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ASYMPTOTIC INDEPENDENCE IN THE SPECTRUM OF THE GAUSSIAN UNITARY ENSEMBLE

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Abstract

Consider a nn matrix from the Gaussian Unitary Ensemble (GUE). Given a finite collection of bounded disjoint real Borel sets $(\Delta_{i,n},\ 1 \le i \le p)$ with positive distance from one another, eventually included in any neighbourhood of the support of Wigner's semi-circle law and properly rescaled (with respective lengths n^{-1} in the bulk and $n^{-2/3}$ around the edges), we prove that the related counting measures $\mathcal{N}_n(\Delta_{i,n}), (1 \le i \le p)$, where $\mathcal{N}_n(\Delta)$ represents the number of eigenvalues within Δ, are asymptotically independent as the size n goes to infinity, p being fixed. As a consequence, we prove that the largest and smallest eigenvalues, properly centered and rescaled, are asymptotically independent; we finally describe the fluctuations of the ratio of the extreme eigenvalues of a matrix from the GUE.

1 Introduction and main result

Denote by $\mathcal{H}_n$ the set of nn random Hermitian matrices endowed with the probability measure

$$P_n(d\,\mathbf{M}) := Z_n^{-1} \exp\left\{-\frac{n}{2}\,\mathrm{Tr}\,(\mathbf{M})^2\right\} d\,\mathbf{M}\,,$$

where Z_n is the normalization constant and where

$$d\,\mathbf{M} = \prod_{i=1}^{n} d\,M_{ii} \prod_{1\le i<j\le n} \Re\left[d\,M_{ij}\right] \prod_{1\le i<j\le n} \Im\left[d\,M_{ij}\right]$$

for every $\mathbf{M} = (M_{ij})_{1\leq i,j\leq n}$ in $\mathscr{H}_n$ ($\Re[z]$ being the real part of $z \in \mathbb{C}$ and $\Im[z]$ its imaginary part). This set is known as the Gaussian Unitary Ensemble (GUE) and corresponds to the case where a nn Hermitian matrix $\mathbf{M}$ has independent, complex, zero mean, Gaussian distributed entries with variance $\mathbb{E}|M_{ij}|^2 = \frac{1}{n}$ above the diagonal while the diagonal entries are independent real Gaussian with the same variance. Much is known about the spectrum of $\mathbf{M}$. Denote by $\lambda_1^{(n)}, \lambda_2^{(n)}, \cdots, \lambda_n^{(n)}$ the eigenvalues of $\mathbf{M}$ (all distinct with probability one), then:

- [1] The joint probability density function of the (unordered) eigenvalues $(\lambda_1^{(n)}, \cdots, \lambda_n^{(n)})$ is given by

$$p_n(x_1, \cdots, x_n) = C_n e^{-\frac{n\sum x_i^2}{2}} \prod_{j<k} |x_j - x_k|^2 ,$$

  where C_n is the normalization constant.

- [19] The empirical distribution of the eigenvalues $\frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i^{(n)}}$ (δ_x stands for the Dirac measure at point x) converges toward Wigner's semi-circle law as $n \to \infty$, whose density is:

$$\frac{1}{2\pi}\mathbf{1}_{(-2,2)}(x)\sqrt{4-x^2} .$$

  Fluctuations of linear statistics of the eigenvalues of large random matrices (and of the GUE in particular) have also been extensively addressed in the literature, see for instance [2, 9] and the references therein; for a determinantal point of view, one can refer to [15].

- [3] The largest eigenvalue $\lambda_{\max}^{(n)}$ (resp. the smallest eigenvalue $\lambda_{\min}^{(n)}$) almost surely converges to 2 (resp. -2), the right-end (resp. left-end) point of the support of the semi-circle law as $n \to \infty$.

- [16] The centered and rescaled quantity $n^{\frac{2}{3}}\left(\lambda_{\max}^{(n)} - 2\right)$ converges in distribution toward Tracy-Widom distribution function F_{GUE}^{+} as $n \to \infty$, which can be defined in the following way

$$F_{GUE}^{+}(s) = \exp\left(-\int_s^\infty (x-s)q^2(x)\,dx\right) ,$$

  where q solves the Painlevé II differential equation

$$\begin{aligned} &q''(x) = xq(x) + 2q^3(x) , \\ &q(x) \sim \mathrm{Ai}(x) \quad \text{as} \quad x \to \infty , \end{aligned}$$

  and $\mathrm{Ai}(x)$ denotes the Airy function. In particular, F_{GUE}^{+} is continuous. Similarly, $n^{\frac{2}{3}}\left(\lambda_{\min}^{(n)} + 2\right) \xrightarrow{\mathscr{D}} F_{GUE}^{-}$ where

$$F_{GUE}^{-}(s) = 1 - F_{GUE}^{+}(-s) .$$

If Δ is a Borel set in $\mathbb{R}$, denote by

$$\mathscr{N}_n(\Delta) = \#\left\{\lambda_i^{(n)} \in \Delta\right\},$$

the number of eigenvalues of $\mathbf{M}$ in Δ. The following theorem is the main result of the article.

Theorem 1. *Let* $\mathbf{M}$ *be a nn matrix from the GUE with eigenvalues* $(\lambda_1^{(n)},\cdots,\lambda_n^{(n)})$*. Let* $p\geq 2$ *be a fixed integer and let* $\boldsymbol{\mu}=(\mu_1,\cdots,\mu_p)\in\mathbb{R}^p$ *be such that* $-2=\mu_1<\mu_2<\cdots<\mu_p=2$*. Denote by* $\boldsymbol{\Delta}=(\Delta_1,\cdots,\Delta_p)$ *a collection of* p *bounded Borel sets in* $\mathbb{R}$ *and consider* $\boldsymbol{\Delta}_n=(\Delta_{1,n},\cdots,\Delta_{p,n})$ *defined by*

$$\begin{array}{llllll}(edge) & \Delta_{1,n} & := & -2+\dfrac{\Delta_1}{n^{2/3}}\,, & \Delta_{p,n} & := \quad 2+\dfrac{\Delta_p}{n^{2/3}}\,,\\ (bulk) & \Delta_{i,n} & := & \mu_i+\dfrac{\Delta_i}{n}\,, & 2\leq i\leq p-1\,. & \end{array}$$

Let $(\ell_1,\cdots,\ell_p)\in\mathbb{N}^p$*, then*

$$\lim_{n\to\infty}\left(\mathbb{P}\left(\mathcal{N}_n(\Delta_{1,n})=\ell_1,\cdots,\mathcal{N}_n(\Delta_{p,n})=\ell_p\right)-\prod_{k=1}^{p}\mathbb{P}\left(\mathcal{N}_n(\Delta_{k,n})=\ell_k\right)\right)=0\,.$$

Remark 1. An important corollary of Theorem 1 is the asymptotic independence of the random variables $n^{\frac{2}{3}}\left(\lambda_{\min}^{(n)}+2\right)$ and $n^{\frac{2}{3}}\left(\lambda_{\max}^{(n)}-2\right)$, where $\lambda_{\min}^{(n)}$ and $\lambda_{\max}^{(n)}$ are the smallest and largest eigenvalues of $\mathbf{M}$. This in turn enables us to describe the fluctuations of the ratio $\frac{\lambda_{\max}^{(n)}}{\lambda_{\min}^{(n)}}$.

Remark 2. For fluctuations of the eigenvalues within the bulk or near the spectrum edges at various scales (different from those studied here), one can refer to [6, 7, 8].

Proof of Theorem 1 is postponed to Section 3. In Section 2, we prove the asymptotic independence of the rescaled smallest and largest eigenvalues of $\mathbf{M}$; we then describe the asymptotic fluctuations of the ratio $\frac{\lambda_{\max}^{(n)}}{\lambda_{\min}^{(n)}}$. Remaining proofs are provided in Section 4.

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2 Asymptotic independence of extreme eigenvalues

In this section, we prove that the random variables $n^{\frac{2}{3}}\left(\lambda_{\max}^{(n)}-2\right)$ and $n^{\frac{2}{3}}\left(\lambda_{\min}^{(n)}+2\right)$ are asymptotically independent as the size of matrix $\mathbf{M}$ goes to infinity. We then apply this result to describe the fluctuations of $\frac{\lambda_{\max}^{(n)}}{\lambda_{\min}^{(n)}}$. For a nice and short operator-theoretic proof of this result (subsequent to the present article, although previously published), one can also refer to [5]. In the sequel, we drop the superscript $^{(n)}$ to lighten the notations.

2.1 Asymptotic independence

Specifying $p=2$, $\mu_1=-2$, $\mu_2=2$ and getting rid of the boundedness condition over Δ_1 and Δ_2 in Theorem 1 yields the following

Corollary 1. *Let* $\mathbf{M}$ *be a nn matrix from the GUE. Denote by* $\lambda_{\min}$ *and* $\lambda_{\max}$ *its smallest and largest eigenvalues, then the following holds true*

$$\mathbb{P}\left(n^{\frac{2}{3}}\,(\lambda_{\min}+2)<x\ ,\ n^{\frac{2}{3}}\,(\lambda_{\max}-2)<y\right) - \mathbb{P}\left(n^{\frac{2}{3}}\,(\lambda_{\min}+2)<x\right)\mathbb{P}\left(n^{\frac{2}{3}}\,(\lambda_{\max}-2)<y\right)\xrightarrow[n\to\infty]{}0\ .$$

Thus

$$\left(n^{\frac{2}{3}}(\lambda_{\min}+2),n^{\frac{2}{3}}(\lambda_{\max}-2)\right)\xrightarrow[n\to\infty]{\mathscr{D}}(\lambda_-,\lambda_+),$$

where λ_- *and* λ_+ *are independent random variables with distribution functions* F^-_{GUE} *and* F^+_{GUE}.

