CLT FOR EIGEN-INFERENCE METHODS IN COGNITIVE RADIOS

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ABSTRACT

This article provides a central limit theorem for a consistent estimator of the population eigenvalues of a class of sample covariance matrices. An exact expression as well as an empirical and asymptotically accurate approximation of the limiting variance is also derived. These results are applied in a cognitive radio context featuring an orthogonal-CDMA primary network and a secondary network whose objective is to maximise the coverage of secondary transmissions under low probability of interference with primary users.

Index Terms— CLT, G-estimation, cognitive radios.

I. INTRODUCTION

Problems of statistical inference based on M independent observations of an N-variate random variable $\mathbf{y}$, with $\mathrm{E}[\mathbf{y}] = 0$ and $\mathrm{E}[\mathbf{y}\mathbf{y}^{\mathsf{H}}] = \mathbf{R}$ have drawn the attention of researchers from many fields for years. If the entries of $\mathbf{y}$ are the monthly market evolutions of N retail products, then the largest eigenvalue and corresponding eigenvector of $\mathbf{R}$ characterise the optimal portfolio for a trader [1]. If $\mathbf{y}$ is the sample of alleles of N genes extracted from a living being, then $\mathbf{R}$ predicts gene coexistence [2]. In wireless communications, if $\mathbf{y}$ are signals transmitted through a multi-dimensional channel, then the eigenvalues of $\mathbf{R}$ are a sufficient statistic for the capacity of this channel [3]. In the context of cognitive radios, if $\mathbf{y}$ is a vector of data observed by a secondary network and arising from signals transmitted by K primary users with respective transmit powers $P_1, \ldots, P_K$, then the eigenvalues of $\mathbf{R}$ contain information about those P_k, e.g. [11]. The present work focuses on this example.

Retrieving spectral properties of the *population covariance matrix* $\mathbf{R}$, based on the observation of M samples $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(M)}$, is therefore paramount to many questions of general science. If M is large compared to N, then it is known that $\|\mathbf{R}_M - \mathbf{R}\| \to 0$, as $M \to \infty$, for any matrix norm, where $\mathbf{R}_M$ is the *sample covariance matrix* $\mathbf{R}_M \triangleq \frac{1}{M}\sum_{m=1}^{M} \mathbf{y}^{(m)}\mathbf{y}^{(m)\mathsf{H}}$. However, one cannot always afford a large number of samples (this requires long delays in finance and wireless communications or too many individuals to sample in biology). In order to cope with this issue, random matrix theory [4] has proposed new tools, mainly spurred by the *G-estimators* of Girko [5]. Other works include convex optimisation methods [6], [7] and free probability tools [8], [9]. Many of those estimators are consistent in the sense that they are asymptotically unbiased as M, N grow large at the same rate. Nonetheless, it is only recently that new techniques have been unearthed which allow to estimate individual eigenvalues and functionals of eigenvectors of $\mathbf{R}$. The main contributor is Mestre [10] who provided an estimator for every eigenvalue of $\mathbf{R}$ under some separability condition, followed by Couillet *et al.* [11] and Vallet *et al.* [12] for more elaborate models.

These estimators, although proven asymptotically unbiased, have nonetheless not been fully characterised in terms of higher order statistics. It is in particular fundamental to evaluate the variance of these estimators for not-too-large M, N. In the context of cognitive radios, evaluating the transmit powers and statistical information about the resulting estimates of primary users allows a secondary network to characterise the optimal coverage that ensures both a low probability of interference towards the primary network and high communication rates for the secondary users.

The rest of the article is structured as follows: in Section II, we introduce the system model and recollect the main required results of random matrix theory. In Section III, we derive the main result of this paper. In Section IV, this result is applied in the context of cognitive radios while a comparative Monte Carlo simulation is performed. Section V concludes this article.

