World Scientific
www.worldscientific.com

A CLT FOR INFORMATION-THEORETIC STATISTICS
OF NON-CENTERED GRAM RANDOM MATRICES

WALID HACHEM[*], MALIKA KHAROUF[†] and JAMAL NAJIM[‡]

CNRS; TELECOM ParisTech
46, rue Barrault, 75634 Paris Cedex 13, France
*walid.hachem@telecom-paristech.fr
†malika.kharouf@telecom-paristech.fr
‡jamal.najim@telecom-paristech.fr

JACK W. SILVERSTEIN

Department of Mathematics, Box 8205
North Carolina State University
Raleigh, NC 27695-8205, USA
jack@math.ncsu.edu

In this article, we study the fluctuations of the random variable:

$$\mathcal{I}_n(\rho) = \frac{1}{N} \log \det(\Sigma_n \Sigma_n^* + \rho I_N), \quad (\rho > 0)$$

where $\Sigma_n = n^{-1/2} D_n^{1/2} X_n \tilde{D}_n^{1/2} + A_n$, as the dimensions of the matrices go to infinity at the same pace. Matrices X_n and A_n are respectively random and deterministic $N \times n$ matrices; matrices D_n and $\tilde{D}_n$ are deterministic and diagonal, with respective dimensions $N \times N$ and $n \times n$; matrix $X_n = (X_{ij})$ has centered, independent and identically distributed entries with unit variance, either real or complex.

We prove that when centered and properly rescaled, the random variable $\mathcal{I}_n(\rho)$ satisfies a Central Limit Theorem and has a Gaussian limit. The variance of $\mathcal{I}_n(\rho)$ depends on the moment $\mathbb{E}X_{ij}^2$ of the variables X_{ij} and also on its fourth cumulant $\kappa = \mathbb{E}|X_{ij}|^4 - 2 - |\mathbb{E}X_{ij}^2|^2$.

The main motivation comes from the field of wireless communications, where $\mathcal{I}_n(\rho)$ represents the mutual information of a multiple antenna radio channel. This article closely follows the companion article "A CLT for Information-theoretic statistics of Gram random matrices with a given variance profile", *Ann. Appl. Probab.* (2008) by Hachem *et al.*, however the study of the fluctuations associated to non-centered large random matrices raises specific issues, which are addressed here.

Keywords: Central Limit Theorem; Shannon capacity; random matrices; spectral measure; Stieltjes transform.

Mathematics Subject Classification 2010: Primary 15B52, 60B20; Secondary 94A05, 94A15

1. Introduction

1.1. *The model, the statistics, and the literature*

Consider an $N \times n$ random matrix $\Sigma_n = (\xi_{ij}^n)$ which has the expression

$$\Sigma_n = \frac{1}{\sqrt{n}} D_n^{\frac{1}{2}} X_n \tilde{D}_n^{\frac{1}{2}} + A_n, \tag{1.1}$$

where $A_n = (a_{ij}^n)$ is a deterministic $N \times n$ matrix, D_n and $\tilde{D}_n$ are diagonal deterministic matrices with non-negative entries, with respective dimensions $N \times N$ and $n \times n$; $X_n = (X_{ij})$ is an $N \times n$ matrix with the entries X_{ij}'s being centered, independent and identically distributed (i.i.d.) random variables with unit variance $\mathbb{E}|X_{ij}|^2 = 1$ and finite 16th moment.

Consider the following linear statistics of the eigenvalues:

$$\mathcal{I}_n(\rho) = \frac{1}{N} \log \det(\Sigma_n \Sigma_n^* + \rho I_N) = \frac{1}{N} \sum_{i=1}^{N} \log(\lambda_i + \rho),$$

where I_N is the $N \times N$ identity matrix, $\rho > 0$ is a given parameter and the λ_i's are the eigenvalues of matrix $\Sigma_n \Sigma_n^*$ (Σ_n^* stands for the Hermitian adjoint of Σ_n). This functional, known as the mutual information for multiple antenna radio channels, is fundamental in wireless communication as it characterizes the performance of a (coherent) communication over a wireless Multiple-Input Multiple-Output (MIMO) channel with gain matrix Σ_n. When Σ_n follows the model described by (1.1), the deterministic matrix A_n accounts for the so-called specular component, while D_n and $\tilde{D}_n$ account for the correlations in certain bases at the receiving and emitting sides, respectively.

Since the seminal work of Telatar [37], the study of the mutual information $\mathcal{I}_n(\rho)$ of an MIMO channel (and other performance indicators) in the regime where the dimensions of the gain matrix grow to infinity at the same pace has turned to be extremely fruitful. However, non-centered channel matrices have been comparatively less studied from this point of view, as their analysis is more difficult due to the presence of the deterministic matrix A_n. First order results can be found in Girko [15, 16]; Dozier and Silverstein [11, 12] established convergence results for the spectral measure; and the systematic study of the convergence of $\mathcal{I}_n(\rho)$ for a correlated Rician channel has been undertaken by Hachem *et al.* in [13, 20], etc. The fluctuations of $\mathcal{I}_n$ are important as well, for the computation of the outage probability of an MIMO channel for instance. With the help of the replica method, Taricco [35, 36] provided a closed-form expression for the asymptotic variance of $\mathcal{I}_n$ when the elements of X_n are Gaussian.

The purpose of this article is to establish a Central Limit Theorem (CLT) for $\mathcal{I}_n(\rho)$ in the following regime

$$n \to \infty \quad \text{and} \quad 0 < \liminf \frac{N}{n} \leq \limsup \frac{N}{n} < \infty,$$

(simply denoted by $n \to \infty$ in the sequel) under mild assumptions for matrices X_n, A_n, D_n and $\tilde{D}_n$.

The contributions of this article are twofold. From a wireless communication perspective, the fluctuations of $\mathcal{I}_n$ are established, regardless of the Gaussianity of the entries and the CLT conjectured by Tarrico is fully proven. Also, this article concludes a series of studies devoted to Rician MIMO channels, initiated in [20] where a deterministic equivalent of the mutual information was provided, and continued in [13] where the computation of the ergodic capacity was addressed and an iterative algorithm proposed.

From a mathematical point of view, the study of the fluctuations of $\mathcal{I}_n$ is the first attempt (up to our knowledge) to establish a CLT for a linear statistic of the eigenvalues of a Gram non-centered matrix (so-called signal plus noise model in [11, 12]). It complements (but does not supersede) the CLT established in [21] for a centered Gram matrix with a given variance profile. The fact that matrix Σ_n is non-centered ($\mathbb{E}\Sigma_n = A_n$) raises specific issues, from a different nature than those addressed in close-by results [1, 4, 21], etc. These issues arise from the presence in the computations of bilinear forms $u_n^* Q_n(z) v_n$ where at least one of the vectors u_n or v_n is deterministic. Often, the deterministic vector is related to the columns of matrix A_n, and has to be dealt with in such a way that the assumption over the spectral norm of A_n is exploited.

Another important contribution of this paper is to establish the CLT regardless of specific assumptions on the real or complex nature of the underlying random variables. It is in particular *not* assumed that the random variables are Gaussian, neither that whenever the random variables X_{ij} are complex, their second moment $\mathbb{E}X_{ij}^2$ is zero; nor is assumed that the random variables are circular.[a] As we shall see, all these assumptions, if assumed, would have resulted in substantial simplifications. As a reward however, we obtain a variance expression which smoothly depends upon $\mathbb{E}X_{ij}^2$ whose value is one in the real case, and zero in the complex case where the real and imaginary parts are not correlated.

Interestingly, the mutual information $\mathcal{I}_n$ has a strong relationship with the Stieltjes transform $f_n(z) = \frac{1}{N}\operatorname{Trace}(\Sigma_n\Sigma_n^* - zI_N)^{-1}$ of the spectral measure of $\Sigma_n\Sigma_n^*$:

$$\mathcal{I}_n(\rho) = \log \rho + \int_\rho^\infty \left(\frac{1}{w} - f_n(-w) \right) \mathbf{d}w.$$

Accordingly, the study of the fluctuations of $\mathcal{I}_n$ is also an important step toward the study of general linear statistics of $\Sigma_n\Sigma_n^*$'s eigenvalues which can be expressed via the Stieltjes transform:

$$\frac{1}{N}\operatorname{Trace} h(\Sigma_n\Sigma_n) = \frac{1}{N}\sum_{i=1}^N h(\lambda_i) = -\frac{1}{2\mathbf{i}\pi}\oint_{\mathcal{C}} h(z) f_n(z)\mathbf{d}z,$$

for some well-chosen contour $\mathcal{C}$ (see for instance [4]).

---

[a]A random variable $X \in \mathbb{C}$ is circular if the distribution of X is equal to the distribution of ρX for every $\rho \in \mathbb{C}$, $|\rho| = 1$. This assumption is very often relevant in wireless communication and has an important consequence; it implies that all the cross moments $\mathbb{E}|X|^k X^\ell$ ($\ell \geq 1$) are zero.

Fluctuations for particular linear statistics (and general classes of linear statistics) of large random matrices have been widely studied: CLTs for Wigner matrices can be traced back to Girko [14] (see also [17]). Results for this class of matrices have also been obtained by Khorunzhy *et al.* [27], Boutet de Monvel and Khorunzhy [7], Johansson [24], Sinai and Sochnikov [33], Soshnikov [34], Cabanal-Duvillard [8], Guionnet [18], Anderson and Zeitouni [1], Mingo and Speicher [29], Chatterjee [9], Lytova and Pastur [28], etc. The case of Gram matrices has been studied in Arharov [2], Jonsson [25], Bai and Silverstein [4], Hachem *et al.* [21], and also in [9, 28, 29]. Fluctuation results dedicated to wireless communication applications have been developed in the centered case ($A_n = 0$) by Debbah and Müller [10] and Tulino and Verdù [38] (based on Bai and Silverstein [4]), Hachem *et al.* [19] (for Gaussian entries) and [21]. Other fluctuation results either based on the replica method or on saddle-point analysis have been developed by Moustakas, Sengupta and coauthors [30, 31], and Tarrico [35, 36].

1.2. *Presentation of the results*

We first introduce the fundamental equations needed to express the deterministic approximation of the mutual information and the variance in the CLT.