Proof. Denote by $(\lambda_{(i)})$ the ordered eigenvalues of $\mathbf{M}$ $\lambda_{\min}=\lambda_{(1)}\le\lambda_{(2)}\le\cdots\le\lambda_{(n)}=\lambda_{\max}$. Let $(x,y)\in\mathbb{R}^2$ and take $\alpha\ge\max(|x|,|y|)$. Let $\Delta_1=(-\alpha,x)$ and $\Delta_2=(y,\alpha)$ so that

$$\Delta_{1,n}\ =\ \left(-2-\frac{\alpha}{n^{\frac{2}{3}}},-2+\frac{x}{n^{\frac{2}{3}}}\right)\quad\text{and}\quad\Delta_{2,n}\ =\ \left(2+\frac{y}{n^{\frac{2}{3}}},2+\frac{\alpha}{n^{\frac{2}{3}}}\right)\ .$$

We have

$$\begin{aligned}\left\{\mathscr{N}(\Delta_{1,n})=0\right\}\ &=\ \left\{n^{\frac{2}{3}}(\lambda_{\min}+2)>x\right\}\\ &\qquad\cup\left\{\exists i\in\{1,\cdots,n\};\ \lambda_{(i)}\le-2-\frac{\alpha}{n^{\frac{2}{3}}}\ ,\ \lambda_{(i+1)}\ge-2+\frac{x}{n^{\frac{2}{3}}}\right\}\ ,\\ &=:\ \left\{n^{\frac{2}{3}}(\lambda_{\min}+2)>x\right\}\cup\ \{\,\Pi(-\alpha,x)\,\}\ ,\end{aligned}\tag{1}$$

with the convention that if $i=n$, the condition simply becomes $\lambda_{\max}\le-2-\alpha n^{-\frac{2}{3}}$. Note that both sets in the right-hand side of the equation are disjoint. Similarly

$$\left\{\mathscr{N}(\Delta_{2,n})=0\right\}\ =\ \left\{n^{\frac{2}{3}}(\lambda_{\max}-2)<y\right\}$$

$$\cup\left\{\exists i\in\{1,\cdots,n\};\ \lambda_{(i-1)}\le2+\frac{y}{n^{\frac{2}{3}}}\ ,\ \lambda_{(i)}\ge2+\frac{\alpha}{n^{\frac{2}{3}}}\right\}\ ,\tag{2}$$

$$=:\ \left\{n^{\frac{2}{3}}(\lambda_{\max}-2)<y\right\}\cup\{\,\tilde{\Pi}(y,\alpha)\,\}\ ,\tag{3}$$

with the convention that if $i=1$, the condition simply becomes $\lambda_{\min}\ge2+\alpha n^{-\frac{2}{3}}$. Gathering the two previous equalities enables to write $\{\mathscr{N}(\Delta_{1,n})=0,\mathscr{N}(\Delta_{2,n})=0\}$ as the following union of disjoint events

$$\begin{aligned}&\left\{\mathscr{N}(\Delta_{1,n})=0\ ,\ \mathscr{N}(\Delta_{2,n})=0\right\}\\ &=\ \left\{\Pi(-\alpha,x)\ ,\ n^{\frac{2}{3}}(\lambda_{\max}-2)<y\right\}\cup\left\{\Pi(-\alpha,x)\ ,\ \tilde{\Pi}(y,\alpha)\right\}\cup\left\{n^{\frac{2}{3}}(\lambda_{\min}+2)>x\ ,\tilde{\Pi}(y,\alpha)\right\}\\ &\qquad\cup\left\{n^{\frac{2}{3}}(\lambda_{\min}+2)>x\ ,\ n^{\frac{2}{3}}(\lambda_{\max}-2)<y\right\}\ .\end{aligned}\tag{4}$$

Define

$$\begin{aligned}u_n\ \ :=\ &\ \mathbb{P}\left\{n^{\frac{2}{3}}(\lambda_{\min}+2)>x\ ,\ n^{\frac{2}{3}}(\lambda_{\max}-2)<y\right\}\\ &-\mathbb{P}\left\{n^{\frac{2}{3}}(\lambda_{\min}+2)>x\right\}\mathbb{P}\left\{n^{\frac{2}{3}}(\lambda_{\max}-2)<y\right\}\ ,\\ =\ &\ \mathbb{P}\left\{\mathscr{N}(\Delta_{1,n})=0\ ,\ \mathscr{N}(\Delta_{2,n})=0\right\}\\ &-\mathbb{P}\left\{\mathscr{N}(\Delta_{1,n})=0\right\}\mathbb{P}\left\{\mathscr{N}(\Delta_{2,n})=0\right\}+\epsilon_n(\alpha)\ ,\end{aligned}\tag{5}$$

where by equations (1), (3) and (4)

$$\epsilon_n(\alpha) := -\mathbb{P}\left\{\Pi(-\alpha,x)\,,\ n^{\frac{2}{3}}(\lambda_{\max}-2)<y\right\} - \mathbb{P}\left\{\Pi(-\alpha,x)\,,\ \tilde{\Pi}(y,\alpha)\right\}$$
$$-\,\mathbb{P}\left\{n^{\frac{2}{3}}(\lambda_{\min}+2)>x\ ,\tilde{\Pi}(y,\alpha)\right\} + \mathbb{P}\left\{\mathcal{N}(\Delta_{1,n})=0\right\}\mathbb{P}\left\{\tilde{\Pi}(y,\alpha)\right\}$$
$$+\,\mathbb{P}\left\{\Pi(-\alpha,x)\right\}\mathbb{P}\left\{\mathcal{N}(\Delta_{2,n})=0\right\} - \mathbb{P}\left\{\Pi(-\alpha,x)\right\}\mathbb{P}\left\{\tilde{\Pi}(y,\alpha)\right\}\ .$$

Using the triangular inequality, we obtain:

$$|\epsilon_n(\alpha)| \le 6\max\left(\mathbb{P}\left\{\Pi(-\alpha,x)\right\},\mathbb{P}\left\{\tilde{\Pi}(y,\alpha)\right\}\right)\ .$$

As $\{\,\Pi(-\alpha,x)\,\}\subset\{n^{\frac{2}{3}}(\lambda_{\min}+2)<-\alpha\}$, we have

$$\mathbb{P}\{\Pi(-\alpha,x)\ \}\le\mathbb{P}\{n^{\frac{2}{3}}(\lambda_{\min}+2)<-\alpha\}\xrightarrow[n\to\infty]{}F^-_{GUE}(-\alpha)\xrightarrow[\alpha\to\infty]{}0\ .$$

We can apply the same arguments to $\{\,\tilde{\Pi}(y,\alpha)\,\}\subset\{n^{\frac{2}{3}}(\lambda_{\max}-2)>\alpha\}$. We thus obtain:

$$\lim_{\alpha\to\infty}\limsup_{n\to\infty}|\epsilon_n(\alpha)| = 0\,. \tag{6}$$

The difference $\mathbb{P}\left\{\mathcal{N}(\Delta_{1,n})=0\ ,\ \mathcal{N}(\Delta_{2,n})=0\right\}-\mathbb{P}\left\{\mathcal{N}(\Delta_{1,n})=0\right\}\mathbb{P}\left\{\mathcal{N}(\Delta_{2,n})=0\right\}$ in the right-hand side of (5) converges to zero as $n\to\infty$ by Theorem 1 for every α large enough. We therefore obtain

$$\limsup_{n\to\infty}|u_n| = \limsup_{n\to\infty}|\epsilon_n(\alpha)|\,.$$

The lefthand side of the above equation is a constant w.r.t. α while the second term (whose behaviour for small α is unknown) converges to zero as $\alpha\to\infty$ by (6). Thus, $\lim_{n\to\infty}u_n=0$. The mere definition of u_n together with Tracy and Widom fluctuation results yields

$$\lim_{n\to\infty}\mathbb{P}\left\{n^{\frac{2}{3}}(\lambda_{\min}+2)>x\ ,\ n^{\frac{2}{3}}(\lambda_{\max}-2)<y\right\} = \left(1-F^-_{GUE}(x)\right)F^+_{GUE}(y)\,.$$

This completes the proof of Corollary 1.

□

2.2 Application: Fluctuations of the ratio of the extreme eigenvalues in the GUE

As a simple consequence of Corollary 1, we can easily describe the fluctuations of the ratio $\frac{\lambda_{\max}}{\lambda_{\min}}$. The counterpart of such a result to Gaussian Wishart matrices is of interest in digital communication (see [4] for an application in digital signal detection).

Corollary 2. *Let* **M** *be a nn matrix from the GUE. Denote by* $\lambda_{\min}$ *and* $\lambda_{\max}$ *its smallest and largest eigenvalues, then*

$$n^{\frac{2}{3}}\left(\frac{\lambda_{\max}}{\lambda_{\min}}+1\right)\xrightarrow[n\to\infty]{\mathcal{D}}-\frac{1}{2}(\lambda_-+\lambda_+)\,,$$

where $\xrightarrow{\mathcal{D}}$ *denotes convergence in distribution,* λ_- *and* λ_+ *are independent random variable with respective distribution* F^-_{GUE} *and* F^+_{GUE}.