II. SYSTEM MODEL

Consider a primary orthogonal uplink CDMA network composed of K transmitters. Transmitter k uses the n_k orthogonal N-chip codes $\mathbf{w}_{k,1}, \ldots, \mathbf{w}_{k,n_k} \in \mathbb{C}^N$. Consider also a secondary sensor that we assume time-synchronised with the primary network. From the sensor viewpoint, primary user k has power P_k. Then, at symbol time m, the sensor receives the N-dimensional data vector

$$\mathbf{y}^{(m)} = \sum_{k=1}^{K} \sqrt{P_k} \sum_{j=1}^{n_k} \mathbf{w}_{k,j} x_{k,j}^{(m)} + \sigma \mathbf{n}^{(m)} \tag{1}$$

with $\sigma\mathbf{n}^{(m)} \in \mathbb{C}^N$ the additive white Gaussian noise received at time m and $x_{k,j}^{(m)}$ the signal transmitted by user k on the carrier code j at time m, which we assume Gaussian as well. We assume that the sensor knows perfectly σ^2 and the number of users, and desires to the transmit powers of each user. The sensor may or may not be aware of the number of codewords employed by each user.

Equation (1) can be compacted under the form

$$\mathbf{y}^{(m)} = \mathbf{W}\mathbf{P}^{\frac{1}{2}}\mathbf{x}^{(m)} + \sigma\mathbf{n}^{(m)}$$

with $\mathbf{W} = [\mathbf{w}_{1,1}, \ldots, \mathbf{w}_{1,n_1}, \mathbf{w}_{2,1}, \ldots, \mathbf{w}_{K,n_K}] \in \mathbb{C}^{N \times n}$, $n \triangleq \sum_{k=1}^{K} n_k$, $\mathbf{P} \in \mathbb{C}^{n \times n}$ the diagonal matrix with entry P_1 of multiplicity n_1, P_2 of multiplicity n_2, etc. and P_K of multiplicity n_K, and $\mathbf{x}^{(m)} = [\mathbf{x}_1^{(m)\mathsf{T}}, \ldots, \mathbf{x}_K^{(m)\mathsf{T}}]^{\mathsf{T}} \in \mathbb{C}^n$ where $\mathbf{x}_k^{(m)} \in \mathbb{C}^{n_k}$ is a column vector with j-th entry $x_{k,j}^{(m)}$.

Gathering M successive independent observations, we obtain the matrix $\mathbf{Y} = [\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(M)}] \in \mathbb{C}^{N \times M}$ given by

$$\mathbf{Y} = \mathbf{W}\mathbf{P}^{\frac{1}{2}}\mathbf{X} + \sigma\mathbf{N} = \begin{bmatrix}\mathbf{W}\mathbf{P}^{\frac{1}{2}} & \sigma\mathbf{I}_N\end{bmatrix} \begin{bmatrix}\mathbf{X} \\ \mathbf{N}\end{bmatrix}$$

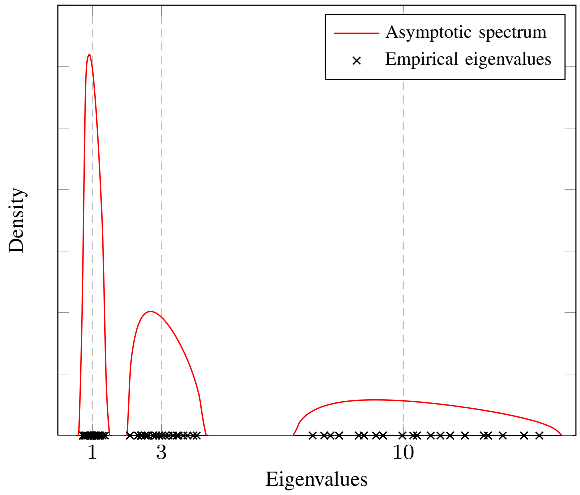

Fig. 1. Empirical and asymptotic eigenvalue distribution of $\mathbf{R}_M$ for $L = 3$, $t_1 = 1$, $t_2 = 3$, $t_3 = 10$, $N/M = c = 0.1$, $N = 60$.

where $\mathbf{X} = [\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(M)}]$ and $\mathbf{N} = [\mathbf{n}^{(1)}, \ldots, \mathbf{n}^{(M)}]$.