1.2.1. *Fundamental equations, deterministic equivalents*

We collect here resuls from [20]. The following system of equations

$$
\begin{cases}
\delta_n(z) = \dfrac{1}{n}\operatorname{Tr} D_n(-z(I_N + \tilde{\delta}_n(z)D_n) + A_n(I_n + \delta_n(z)\tilde{D}_n)^{-1}A_n^*)^{-1} \\[2mm]
\tilde{\delta}_n(z) = \dfrac{1}{n}\operatorname{Tr} \tilde{D}_n(-z(I_n + \delta_n(z)\tilde{D}_n) + A_n^*(I_N + \tilde{\delta}_n(z)D_n)^{-1}A_n)^{-1}
\end{cases}, \quad z \in \mathbb{C} - \mathbb{R}^+
$$

(1.2)

admits a unique solution $(\delta_n, \tilde{\delta}_n)$ in the class of Stieltjes transforms of non-negative measures[b] with support in $\mathbb{R}^+$. Matrices $T_n(z)$ and $\tilde{T}_n(z)$ defined by

$$
\begin{cases}
T_n(z) = (-z(I_N + \tilde{\delta}_n(z)D_n) + A_n(I_n + \delta_n\tilde{D}_n)^{-1}A_n^*)^{-1} \\[2mm]
\tilde{T}_n(z) = (-z(I_n + \delta_n(z)\tilde{D}_n) + A_n^*(I_N + \tilde{\delta}_n D_n)^{-1}A_n)^{-1}
\end{cases}
$$

(1.3)

are approximations of the resolvent $Q_n(z) = (\Sigma_n\Sigma_n^* - zI_N)^{-1}$ and the co-resolvent $\tilde{Q}_n(z) = (\Sigma_n^*\Sigma_n - zI_n)^{-1}$ in the sense that ($\xrightarrow{a.s.}$ stands for almost sure convergence):

$$
\frac{1}{n}\operatorname{Tr}(Q_n(z) - T_n(z)) \xrightarrow[n\to\infty]{a.s.} 0, \quad z \in \mathbb{C} - \mathbb{R}^+
$$

which readily gives a deterministic approximation of the Stieltjes transform $N^{-1}\operatorname{Tr} Q_n(z)$ of the spectral measure of $\Sigma_n\Sigma_n^*$ in terms of T_n (and similarly for

---

[b]In fact, δ_n is the Stieltjes transform of a measure with total mass equal to $n^{-1}\operatorname{Tr} D_n$ while $\tilde{\delta}_n$ is the Stieltjes transform of a measure with total mass equal to $n^{-1}\operatorname{Tr} \tilde{D}_n$.

$\tilde{Q}_n$ and $\tilde{T}_n$). Also proven in [22] is the convergence of bilinear forms

$$u_n^*(Q_n(z) - T_n(z))v_n \xrightarrow[n\to\infty]{a.s.} 0, \quad z \in \mathbb{C} - \mathbb{R}^+, \tag{1.4}$$

where (u_n) and (v_n) are sequences of $N \times 1$ deterministic vectors with bounded Euclidean norms, which complements the picture of T_n approximating Q_n.

Matrices $T_n = (t_{ij}; 1 \le i, j \le N)$ and $\tilde{T}_n = (\tilde{t}_{ij}; 1 \le i, j \le n)$ will play a fundamental role in the sequel and enable us to express a deterministic equivalent to $\mathbb{E}\,\mathcal{I}_n(\rho)$. Define $V_n(\rho)$ by:

$$V_n(\rho) = \frac{1}{N} \log \det(\rho(I_N + \tilde{\delta}_n D_n) + A_n(I_n + \delta_n \tilde{D}_n)^{-1} A_n^*)$$

$$+ \frac{1}{N} \log \det(I_n + \delta_n \tilde{D}_n) - \frac{\rho n}{N} \delta_n \tilde{\delta}_n, \tag{1.5}$$

where δ_n and $\tilde{\delta}_n$ are evaluated at $z = -\rho$. Then the difference $\mathbb{E}\,\mathcal{I}_n(\rho) - V_n(\rho)$ goes to zero as $n \to \infty$.

In order to study the fluctuations $N(\mathcal{I}_n(\rho) - V_n(\rho))$ and to establish a CLT, we study separately the quantity $N(\mathcal{I}_n(\rho) - \mathbb{E}\,\mathcal{I}_n(\rho))$ from which the fluctuations arise and the quantity $N(\mathbb{E}\,\mathcal{I}_n(\rho) - V_n(\rho))$ which yields a bias.

1.2.2. *The fluctuations*

In every case where the fluctuations of the mutual information have been studied, the variance of $N(\mathcal{I}_n(\rho) - V_n(\rho))$ always proved to take a remarkably simple closed-form expression (see, for instance [30, 36, 38] and in a more mathematical flavor [19, 21]). The same phenomenon again occurs for the matrix model Σ_n under consideration. Drop the subscripts N, n and let

$$\gamma = \frac{1}{n}\operatorname{Tr} DTDT, \quad \tilde{\gamma} = \frac{1}{n}\operatorname{Tr} \tilde{D}\tilde{T}\tilde{D}\tilde{T},$$

$$\underline{\gamma} = \frac{1}{n}\operatorname{Tr} DTD\bar{T}, \quad \underline{\tilde{\gamma}} = \frac{1}{n}\operatorname{Tr} \tilde{D}\tilde{T}\tilde{D}\bar{\tilde{T}}, \tag{1.6}$$

where $\bar{M}$ stands for the (elementwise) conjugate of matrix M. Let

$$\vartheta = \mathbb{E}(X_{ij})^2 \quad \text{and} \quad \kappa = \mathbb{E}|X_{ij}|^4 - 2 - |\vartheta|^2.$$

Let

$$\Theta_n = -\log\left(\left(1 - \frac{1}{n}\operatorname{Tr} D^{\frac{1}{2}}TA(I + \delta\tilde{D})^{-1}\tilde{D}(I + \delta\tilde{D})^{-1}A^*TD^{\frac{1}{2}}\right)^2 - \rho^2\gamma\tilde{\gamma}\right)$$

$$- \log\left(\left|1 - \vartheta\frac{1}{n}\operatorname{Tr} D^{\frac{1}{2}}\bar{T}\bar{A}(I + \delta\tilde{D})^{-1}\tilde{D}(I + \delta\tilde{D})^{-1}A^*TD^{\frac{1}{2}}\right|^2 - |\vartheta|^2\rho^2\underline{\gamma}\,\underline{\tilde{\gamma}}\right)$$

$$+ \kappa\frac{\rho^2}{n^2}\sum_i d_i^2 t_{ii}^2 \sum_j \tilde{d}_j^2 \tilde{t}_{jj}^2,$$

where $d_i = [D_n]_{ii}$, $\tilde{d}_j = [\tilde{D}_n]_{jj}$, and all the needed quantities are evaluated at $z = -\rho$. The CLT can then be expressed as:

$$\frac{N}{\sqrt{\Theta_n}}(\mathcal{I}_n - \mathbb{E}\,\mathcal{I}_n) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1),$$

where $\xrightarrow{\mathcal{D}}$ stands for convergence in distribution. Although complicated at first sight, variance Θ_n encompasses the case of standard real random variables ($\vartheta = 1$), standard complex random variables ($\vartheta = 0$) and all the intermediate cases $0 < |\vartheta| < 1$. Moreover, Θ_n often takes simpler forms if the variables are Gaussian, real, etc. (see for instance Remark 2.2).

1.2.3. *The bias*

When the entries of X_n are complex Gaussian with independent and identically distributed real and imaginary parts, $\kappa = \vartheta = 0$, and it has already been proven in [13] that $\mathbb{E}\,\mathcal{I}_n(\rho) - V_n(\rho) = \mathcal{O}(n^{-2})$. When any of κ or ϑ is non-zero, a bias term $\mathcal{B}_n(\rho) \neq 0$ appears in the sense that

$$N(\mathbb{E}\,\mathcal{I}_n(\rho) - V_n(\rho)) - \mathcal{B}_n(\rho) \xrightarrow[n\to\infty]{} 0.$$

We establish the existence of this bias and provide its expression in the case where $A = 0$.

1.3. *Outline of the article*

In Sec. 2, we provide the main assumptions and state the main results of the paper: Definition of the variance Θ_n and asymptotic fluctuations of $N(\mathcal{I}_n(\rho) - \mathbb{E}\,\mathcal{I}_n(\rho))$ (Theorem 2.2), asymptotic bias of $N(\mathbb{E}\,\mathcal{I}_n(\rho) - V_n(\rho))$ (Proposition 2.1). Notations, important estimates and classical results are provided in Sec. 3. Sections 4, 5 and 6 are devoted to the proof of Theorem 2.2. In Sec. 4, the general framework of the proof is exposed; in Sec. 5, the central part of the CLT and of the identification of the variance are established; remaining proofs are provided in Sec. 6. Finally, proof of Proposition 2.1 (bias) is provided in Sec. 7.

2. The Central Limit Theorem for $\mathcal{I}_n(\rho)$

2.1. *Notations, assumptions and first-order results*

Let $\mathbf{i} = \sqrt{-1}$. As usual, $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. Denote by $\xrightarrow{\mathcal{P}}$ the convergence in probability of random variables and by $\xrightarrow{\mathcal{D}}$ the convergence in distribution of probability measures. Denote by $\mathrm{diag}(a_i; 1 \leq i \leq k)$ the $k \times k$ diagonal matrix whose diagonal entries are the a_i's. Element (i,j) of matrix M will be either denoted m_{ij} or $[M]_{ij}$ depending on the notational context. If M is an $n \times n$ square matrix, $\mathrm{diag}(M) = \mathrm{diag}(m_{ii}; 1 \leq i \leq n)$. Denote by M^T the matrix transpose of M, by M^* its Hermitian adjoint, by $\bar{M}$ the (elementwise) conjugate of matrix M, by $\mathrm{Tr}(M)$

its trace and $\det(M)$ its determinant (if M is square). When dealing with vectors, $\|\cdot\|$ will refer to the Euclidean norm. In the case of matrices, $\|\cdot\|$ will refer to the spectral norm. We shall denote by K a generic constant that does not depend on n and that might change from a line to another. If (u_n) is a sequence of real numbers, then $u_n = \mathcal{O}(v_n)$ stands for $|u_n| \leq K|v_n|$ where constant K does not depend on n.

Recall that

$$\Sigma_n = \frac{1}{\sqrt{n}} D_n^{1/2} X_n \tilde{D}_n^{1/2} + A_n, \tag{2.1}$$

denote $D_n = \operatorname{diag}(d_i, 1 \leq i \leq N)$ and $\tilde{D}_n = \operatorname{diag}(\tilde{d}_j, 1 \leq j \leq n)$. When no confusion can occur, we shall often drop subscripts and superscripts n for readability. Recall also that the asymptotic regime of interest is:

$$n \to \infty \quad \text{and} \quad 0 < \liminf \frac{N}{n} \leq \limsup \frac{N}{n} < \infty,$$

and will be simply denoted by $n \to \infty$ in the sequel. We can assume without loss of generality that there exist non-negative real numbers $\boldsymbol{\ell}^-$ and $\boldsymbol{\ell}^+$ such that:

$$0 < \boldsymbol{\ell}^- \leq \frac{N}{n} \leq \boldsymbol{\ell}^+ < \infty \quad \text{as} \quad n \to \infty. \tag{2.2}$$

Assumption 2.1. The random variables $(X_{ij}^n; 1 \leq i \leq N, 1 \leq j \leq n, n \geq 1)$ are complex, independent and identically distributed. They satisfy

$$\mathbb{E} X_{11}^n = 0, \quad \mathbb{E}|X_{11}^n|^2 = 1 \quad \text{and} \quad \mathbb{E}|X_{11}^n|^{16} < \infty.$$

Remark 2.1 (Gaussian distributions). If X_{11} is a standard complex or real Gaussian random variable, then $\kappa = 0$. More precisely, in the complex case, $\operatorname{Re}(X_{11})$ and $\operatorname{Im}(X_{11})$ are independent real Gaussian random variables, then $\vartheta = \kappa = 0$; in the real case, then $\vartheta = 1$ while $\kappa = 0$.