Proof. The proof is a mere application of Slutsky's lemma (see for instance [18, Lemma 2.8]). Write

$$n^{\frac{2}{3}}\left(\frac{\lambda_{\max}}{\lambda_{\min}}+1\right)=\frac{1}{\lambda_{\min}}\left[n^{\frac{2}{3}}(\lambda_{\max}-2)+n^{\frac{2}{3}}(\lambda_{\min}+2)\right]\ . \tag{7}$$

Now, $\left(\lambda_{\min}\right)^{-1}$ goes almost surely to -2 as $n\to\infty$, and $n^{\frac{2}{3}}(\lambda_{\max}-2)+n^{\frac{2}{3}}(\lambda_{\min}+2)$ converges in distribution to the convolution of F^-_{GUE} and F^+_{GUE} by Corollary 1. Thus, Slutsky's lemma yields the convergence (in distribution) of the right-hand side of (7) to $-\frac{1}{2}(\lambda^-+\lambda^+)$ with λ^- and λ^+ independent and distributed according to F^-_{GUE} and F^+_{GUE}. Proof of Corollary 2 is completed. □

3 Proof of Theorem 1

3.1 Useful results

3.1.1 Kernels

Let $\{H_k(x)\}_{k\geq 0}$ be the classical Hermite polynomials $H_k(x):=e^{x^2}\left(-\frac{d}{dx}\right)^k e^{-x^2}$ and consider the function $\psi_k^{(n)}(x)$ defined for $0\leq k\leq n-1$ by:

$$\psi_k^{(n)}(x):=\left(\frac{n}{2}\right)^{\frac{1}{4}}\frac{e^{-\frac{nx^2}{4}}}{(2^k k!\sqrt{\pi})^{\frac{1}{2}}}H_k\left(\sqrt{\frac{n}{2}}x\right)\ .$$

Denote by $K_n(x,y)$ the following kernel on $\mathbb{R}^2$

$$K_n(x,y) := \sum_{k=0}^{n-1}\psi_k^{(n)}(x)\psi_k^{(n)}(y)\ , \tag{8}$$

$$= \frac{\psi_n^{(n)}(x)\psi_{n-1}^{(n)}(y)-\psi_n^{(n)}(y)\psi_{n-1}^{(n)}(x)}{x-y}\ . \tag{9}$$

Equation (9) is obtained from (8) by the Christoffel-Darboux formula. We recall the two well-known asymptotic results

Proposition 1. *a)* Bulk of the spectrum. *Let $\mu\in(-2,2)$.*

$$\forall(x,y)\in\mathbb{R}^2,\ \lim_{n\to\infty}\frac{1}{n}K_n\left(\mu+\frac{x}{n},\mu+\frac{y}{n}\right)=\frac{\sin\pi\rho(\mu)(x-y)}{\pi(x-y)}\ , \tag{10}$$

where $\rho(\mu)=\frac{\sqrt{4-\mu^2}}{2\pi}$. Furthermore, the convergence (10) *is uniform on every compact set of $\mathbb{R}^2$.*

b) Edge of the spectrum.

$$\forall(x,y)\in\mathbb{R}^2,\ \lim_{n\to\infty}\frac{1}{n^{2/3}}K_n\left(2+\frac{x}{n^{2/3}},2+\frac{y}{n^{2/3}}\right)=\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y)-\mathrm{Ai}(y)\mathrm{Ai}'(x)}{x-y}\ , \tag{11}$$

where $Ai(x)$ is the Airy function. Furthermore, the convergence (11) *is uniform on every compact set of $\mathbb{R}^2$.*

We will need as well the following result on the asymptotic behavior of functions $\psi_k^{(n)}$.

Proposition 2. *Let $\mu \in (-2,2)$, $k \in \{0,1\}$ and denote by K a compact set of $\mathbb{R}$.*

a) Bulk of the spectrum. *There exists a constant C such that*

$$\sup_{x\in K}\left|\psi_{n-k}^{(n)}\left(\mu+\frac{x}{n}\right)\right| \le C\,. \tag{12}$$

b) Edge of the spectrum. *There exists a constant C such that*

$$\sup_{x\in K}\left|\psi_{n-k}^{(n)}\left(2\frac{x}{n^{2/3}}\right)\right| \le n^{1/6}C\,. \tag{13}$$

The proof of these results can be found in [11, Chapter 7], see also [1, Chapter 3].

3.1.2 Determinantal representations, Fredholm determinants

There are determinantal representations using kernel $K_n(x,y)$ for the joint density p_n of the eigenvalues $(\lambda_i^{(n)}; 1\le i\le n)$, and for its marginals (see for instance [10, Chapter 6]):

$$p_n(x_1,\cdots,x_n) = \frac{1}{n!}\det\left\{K_n(x_i,x_j)\right\}_{1\le i,j\le n}\,, \tag{14}$$

$$\int_{\mathbb{R}^{n-m}} p_n(x_1,\cdots,x_n)dx_{m+1}\cdots dx_n = \frac{(n-m)!}{n!}\det\left\{K_n(x_i,x_j)\right\}_{1\le i,j\le m} \quad (m\le n)\,. \tag{15}$$

Definition 1. *Consider a linear operator S defined for any bounded integrable function $f:\mathbb{R}\to\mathbb{R}$ by*

$$Sf : x \mapsto \int_{\mathbb{R}} S(x,y)f(y)dy\,,$$

where $S(x,y)$ is a bounded integrable Kernel on $\mathbb{R}^2\to\mathbb{R}$ with compact support. The Fredholm determinant $D(z)$ associated with operator S is defined as follows

$$\forall z\in\mathbb{C},\quad D(z) := \det(1-zS) = 1+\sum_{k=1}^{\infty}\frac{(-z)^k}{k!}\int_{\mathbb{R}^k}\det\left\{S(x_i,x_j)\right\}_{1\le i,j\le k}dx_1\cdots dx_k\,. \tag{16}$$

It is in particular an entire function and its logarithmic derivative has a simple expression [17, Section 2.5] given by

$$\frac{D'(z)}{D(z)} = -\sum_{k=0}^{\infty}T(k+1)z^k\,, \tag{17}$$

where

$$T(k) = \int_{\mathbb{R}^k} S(x_1,x_2)S(x_2,x_3)\cdots S(x_k,x_1)\,dx_1\cdots dx_k\,. \tag{18}$$

For details related to Fredholm determinants, see for instance [14, 17].
The following kernel will be of constant use in the sequel

$$S_n(x,y;\boldsymbol{\lambda},\boldsymbol{\Delta}) := \sum_{i=1}^{p}\lambda_i\mathbf{1}_{\Delta_i}(x)K_n(x,y), \tag{19}$$

where $\boldsymbol{\lambda}=(\lambda_1,\cdots,\lambda_p)\in\mathbb{R}^p$ or $\boldsymbol{\lambda}\in\mathbb{C}^p$, depending on the need, and $\boldsymbol{\Delta}=(\Delta_1,\cdots,\Delta_p)$ is a collection of p bounded Borel sets in $\mathbb{R}$.

Remark 3. The kernel $K_n(x,y)$ is unbounded and one cannot consider its Fredholm determinant without caution. The kernel $S_n(x,y)$ is bounded in x since the kernel is zero if x is outside the compact closure of the set $\cup_{i=1}^p \Delta_i$, but a priori unbounded in y. In all the forthcoming computations, one may replace S_n with the bounded kernel $\tilde{S}_n(x,y) = \sum_{i,\ell=1}^p \lambda_i \mathbf{1}_{\Delta_i}(x)\mathbf{1}_{\Delta_\ell}(y) K_n(x,y)$ and get exactly the same results. For notational convenience, we keep on working with S_n.

Proposition 3. *Let $p \geq 1$ be a fixed integer, $\boldsymbol{\ell} = (\ell_1, \cdots, \ell_p) \in \mathbb{N}^p$ and denote $\boldsymbol{\Delta} = (\Delta_1, \cdots, \Delta_p)$, where every Δ_i is a bounded Borel set. Assume that the Δ_i's are pairwise disjoint. Then the following identity holds true*

$$\mathbb{P}\left\{\mathcal{N}(\Delta_1) = \ell_1, \cdots, \mathcal{N}(\Delta_p) = \ell_p\right\} = \frac{1}{\ell_1! \cdots \ell_p!}\left(-\frac{\partial}{\partial \lambda_1}\right)^{\ell_1} \cdots \left(-\frac{\partial}{\partial \lambda_p}\right)^{\ell_p} \det\left(1 - S_n(\boldsymbol{\lambda}, \boldsymbol{\Delta})\right)\Big|_{\lambda_1 = \cdots = \lambda_p = 1}, \quad (20)$$

where $S_n(\boldsymbol{\lambda}, \boldsymbol{\Delta})$ is the operator associated to the kernel defined in (19).

Proof of Proposition 3 is postponed to Section 4.1.

3.1.3 Useful estimates for kernel $S_n(x,y;\boldsymbol{\lambda},\boldsymbol{\Delta})$ and its iterations

Consider $\boldsymbol{\mu}$, $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}_n$ as in Theorem 1. Assume moreover that n is large enough so that the Borel sets $(\Delta_{i,n}; 1 \leq i \leq p)$ are pairwise disjoint. For $i \in \{1, \cdots, p\}$, define κ_i as

$$\kappa_i = \begin{cases} 1 & \text{if } -2 < \mu_i < 2 \\ \frac{2}{3} & \text{if } \mu_i = 2 \end{cases} . \quad (21)$$

Otherwise stated, $\kappa_1 = \kappa_p = \frac{2}{3}$ and $\kappa_i = 1$ for $1 < i < p$.
Let $\boldsymbol{\lambda} \in \mathbb{C}^p$. With a slight abuse of notation, denote by $S_n(x,y;\boldsymbol{\lambda})$ the kernel

$$S_n(x,y;\boldsymbol{\lambda}) := S_n(x,y;\boldsymbol{\lambda},\boldsymbol{\Delta}_n) . \quad (22)$$

For $1 \leq m, \ell \leq p$ and $\boldsymbol{\Lambda} \subset \mathbb{C}^p$, define

$$\mathcal{M}_{m\ell,n}(\boldsymbol{\Lambda}) := \sup_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}} \sup_{(x,y) \in \Delta_{m,n}\Delta_{\ell,n}} \left|S_n(x,y;\boldsymbol{\lambda})\right| , \quad (23)$$

where $S_n(x,y;\boldsymbol{\lambda})$ is given by (22).