The $\mathbf{y}^{(m)}$ are therefore independent Gaussian vectors of zero mean and covariance $\mathbf{R} \triangleq \mathbf{WPW}^{\mathsf{H}} + \sigma^2 \mathbf{I}_N$. Since the question is to retrieve the powers P_k, while σ^2 is known, the problem boils down to finding the eigenvalues of $\mathbf{WPW}^{\mathsf{H}} + \sigma^2 \mathbf{I}_N$. However, the sensor only has access to $\mathbf{Y}$, or equivalently to the sample covariance matrix

$$\mathbf{R}_M \triangleq \frac{1}{M}\mathbf{Y}\mathbf{Y}^{\mathsf{H}} = \frac{1}{M}\sum_{m=1}^{M} \mathbf{y}^{(m)}\mathbf{y}^{(m)\mathsf{H}}.$$

The problem of retrieving the eigenvalues of $\mathbf{R}$ based on $\mathbf{R}_M$ was tackled by Mestre in [10], who proved the following:

Proposition 1 ([10]): Let $\mathbf{R}_M = \frac{1}{M}\mathbf{T}_M^{\frac{1}{2}}\mathbf{X}_M\mathbf{X}_M^{\mathsf{H}}\mathbf{T}_M^{\frac{1}{2}}$ where the eigenvalue distribution function $F^{\mathbf{T}_M}$ of $\mathbf{T}_M \in \mathbb{C}^{N\times N}$ converges to the distribution function T, composed of L masses in $t_1 < \ldots < t_L$ with weights $N_1/N, \ldots, N_L/N$, respectively, and $\mathbf{X}_M \in \mathbb{C}^{N\times M}$ has independent $\mathcal{CN}(0,1)$ entries X_{ij}. Denote $\lambda_1 \leq \ldots \leq \lambda_N$ the eigenvalues of $\mathbf{R}_M$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)^{\mathsf{T}}$. We further assume that $F^{\mathbf{T}_M} = T$ and $N/M = c$, $N_i/M = c_i$ for all large M considered. Then, as $M, N \to \infty$, if the limiting support $\mathcal{S}$ of the eigenvalue distribution of $\mathbf{R}_M$ is formed of L compact disjoint subsets, we have

$$\hat{t}_k - t_k \to 0$$

almost surely, where

$$\hat{t}_k = \frac{M}{N_k}\sum_{m\in\mathcal{N}_k}(\lambda_m - \mu_m) \tag{2}$$

with $\mathcal{N}_k = \{\sum_{j=1}^{k-1} N_j + 1, \ldots, \sum_{j=1}^{k} N_j\}$ and $\mu_1 \leq \ldots \leq \mu_N$ are the ordered eigenvalues of $\operatorname{diag}(\boldsymbol{\lambda}) - \frac{1}{M}\sqrt{\boldsymbol{\lambda}}\sqrt{\boldsymbol{\lambda}}^{\mathsf{T}}$.

Figure 1 depicts the eigenvalues of $\mathbf{R}_M$ and the associated limiting distribution as N, M grow large, for $t_1 = 1$, $t_2 = 3$, $t_3 = 10$ of equal multiplicity. Notice that we are here in a scenario where the limiting eigenvalue distribution of $\mathbf{R}_M$ is formed of L compact disjoint subsets as required by Proposition 1. In the present scenario, extending $[\mathbf{WP}^{\frac{1}{2}}\ \sigma\mathbf{I}_N]$ into an $(N+n)\times(N+n)$ matrix filled with zeros, $\mathbf{T}_M$ is the matrix $\mathbf{WPW}^{\mathsf{H}} + \sigma^2\mathbf{I}_N$, with $t_k = P_k + \sigma^2$ of multiplicity $N_k = n_k$ for each k, possibly an eigenvalue equal to σ^2 of multiplicity $N - n$ and a last eigenvalue $t_L = 0$ of multiplicity $N_L = n$.

The objective of the article is to study the performance of the estimator of Proposition 1 and apply it to the model (1). We will precisely show that, as $N, M \to \infty$, the random vector $\left(M(\hat{t}_k - t_k)\right)_{1\leq k\leq K}$ is asymptotically distributed as $\mathcal{N}(0, \boldsymbol{\Theta})$, where $\boldsymbol{\Theta}$ will be characterised exactly and will be given an approximation $\hat{\boldsymbol{\Theta}}$ based on the observation of $\mathbf{Y}$, such that, as $N, M \to \infty$, $\hat{\Theta}_{ij} - \Theta_{ij} \xrightarrow{\text{a.s.}} 0$.