Assumption 2.2. The family of deterministic $N \times n$ complex matrices $(A_n, n \geq 1)$ is bounded for the spectral norm:

$$\boldsymbol{a}_{\max} = \sup_{n \geq 1} \|A_n\| < \infty.$$

Assumption 2.3. The families of real deterministic $N \times N$ and $n \times n$ matrices (D_n) and $(\tilde{D}_n)$ are diagonal with non-negative diagonal elements, and are bounded for the spectral norm as $n \to \infty$:

$$\boldsymbol{d}_{\max} = \sup_{n \geq 1} \|D_n\| < \infty \quad \text{and} \quad \tilde{\boldsymbol{d}}_{\max} = \sup_{n \geq 1} \|\tilde{D}_n\| < \infty.$$

Moreover,

$$\boldsymbol{d}_{\min} = \inf_n \frac{1}{n} \operatorname{Tr} D_n > 0 \quad \text{and} \quad \tilde{\boldsymbol{d}}_{\min} = \inf_n \frac{1}{n} \operatorname{Tr} \tilde{D}_n > 0.$$

Theorem 2.1 (First order results [13, 20]). *Consider the $N \times n$ matrix Σ_n given by (2.1) and let Assumptions 2.1–2.3 hold true. Then the system (1.2) admits a unique solution $(\delta_n, \tilde{\delta}_n)$ in the class of Stieltjes transforms of non-negative measures. Moreover,*

$$\frac{1}{n}\operatorname{Tr}(Q_n(z) - T_n(z)) \xrightarrow[n\to\infty]{a.s.} 0 \quad and \quad \frac{1}{n}\operatorname{Tr}(\tilde{Q}_n(z) - \tilde{T}_n(z)) \xrightarrow[n\to\infty]{a.s.} 0$$

for any $z \in \mathbb{C} - \mathbb{R}^+$.

2.2. *The central limit theorem*

In this section, we state the CLT then provide the asymptotic bias in some particular cases.

Theorem 2.2 (The CLT). *Consider the $N \times n$ matrix Σ_n given by (2.1) and let Assumptions 2.1–2.3 hold true. Recall the definitions of δ and $\tilde{\delta}$ given by (1.2), T and $\tilde{T}$ given by (1.3), γ, $\tilde{\gamma}$, $\underline{\gamma}$ and $\underline{\tilde{\gamma}}$ given by (1.6). Let $\rho > 0$. All the considered quantities are evaluated at $z = -\rho$. Define Δ_n and $\underline{\Delta}_n$ as*

$$\Delta_n = \left(1 - \frac{1}{n}\operatorname{Tr} D^{\frac{1}{2}}TA(I + \delta\tilde{D})^{-2}\tilde{D}A^*TD^{\frac{1}{2}}\right)^2 - \rho^2\gamma\tilde{\gamma}$$

and

$$\underline{\Delta}_n = \left|1 - \vartheta\frac{1}{n}\operatorname{Tr} D^{\frac{1}{2}}\bar{T}\bar{A}(I + \delta\tilde{D})^{-2}\tilde{D}A^*TD^{\frac{1}{2}}\right|^2 - |\vartheta|^2\rho^2\underline{\gamma}\underline{\tilde{\gamma}}.$$

Then the real numbers

$$\Theta_n = -\log\Delta_n - \log\underline{\Delta}_n + \kappa\frac{\rho^2}{n^2}\sum_{i=1}^{N}d_i^2t_{ii}^2\sum_{j=1}^{n}\tilde{d}_j^2\tilde{t}_{jj}^2 \tag{2.3}$$

are well-defined and satisfy:

$$0 < \liminf_n \Theta_n \leq \limsup_n \Theta_n < \infty \tag{2.4}$$

as $n \to \infty$. Let

$$\mathcal{I}_n(\rho) = \frac{1}{N}\log\det(\Sigma_n\Sigma_n^* + \rho I_N),$$

then the following convergence holds true:

$$\frac{N}{\sqrt{\Theta_n}}(\mathcal{I}_n(\rho) - \mathbb{E}\,\mathcal{I}_n(\rho)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1).$$

Remark 2.2 (Simpler forms for the variance). We consider here special cases where the variance Θ_n takes a simpler form.

(1) The standard complex Gaussian case. Assume that the X_{ij}'s are standard complex Gaussian random variables, i.e. that both the real and imaginary parts of

X_{ij} are independent real Gaussian random variables, each with variance $1/2$. In this case, $\vartheta = \kappa = 0$ and Θ_n is equal to $-\log \Delta_n$, and we in particular recover the variance formula given in [36].

(2) The standard real case. Assume that the X_{ij}'s are standard real random variables, assume also that A has real entries. Then Δ_n and $\underline{\Delta}_n$ are equal.

(3) The "signal plus noise" model. In this case, $D_n = I_N$ and $\tilde{D}_n = I_n$, which already yields simplifications in the variance expression. In the case where $\vartheta = 0$, the variance is:

$$\Theta_n = -\log((1 - n^{-1}(1 + \delta)^{-2} \operatorname{Tr} TAA^*T)^2 - \rho^2\gamma\tilde{\gamma}) + \frac{\kappa\rho^2}{n^2} \sum_i d_i^2 t_{ii}^2 \sum_j \tilde{d}_j^2 \tilde{t}_{jj}^2.$$

As one may easily check, the first term of the variance only depends upon the spectrum of AA^*. The second term however also depends on the eigenvectors of AA^* (see, for instance [26]).

A full study of the asymptotic bias turns out to be extremely involved and would have substantially increased the volume of this paper. In the following proposition, we restrict our study to two important particular cases:

Proposition 2.1 (The bias — particular cases). *Assume that the setting of Theorem 2.2 holds true.*

(i) *If the random variables $(X_{ij}^n; i, j, n)$ are complex with $\operatorname{Re}(X_{ij}^n)$ and $\operatorname{Im}(X_{ij}^n)$ independent, both with distribution $\mathcal{N}(0, 1/2)$, then:*

$$N(\mathbb{E}\,\mathcal{I}_n(\rho) - V_n(\rho)) = \mathcal{O}\left(\frac{1}{N}\right).$$

(ii) *If $A_n = 0$, let the quantities γ and $\tilde{\gamma}$ be evaluated at $z = -\rho$ and consider*

$$\mathcal{B}_n = -\frac{\kappa}{2}\rho^2\gamma\tilde{\gamma} + \frac{1}{2}\log(1 - |\vartheta|^2\rho^2\gamma\tilde{\gamma}). \tag{2.5}$$

Then

$$N(\mathbb{E}\,\mathcal{I}_n(\rho) - V_n(\rho)) - \mathcal{B}_n \xrightarrow[n\to\infty]{} 0.$$

Remark 2.3. Observe that $T(z) = [-z(I + \tilde{\delta}(z)D)]^{-1}$ and $\tilde{T}(z) = [-z(I + \delta(z)\tilde{D})]^{-1}$ when $A = 0$. It is interesting to notice that $\mathcal{B}_n$ coincides in that case with $-0.5\times$ the sum of the two last terms at the right-hand side of (2.3).

Proof of Proposition 2.1 is deferred to Sec. 7.

3. Notations and Classical Results

3.1. *Further notations*

We denote by Y the $N \times n$ matrix $n^{-1/2}D^{1/2}X\tilde{D}^{1/2}$; by (η_j), (a_j) and (y_j) the columns of matrices Σ, A and Y. Denote by Σ_j, A_j, and Y_j the matrices Σ, A,

and Y where column j has been removed. The associated resolvent is $Q_j(z) = (\Sigma_j \Sigma_j^* - z I_N)^{-1}$. We shall often write Q, Q_j, T for $Q(z), Q_j(z), T(z)$, etc. We denote by $\tilde{D}_j$ matrix $\tilde{D}$ where row and column j have been removed. We also denote by $A_{1:j}$ and $\Sigma_{1:j}$ the $N \times j$ matrices $A_{1:j} = [a_1, \ldots, a_j]$ and $\Sigma_{1:j} = [\eta_1, \ldots, \eta_j]$. Denote by $\mathbb{E}_j$ the conditional expectation with respect to the σ-field $\mathcal{F}_j$ generated by the vectors $(y_\ell, 1 \leq \ell \leq j)$. By convention, $\mathbb{E}_0 = \mathbb{E}$.

We introduce here intermediate quantities of constant use in the rest of the paper. For $1 \leq j \leq n$, let:

$$\tilde{b}_j(z) = \frac{-1}{z(1 + a_j^* Q_j(z) a_j + \frac{\tilde{d}_j}{n} \operatorname{Tr} D Q_j(z))},$$

$$e_j(z) = \eta_j^* Q_j(z) \eta_j - \left(\frac{\tilde{d}_j}{n} \operatorname{Tr} D Q_j(z) + a_j^* Q_j(z) a_j \right)$$

$$= y_j^* Q_j(z) y_j - \frac{\tilde{d}_j}{n} \operatorname{Tr} D Q_j(z) + a_j^* Q_j(z) y_j + y_j^* Q_j(z) a_j. \qquad (3.1)$$

3.2. *Important identities*

Recall the following classical identity for the inverse of a perturbed matrix (see [23, Sec. 0.7.4]):

$$(A + XRY)^{-1} = A^{-1} - A^{-1} X (R^{-1} + Y A^{-1} X)^{-1} Y A^{-1}. \qquad (3.2)$$

3.2.1. *Identities involving the resolvents*

The following identity expresses the diagonal elements $\tilde{q}_{jj}(z) = [\tilde{Q}(z)]_{jj}$ of the co-resolvent; the two following ones are obtained from (3.2).

$$\tilde{q}_{jj}(z) = \frac{-1}{z(1 + \eta_j^* Q_j(z) \eta_j)}, \qquad (3.3)$$

$$Q(z) = Q_j(z) - \frac{Q_j(z) \eta_j \eta_j^* Q_j(z)}{1 + \eta_j^* Q_j(z) \eta_j}$$

$$= Q_j(z) + z \tilde{q}_{jj}(z) Q_j(z) \eta_j \eta_j^* Q_j(z), \qquad (3.4)$$

$$Q_j(z) = Q(z) + \frac{Q(z) \eta_j \eta_j^* Q(z)}{1 - \eta_j^* Q(z) \eta_j}, \qquad (3.5)$$

$$1 + \eta_j^* Q_j(z) \eta_j = \frac{1}{1 - \eta_j^* Q(z) \eta_j}. \qquad (3.6)$$

