Complex Analysis*. McGraw-Hill, 3rd edition, 1986.

[19] J. W. Silverstein. Strong convergence of the empirical distribution of eigenvalues of large-dimensional random matrices. *J. Multivariate Anal.*, 55(2):331–339, 1995.

[20] J. W. Silverstein and Z. D. Bai. On the empirical distribution of eigenvalues of a class of large-dimensional random matrices. *J. Multivariate Anal.*, 54(2):175–192, 1995.

[21] P. Vallet, P. Loubaton, and X. Mestre. Improved subspace DoA estimation methods with large arrays: The deterministic signals case. In *Proceedings of IEEE International Conference on Acoustics, Speech and Signal Processing.*, pages 2137–2140, 2009.

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