Proposition 4. *Let $\boldsymbol{\Lambda} \subset \mathbb{C}^p$ be a compact set. There exist two constants $R := R(\boldsymbol{\Lambda}) > 0$ and $C := C(\boldsymbol{\Lambda}) > 0$, independent from n, such that for n large enough,*

$$\begin{cases} \mathcal{M}_{ii,n}(\boldsymbol{\Lambda}) \leq R^{-1} n^{\kappa_i} , & 1 \leq i \leq p \\ \mathcal{M}_{ij,n}(\boldsymbol{\Lambda}) \leq C n^{1 - \frac{\kappa_i + \kappa_j}{2}} , & 1 \leq i,j \leq p,\ i \neq j \end{cases} . \quad (24)$$

Proposition 4 is proved in Section 4.2.
Consider the iterated kernel $|S_n|^{(k)}(x,y;\boldsymbol{\lambda})$ defined by

$$\begin{cases} |S_n|^{(1)}(x,y;\boldsymbol{\lambda}) = |S_n(x,y;\boldsymbol{\lambda})| \\ |S_n|^{(k)}(x,y;\boldsymbol{\lambda}) = \int_{\mathbb{R}^{k-1}} |S_n(x,u;\boldsymbol{\lambda})||S_n|^{(k-1)}(u,y;\boldsymbol{\lambda})\, du \quad k \geq 2 \end{cases} , \quad (25)$$

where $S_n(x,y;\boldsymbol{\lambda})$ is given by (22). The next estimates will be stated with $\boldsymbol{\lambda} \in \mathbb{C}^p$ fixed. Note that $|S_n|^{(k)}$ is nonnegative and write

$$\int_{\mathbb{R}^{k-1}} |S_n(x,u_1;\boldsymbol{\lambda})S_n(u_1,u_2;\boldsymbol{\lambda})\cdots S_n(u_{k-1},y;\boldsymbol{\lambda})|du_1\cdots du_{k-1} .$$

As previously, define for $1 \le m,\ell \le p$

$$\mathscr{M}^{(k)}_{m\ell,n}(\boldsymbol{\lambda}) := \sup_{(x,y)\in\Delta_{m,n}\Delta_{\ell,n}} |S_n|^{(k)}(x,y;\boldsymbol{\lambda}) .$$

The following estimates hold true

Proposition 5. *Consider the compact set $\boldsymbol{\Lambda} = \{\boldsymbol{\lambda}\}$ and the associated constants $R = R(\boldsymbol{\lambda})$ and $C = C(\boldsymbol{\lambda})$ as given by Prop. 4. Let $\beta > 0$ be such that $\beta > R^{-1}$ and consider $\epsilon \in (0,\frac{1}{3})$. There exists an integer $N_0 := N_0(\beta,\epsilon)$ such that for every $n \ge N_0$ and for every $k \ge 1$,*

$$\left\{\begin{array}{lllll} \mathscr{M}^{(k)}_{mm,n}(\boldsymbol{\lambda}) & \le & \beta^k n^{\kappa_m} , & & 1 \le m \le p \\ \mathscr{M}^{(k)}_{m\ell,n}(\boldsymbol{\lambda}) & \le & C\beta^{k-1}\, n^{\left(1+\epsilon-\frac{\kappa_m+\kappa_\ell}{2}\right)} , & & 1\le m,\ell\le p,\ m\neq \ell \end{array}\right. . \tag{26}$$

Proposition 5 is proved in Section 4.3.

3.2 End of proof

Consider $\boldsymbol{\mu}$, $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}_n$ as in Theorem 1. Assume moreover that n is large enough so that the Borel sets $(\Delta_{i,n}; 1 \le i \le p)$ are pairwise disjoint. As previously, denote $S_n(x,y;\boldsymbol{\lambda}) = S_n(x,y;\boldsymbol{\lambda},\boldsymbol{\Delta}_n)$; denote also $S_{i,n}(x,y;\lambda_i) = S_n(x,y;\lambda_i,\Delta_{i,n}) = \lambda_i \mathbf{1}_{\Delta_i}(x)K_n(x,y)$, for $1 \le i \le p$. Note that $S_n(x,y;\boldsymbol{\lambda}) = S_{i,n}(x,y;\lambda_i)$ if $x \in \Delta_{i,n}$.
For every $z \in \mathbb{C}$ and $\boldsymbol{\lambda} \in \mathbb{C}^p$, we use the following notations

$$D_n(z,\boldsymbol{\lambda}) := \det(1 - zS_n(\boldsymbol{\lambda},\boldsymbol{\Delta}_n)) \quad \text{and} \quad D_{n,i}(z,\lambda_i) := \det(1 - zS_n(\lambda_i,\Delta_{i,n})) . \tag{27}$$

The following controls will be of constant use in the sequel.

Proposition 6. *1. Let $\boldsymbol{\lambda} \in \mathbb{C}^p$ be fixed. The sequences of functions*

$$z \mapsto D_n(z,\boldsymbol{\lambda}) \qquad \textit{and} \qquad z \mapsto D_{i,n}(z,\lambda_i), \quad 1 \le i \le p$$

are uniformly bounded in n on every compact subset of $\mathbb{C}$.

2. Let $z = 1$. The sequences of functions

$$\boldsymbol{\lambda} \mapsto D_n(1,\boldsymbol{\lambda}) \qquad \textit{and} \qquad \boldsymbol{\lambda} \mapsto D_{1,n}(1,\lambda_i), \quad 1 \le i \le p$$

are uniformly bounded in n on every compact subset of $\mathbb{C}^p$.

3. Let $\boldsymbol{\lambda} \in \mathbb{C}^p$ be fixed. For every $\delta > 0$, there exists $r > 0$ such that

$$\begin{array}{llll} \sup_n \sup_{z\in B(0,r)} |D_n(z,\boldsymbol{\lambda}) - 1| & < & \delta , & \\ \sup_n \sup_{z\in B(0,r)} |D_{i,n}(z,\lambda_i) - 1| & < & \delta , & 1 \le i \le p , \end{array}$$

where $B(0,r) = \{z \in \mathbb{C},\ |z| < r\}$.

The proof of Proposition 6 is provided in Section 4.4.
We introduce the following functions

$$d_n : (z, \boldsymbol{\lambda}) \mapsto \det\left(1 - zS_n(\boldsymbol{\lambda}, \boldsymbol{\Delta}_n)\right) - \prod_{i=1}^{p} \det\left(1 - zS_n(\lambda_i, \Delta_{i,n})\right) , \tag{28}$$

$$f_n : (z, \boldsymbol{\lambda}) \mapsto \frac{D_n'(z, \boldsymbol{\lambda})}{D_n(z, \boldsymbol{\lambda})} - \sum_{i=1}^{p} \frac{D_{i,n}'(z, \lambda_i)}{D_{i,n}(z, \lambda_i)} , \tag{29}$$

where $'$ denotes the derivative with respect to $z \in \mathbb{C}$. We first prove that f_n goes to zero as $n \to \infty$.

3.2.1 Asymptotic study of f_n in a neighbourhood of $z = 0$

In this section, we mainly consider the dependence of f_n in $z \in \mathbb{C}$ while $\boldsymbol{\lambda} \in \mathbb{C}^p$ is kept fixed. We therefore drop the dependence in $\boldsymbol{\lambda}$ for readability. Equality (17) yields

$$\frac{D_n'(z)}{D_n(z)} = -\sum_{k=0}^{\infty} T_n(k+1) z^k \quad \text{and} \quad \frac{D_{i,n}'(z)}{D_{i,n}(z)} = -\sum_{k=0}^{\infty} T_{i,n}(k+1) z^k \quad (1 \le i \le p) , \tag{30}$$

where $'$ denotes the derivative with respect to $z \in \mathbb{C}$ and $T_n(k)$ and $T_{i,n}(k)$ are as in (18), respectively defined by

$$T_n(k) := \int_{\mathbb{R}^k} S_n(x_1, x_2) S_n(x_2, x_3) \cdots S_n(x_k, x_1) dx_1 \cdots dx_k , \tag{31}$$

$$T_{i,n}(k) := \int_{\mathbb{R}^k} S_{i,n}(x_1, x_2) S_{i,n}(x_2, x_3) \cdots S_{i,n}(x_k, x_1) dx_1 \cdots dx_k . \tag{32}$$

Recall that D_n and $D_{i,n}$ are entire functions (of $z \in \mathbb{C}$). The functions $\frac{D_n'}{D_n}$ and $\frac{D_{i,n}'}{D_{i,n}}$ admit a power series expansion around zero given by (30). Therefore, the same holds true for $f_n(z)$. Moreover

Lemma 1. *Define R as in Proposition 4. For n large enough, $f_n(z)$ defined by (29) is holomorphic on $B(0, R) := \{z \in \mathbb{C},\ |z| < R\}$, and converges uniformly to zero as $n \to \infty$ on each compact subset of $B(0, R)$.*