Notice that

$$\tilde{q}_{jj}(z) = \tilde{b}_j(z) + z \tilde{q}_{jj}(z) \tilde{b}_j(z) e_j(z) \qquad (3.7)$$

and that $0 < \tilde{b}_j(-\rho), \tilde{q}_{jj}(-\rho) < \rho^{-1}$. These facts will be repeatedly used in the remainder. A useful consequence of (3.4) is:

$$\eta_j^* Q(z) = \frac{\eta_j^* Q_j(z)}{1 + \eta_j^* Q_j(z) \eta_j} = -z \tilde{q}_{jj}(z) \eta_j^* Q_j(z). \tag{3.8}$$

3.2.2. *Identities involving the deterministic equivalents T and $\tilde{T}$*

Define the $N \times N$ matrix $\mathcal{T}_j(z)$ as

$$\mathcal{T}_j(z) = (-z(I_N + \tilde{\delta}(z)D) + A_j(I_{n-1} + \delta(z)\tilde{D}_j)^{-1}A_j^*)^{-1}, \tag{3.9}$$

where δ and $\tilde{\delta}$ are defined in (1.2). Notice that matrix $\mathcal{T}_j$ is not obtained in general by solving the analogue of system (1.3) where A is replaced with A_j and when $\tilde{D}$ is truncated accordingly. This matrix naturally pops up when expressing the diagonal elements $\tilde{t}_{jj}$ of $\tilde{T}$. Indeed, we obtain (see Appendix A.1):

$$\tilde{t}_{jj}(z) = \frac{-1}{z(1 + a_j^* \mathcal{T}_j(z)a_j + \tilde{d}_j\delta(z))}. \tag{3.10}$$

Let b be a given $N \times 1$ vector. The following identity is also shown in Appendix A.1:

$$-z\tilde{t}_{\ell\ell}(z)a_\ell^* \mathcal{T}_\ell(z)b = \frac{a_\ell^* T(z)b}{1 + \tilde{d}_\ell \delta(z)}. \tag{3.11}$$

Thanks to (3.2), we also have

$$\tilde{T}(z) = -z^{-1}(I + \delta(z)\tilde{D})^{-1} + z^{-1}(I + \delta(z)\tilde{D})^{-1}A^* T(z)A(I + \delta(z)\tilde{D})^{-1}. \tag{3.12}$$

3.3. *Important estimates*

We gather in this section matrix estimates which will be of constant use in the sequel. In all the remainder, z will belong to the open negative real axis, and will be fixed to $z = -\rho$ until Sec. 7.

Let A and B be two square matrices. Then

$$|\mathrm{Tr}(AB)| \leq \sqrt{\mathrm{Tr}(AA^*)}\sqrt{\mathrm{Tr}(BB^*)} \tag{3.13}$$

When B is Hermitian non-negative, then a consequence of Von Neumann's trace theorem is

$$|\mathrm{Tr}(AB)| \leq \|A\|\mathrm{Tr}\,B. \tag{3.14}$$

The following lemma gives an estimate for a rank-one perturbation of the resolvent ([21, Lemma 6.3] and [32, Lemma 2.6]):

Lemma 3.1. *The resolvents Q and the perturbed resolvent Q_j satisfy for $z = -\rho$:*

$$|\mathrm{Tr}\,A(Q - Q_j)| \leq \frac{\|A\|}{\rho}$$

for any $N \times N$ matrix A.

The following results describe the asymptotic behavior of quadratic forms based on the resolvent.

Lemma 3.2 (Bai and Silverstein [3, Lemma 2.7]). *Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an $n \times 1$ vector where the x_i are centered i.i.d. complex random variables with unit variance. Let M be an $n \times n$ deterministic complex matrix. Then for any $p \geq 2$, there exists a constant K_p depending on the 2p-th moment of x_1 for which*

$$\mathbb{E}|\boldsymbol{x}^* M \boldsymbol{x} - \operatorname{Tr} M|^p \leq K_p((\mathbb{E}|x_1|^4 \operatorname{Tr} MM^*)^{p/2} + \mathbb{E}|x_1|^{2p} \operatorname{Tr}(MM^*)^{p/2}).$$

Remark 3.1. There are some important consequences of the previous lemma. Let (M_n) be a sequence of $n \times n$ deterministic matrices with bounded spectral norm and $(\boldsymbol{x}_n)$ be a sequence of random vectors as in the statement of Lemma 3.2. Then for any $p \in [2; 8]$,

$$\max\left(\mathbb{E}\left|\frac{\boldsymbol{x}_n^* M_n \boldsymbol{x}_n}{n} - \frac{\operatorname{Tr} M_n}{n}\right|^p, \mathbb{E}|e_j|^p\right) \leq \frac{K}{n^{p/2}} \tag{3.15}$$

where e_j is given by (3.1) (the estimate $\mathbb{E}|e_j|^p = \mathcal{O}(n^{-p/2})$ is proven in Appendix A.2).

Remark 3.2. By replacing $\mathbb{E}|x_1|^4$ with $\max_i \mathbb{E}|x_i|^4$ and $\mathbb{E}|x_1|^{2p}$ with $\max_i \mathbb{E}|x_i|^{2p}$, Lemma 3.2 can be extended to the case where elements of vector $\boldsymbol{x}$ are independent but not necessarily identically distributed [5, Lemma B.26]. Accordingly, the results of this paper remain true when the X_{ij} are independent but not necessarily identically distributed, provided $\mathbb{E}|X_{ij}|^2 = 1$, $\mathbb{E}X_{ij}^2 = \vartheta$, and $\mathbb{E}|X_{ij}|^4 - 2 - |\vartheta|^2 = \kappa$ for all i, j, and $\sup_n \max_{i,j} \mathbb{E}|X_{ij}|^{16} < \infty$.

The following theorem is proven is Appendix A.3:

Theorem 3.1. *Assume that the setting of Theorem 2.2 holds true. Let (u_n) and (v_n) be two sequences of deterministic complex $N \times 1$ vectors bounded in the Euclidean norm:*

$$\sup_{n \geq 1} \max(\|u_n\|, \|v_n\|) < \infty,$$

and let (U_n) be a sequence of deterministic $N \times N$ matrices with bounded spectral norms:

$$\sup_{n \geq 1} \|U_n\| < \infty.$$

Then,

(1) There exists a constant K for which

$$\sum_{j=1}^{n} \mathbb{E}|u_n^* Q_j a_j|^2 \leq K.$$

(2) *The following holds true*:

$$\left| \frac{1}{n} \operatorname{Tr} U(T - \mathbb{E}Q) \right| \le \frac{K}{n}.$$

(3) *For every $p \in [1,2]$, there exists a constant K_p such that*:

$$\mathbb{E}|u_n^*(Q - T)v_n|^{2p} \le \frac{K_p}{n^p}.$$

(4) *For every $p \in [1,2]$, there exists a constant K_p such that*:

$$\mathbb{E}|u_n^*(Q_j - \mathcal{T}_j)v_n|^{2p} \le \frac{K_p}{n^p}.$$

(5) *There exists a constant K such that*

$$\mathbb{E}|\operatorname{Tr} U(Q - \mathbb{E}Q)|^2 < K.$$

The following results stem from Lemma 3.2 and Theorem 3.1 and will be of constant use in the sequel. Recalling (3.7) and (3.15) along with the bounds on $\tilde{q}_{jj}$ and $\tilde{b}_j$, we have for any $p \in [2,8]$

$$\mathbb{E}|\tilde{q}_{jj} - \tilde{b}_j|^p \le \frac{K}{n^{p/2}}. \tag{3.16}$$

Of course, the counterpart of Theorem 3.1 for the co-resolvent $\tilde{Q}$ and matrix $\tilde{T}$ holds true. In particular, taking the vectors u_n and v_n as the jth canonical vector of $\mathbb{C}^n$ yields the following estimate for any $p \in [2,4]$:

$$\mathbb{E}|\tilde{q}_{jj} - \tilde{t}_{jj}|^p \le \frac{K}{n^{p/2}}. \tag{3.17}$$

The following two lemmas, proven in Appendices A.4 and A.5, provide some important bounds:

Lemma 3.3. *Assume that the setting of Theorem 2.2 holds true. Then, the following quantities satisfy*:

$$\frac{\boldsymbol{d}_{\min}}{\rho + \boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max} + \boldsymbol{a}_{\max}^2} \le \delta_n \le \frac{\boldsymbol{\ell}^+\boldsymbol{d}_{\max}}{\rho},$$

$$\frac{\tilde{\boldsymbol{d}}_{\min}}{\rho + \boldsymbol{\ell}^+\boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max} + \boldsymbol{a}_{\max}^2} \le \tilde{\delta}_n \le \frac{\tilde{\boldsymbol{d}}_{\max}}{\rho},$$

$$\frac{\boldsymbol{d}_{\min}}{(\rho + \boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max} + \boldsymbol{a}_{\max}^2)^2} \le \frac{1}{n} \operatorname{Tr} DT^2 \le \frac{\boldsymbol{\ell}^+\boldsymbol{d}_{\max}}{\rho^2},$$

$$\frac{\tilde{\boldsymbol{d}}_{\min}}{(\rho + \boldsymbol{\ell}^+\boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max} + \boldsymbol{a}_{\max}^2)^2} \le \frac{1}{n} \operatorname{Tr} \tilde{D}\tilde{T}^2 \le \frac{\tilde{\boldsymbol{d}}_{\max}}{\rho^2},$$

$$\frac{d_{\min}^2}{\ell^+(\rho + d_{\max}\tilde{d}_{\max} + a_{\max}^2)^2} \le \gamma_n \le \frac{\ell^+ d_{\max}^2}{\rho^2},$$

$$\frac{\tilde{d}_{\min}^2}{(\rho + \ell^+ d_{\max}\tilde{d}_{\max} + a_{\max}^2)^2} \le \tilde{\gamma}_n \le \frac{\tilde{d}_{\max}^2}{\rho^2}$$

$$\frac{d_{\min}^2}{\ell^+(\rho + d_{\max}\tilde{d}_{\max} + a_{\max}^2)^2} \le \frac{1}{n}\sum_{i=1}^{N} d_i^2 t_{ii}^2 \le \frac{\ell^+ d_{\max}^2}{\rho^2},$$

$$\frac{\tilde{d}_{\min}^2}{(\rho + \ell^+ d_{\max}\tilde{d}_{\max} + a_{\max}^2)^2} \le \frac{1}{n}\sum_{j=1}^{n} \tilde{d}_j^2 \tilde{t}_{jj}^2 \le \frac{\tilde{d}_{\max}^2}{\rho^2}.$$