III. CENTRAL LIMIT THEOREM

III-A. Further discussion on Proposition 1

The work of Mestre relies on tools of random matrix theory, among which the *Stieltjes transform* of distribution functions. The Stieltjes transform $m_{\mathbf{Z}}(z)$ of the distribution function $F^{\mathbf{Z}}$ of the eigenvalues of a nonnegative Hermitian matrix $\mathbf{Z} \in \mathbb{C}^{N\times N}$, with eigenvalues $\lambda_1, \ldots, \lambda_N$, is defined for $z \in \mathbb{C}\setminus\mathbb{R}^+$ as

$$m_{\mathbf{Z}}(z) \triangleq \int \frac{1}{\lambda - z}dF^{\mathbf{Z}}(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\lambda_i - z}.$$

The proof of Proposition 1 is based on the work of Silverstein and Bai [13] who prove that the Stieltjes transform $m_{\mathbf{R}_M}(z)$ of the sample covariance matrix $\mathbf{R}_M$ converges almost surely to a function $m(z)$ as $M, N \to \infty$ with $N/M \to c$, $0 < c < \infty$, where, for $z \in \mathbb{C}^+$, $m(z)$ is defined as the unique solution in $\mathbb{C}^+$ of [13]

$$\begin{aligned}\underline{m}(z) &= cm(z) + (c-1)\frac{1}{z}\\ \underline{m}(z) &= -\left(z - c\int\frac{t}{1 + t\underline{m}(z)}dT(t)\right)^{-1},\end{aligned}$$

with T the limiting distribution function of the eigenvalues of $\mathbf{T}_M$. Moreover, $m(z)$ and $\underline{m}(z)$ are the Stieltjes transform of distribution functions F and $\underline{F}$, respectively.

When dT is composed of L masses in $t_1, \ldots, t_L$, based on the link between $m(z)$ and T and under the condition that $\mathcal{S}$ is formed of L compact disjoint subsets, Mestre writes t_k explicitly as the following complex integral of $m(z)$ [4, Chapter 6]

$$t_k = \frac{1}{2\pi i c_k}\oint_{\mathcal{C}_k} z\frac{\underline{m}'(z)}{\underline{m}(z)}dz$$

with $\mathcal{C}_k$ a negatively oriented contour that circles around the k-th cluster in $\mathcal{S}$ only. Denote now $\underline{\mathbf{R}}_M \triangleq \mathbf{X}_M^{\mathsf{H}}\mathbf{T}_M\mathbf{X}_M$. Defining

$$\hat{t}_k \triangleq \frac{N}{2\pi i N_k}\oint_{\mathcal{C}_k} z\frac{m'_{\underline{\mathbf{R}}_M}(z)}{m_{\underline{\mathbf{R}}_M}(z)}dz \tag{5}$$

with $m_{\underline{\mathbf{R}}_M}(z) = \frac{N}{M}m_{\mathbf{R}_M}(z) + \frac{N-M}{M}\frac{1}{z}$ the Stieltjes transform of $\underline{\mathbf{R}}_M$, dominated convergence arguments ensure that $t_k - \hat{t}_k \xrightarrow{\text{a.s.}} 0$. The integral form of $\hat{t}_k$ can then be explicitly computed thanks to residue calculus [14] and we obtain (2).

III-B. Main results

In [15], Bai and Silverstein extend the limiting result on $F^{\mathbf{R}_M}$ to a central limit theorem, when $\mathbf{X}_M$ has entries with fourth order moment $\mathrm{E}[|X_{ij}|^4] = 2$, which is the case for complex Gaussian X_{ij}.

$$\Theta_{ij} \triangleq -\frac{1}{4\pi^2 c^2 c_i c_j} \oint_{\mathcal{C}_i} \oint_{\mathcal{C}_j} \left[\frac{\underline{m}'(z_1)\underline{m}'(z_2)}{(\underline{m}(z_1)-\underline{m}(z_2))^2} - \frac{1}{(z_1-z_2)^2} \right] \frac{1}{\underline{m}(z_1)\underline{m}(z_2)} dz_1 dz_2 \tag{3}$$

$$\hat{\Theta}_{ij} \triangleq \frac{M^2}{N_i N_j} \left[\sum_{(a,b)\in\mathcal{N}_i\times\mathcal{N}_j, a\neq b} -\frac{1}{(\mu_a-\mu_b)^2 m'_{\underline{\mathbf{R}}_M}(\mu_a) m'_{\underline{\mathbf{R}}_M}(\mu_b)} + \delta_{ij} \sum_{a\in\mathcal{N}_i} \frac{m'''_{\underline{\mathbf{R}}_M}(\mu_a)}{6 m'_{\underline{\mathbf{R}}_M}(\mu_a)^3} - \frac{m''_{\underline{\mathbf{R}}_M}(\mu_a)^2}{4 m'_{\underline{\mathbf{R}}_M}(\mu_a)^4} \right] \tag{4}$$