Proof. Denote by $\xi_i^{(n)}(x) := \lambda_i \mathbf{1}_{\Delta_{i,n}}(x)$ and recall that $T_n(k)$ is defined by (31). Using the identity

$$\prod_{m=1}^{k} \sum_{i=1}^{p} a_{im} = \sum_{\sigma \in \{1, \cdots, p\}^k} \prod_{m=1}^{k} a_{\sigma(m)m}, \tag{33}$$

where a_{im} are complex numbers, $T_n(k)$ writes ($k \ge 2$)

$$\begin{aligned} T_n(k) &= \int_{\mathbb{R}^k} \left(\prod_{m=1}^{k} \sum_{i=1}^{p} \xi_i^{(n)}(x_m) \right) K_n(x_1, x_2) \cdots K_n(x_k, x_1) dx_1 \cdots dx_k , \\ &= \sum_{\sigma \in \{1, \cdots, p\}^k} j_{n,k}(\sigma) , \end{aligned} \tag{34}$$

where we define

$$j_{n,k}(\sigma) := \int_{\mathbb{R}^k} \left(\prod_{m=1}^{k} \xi^{(n)}_{\sigma(m)}(x_m) \right) K_n(x_1,x_2)\cdots K_n(x_k,x_1)dx_1\cdots dx_k\,. \tag{35}$$

We split the sum in the right-hand side of (34) into two subsums. The first is obtained by gathering the terms with k-tuples $\sigma = (i, i, \cdots, i)$ for $1 \le i \le p$ and writes

$$\sum_{i=1}^{p} \int_{\mathbb{R}^k} \left(\prod_{m=1}^{k} \lambda_i \mathbf{1}_{\Delta_{i,n}}(x_m) \right) K_n(x_1,x_2)\cdots K_n(x_k,x_1)dx_1\cdots dx_k = \sum_{i=1}^{p} T_{i,n}(k)\,,$$

where $T_{i,n}(k)$ is defined by (32). The remaining sum consists of those terms for which there exists at least one couple $(m,\ell) \in \{1,\cdots,k\}^2$ such that $\sigma(m) \neq \sigma(\ell)$. Let

$$\mathscr{S} = \left\{ \sigma \in \{1,\cdots,p\}^k \,:\, \exists (m,\ell) \in \{1,\cdots,k\}^2, \sigma(m) \neq \sigma(\ell) \right\}\ .$$

We obtain $T_n(k) = \sum_{i=1}^{p} T_{i,n}(k) + s_n(k)$ where

$$s_n(k) := \sum_{\sigma \in \mathscr{S}} j_{n,k}(\sigma)$$

for every $k \ge 2$. For each $q \in \{1,\ldots,k-1\}$, denote by π_q the following permutation for any k-tuple $(a_1,\ldots,a_k)$

$$\pi_q(a_1,\ldots,a_k) = (a_q, a_{q+1},\ldots,a_k,a_1,\ldots,a_{q-1})\,.$$

In other words, π_q operates a circular shift of $q-1$ elements to the left. Clearly, any k-tuple $\sigma \in \mathscr{S}$ can be written as $\sigma = \pi_q(m,\ell,\tilde{\sigma})$ for some $q \in \{1,\ldots,k-1\}$, $(m,\ell) \in \{1,\ldots,p\}^2$ such that $m \neq \ell$, and $\tilde{\sigma} \in \{1,\ldots,p\}^{k-2}$. This simply expresses the fact that if $\sigma \in \mathscr{S}$, there exists two consecutive elements that differ at some point. Thus

$$|s_n(k)| \quad \le \quad \sum_{q=1}^{k-1} \sum_{\substack{(m,\ell)\in\{1\cdots p\}^2 \\ m\neq\ell}} \sum_{\tilde{\sigma}\in\{1\cdots p\}^{k-2}} |j_{n,k}(\pi_q(m,\ell,\tilde{\sigma}))|\,.$$

From (35), function $j_{n,k}$ is invariant up to any circular shift π_q, so that $j_{n,k}(\sigma)$ coincides with $j_{n,k}(\pi_q(m,\ell,\tilde{\sigma}))$ for any $\sigma = \pi_q(m,\ell,\tilde{\sigma})$ as above. Therefore, $|s_n(k)|$ writes

$$\begin{aligned} |s_n(k)| \quad &\le \quad \sum_{q=1}^{k-1} \sum_{\substack{(m,\ell)\in\{1\cdots p\}^2 \\ m\neq\ell}} \sum_{\tilde{\sigma}\in\{1\cdots p\}^{k-2}} |j_{n,k}(\pi_q(m,\ell,\tilde{\sigma}))|\,, \\ &\le \quad k \sum_{\substack{(m,\ell)\in\{1\cdots p\}^2 \\ m\neq\ell}} \sum_{\tilde{\sigma}\in\{1\cdots p\}^{k-2}} \int_{\mathbb{R}^k} |\xi^{(n)}_m(x_1)\xi^{(n)}_\ell(x_2)\xi^{(n)}_{\tilde{\sigma}(1)}(x_3)\cdots\xi^{(n)}_{\tilde{\sigma}(k-2)}(x_k)| \\ &\qquad |K_n(x_1,x_2)\cdots K_n(x_k,x_1)|dx_1\cdots dx_k\,. \end{aligned}$$

The latter writes

$$\begin{aligned}|s_n(k)| \quad &\le \quad k \sum_{\substack{1\le m,\ell\le p\\ m\neq\ell}} \int_{\Delta_{m,n}\Delta_{\ell,n}} \left|K_n(x_1,x_2)\xi_m^{(n)}(x_1)\xi_\ell^{(n)}(x_2)\right| \\ &\left(\int_{\mathbb{R}^{k-2}} \sum_{\tilde{\sigma}\in\{1\cdots p\}^{k-2}} \left|\xi_{\tilde{\sigma}(1)}^{(n)}(x_3)\cdots\xi_{\tilde{\sigma}(k-2)}^{(n)}(x_k)\right| \left|K_n(x_2,x_3)\cdots K_n(x_k,x_1)\right| dx_3\cdots dx_k\right) dx_1 dx_2\ , \\ &= \quad k \sum_{\substack{1\le m,\ell\le p\\ m\neq\ell}} \int_{\Delta_{m,n}\Delta_{\ell,n}} \left|K_n(x_1,x_2)\sum_{i=1}^{p}\xi_i^{(n)}(x_1)\right| \sum_{i=1}^{p}\left|\xi_i^{(n)}(x_2)\right| \\ &\left(\int_{\mathbb{R}^{k-2}} \sum_{\tilde{\sigma}\in\{1\cdots p\}^{k-2}} \left|\xi_{\tilde{\sigma}(1)}^{(n)}(x_3)\cdots\xi_{\tilde{\sigma}(k-2)}^{(n)}(x_k)\right| \left|K_n(x_2,x_3)\cdots K_n(x_k,x_1)\right| dx_3\cdots dx_k\right) dx_1 dx_2\ .\end{aligned}$$

It remains to notice that

$$\begin{aligned}&\sum_{i=1}^{p}\left|\xi_i^{(n)}(x_2)\right| \int_{\mathbb{R}^{k-2}} \sum_{\tilde{\sigma}\in\{1\cdots p\}^{k-2}} \prod_{m=3}^{k} \left|\xi_{\tilde{\sigma}(m-2)}^{(n)}(x_m)\right| \left|K_n(x_2,x_3)\cdots K_n(x_k,x_1)\right| dx_3\cdots dx_k \\ &\overset{(a)}{=} \quad \sum_{i=1}^{p}\left|\xi_i^{(n)}(x_2)\right| \int_{\mathbb{R}^{k-2}} \left(\prod_{m=3}^{k}\sum_{i=1}^{p}\left|\xi_i^{(n)}(x_m)\right|\right) \left|K_n(x_2,x_3)\cdots K_n(x_k,x_1)\right| dx_3\cdots dx_k\ , \\ &= \quad \int_{\mathbb{R}^{k-2}} |S_n(x_2,x_3)S_n(x_3,x_4)\cdots S_n(x_k,x_1)| dx_3\cdots dx_k\ , \\ &\overset{(b)}{=} \quad |S_n|^{(k-1)}(x_2,x_1)\ ,\end{aligned}$$

where (a) follows from (33), and (b) from the mere definition of the iterated kernel (25). Thus, for $k \ge 2$, the following inequality holds true

$$|s_n(k)| \le k \sum_{\substack{1\le m,\ell\le p\\ m\neq\ell}} \int_{\Delta_{m,n}\Delta_{\ell,n}} |S_n(x_1,x_2)||S_n|^{(k-1)}(x_2,x_1) dx_1 dx_2\ . \tag{36}$$