Lemma 3.4. *Assume that the setting of Theorem* 2.2 *holds true. Then*

$$\sup_n \frac{1}{n}\operatorname{Tr} D^{1/2} T A (I + \delta\tilde{D})^{-2} \tilde{D} A^* T D^{\frac{1}{2}} < 1.$$

Moreover, the sequence (Δ_n) *as defined in Theorem* 2.2 *satisfies*

$$\Delta_n \ge \frac{\rho}{n\delta}\operatorname{Tr} DT^2 \frac{\rho}{n\tilde{\delta}}\operatorname{Tr} \tilde{D}\tilde{T}^2$$

and

$$\liminf_n \Delta_n > 0.$$

3.4. *Other important results*

The main result we shall rely on to establish the Central Limit Theorem is the following CLT for martingales:

Theorem 3.2 (CLT for martingales, [6, Theorem 35.12]). *Let* $\gamma_1^{(n)}$, $\gamma_2^{(n)}, \ldots, \gamma_n^{(n)}$ *be a martingale difference sequence with respect to the increasing filtration* $\mathcal{F}_1^{(n)}, \ldots, \mathcal{F}_n^{(n)}$. *Assume that there exists a sequence of real positive numbers* Υ_n^2 *such that*

$$\frac{1}{\Upsilon_n^2}\sum_{j=1}^{n} \mathbb{E}_{j-1} \gamma_j^{(n)\,2} \xrightarrow[n\to\infty]{\mathcal{P}} 1. \tag{3.18}$$

Assume further that the Lyapounov condition ([6, Sec. 27]) holds true:

$$\exists \delta > 0, \quad \frac{1}{\Upsilon_n^{2(1+\delta)}}\sum_{j=1}^{n} \mathbb{E}|\gamma_j^{(n)}|^{2+\delta} \xrightarrow[n\to\infty]{} 0.$$

Then $\Upsilon_n^{-1}\sum_{j=1}^{n} \gamma_j^{(n)}$ *converges in distribution to* $\mathcal{N}(0,1)$.

Remark 3.3. Note that if moreover $\liminf_n \Upsilon_n^2 > 0$, it is sufficient to prove:

$$\sum_{j=1}^{n} \mathbb{E}_{j-1}\gamma_j^{(n)^2} - \Upsilon_n^2 \xrightarrow[n\to\infty]{\mathcal{P}} 0, \tag{3.19}$$

instead of (3.18).

We now state a covariance identity (the proof of which is straightforward and therefore omitted) for quadratic forms based on non-centered vectors. This identity explains to some extent the various terms obtained in the variance.

Let $\boldsymbol{x} = (x_1, \ldots, x_N)^T$ be an $N \times 1$ vector where the x_i are centered i.i.d. complex random variables with unit variance. Let $\boldsymbol{y} = N^{-1/2}D^{1/2}\boldsymbol{x}$ where D is an $N \times N$ diagonal non-negative deterministic matrix. Let $M = (m_{ij})$ and $P = (p_{ij})$ be $N \times N$ deterministic complex matrices and let $\boldsymbol{u}$ be an $N \times 1$ deterministic vector.

If M is an $N \times N$ matrix, vdiag(M) stands for the $N \times 1$ vector $[M_{11}, \ldots, M_{NN}]^T$. Denote by $\Upsilon(M)$ the random variable:

$$\Upsilon(M) = (\boldsymbol{y} + \boldsymbol{u})^* M (\boldsymbol{y} + \boldsymbol{u}).$$

Then $\mathbb{E}\Upsilon(M) = \frac{1}{N}\operatorname{Tr} DM + \boldsymbol{u}^* M \boldsymbol{u}$ and the covariance between $\Upsilon(M)$ and $\Upsilon(P)$ is:

$$\mathbb{E}[(\Upsilon(M) - \mathbb{E}\Upsilon(M))(\Upsilon(P) - \mathbb{E}\Upsilon(P))]$$

$$= \frac{1}{N^2}\operatorname{Tr}(MDPD) + \frac{1}{N}(\boldsymbol{u}^* MDP\boldsymbol{u} + \boldsymbol{u}^* PDM\boldsymbol{u})$$

$$+ \frac{|\mathbb{E}[x_1^2]|^2}{N^2}\operatorname{Tr}(MDP^T D) + \frac{\mathbb{E}[x_1^2]}{N}\boldsymbol{u}^* PDM^T \bar{\boldsymbol{u}} + \frac{\mathbb{E}[\bar{x}_1^2]}{N}\boldsymbol{u}^T M^T DP\boldsymbol{u}$$

$$+ \frac{\mathbb{E}[|x_1|^2 x_1]}{N^{3/2}}(\boldsymbol{u}^* PD^{3/2}\operatorname{vdiag}(M) + \boldsymbol{u}^* MD^{3/2}\operatorname{vdiag}(P))$$

$$+ \frac{\mathbb{E}[|x_1|^2 \bar{x}_1]}{N^{3/2}}(\operatorname{vdiag}(P)^T D^{3/2} M\boldsymbol{u} + \operatorname{vdiag}(M)^T D^{3/2} P\boldsymbol{u})$$

$$+ \frac{\kappa}{N^2}\sum_{i=1}^{N} d_{ii}^2 m_{ii}p_{ii}, \tag{3.20}$$

where $\kappa = \mathbb{E}|x_1|^4 - 2 - |\mathbb{E}x_1^2|^2$.

Remark 3.4. Identity (3.20) is the cornerstone for the proof of the CLT; it is the counterpart of Identity (1.15) in [4]. The complexity of Identity (3.20) with respect to [4, Identity (1.15)] lies in 8 extra terms and stems from two elements:

(1) The fact that matrix Σ is non-centered.
(2) The fact that the random variables X_{ij}'s are either real and complex with no particular assumption on their second moment (in particular, $\mathbb{E}X_{ij}^2$ can be non-zero in the complex case).

It is this identity which induces to a large extent all the computations in the present article.

4. Proof of Theorem 2.2 (Part I)

Decomposition of $\mathcal{I}_n - \mathbb{E}\,\mathcal{I}_n$, Cumulant and cross-moments terms in the variance.

4.1. Decomposition of $\mathcal{I}_n - \mathbb{E}\,\mathcal{I}_n$ as a sum of martingale differences

Denote by

$$\Gamma_j = \frac{\eta_j^* Q_j \eta_j - (\frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j + a_j^* Q_j a_j)}{1 + \frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j + a_j^* Q_j a_j}.$$

With this notation at hand, the decomposition of $\mathcal{I}_n - \mathbb{E}\,\mathcal{I}_n$ as

$$\mathcal{I}_n - \mathbb{E}\,\mathcal{I}_n = \sum_{j=1}^{n}(\mathbb{E}_j - \mathbb{E}_{j-1})\log(1 + \Gamma_j) \tag{4.1}$$

follows verbatim from [21, Sec. 6.2]. Moreover, it is a matter of bookkeeping to establish the following (cf. [21, Sec. 6.4]):

$$\sum_{j=1}^{n}\mathbb{E}_{j-1}((\mathbb{E}_j - \mathbb{E}_{j-1})\log(1 + \Gamma_j))^2 - \sum_{j=1}^{n}\mathbb{E}_{j-1}(\mathbb{E}_j\Gamma_j)^2 \xrightarrow[n\to\infty]{\mathcal{P}} 0. \tag{4.2}$$

Hence, the details are omitted. In view of Theorem 3.2, Eqs. (3.19), (4.1) and (4.2), the CLT will be established if one proves the following 3 results:

(1) (Lyapounov condition)

$$\exists \delta > 0, \quad \sum_{j=1}^{n}\mathbb{E}|\mathbb{E}_j\Gamma_j|^{2+\delta} \xrightarrow[n\to\infty]{} 0.$$

(2) (Martingale increments and variance)

$$\sum_{j=1}^{n}\mathbb{E}_{j-1}(\mathbb{E}_j\Gamma_j)^2 - \Theta_n \xrightarrow[n\to\infty]{\mathcal{P}} 0.$$

(3) (Estimates over the variance)

$$0 < \liminf_n \Theta_n \leq \limsup_n \Theta_n < \infty.$$

It is straightforward (and hence omitted) to verify Lyapounov condition. The convergence toward the variance is the cornerstone of the proof of the CLT: The rest of this section together with much of Sec. 5 are devoted to establish it. The estimates over the variance Θ_n, also central to apply Theorem 3.2, are established in Sec. 6.2.

Notice that $\mathbb{E}_{j-1}(\mathbb{E}_j \Gamma_j)^2 = \mathbb{E}_{j-1}(\mathbb{E}_j \rho \tilde{b}_j e_j)^2$. We prove hereafter that

$$\sum_{j=1}^{n} \mathbb{E}_{j-1}(\mathbb{E}_j \rho \tilde{b}_j e_j)^2 - \sum_{j=1}^{n} \rho^2 \tilde{t}_{jj}^2 \mathbb{E}_{j-1}(\mathbb{E}_j e_j)^2 \xrightarrow[n\to\infty]{\mathcal{P}} 0. \qquad (4.3)$$

The inequality $\mathbb{E}|\tilde{b}_j - \tilde{t}_{jj}|^2 \le 2\mathbb{E}|\tilde{b}_j - \tilde{q}_{jj}|^2 + 2\mathbb{E}|\tilde{q}_{jj} - \tilde{t}_{jj}|^2$ in conjunction with Estimates (3.16) and (3.17) yield $\mathbb{E}|\tilde{b}_j - \tilde{t}_{jj}|^2 = \mathcal{O}(n^{-1})$. Moreover,

$$\mathbb{E}|\mathbb{E}_{j-1}(\mathbb{E}_j \rho \tilde{b}_j e_j)^2 - \mathbb{E}_{j-1}(\rho \tilde{t}_{jj} \mathbb{E}_j e_j)^2| \le \mathbb{E}|(\mathbb{E}_j \rho \tilde{b}_j e_j)^2 - (\mathbb{E}_j \rho \tilde{t}_{jj} e_j)^2|$$

$$= \mathbb{E}|(\mathbb{E}_j(\rho \tilde{b}_j - \rho \tilde{t}_{jj})e_j)(\mathbb{E}_j(\rho \tilde{b}_j + \rho \tilde{t}_{jj})e_j)|$$

$$\le \mathbb{E}|(\rho \tilde{b}_j - \rho \tilde{t}_{jj})e_j(\mathbb{E}_j(\rho \tilde{b}_j + \rho \tilde{t}_{jj})e_j)|$$

$$\le K n^{-3/2}$$

using Cauchy–Schwarz inequality and (3.15). This implies (4.3). Let $\varsigma = \mathbb{E}(|X_{11}^2|X_{11})$. Using Identity (3.20), we develop the quantity $\mathbb{E}_{j-1}(\mathbb{E}_j e_j)^2$:

$$\sum_{j=1}^{n} \rho^2 \tilde{t}_{jj}^2 \mathbb{E}_{j-1}(\mathbb{E}_j e_j)^2$$

$$= \frac{\kappa}{n^2} \sum_{j=1}^{n} \rho^2 \tilde{d}_j^2 \tilde{t}_{jj}^2 \sum_{i=1}^{N} d_i^2 [\mathbb{E}_j Q_j]_{ii}^2$$

$$+ \frac{4}{n} \sum_{j=1}^{n} \rho^2 \tilde{d}_j^{3/2} \tilde{t}_{jj}^2 \operatorname{Re}\left(\varsigma \frac{a_j^*(\mathbb{E}_j Q_j) D^{3/2} \operatorname{vdiag}(\mathbb{E}_j Q_j)}{\sqrt{n}} \right)$$

$$+ \frac{1}{n} \sum_{j=1}^{n} \rho^2 \tilde{t}_{jj}^2 \left(\frac{\tilde{d}_j^2}{n} \operatorname{Tr}(\mathbb{E}_j Q_j) D(\mathbb{E}_j Q_j) D + 2\tilde{d}_j a_j^*(\mathbb{E}_j Q_j) D(\mathbb{E}_j Q_j) a_j \right)$$

$$+ \frac{1}{n} \sum_{j=1}^{n} \rho^2 \tilde{t}_{jj}^2 \left(|\vartheta|^2 \frac{\tilde{d}_j^2}{n} \operatorname{Tr}(\mathbb{E}_j Q_j) D(\mathbb{E}_j \bar{Q}_j) D + 2\operatorname{Re}(\vartheta \tilde{d}_j a_j^*(\mathbb{E}_j Q_j) D(\mathbb{E}_j \bar{Q}_j) \bar{a}_j) \right)$$

$$\overset{\triangle}{=} \sum_{j=1}^{n} \chi_{1j} + \sum_{j=1}^{n} \chi_{2j} + \sum_{j=1}^{n} \chi_{3j} + \sum_{j=1}^{n} \chi_{4j}.$$

4.2. *Key lemmas for the identification of the variance*

The remainder of the proof of Theorem 2.2 is devoted to find deterministic equivalents for the terms $\sum_{j=1}^{n} \chi_{\ell j}$ for $\ell = 1, 2, 3, 4$.

Lemma 4.1. *Assume that the setting of Theorem* 2.2 *holds true, then*:

$$\sum_{j=1}^{n} \chi_{1j} - \frac{\kappa \rho^2}{n^2} \sum_{i=1}^{N} \sum_{j=1}^{n} d_i^2 t_{ii}^2 \tilde{d}_j^2 \tilde{t}_{jj}^2 \xrightarrow[n\to\infty]{\mathcal{P}} 0.$$

Proof. Write

$$\frac{1}{n}\sum_{i=1}^{N}d_i^2[\mathbb{E}_jQ_j]_{ii}^2 - \frac{1}{n}\sum_{i=1}^{N}d_i^2[\mathbb{E}_jQ_j]_{ii}t_{ii}$$

$$= \frac{1}{n}\sum_{i=1}^{N}d_i^2[\mathbb{E}_jQ_j]_{ii}\mathbb{E}_j([Q_j]_{ii}-[Q]_{ii}) + \frac{1}{n}\sum_{i=1}^{N}d_i^2[\mathbb{E}_jQ_j]_{ii}([\mathbb{E}_jQ]_{ii}-t_{ii})$$

$$= \varepsilon_{1,j} + \varepsilon_{2,j}.$$

The term $|\varepsilon_{1,j}| = n^{-1}|\mathbb{E}_j[\operatorname{Tr} D^2\operatorname{diag}(\mathbb{E}_jQ_j)(Q_j-Q)]|$ is of order $\mathcal{O}(n^{-1})$ thanks to Lemma 3.1. Moreover, $\mathbb{E}|\varepsilon_{2,j}| = \mathcal{O}(n^{-1/2})$ by the analogue of (3.17) for the diagonal elements of the resolvent. Hence,

$$\sum_{j=1}^{n}\chi_{1j} - \frac{\kappa\rho^2}{n}\sum_{j=1}^{n}\tilde{d}_j^2\tilde{t}_{jj}^2\left(\frac{1}{n}\sum_{i=1}^{N}d_i^2[\mathbb{E}_jQ_j]_{ii}t_{ii}\right) \xrightarrow[n\to\infty]{\mathcal{P}} 0.$$

Iterating the same arguments, we can replace the remaining term $\mathbb{E}_j[Q_j]_{ii}$ by t_{ii} to obtain the desired result. $\quad\square$

Lemma 4.2. *Assume that the setting of Theorem* 2.2 *holds true. Then:*

$$\sum_{j=1}^{n}\chi_{2j} \xrightarrow[n\to\infty]{\mathcal{P}} 0.$$

Proof. We have

$$\mathbb{E}\left|\sum_{j=1}^{n}\chi_{2j}\right| \leq \frac{K}{n}\sum_{j=1}^{n}\mathbb{E}\left|a_j^*(\mathbb{E}_jQ_j)D^{3/2}\frac{\operatorname{vdiag}(Q_j)}{\sqrt{n}}\right|$$

$$\leq \frac{K}{n}\sum_{j=1}^{n}\mathbb{E}\left|a_j^*Q_jD^{3/2}\frac{\operatorname{vdiag}(T)}{\sqrt{n}}\right|$$

$$+ \frac{K}{n}\sum_{j=1}^{n}\mathbb{E}\left|a_j^*(\mathbb{E}_jQ_j)D^{3/2}\frac{\operatorname{vdiag}(Q-T)}{\sqrt{n}}\right|$$

$$+ \frac{K}{n}\sum_{j=1}^{n}\mathbb{E}\left|a_j^*(\mathbb{E}_jQ_j)D^{3/2}\frac{\operatorname{vdiag}(Q_j-Q)}{\sqrt{n}}\right|. \qquad (4.4)$$

The first term satisfies

$$\sum_{j=1}^{n}\mathbb{E}\left|a_j^*Q_jD^{3/2}\frac{\operatorname{vdiag}(T)}{\sqrt{n}}\right| \leq \sqrt{n}\left(\sum_{j=1}^{n}\mathbb{E}\left|a_j^*Q_jD^{3/2}\frac{\operatorname{vdiag}(T)}{\sqrt{n}}\right|^2\right)^{1/2}.$$

As $\|n^{-1/2}D^{3/2}\operatorname{vdiag}(T)\| = (n^{-1}\sum_{i=1}^{N}d_i^3t_{ii}^2)^{1/2} \leq K$, Theorem 3.1(1) can be applied, and the first term at the right-hand side of (4.4) is of order $n^{-1/2}$. We

now deal with the second term at the right-hand side

$$\mathbb{E}\left|a_j^*(\mathbb{E}_j Q_j) D^{3/2} \frac{\text{vdiag}(Q-T)}{\sqrt{n}}\right| \leq K\mathbb{E}\left\|\frac{\text{vdiag}(Q-T)}{\sqrt{n}}\right\|$$

$$\leq \left(\frac{K}{n}\sum_{i=1}^{N}\mathbb{E}|q_{ii}-t_{ii}|^2\right)^{1/2}$$

$$\leq \frac{K}{\sqrt{n}}$$

by (3.17). We now consider the third term. Since $\|a_j^*(\mathbb{E}_j Q_j)D^{3/2}\|$ is uniformly bounded,

$$\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\left|a_j^*(\mathbb{E}_j Q_j)D^{3/2}\frac{\text{vdiag}(Q_j-Q)}{\sqrt{n}}\right|$$

$$= \frac{1}{n^{3/2}}\sum_{j=1}^{n}\mathbb{E}|\text{Tr}(\text{diag}(a_j^*(\mathbb{E}_j Q_j)D^{3/2})(Q_j-Q))| \leq \frac{K}{\sqrt{n}}$$

by Lemma 3.1. $\qquad\qquad\square$

Lemma 4.3. *Assume that the setting of Theorem* 2.2 *holds true, then*:

$$\sum_{j=1}^{n}\chi_{3j}+\log\left(\left(1-\frac{1}{n}\text{Tr}\,D^{\frac{1}{2}}TA(I+\delta\tilde{D})^{-2}\tilde{D}A^*TD^{\frac{1}{2}}\right)^2-\rho^2\gamma\tilde{\gamma}\right)\xrightarrow[n\to\infty]{\mathcal{P}} 0.$$

Lemma 4.4. *Assume that the setting of Theorem* 2.2 *holds true, then*:

$$\sum_{j=1}^{n}\chi_{4j}+\log\left(\left|1-\vartheta\frac{1}{n}\text{Tr}\,D^{\frac{1}{2}}\bar{T}\bar{A}(I+\delta\tilde{D})^{-2}\tilde{D}A^*TD^{\frac{1}{2}}\right|^2-|\vartheta|^2\rho^2\underline{\gamma}\,\tilde{\gamma}\right)\xrightarrow[n\to\infty]{\mathcal{P}} 0.$$

The core of the paper is devoted to the proof of Lemma 4.3. This proof is provided in Sec. 5. The proof of Lemma 4.4 follows the same canvas with minor differences. Elements of this proof are given in Sec. 6.

5. Proof of Theorem 2.2 (Part II)

This section is devoted to the proof of Lemma 4.3. We begin with the following lemma which implies that $\sum_{j=1}^{n}\chi_{3j}$ can be replaced by its expectation.