Proposition 2 ([15]): Under these conditions, for $f_1,\ldots,f_p$ analytic on $\mathbb{R}$,

$$\left(N \int f_i(x) d(F^{\mathbf{R}_M} - F)(x) \right)_{1\leq i\leq p} \Rightarrow X \sim \mathcal{N}(0,\mathbf{V}),$$
$$V_{ij} = -\frac{1}{4\pi^2} \oint\oint f_i(z_1) f_j(z_2) v_{ij}(z_1,z_2) dz_1 dz_2,$$
$$v_{ij}(z_1,z_2) = \frac{\underline{m}'(z_1)\underline{m}'(z_2)}{(\underline{m}(z_1)-\underline{m}(z_2))^2} - \frac{1}{(z_1-z_2)^2} \tag{6}$$

where the integration is over positively oriented contours that circle around $\mathcal{S}$.

Similar to Mestre who transposed the first order limit of $F^{\mathbf{R}_M}$ into a limiting result on the estimator $\hat{t}_k$ of t_k, the present article transposes the second order limit of functionals of $\mathbf{R}_M$ into a central limit of the variations of $\hat{t}_k$ around t_k.

To this end, the fundamental tool we use here is the *delta-method* [16],

Lemma 1: Let $X_1, X_2, \ldots \in \mathbb{R}^n$ be a random sequence such that

$$a_n(X_n - \mu) \Rightarrow X \sim \mathcal{N}(0,\mathbf{V})$$

for some $a_n \to \infty$. Then for $f: \mathbb{R}^n \mapsto \mathbb{R}^N$, differentiable at μ,

$$a_n(f(X_n) - f(\mu)) \Rightarrow J(f)X$$

with $J(f)$ the Jacobian matrix of f.

The basic idea is the following: since (i) $\hat{t}_k$ is a function of $m_{\mathbf{R}_M}(z)$, itself being a functional of $F^{\mathbf{R}_M}$, and (ii) the limiting variations of well-behaved functionals of $F^{\mathbf{R}_M}$ are Gaussian, we can apply (with some technical care) the delta-method to $\hat{t}_k$.

The outcome of this method are the following two theorems

Theorem 1: Let $\mathbf{R}_M$ be defined as in Proposition 1 with $\mathrm{E}[|X_{ij}|^4] = 2$. Then,

$$\left(M(\hat{t}_k - t_k) \right)_{1\leq k\leq K} \Rightarrow X \sim \mathcal{N}(0,\mathbf{\Theta})$$

with Θ_{ij}, the entry (i,j) of $\mathbf{\Theta}$, given by (3), where the contour $\mathcal{C}_k$ encloses the k-th cluster of $\mathcal{S}$ only.

Similar to Proposition 1, it is possible to provide a consistent estimate $\hat{\Theta}_{ij}$ for Θ_{ij}, $1 \leq i,j \leq K$. This is given as follows:

Theorem 2: Let Θ_{ij} be defined as in Theorem 1. Then,

$$\hat{\Theta}_{ij} - \Theta_{ij} \xrightarrow{\text{a.s.}} 0$$

as $N, M \to \infty$, where $\hat{\Theta}_{ij}$ is defined in (4), with the quantities $\mathcal{N}_k$ and $\mu_1,\ldots,\mu_N$ defined as in Proposition 1.

Theorem 1 describes the limiting performance of the estimator of Proposition 1 with an exact characterisation of its variance, while Theorem 2 introduces an estimator of this variance based on the observation of the random $\mathbf{R}_M$. Theorem 2 is useful in practice in that one can obtain simultaneously an estimate $\hat{t}_k$ of the values of t_k as well as an estimation of the degree of confidence for each $\hat{t}_k$.