For $k = 1$, let $s_n(1) = 0$ so that equation $T_n(k) = \sum_i T_{i,n}(k) + s_n(k)$ holds for every $k \ge 1$. According to (29), $f_n(z)$ writes:

$$f_n(z) = -\sum_{k=1}^{\infty} s_n(k+1) z^k\ .$$

Let us now prove that $f_n(z)$ is well-defined on the desired neighbourhood of zero and converges uniformly to zero as $n \to \infty$. Let $\beta > R^{-1}$, then Propositions 4 and 5 yield

$$\begin{aligned}
|s_n(k)| &\leq k \sum_{\substack{1\le m,\ell\le p \\ m\neq\ell}} \int_{\Delta_{m,n}\Delta_{\ell,n}} |S_n(x,y)||S_n|^{(k-1)}(y,x)dxdy , \\
&\leq k \sum_{\substack{1\le m,\ell\le p \\ m\neq\ell}} \mathcal{M}_{m\ell,n}\mathcal{M}_{\ell m,n}^{(k-1)}|\Delta_{m,n}||\Delta_{\ell,n}| , \\
&\leq k\,\beta^{k-2} \sum_{\substack{1\le m,\ell\le p \\ m\neq\ell}} C^2 n^{\left(1-\frac{\kappa_m+\kappa_\ell}{2}\right)} n^{\left(1+\epsilon-\frac{\kappa_m+\kappa_\ell}{2}\right)} n^{-(\kappa_m+\kappa_\ell)}|\Delta_m\Delta_\ell| , \\
&\leq k\,\beta^{k-2} \sum_{\substack{1\le m,\ell\le p \\ m\neq\ell}} \frac{C^2\,|\Delta_m\Delta_\ell|}{n^{2(\kappa_m+\kappa_\ell-1)-\epsilon}} , \\
&\overset{(a)}{\leq} k\,\beta^{k-2}\left(\max_{1\le m\le p}|\Delta_m|\right)^2 \frac{p(p-1)C^2}{n^{\frac{2}{3}-\epsilon}} ,
\end{aligned}$$

where (a) follows from the fact that $\kappa_m+\kappa_\ell-1 \geq \frac{1}{3}$. Clearly, the power series $\sum_{k=1}^{\infty}(k+1)\beta^{k-1}z^k$ converges for $|z| < \beta^{-1}$. As β^{-1} is arbitrarily lower than R, this implies that $f_n(z)$ is holomorphic in $B(0,R)$. Moreover, for each compact subset K included in the open disk $B(0,\beta^{-1})$ and for each $z \in K$,

$$|f_n(z)| \leq \left(\sum_{k=1}^{\infty}(k+1)\beta^{k-1}(\sup_{z\in K}|z|)^k\right)\left(\max_{1\le m\le p}|\Delta_m|\right)^2 \frac{p(p-1)C^2}{n^{\frac{2}{3}-\epsilon}} .$$

The right-hand side of the above inequality converges to zero as $n \to \infty$. Thus, the uniform convergence of $f_n(z)$ to zero on K is proved; in particular, as $\beta^{-1} < R$, $f_n(z)$ converges uniformly to zero on $B(0,R)$. Lemma 1 is proved.

□

3.2.2 Convergence of d_n to zero as $n \to \infty$

In this section, $\boldsymbol{\lambda} \in \mathbb{C}^p$ is fixed. We therefore drop the dependence in $\boldsymbol{\lambda}$ in the notations. Consider function F_n defined by

$$F_n(z) := \log \frac{D_n(z)}{\prod_{i=1}^{p} D_{i,n}(z)} , \tag{37}$$

where log corresponds to the principal branch of the logarithm and D_n and $D_{i,n}$ are defined in (30). As $D_n(0) = D_{i,n}(0) = 1$, there exists a neighbourhood of zero where F_n is holomorphic. Moreover, using Proposition 6-3), one can prove that there exists a neighbourhood of zero, say $B(0,\rho)$, where $(F_n(z))$ is a uniformly compactly bounded family, hence a normal family (see for instance [13]). Assume that this neighbourhood is included in $B(0,R)$, where R is defined in Proposition 4 and notice that in this neighbourhood, $F_n'(z) = f_n(z)$ as defined in (29). Consider a compactly converging subsequence $F_{\phi(n)} \to F_\phi$ in $B(0,\rho)$ (by compactly, we mean that the convergence is uniform over any compact set $\boldsymbol{K} \subset B(0,\rho)$), then one has in particular $F'_{\phi(n)}(z) \to F'_\phi$ but $F'_{\phi(n)}(z) = f_{\phi(n)}(z) \to 0$. Therefore, F_ϕ is a constant over $B(0,\rho)$, in particular, $F_\phi(z) = F_\phi(0) = 0$. We have proved that every converging subsequence of F_n converges to zero in $B(0,\rho)$. This yields the convergence (uniform on every compact of $B(0,\rho)$) of F_n to zero in $B(0,\rho)$. This yields the existence of a neighbourhood of zero, say $B(0,\rho')$ where

$$\frac{D_n(z)}{\prod_{i=1}^{p} D_{i,n}(z)} \xrightarrow[n\to\infty]{} 1 \tag{38}$$

uniformly on every compact of $B(0,\rho')$. Recall that $d_n(z) = D_n(z) - \prod_{i=1}^{p} D_{i,n}(z)$.
Combining (38) with Proposition 6-3) yields the convergence of $d_n(z)$ to zero in a small neighbourhood of zero. Now, Proposition 6-1) implies that $(d_n(z))$ is a normal family in $\mathbb{C}$. In particular, every subsequence $(d_{\phi(n)})$ compactly converges to a holomorphic function which coincides with 0 in a small neighbourhood of the origin, and thus is equal to 0 over $\mathbb{C}$. We have proved that

$$d_n(z) \xrightarrow[n\to\infty]{} 0, \qquad \forall z \in \mathbb{C}\ ,$$

with $\boldsymbol{\lambda} \in \mathbb{C}^p$ fixed.

3.2.3 Convergence of the partial derivatives of $\boldsymbol{\lambda} \mapsto d_n(1,\boldsymbol{\lambda})$ to zero

In order to establish Theorem 1, we shall rely on Proposition 3 where the probabilities of interest are expressed in terms of partial derivatives of Fredholm determinants. We therefore need to establish that the partial derivatives of $d_n(1,\boldsymbol{\lambda})$ with respect to $\boldsymbol{\lambda}$ converge to zero as well. This is the aim of this section.
In the previous section, we have proved that $\forall (z,\boldsymbol{\lambda}) \in \mathbb{C}^{p+1}$, $d_n(z,\boldsymbol{\lambda}) \to 0$ as $n \to \infty$. In particular,

$$d_n(1,\boldsymbol{\lambda}) \to 0, \quad \forall \boldsymbol{\lambda} \in \mathbb{C}^p\ .$$

We now prove the following facts (with a slight abuse of notation, write $d_n(\boldsymbol{\lambda})$ instead of $d_n(1,\boldsymbol{\lambda})$)

1. As a function of $\boldsymbol{\lambda} \in \mathbb{C}^p$, $d_n(\boldsymbol{\lambda})$ is holomorphic.
2. The sequence $\left(\boldsymbol{\lambda} \mapsto d_n(\boldsymbol{\lambda})\right)_{n\geq 1}$ is a normal family on $\mathbb{C}^p$.
3. The convergence $d_n(\boldsymbol{\lambda}) \to 0$ is uniform over every compact set $\boldsymbol{\Lambda} \subset \mathbb{C}^p$.

Proof of Fact 1) is straightforward and is thus omitted. Proof of Fact 2) follows from Proposition 6-2). Let us now turn to the proof of Fact 3). As (d_n) is a normal family, one can extract from every subsequence a compactly converging one in $\mathbb{C}^p$ (see for instance [12, Theorem 1.13])[1].
But for every $\boldsymbol{\lambda} \in \mathbb{C}^p$, $d_n(\boldsymbol{\lambda}) \to 0$, therefore every compactly converging subsequence converges toward 0. In particular, d_n itself compactly converges toward zero, which proves Fact 3).
In order to conclude the proof, it remains to apply standard results related to the convergence of partial derivatives of compactly converging holomorphic functions of several complex variables, as for instance [12, Theorem 1.9]. As $d_n(\boldsymbol{\lambda})$ compactly converges to zero, the following convergence holds true: Let $(\ell_1,\cdots,\ell_p) \in \mathbb{N}^p$, then

$$\forall \boldsymbol{\lambda} \in \mathbb{C}^p, \qquad \left(\frac{\partial}{\partial \lambda_1}\right)^{\ell_1} \cdots \left(\frac{\partial}{\partial \lambda_p}\right)^{\ell_p} d_n(\boldsymbol{\lambda}) \xrightarrow[n\to\infty]{} 0\ .$$

This, together with Proposition 3, completes the proof of Theorem 1.

[1]Notice that in the case of holomorphic functions in several complex variables, the result in reference [13] does not apply any more.

4 Remaining proofs

4.1 Proof of Proposition 3

Denote by $E_n(\boldsymbol{\ell},\boldsymbol{\Delta})$ the probability that for every $i\in\{1,\cdots,p\}$, the set Δ_i contains exactly ℓ_i eigenvalues

$$E_n(\boldsymbol{\ell},\boldsymbol{\Delta})=\mathbb{P}\left\{\mathcal{N}(\Delta_1)=\ell_1,\cdots,\mathcal{N}(\Delta_p)=\ell_p\right\}\ . \tag{39}$$

Let $\mathscr{P}_n(m)$ be the set of subsets of $\{1,\cdots,n\}$ with exactly m elements. If $A\in\mathscr{P}_n(m)$, denote by A^c its complementary subset in $\{1,\cdots,n\}$. The mere definition of $E_n(\boldsymbol{\ell},\boldsymbol{\Delta})$ yields

$$E_n(\boldsymbol{\ell},\boldsymbol{\Delta})=\int_{\mathbb{R}^n}\sum_{\substack{(A_1,\cdots,A_p)\in\\ \mathscr{P}_n(\ell_1)\cdots\mathscr{P}_n(\ell_p)}}\prod_{k=1}^p\left\{\prod_{i\in A_k}\mathbf{1}_{\Delta_k}(x_i)\prod_{j\in A_k^c}(1-\mathbf{1}_{\Delta_k}(x_j))\right\}p_n(x_1\cdots x_n)dx_1\cdots dx_n\ .$$