Lemma 5.1. *For any $N \times 1$ vector a with bounded Euclidean norm, we have,*

$$\max_{j}\text{var}(a^*(\mathbb{E}_j Q)D(\mathbb{E}_j Q)a) = \mathcal{O}(n^{-1})\quad\text{and}$$

$$\max_{j}\text{var}(\text{Tr}(\mathbb{E}_j Q)D(\mathbb{E}_j Q)D) = \mathcal{O}(1).$$

Proof of Lemma 5.1 is postponed to Appendix B.1. Observe that:

$$\sum_{i=1}^{n} \chi_{3j} = \frac{1}{n} \sum_{j=1}^{n} \rho^2 \tilde{t}_{jj}^2 \left(\frac{\tilde{d}_j^2}{n} \operatorname{Tr}(\mathbb{E}_j Q_j) D (\mathbb{E}_j Q_j) D + 2\tilde{d}_j a_j^* (\mathbb{E}_j Q_j) D (\mathbb{E}_j Q_j) a_j \right)$$

$$= \frac{1}{n} \sum_{j=1}^{n} \rho^2 \tilde{t}_{jj}^2 \left(\frac{\tilde{d}_j^2}{n} \operatorname{Tr}(\mathbb{E}_j Q) D (\mathbb{E}_j Q) D + 2\tilde{d}_j a_j^* (\mathbb{E}_j Q_j) D (\mathbb{E}_j Q_j) a_j \right) + \mathcal{O}(n^{-1}),$$

due to Lemma 3.1. Consider the following notations:

$$\psi_j = \frac{1}{n} \operatorname{Tr} \mathbb{E}[(\mathbb{E}_j Q) D (\mathbb{E}_j Q) D] = \frac{1}{n} \operatorname{Tr} \mathbb{E}[(\mathbb{E}_j Q) D Q D],$$

$$\zeta_{kj} = \mathbb{E}[a_k^* (\mathbb{E}_j Q) D (\mathbb{E}_j Q) a_k] = \mathbb{E}[a_k^* (\mathbb{E}_j Q) D Q a_k],$$

$$\theta_{kj} = \mathbb{E}[a_k^* (\mathbb{E}_j Q_k) D (\mathbb{E}_j Q_k) a_k] = \mathbb{E}[a_k^* (\mathbb{E}_j Q_k) D Q_k a_k],$$

$$\varphi_j = \frac{1}{n} \sum_{k=1}^{j} \rho^2 \tilde{d}_k \tilde{t}_{kk}^2 \theta_{kj}.$$

Thanks to Lemma 5.1, we only need to show that

$$\frac{1}{n} \sum_{j=1}^{n} (\rho^2 \tilde{t}_{jj}^2 \tilde{d}_j^2 \psi_j + 2\rho^2 \tilde{d}_j \tilde{t}_{jj}^2 \theta_{jj}) + \log \Delta_n \xrightarrow[n \to \infty]{} 0. \tag{5.1}$$

There are structural links between the various quantities ψ_j, ζ_{kj}, θ_{kj} and φ_j. The idea behind the proof is to establish the equations between these quantities. Solving these equations will yield explicit expressions which will enable to identify $\frac{1}{n} \sum_{j=1}^{n} (\rho^2 \tilde{t}_{jj}^2 \tilde{d}_j^2 \psi_j + 2\rho^2 \tilde{d}_j \tilde{t}_{jj}^2 \theta_{jj})$ as the deterministic quantity $-\log \Delta_n$ up to a vanishing error term.

Proof of (5.1) is broken down into four steps. In the first step, we establish an equation between ζ_{kj}, ψ_j and φ_j (up to $\mathcal{O}(n^{-1/2})$): Eq. (5.7). In the second step, we establish an equation between ψ_j and φ_j: Eq. (5.11). In the third step, we establish an equation between ζ_{kj}, ψ_j and θ_{kj}: Eq. (5.12). Gathering these results, we obtain a 2×2 linear system (5.15) whose solutions are ψ_j and φ_j. In the fourth step, we solve this system and finally establish (5.1).

5.1. Step 1: Expression of $\zeta_{kj} = \mathbb{E}[a_k^*(\mathbb{E}_j Q)DQa_k]$

Writing

$$Q = T + T(T^{-1} - Q^{-1})Q = T + T(\rho \tilde{\delta} D + A(I + \delta \tilde{D})^{-1} A^* - \Sigma \Sigma^*) Q, \tag{5.2}$$

we have:

$$\zeta_{kj} = \mathbb{E}[a_k^* \mathbb{E}_j [T + T(\rho \tilde{\delta} D + A(I + \delta \tilde{D})^{-1} A^* - \Sigma \Sigma^*) Q] D Q a_k],$$

$$= \mathbb{E}[a_k^* T D Q a_k] + \rho \tilde{\delta} \mathbb{E}[a_k^* T D (\mathbb{E}_j Q) D Q a_k]$$

$$\quad + \mathbb{E}[a_k^* T A(I + \delta \tilde{D})^{-1} A^* (\mathbb{E}_j Q) D Q a_k] - \mathbb{E}[a_k^* T (\mathbb{E}_j \Sigma \Sigma^* Q) D Q a_k], \tag{5.3}$$

$$\stackrel{\triangle}{=} a_k^* T D T a_k + \rho \tilde{\delta} \mathbb{E}[a_k^* T D (\mathbb{E}_j Q) D Q a_k] + X + Z + \varepsilon, \tag{5.4}$$

where X and Z are the last two terms at the right-hand side of (5.3) and where $|\varepsilon| = \mathcal{O}(n^{-1/2})$ by Theorem 3.1(3). Beginning with X, we have

$$X = \sum_{\ell=1}^{n} \frac{\mathbb{E}[a_k^* T a_\ell a_\ell^* (\mathbb{E}_j Q) D Q a_k]}{1 + \delta \tilde{d}_\ell}$$

$$= \sum_{\ell=1}^{n} \frac{\mathbb{E}[a_k^* T a_\ell a_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell} - \sum_{\ell=1}^{n} \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_k^* T a_\ell a_\ell^* (\mathbb{E}_j Q_\ell \eta_\ell \eta_\ell^* Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell} + \varepsilon_1$$

$$= \sum_{\ell=1}^{n} \frac{\mathbb{E}[a_k^* T a_\ell a_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell} - \sum_{\ell=1}^{n} \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_k^* T a_\ell a_\ell^* (\mathbb{E}_j Q_\ell a_\ell \eta_\ell^* Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell}$$

$$\quad + \varepsilon_1 + \varepsilon_2$$

$$= \sum_{\ell=1}^{n} \frac{\mathbb{E}[a_k^* T a_\ell a_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell} - \sum_{\ell=1}^{n} \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell (\mathbb{E}_j \eta_\ell^* Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell}$$

$$\quad + \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

$$\triangleq X_1 + X_2 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3,$$

where

$$\varepsilon_1 = -\sum_{\ell=1}^{n} \frac{\mathbb{E}[a_k^* T a_\ell (\mathbb{E}_j (\rho \tilde{q}_{\ell\ell} - \rho \tilde{t}_{\ell\ell}) a_\ell^* Q_\ell \eta_\ell \eta_\ell^* Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell}, \tag{5.5}$$

$$\varepsilon_2 = -\sum_{\ell=1}^{n} \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_k^* T a_\ell a_\ell^* (\mathbb{E}_j Q_\ell y_\ell \eta_\ell^* Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell},$$

$$\varepsilon_3 = -\sum_{\ell=1}^{n} \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_k^* T a_\ell (\mathbb{E}_j a_\ell (Q_\ell - \mathcal{T}_\ell) a_\ell \eta_\ell^* Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell}.$$

Using (3.3) and (3.8), ε_1 can be written as:

$$\varepsilon_1 = \mathbb{E}[\mathbb{E}_j (a_k^* T A \operatorname{diag}(\xi_\ell)(I + \delta \tilde{D})^{-1} \Sigma^* Q) D Q a_k],$$

where $\xi_\ell = \rho(\tilde{q}_{\ell\ell} - \tilde{t}_{\ell\ell})(1 + \eta_\ell^* Q_\ell \eta_\ell) a_\ell^* Q_\ell \eta_\ell$. Recalling that $\|\Sigma^* Q\|$ is bounded, we obtain $|\varepsilon_1| \leq K \mathbb{E}\|a_k^* T A \operatorname{diag}(\xi_\ell)\| \leq K (\sum_{\ell=1}^{n} |[a_k^* T A]_\ell|^2 \mathbb{E} \xi_\ell^2)^{1/2} \leq K/\sqrt{n}$ by (3.17) and the boundedness of $\mathbb{E}|X_{11}|^{16}$ (Assumption 2.1). We show similarly that ε_2 and ε_3 (with the help of Theorem 3.1(4)) are of order $\mathcal{O}(n^{-1/2})$. We now develop X_2 as:

$$X_2 = -\sum_{\ell=1}^{n} \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell a_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell}$$

$$\quad - \sum_{\ell=1}^{j} \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell}$$

$$\triangleq U_1 + U_2.$$

The term U_2 can be expressed as:

$$U_2 = \sum_{\ell=1}^{j} \frac{\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell} \tilde{q}_{\ell\ell} a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell \eta_\ell^* Q_\ell a_k]}{1 + \delta \tilde{d}_\ell}$$

$$= \sum_{\ell=1}^{j} \frac{\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2 a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell \eta_\ell^* Q_\ell a_k]}{1 + \delta \tilde{d}_\ell} + \mathcal{O}(n^{-1/2}).$$

Write $\eta_\ell \eta_\ell^* = a_\ell a_\ell^* + a_\ell y_\ell^* + y_\ell y_\ell^* + y_\ell a_\ell^*$. The term in $a_\ell a_\ell^*$ is zero. Turning to the term in $a_\ell y_\ell^*$, we have $\mathbb{E}|y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell a_\ell y_\ell^* Q_\ell a_k| = \mathcal{O}(n^{-1})$, hence

$$\sum_{\ell=1}^{j} \frac{|\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2 a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell a_\ell y_\ell^* Q_\ell a_k]|}{1 + \delta \tilde{d}_\ell} \le \frac{K}{n} \sum_{\ell=1}^{j} |a_k^* T a_\ell| \le \frac{K}{\sqrt{n}}.$$

Moreover,

$$\sum_{\ell=1}^{j} \frac{|\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2 a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell y_\ell^* Q_\ell a_k]|}{1 + \delta \tilde{d}_\ell}$$

$$= \sum_{\ell=1}^{j} \frac{|\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2 a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell (y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell - \tilde{d}_\ell n^{-1} \operatorname{Tr} D(\mathbb{E}_j Q_\ell) D Q_\ell) y_\ell^* Q_\ell a_k]|}{1 + \delta \tilde{d}_\ell}$$

$$\le \frac{K}{n} \sum_{\ell=1}^{j} |a_k^* T a_\ell|$$

$$= \mathcal{O}(n^{-1/2}).$$

The term in $y_\ell a_\ell^*$ is written as

$$\sum_{\ell=1}^{j} \frac{\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2 a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell a_\ell^* Q_\ell a_k]}{1 + \delta \tilde{d}_\ell}$$

$$= \psi_j \sum_{\ell=1}^{j} \frac{\mathbb{E}[\rho^2 \tilde{d}_\ell \tilde{t}_{\ell\ell}^2 a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell a_\ell^* Q_\ell a_k]}{1 + \delta \tilde{d}_\ell} + \varepsilon$$

where $\varepsilon = \mathcal{O}(n^{-1})$ by Lemmas 5.1 and 3.1. The remaining term in the right-hand side can be handled by the following lemma which is proven in Appendix B.2:

Lemma 5.2. *Let* $(u) = (u_n)_{n \in \mathbb{N}}$ *be a sequence of vectors with bounded Euclidean norms. Let* $(\alpha_\ell)_{1 \le \ell \le n} = (\alpha_{\ell,n})_{1 \le \ell \le n}$ *be an array of bounded real numbers. Then:*

$$\sum_{\ell=1}^{j} \alpha_\ell u^* T a_\ell \mathbb{E}[a_\ell Q_\ell u] = \sum_{\ell=1}^{j} \frac{\alpha_\ell u^* T a_\ell a_\ell^* T u}{\rho \tilde{t}_{\ell\ell} (1 + \tilde{d}_\ell \delta)} + \mathcal{O}(n^{-1/2}).$$