We provide hereafter a sketch of proof of the above results.

Proof: The proof is composed of three steps. The first step consists in applying the delta method to prove that the terms

$$\left(M \left[z_i \frac{m'_{\underline{\mathbf{R}}_M}(z_i)}{m_{\underline{\mathbf{R}}_M}(z_i)} - z_i \frac{\underline{m}'(z_i)}{\underline{m}(z_i)} \right] \right)_{1\leq i\leq p},$$

i.e. the deviation of p points of the integrands in (5), are asymptotically jointly Gaussian. This unfolds first from Proposition 2 applied to $f_i(x) = (x-z_i)^{-1}$, $1 \leq i \leq m$, and $f_i(x) = (x-z_i)^{-2}$, $m+1 \leq i \leq 2m$, which ensures the joint Gaussianity of the deviations of $N(m_{\underline{\mathbf{R}}_M}(z_i) - \underline{m}(z_i))$ and $N(m'_{\underline{\mathbf{R}}_M}(z_i) - \underline{m}'(z_i))$. Then, notice that

$$M \left[z \frac{m'_{\underline{\mathbf{R}}_M}(z_i)}{m_{\underline{\mathbf{R}}_M}(z_i)} - z \frac{\underline{m}'(z_i)}{\underline{m}(z_i)} \right]$$
$$= M \left[\frac{z_i m'_{\underline{\mathbf{R}}_M}(z_i)\underline{m}(z_i) - z_i \underline{m}'(z_i) m_{\underline{\mathbf{R}}_M}(z_i)}{m_{\underline{\mathbf{R}}_M}(z_i)\underline{m}(z_i)} \right]$$
$$\Rightarrow z_i \frac{\underline{m}'(z_i)X - \underline{m}(z_i)Y}{\underline{m}(z_i)^2}$$

where (X,Y) is a random variable with law the (Gaussian) weak limit of $(M(m_{\underline{\mathbf{R}}_M}(z_i) - \underline{m}(z_i)), M(m'_{\underline{\mathbf{R}}_M}(z_i) - \underline{m}'(z_i)))$ and the last line unfolds from Slutsky's lemma [16]. This last form can be rewritten $f(X,Y) = f(X-0, Y-0) = z_i \frac{\underline{m}'(z_i)X - \underline{m}(z_i)Y}{\underline{m}(z_i)^2}$, where f is differentiable at $(0,0)$. Using Lemma 1 on $M(m_{\underline{\mathbf{R}}_M}(z_i) - \underline{m}(z_i))$ and $M(m'_{\underline{\mathbf{R}}_M}(z_j) - \underline{m}'(z_j))$ for different i,j, applied to the function f, leads to the result.

In order to propagate the Gaussianity of the deviations in the integrands of (5) to the deviations in $\hat{t}_k$, it suffices to study the behaviour of the sum of Gaussian variables over the integration contour. Since (i) the integral can be written as the limit of a finite Riemann sum and (ii) a finite Riemann sum of Gaussian random variable is still Gaussian, it suffices to ensure that the Riemann sum is still Gaussian in the limit. This requires an additional ingredient: the tightness of the sequences $M(z\frac{m'_{\underline{\mathbf{R}}_M}(z)}{m_{\underline{\mathbf{R}}_M}(z)} - z\frac{\underline{m}'(z)}{\underline{m}(z)})$ for growing M and for all z, see [17, Theorem 13.1]. This naturally unfolds from a direct application of [17, Theorem 13.2], following a similar idea to [15].

The last step of the proof is the calculus of the covariance of the Gaussian limit. This requires to evaluate for all i,j

$$M^2 \mathrm{E} \oint_{\mathcal{C}_i} \oint_{\mathcal{C}_j} \left(z_i \frac{m'_{\underline{\mathbf{R}}_M}(z_i)}{m_{\underline{\mathbf{R}}_M}(z_i)} - z_i \frac{\underline{m}'(z_i)}{\underline{m}(z_i)} \right) \times \left(z_j \frac{m'_{\underline{\mathbf{R}}_M}(z_j)}{m_{\underline{\mathbf{R}}_M}(z_j)} - z_j \frac{\underline{m}'(z_j)}{\underline{m}(z_j)} \right) dz_i dz_j.$$