Using the following formula

$$\frac{1}{\ell!}\left(-\frac{d}{d\lambda}\right)^{\ell}\prod_{i=1}^n(1-\lambda\alpha_i)=\sum_{A\in\mathscr{P}_n(\ell)}\prod_{i\in A}\alpha_i\prod_{j\in A^c}(1-\lambda\alpha_j)\ ,$$

we obtain

$$E_n(\boldsymbol{\ell},\boldsymbol{\Delta})\quad=\quad\frac{1}{\ell_1!\cdots\ell_p!}\left(-\frac{\partial}{\partial\lambda_1}\right)^{\ell_1}\cdots\left(-\frac{\partial}{\partial\lambda_p}\right)^{\ell_p}\Gamma(\boldsymbol{\lambda},\boldsymbol{\Delta})\Big|_{\lambda_1=\cdots=\lambda_p=1}\ ,$$

where

$$\Gamma(\boldsymbol{\lambda},\boldsymbol{\Delta})=\int_{\mathbb{R}^n}\prod_{i=1}^n(1-\lambda_1\mathbf{1}_{\Delta_1}(x_i))\cdots(1-\lambda_p\mathbf{1}_{\Delta_p}(x_i))\,p_n(x_1\cdots x_n)\,dx_1\cdots dx_n\ .$$

Expanding the inner product and using the fact that the Δ_k's are pairwise disjoint yields

$$(1-\lambda_1\mathbf{1}_{\Delta_1}(x))\cdots(1-\lambda_p\mathbf{1}_{\Delta_p}(x))=\left(1-\sum_{k=1}^p\lambda_k\mathbf{1}_{\Delta_k}(x)\right)\ .$$

Thus

$$\begin{aligned}
\Gamma(\boldsymbol{\lambda},\boldsymbol{\Delta})\quad &=\quad \int_{\mathbb{R}^n}\prod_{i=1}^n\left(1-\sum_{k=1}^p\lambda_k\mathbf{1}_{\Delta_k}(x_i)\right)\,p_n(x_1\cdots x_n)\,dx_1\cdots dx_n\ ,\\
&\overset{(a)}{=}\quad 1+\int_{\mathbb{R}^n}\sum_{m=1}^n(-1)^m\sum_{A\in\mathscr{P}_n(m)}\prod_{i\in A}\left(\sum_{k=1}^p\lambda_k\mathbf{1}_{\Delta_k}(x_i)\right)\,p_n(x_1\cdots x_n)\,dx_1\cdots dx_n\ ,\\
&=\quad 1+\sum_{m=1}^n(-1)^m\sum_{A\in\mathscr{P}_n(m)}\int_{\mathbb{R}^n}\prod_{i\in A}\left(\sum_{k=1}^p\lambda_k\mathbf{1}_{\Delta_k}(x_i)\right)\,p_n(x_1\cdots x_n)\,dx_1\cdots dx_n\ ,\\
&\overset{(b)}{=}\quad 1+\sum_{m=1}^n(-1)^m\binom{n}{m}\int_{\mathbb{R}^n}\prod_{i=1}^m\left(\sum_{k=1}^p\lambda_k\mathbf{1}_{\Delta_k}(x_i)\right)\,p_n(x_1\cdots x_n)\,dx_1\cdots dx_n\ ,\\
&\overset{(c)}{=}\quad 1+\sum_{m=1}^n\frac{(-1)^m}{m!}\int_{\mathbb{R}^m}\prod_{i=1}^m\left(\sum_{k=1}^p\lambda_k\mathbf{1}_{\Delta_k}(x_i)\right)\det\left\{K_n(x_i,x_j)\right\}_{1\le i,j\le m}dx_1\cdots dx_m\ ,
\end{aligned}$$

where (a) follows from the expansion of $\prod_i \left(1-\sum_k \lambda_k \mathbf{1}_{\Delta_k}(x_i)\right)$, (b) from the fact that the inner integral in the third line of the previous equation does not depend upon E due to the invariance of p_n with respect to any permutation of the x_i's, and (c) follows from the determinantal representation (15).

Therefore, $\Gamma(\boldsymbol{\lambda}, \boldsymbol{\Delta})$ writes

$$\Gamma(\boldsymbol{\lambda}, \boldsymbol{\Delta}) = 1 + \sum_{m=1}^{n} \frac{(-1)^m}{m!} \int_{\mathbb{R}^m} \det\left\{S_n(x_i, x_j; \boldsymbol{\lambda}, \boldsymbol{\Delta})\right\}_{1\le i,j\le m} dx_1 \cdots dx_m \;, \tag{40}$$

where $S_n(x,y;\boldsymbol{\lambda},\boldsymbol{\Delta})$ is the kernel defined in (19). As the operator $S_n(\boldsymbol{\lambda},\boldsymbol{\Delta})$ has finite rank n, (40) coincides with the Fredholm determinant $\det(1 - S_n(\boldsymbol{\lambda},\boldsymbol{\Delta}))$ (see [17] for details). Proof of Proposition 3 is completed.

4.2 Proof of Proposition 4

In the sequel, $C > 0$ will be a constant independent from n, but whose value may change from line to line. First consider the case $i = j$. Denote by $S_{\mu_i}(x,y)$ the following limiting kernel

$$S_{\mu_i}(x,y) := \begin{cases} \dfrac{\sin \pi\rho(\mu_i)(x-y)}{\pi(x-y)} & \text{if } -2 < \mu_i < 2 \\ \dfrac{Ai(x)Ai'(y) - Ai(y)Ai'(x)}{x-y} & \text{if } \mu_i = 2 \\ \dfrac{Ai(-x)Ai'(-y) - Ai(-y)Ai'(-x)}{-x+y} & \text{if } \mu_i = -2 \end{cases} .$$

Proposition 1 implies that $n^{-\kappa_i} K_n(\mu_i + x/n^{\kappa_i}, \mu_i + y/n^{\kappa_i})$ converges uniformly to $S_{\mu_i}(x,y)$ on every compact subset of $\mathbb{R}^2$, where κ_i is defined by (21). Moreover, $S_{\mu_i}(x,y)$ being bounded on every compact subset of $\mathbb{R}^2$, there exists a constant C_i such that

$$\begin{aligned} \mathcal{M}_{ii,n}(\boldsymbol{\Lambda}) &= \left(\sup_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} |\lambda_i|\right) \sup_{(x,y)\in\Delta_{i,n}^2} \left|K_n\left(x,y\right)\right| \;, \\ &= \left(\sup_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} |\lambda_i|\right) \sup_{(x,y)\in\Delta_i^2} \left|K_n\left(\mu_i + \frac{x}{n^{\kappa_i}}, \mu_i + \frac{y}{n^{\kappa_i}}\right)\right| \;, \\ &\le \left(\sup_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} |\lambda_i|\right) n^{\kappa_i} \left(\sup_{(x,y)\in\Delta_i^2} \left| \frac{1}{n^{\kappa_i}} K_n\left(\mu_i + \frac{x}{n^{\kappa_i}}, \mu_i + \frac{y}{n^{\kappa_i}}\right) - S_{\mu_i}(x,y)\right| + \sup_{(x,y)\in\Delta_i^2} \left|S_{\mu_i}(x,y)\right| \right) \;, \\ &\le n^{\kappa_i} C_i \;. \end{aligned} \tag{41}$$

It remains to take R as $R^{-1} = \max(C_1, \cdots, C_p)$ to get the desired estimate.

Consider now the case where $i \neq j$. Using notation κ_i, inequalities (12) and (13) can be conveniently merged as follows There exists a constant C such that

$$\sup_{x\in\Delta_{i,n}} \left|\psi_{n-k}^{(n)}(x)\right| \le n^{\frac{1-\kappa_i}{2}} C \tag{42}$$

for $1 \le i \le p$ and $k = 0, 1$. For n large enough, we obtain, using (9)

$$
\begin{aligned}
\mathcal{M}_{ij,n}(\mathbf{\Lambda}) &\overset{(a)}{\le} \left(\sup_{\lambda\in\mathbf{\Lambda}}|\lambda_i|\right) \sup_{(x,y)\in\Delta_{i,n}\Delta_{j,n}} \frac{|\psi_n^{(n)}(x)||\psi_{n-1}^{(n)}(y)| + |\psi_n^{(n)}(y)||\psi_{n-1}^{(n)}(x)|}{|x-y|} ,\\
&\overset{(b)}{\le} \left(\sup_{\lambda\in\mathbf{\Lambda}}|\lambda_i|\right) n^{\frac{1-\kappa_i}{2}+\frac{1-\kappa_j}{2}} \frac{2C^2}{\inf_{(x,y)\in\Delta_{i,n}\Delta_{j,n}}|x-y|} ,\\
&\overset{(c)}{\le} C\, n^{1-\frac{\kappa_i+\kappa_j}{2}} ,
\end{aligned}
$$

where (a) follows from (9), (b) from (42) and (c) from the fact that

$$
\liminf_{n\to\infty} \inf_{(x,y)\in\Delta_{i,n}\Delta_{j,n}} |x-y| = |\mu_i - \mu_j| > 0 .
$$

Proposition 4 is proved.

4.3 Proof of Proposition 5

Let $\mathbf{\Lambda} = \{\boldsymbol{\lambda}\}$ be fixed. We drop, in the rest of the proof, the dependence in $\boldsymbol{\lambda}$ in the notations. The mere definition of $|S_n|^{(k)}$ yields

$$
\begin{aligned}
0 \le |S_n|^{(k)}(x,y) &\le \int_{\mathbb{R}} |S_n(x,u)||S_n|^{(k-1)}(u,y)du ,\\
&= \sum_{i=1}^{p} \int_{\Delta_{i,n}} |S_n(x,u)||S_n|^{(k-1)}(u,y)du .
\end{aligned}
$$

From the above inequality, the following is straightforward

$$
\forall (x,y) \in \Delta_{m,n}\Delta_{\ell,n},\ \ |S_n|^{(k)}(x,y) \le \sum_{i=1}^{p} |\Delta_{i,n}|\mathcal{M}_{mi,n}\mathcal{M}_{i\ell,n}^{(k-1)} .
$$

Using Proposition 4, we obtain

$$
\mathcal{M}_{m\ell,n}^{(k)} \le R^{-1}\mathcal{M}_{m\ell,n}^{(k-1)} + \alpha \sum_{i\ne m} n^{(1-\frac{\kappa_m+3\kappa_i}{2})}\mathcal{M}_{i\ell,n}^{(k-1)} , \tag{43}
$$

where $\alpha := \max(C|\Delta_1|, \cdots, C|\Delta_p|)$. Now take $\beta > R^{-1}$ and $\epsilon \in (0, \frac{1}{3})$. Property (26) holds for $k = 1$ since

$$
\mathcal{M}_{mm,n} \le R^{-1} n^{\kappa_m} \le \beta n^{\kappa_m} \quad \text{and} \quad \mathcal{M}_{m\ell,n} \le C n^{\left(1-\frac{\kappa_m+\kappa_\ell}{2}\right)} \le C n^{\left(1+\epsilon-\frac{\kappa_m+\kappa_\ell}{2}\right)}
$$

for every $m \ne \ell$ by Proposition 4. Assume that the same holds at step $k-1$.
Consider first the case where $m = \ell$. Eq. (43) becomes

$$
\begin{aligned}
\mathcal{M}_{mm,n}^{(k)} &\le R^{-1}\beta^{k-1}n^{\kappa_m} + \alpha C \beta^{k-2} \sum_{i\ne m} n^{(1-\frac{\kappa_m}{2}-\frac{3\kappa_i}{2})} n^{(1+\epsilon-\frac{\kappa_i}{2}-\frac{\kappa_m}{2})} ,\\
&\le \beta^k n^{\kappa_m} \left(\frac{R^{-1}}{\beta} + \sum_{i\ne m} \frac{\alpha C}{\beta^2} n^{\left(2+\epsilon-2\kappa_m-2\kappa_i\right)} \right) ,\\
&\le \beta^k n^{\kappa_m} \qquad \text{for } n \text{ large enough} ,
\end{aligned}
$$

where the last inequality follows from the fact that $2+\epsilon-2\kappa_m-2\kappa_i<0$, which implies that $n^{2+\epsilon-2\kappa_m-2\kappa_i}\to 0$, which in turn implies that the term inside the parentheses is lower than one for n large enough.

Now if $m\neq\ell$, Eq. (43) becomes

$$\begin{aligned}
\mathscr{M}^{(k)}_{m\ell,n} &\leq R^{-1}C\beta^{k-2}n^{\left(1+\epsilon-\frac{\kappa_\ell+\kappa_m}{2}\right)}+\alpha\beta^{k-1}n^{\left(1-\frac{\kappa_\ell+\kappa_m}{2}\right)} \\
&\quad +\sum_{i\neq m,\ell} C\alpha\beta^{k-2}n^{\left(1-\frac{\kappa_m+3\kappa_i}{2}\right)}n^{\left(1+\epsilon-\frac{\kappa_i+\kappa_\ell}{2}\right)} , \\
&= C\beta^{k-1}n^{\left(1+\epsilon-\frac{\kappa_\ell+\kappa_m}{2}\right)}\left(\frac{R^{-1}}{\beta}+\frac{\alpha}{Cn^\epsilon}+\frac{\alpha}{\beta}\sum_{i\neq m,\ell}n^{1-2\kappa_i}\right) , \\
&\leq C\beta^{k-1}n^{\left(1+\epsilon-\frac{\kappa_\ell+\kappa_m}{2}\right)}\left(\frac{R^{-1}}{\beta}+\frac{\alpha}{Cn^\epsilon}+\frac{\alpha p^2}{\beta n^{\frac{1}{3}}}\right) , \\
&\leq C\beta^{k-1}n^{\left(1+\epsilon-\frac{\kappa_\ell+\kappa_m}{2}\right)} ,
\end{aligned}$$

where the last inequality follows from the fact that the term inside the parentheses is lower than one for n large enough. Therefore, (26) holds for each $k\geq 1$ and for n large enough.

4.4 Proof of Proposition 6

Define $U_n(k,\boldsymbol{\lambda}):=\int_{\mathbb{R}^k}\left|\det\left\{S_n(x_i,x_j;\boldsymbol{\lambda})\right\}_{i,j=1\cdots k}\right|dx_1\cdots dx_k$. Using Hadamard's inequality, we obtain

$$\begin{aligned}
U_n(k,\boldsymbol{\lambda}) &\leq \int_{\mathbb{R}^k}\prod_{i=1}^k\sqrt{\sum_{j=1}^k|S_n(x_i,x_j;\boldsymbol{\lambda})|^2}dx_1\cdots dx_k , \\
&\leq \int_{\mathbb{R}^k}\prod_{i=1}^k\sqrt{\sum_{j=1}^k\left|\sum_{m=1}^p\lambda_m\mathbf{1}_{\Delta_{m,n}}(x_i)\right|^2|K_n(x_i,x_j)|^2}dx_1\cdots dx_k .
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
U_n(k,\boldsymbol{\lambda}) &\leq \int_{\mathbb{R}^k}\prod_{i=1}^k\left|\sum_{m=1}^p\lambda_m\mathbf{1}_{\Delta_{m,n}}(x_i)\right|\sqrt{\sum_{j=1}^k|K_n(x_i,x_j)|^2}dx_1\cdots dx_k , \\
&\leq \int_{\mathbb{R}^k}\sum_{\sigma\in\{1\cdots p\}^k}\prod_{i=1}^k\left|\lambda_{\sigma(i)}\right|\mathbf{1}_{\Delta_{\sigma(i),n}}(x_i)\sqrt{\sum_{j=1}^k|K_n(x_i,x_j)|^2}dx_1\cdots dx_k , \\
&= \sum_{\sigma\in\{1\cdots p\}^k}\int_{\mathbb{R}^k}\prod_{i=1}^k\sqrt{\sum_{j=1}^k|\lambda_{\sigma(i)}\mathbf{1}_{\Delta_{\sigma(i),n}}(x_i)K_n(x_i,x_j)|^2}dx_1\cdots dx_k .
\end{aligned}$$

In the above equation, integral $\int_{\mathbb{R}^k}$ clearly reduces to an integral on the set $\Delta_{\sigma(1),n} \cdots \Delta_{\sigma(p),n}$. Thus

$$
\begin{aligned}
\sup_{\boldsymbol{\lambda}\in\Lambda} U_n(k,\boldsymbol{\lambda}) &\leq \sum_{\sigma\in\{1\cdots p\}^k} \int_{\Delta_{\sigma(1),n}\cdots\Delta_{\sigma(p),n}} \prod_{i=1}^{k} \sqrt{\sum_{j=1}^{k} \mathscr{M}^2_{\sigma(i)\sigma(j),n}(\boldsymbol{\Lambda})} dx_1 \cdots dx_k \,, \\
&= \sum_{\sigma\in\{1\cdots p\}^k} \prod_{i=1}^{k} \sqrt{\sum_{j=1}^{k} \left(|\Delta_{\sigma(i),n}| \mathscr{M}_{\sigma(i)\sigma(j),n}(\boldsymbol{\Lambda})\right)^2} \,. \qquad (44)
\end{aligned}
$$

We now use Proposition 4 to bound the right-hand side. Clearly, when $\sigma(i) = \sigma(j)$, Proposition 4 implies that $|\Delta_{\sigma(i),n}|\mathscr{M}_{\sigma(i)\sigma(i),n}(\boldsymbol{\Lambda}) \leq R_{\boldsymbol{\Lambda}}^{-1}\Delta_{\max}$, where $\Delta_{\max} = \max_{1\leq i\leq p}|\Delta_i|$. This inequality still holds when $\sigma(i) \neq \sigma(j)$ as a simple application of Proposition 4. Therefore, we obtain

$$
\sup_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} U_n(k,\boldsymbol{\lambda}) \leq \sum_{\sigma\in\{1,\cdots,p\}^k} k^{\frac{k}{2}} \Delta_{\max}^k R_{\boldsymbol{\Lambda}}^{-k} = \left(\frac{p\,\Delta_{\max}\sqrt{k}}{R_{\boldsymbol{\Lambda}}}\right)^k \,.
$$

Using this inequality, it is straightforward to show that the series $\sum_k \frac{\sup_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} U_n(k,\boldsymbol{\lambda})}{k!} z^k$ converges for every $z \in \mathbb{C}$ and every compact set $\boldsymbol{\Lambda} \subset \mathbb{C}^p$. Parts 1) and 2) of the proposition are proved. Based on the definition of $D_n(z,\boldsymbol{\lambda})$ and $D_{i,n}(z,\lambda_i)$, we obtain

$$
\max\left(|D_n(z,\boldsymbol{\lambda}) - 1|, |D_{i,n}(z,\lambda_i) - 1|,\ 1 \leq i \leq p\right) \leq |z| \sum_{k=1}^{\infty} \frac{|z|^{k-1}}{k!} U_n(k,\boldsymbol{\lambda}) \,,
$$

which completes the proof of Proposition 6.

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