Integrations by parts simplify the result and lead to (3). In order to obtain (4), residue calculus is finally performed [14]. ■

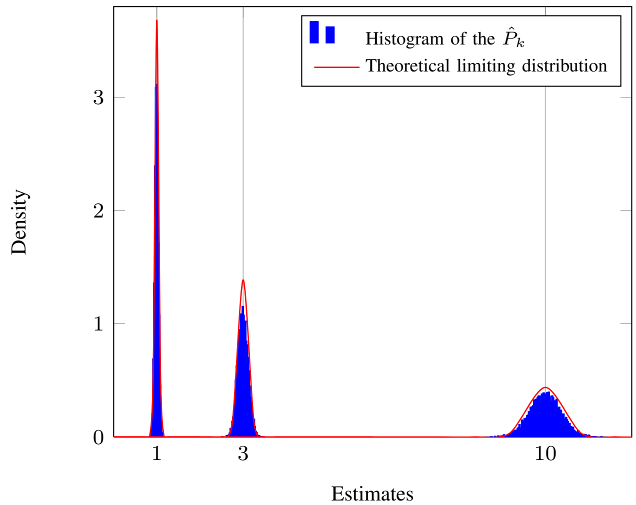

Fig. 2. Comparison of empirical against theoretical variances for three users, $P_1 = 1$, $P_2 = 3$, $P_3 = 10$, $n_1 = n_2 = n_3 = 20$ codes per user, $N = 60$, $M = 600$ and SNR= 20 dB.

IV. PERFORMANCE OF COGNITIVE RADIOS

We consider the system model (1). Assuming the spectrum of $\mathbf{R}_M$ allows one to clearly distinguish the successive clusters (as in Figure 1), Proposition 1 enables the detection of primary transmitters and the estimation of their transmit powers $P_1, \ldots, P_K$; this boils down to estimating the largest K eigenvalues of $\mathbf{WPW}^{\mathsf{H}} + \sigma^2 \mathbf{I}_N$, i.e. the $P_k + \sigma^2$, and to subtract σ^2 (optionally estimated from the smallest eigenvalue of $\mathbf{WPW}^{\mathsf{H}} + \sigma^2 \mathbf{I}_N$ if $n < N$). Call $\hat{P}_k$ the estimate of P_k.

Based on these power estimates, the sensor can determine the optimal coverage for secondary communications that ensures no interference to the primary network. A basic idea for instance is to ensure that the closest primary user, i.e. that with strongest received power, is not interfered. Our interest is then cast on P_K. Now, since the power estimator is imperfect, it is hazardous for the secondary network to state that K has power $\hat{P}_K$ or to add some empirical security margin to $\hat{P}_K$. The results of Section III partially answer this problem.

Theorems 1 and 2 enable the secondary sensor to evaluate the accuracy of $\hat{P}_k$. In particular, assume that the cognitive radio protocol allows the secondary network to interfere the primary network with probability q and denote A the value

$$A \triangleq \inf_a \{\Pr(P_K - \hat{P}_K > a) \leq q\}.$$

According to Theorem 1, for N, M large, A is well approximated by $\bar{\Theta}_{K,K} Q^{-1}(q)$, with Q the Gaussian *Q-function*. If the sensor detects a user with power P_K, estimated by $\hat{P}_K$, $\Pr(\hat{P}_K + A < P_K) < q$ and then it is safe for the secondary network to assume the worst case scenario where user K transmits at power $\hat{P}_K + A \simeq \hat{P}_K + \hat{\Theta}_{K,K} Q^{-1}(q)$.

In Figure 2, the performance of Theorem 1 is compared against $10,000$ Monte Carlo simulations of a scenario of three users, with $n_1 = n_2 = n_3 = 20$, $N = 60$ and $M = 600$. It appears that the limiting distribution is very accurate for these values of N, M. We also performed simulations to obtain empirical estimates $\hat{\Theta}_{k,k}$ of $\Theta_{k,k}$ from Theorem 2, which suggest that $\bar{\Theta}_{k,k}$ is an accurate estimator as well.

V. CONCLUSION

In this paper, we derived an exact expression and an approximation of the limiting performance of a statistical inference method that estimates the population eigenvalues of a class of sample covariance matrices. These results are applied in the context of cognitive radios to optimize secondary network coverage based on measures of the primary network activity.

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