Applying this lemma with $u = a_k$ and $\alpha_\ell = \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell (1 + \tilde{d}_\ell)^{-1} a_\ell \mathcal{T}_\ell a_\ell$, we obtain

$$U_2 = \psi_j \sum_{\ell=1}^{j} \frac{\rho \tilde{d}_\ell \tilde{t}_{\ell\ell} a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell a_\ell^* T a_k}{(1 + \delta \tilde{d}_\ell)^2} + \mathcal{O}(n^{-1/2}).$$

Gathering these results, and using the identity $(1 - \rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell) = \rho\tilde{t}_{\ell\ell}(1 + \tilde{d}_\ell\delta)$ (see (3.10)), we obtain

$$X = \sum_{\ell=1}^{n} \rho\tilde{t}_{\ell\ell}a_k^*Ta_\ell\mathbb{E}[a_\ell^*(\mathbb{E}_jQ_\ell)DQa_k]$$

$$+ \psi_j \sum_{\ell=1}^{j} \frac{\rho\tilde{d}_\ell\tilde{t}_{\ell\ell}a_k^*Ta_\ell a_\ell^*\mathcal{T}_\ell a_\ell a_\ell^*Ta_k}{(1 + \tilde{d}_\ell\delta)^2} + \mathcal{O}(n^{-1/2}). \qquad (5.6)$$

We now turn to the term Z in (5.4).

$$Z = -\sum_{\ell=1}^{n} \mathbb{E}[a_k^*T(\mathbb{E}_j\rho\tilde{q}_{\ell\ell}\eta_\ell\eta_\ell^*Q_\ell)DQa_k] = -\sum_{\ell=1}^{n} \rho\tilde{t}_{\ell\ell}\mathbb{E}[a_k^*T(\mathbb{E}_j\eta_\ell\eta_\ell^*Q_\ell)DQa_k] + \varepsilon,$$

where

$$\varepsilon = \sum_{\ell=1}^{n} \mathbb{E}[a_k^*T(\mathbb{E}_j\rho(\tilde{q}_{\ell\ell} - \tilde{t}_{\ell\ell})\eta_\ell\eta_\ell^*Q_\ell)DQa_k]$$

satisfies $\varepsilon = \mathcal{O}(n^{-1/2})$ (same arguments as for ε_1 in (5.5)). Writing $\eta_\ell\eta_\ell^* = a_\ell a_\ell^* + y_\ell y_\ell^* + a_\ell y_\ell^* + y_\ell a_\ell^*$, we obtain:

$$Z = -\sum_{\ell=1}^{n} \rho\tilde{t}_{\ell\ell}a_k^*Ta_\ell\mathbb{E}[a_\ell^*(\mathbb{E}_jQ_\ell)DQa_k]$$

$$- \left(\sum_{\ell=1}^{j} \rho\tilde{t}_{\ell\ell}\mathbb{E}[a_k^*Ty_\ell y_\ell^*(\mathbb{E}_jQ_\ell)DQa_k] + \frac{1}{n}\sum_{\ell=j+1}^{n} \rho\tilde{t}_{\ell\ell}\tilde{d}_\ell\mathbb{E}[a_k^*TD(\mathbb{E}_jQ_\ell)DQa_k] \right)$$

$$- \sum_{\ell=1}^{j} \rho\tilde{t}_{\ell\ell}a_k^*Ta_\ell\mathbb{E}[y_\ell^*(\mathbb{E}_jQ_\ell)DQa_k] - \sum_{\ell=1}^{j} \rho\tilde{t}_{\ell\ell}\mathbb{E}[a_k^*Ty_\ell a_\ell^*(\mathbb{E}_jQ_\ell)DQa_k]$$

$$+ \mathcal{O}(n^{-1/2})$$

$$\triangleq Z_1 + Z_2 + Z_3 + Z_4 + \mathcal{O}(n^{-1/2}).$$

The term Z_1 cancels with the first term in the decomposition of X (first term at the right-hand side of (5.6)). The term Z_2 can be written as:

$$Z_2 = -\left(\sum_{\ell=1}^{j} \rho\tilde{t}_{\ell\ell}\mathbb{E}[a_k^*Ty_\ell y_\ell^*(\mathbb{E}_jQ_\ell)DQ_\ell a_k] + \frac{1}{n}\sum_{\ell=j+1}^{n} \rho\tilde{t}_{\ell\ell}\tilde{d}_\ell\mathbb{E}[a_k^*TD(\mathbb{E}_jQ_\ell)DQa_k] \right)$$

$$+ \sum_{\ell=1}^{j} \rho^2\tilde{t}_{\ell\ell}^2\mathbb{E}[a_k^*Ty_\ell y_\ell^*(\mathbb{E}_jQ_\ell)DQ_\ell\eta_\ell\eta_\ell^*Q_\ell a_k] + \varepsilon$$

$$\triangleq W_1 + W_2 + \varepsilon,$$

where ε follows from the substitution of $\rho\tilde{q}_{\ell\ell}$ with $\rho\tilde{t}_{\ell\ell}$ and satisfies $\varepsilon = \mathcal{O}(n^{-1/2})$ as in (5.5). Consider first W_1:

$$W_1 = -\left(\frac{1}{n}\sum_{\ell=1}^{j}\rho\tilde{t}_{\ell\ell}\tilde{d}_\ell\mathbb{E}[a_k^*TD(\mathbb{E}_jQ_\ell)DQ_\ell a_k]\right.$$

$$\left. + \frac{1}{n}\sum_{\ell=j+1}^{n}\rho\tilde{t}_{\ell\ell}\tilde{d}_\ell\mathbb{E}[a_k^*TD(\mathbb{E}_jQ_\ell)DQa_k]\right).$$

Write:

$$(\mathbb{E}_jQ_\ell)DQ_\ell - (\mathbb{E}_jQ)DQ = (\mathbb{E}_jQ_\ell)D(Q_\ell - Q) + (\mathbb{E}_jQ_\ell - \mathbb{E}_jQ)DQ.$$

Using (3.5) and (3.6),

$$\frac{1}{n}\sum_{\ell=1}^{j}\rho\tilde{t}_{\ell\ell}\tilde{d}_\ell|\mathbb{E}[a_k^*TD(\mathbb{E}_jQ_\ell)D(Q_\ell - Q)a_k]|$$

$$\leq \frac{K}{n}\sum_{\ell=1}^{j}(\mathbb{E}|(1 + \eta_\ell^*Q_\ell\eta_\ell)|^2)^{1/2}(\mathbb{E}|\eta_\ell^*Qa_k|^2)^{1/2}$$

$$\leq \frac{K}{\sqrt{n}}(\mathbb{E}a_k^*Q\Sigma_{1:j}\Sigma_{1:j}^*Qa_k)^{1/2}$$

$$= \mathcal{O}(n^{-1/2}),$$

and the same arguments apply to the term $(\mathbb{E}_jQ_\ell - \mathbb{E}_jQ)DQ$. Hence,

$$W_1 = -\rho\tilde{\delta}\mathbb{E}[a_k^*TD(\mathbb{E}_jQ)DQa_k] + \mathcal{O}(n^{-1/2}).$$

Turning to W_2, we have:

$$\sum_{\ell=1}^{j}\rho^2\tilde{t}_{\ell\ell}^2\mathbb{E}[a_k^*Ty_\ell y_\ell^*(\mathbb{E}_jQ_\ell)DQ_\ell y_\ell a_\ell^*Q_\ell a_k]$$

$$= \sum_{\ell=1}^{j}\rho^2\tilde{t}_{\ell\ell}^2\mathbb{E}\left[a_k^*Ty_\ell\left(y_\ell^*(\mathbb{E}_jQ_\ell)DQ_\ell y_\ell - \frac{\tilde{d}_\ell}{n}\operatorname{Tr}D(\mathbb{E}_jQ_\ell)DQ_\ell\right)a_\ell^*Q_\ell a_k\right]$$

whose modulus is of order $\mathcal{O}(n^{-1/2})$. The term

$$\sum_{\ell=1}^{j}\rho^2\tilde{t}_{\ell\ell}^2\mathbb{E}[a_k^*Ty_\ell y_\ell^*(\mathbb{E}_jQ_\ell)DQ_\ell a_\ell y_\ell^*Q_\ell a_k]$$

can be handled similarly.

The term

$$\sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \mathbb{E}[a_k^* T y_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell a_\ell a_\ell^* Q_\ell a_k]$$

$$= \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell \mathbb{E}[a_k^* T D (\mathbb{E}_j Q_\ell) D Q_\ell a_\ell a_\ell^* Q_\ell a_k]$$

is bounded by $Kn^{-1/2}$. Finally,

$$W_2 = \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \mathbb{E}[y_\ell^* Q_\ell a_k a_k^* T y_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell] + \mathcal{O}(n^{-1/2})$$

$$\overset{(a)}{=} \psi_j a_k^* T D T a_k \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 + \mathcal{O}(n^{-1/2}),$$

where (a) follows by standard arguments as those already developed.

The term Z_3 satisfies

$$Z_3 = \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 a_k^* T a_\ell \mathbb{E}[y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell \eta_\ell^* Q_\ell a_k] + \mathcal{O}(n^{-1/2}).$$

Writing $\eta_\ell \eta_\ell^* = y_\ell y_\ell^* + a_\ell y_\ell^* + a_\ell a_\ell^* + y_\ell a_\ell^*$ and relying on arguments as those already developed, one can check that the only non-negligible contribution stems from the term containing $y_\ell a_\ell^*$. Hence,

$$Z_3 = \psi_j \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell a_k^* T a_\ell \mathbb{E}[a_\ell^* Q_\ell a_k] + \mathcal{O}(n^{-1/2})$$

$$= \psi_j \sum_{\ell=1}^{j} \frac{\rho \tilde{t}_{\ell\ell} \tilde{d}_\ell a_k^* T a_\ell a_\ell^* T a_k}{1 + \tilde{d}_\ell \delta} + \mathcal{O}(n^{-1/2}),$$

by Lemma (5.2). Similarly,

$$Z_4 = \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \mathbb{E}[a_k^* T y_\ell a_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell \eta_\ell^* Q_\ell a_k] + \mathcal{O}(n^{-1/2})$$

$$= a_k^* T D T a_k \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{d}_\ell \tilde{t}_{\ell\ell}^2 \mathbb{E}[a_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell a_\ell] + \mathcal{O}(n^{-1/2})$$

$$= a_k^* T D T a_k \varphi_j + \mathcal{O}(n^{-1/2}).$$

Gathering these results, we obtain

$$Z = -\sum_{\ell=1}^{n} \rho \tilde{t}_{\ell\ell} a_k^* T a_\ell \mathbb{E}[a_\ell^* (\mathbb{E}_j Q_\ell) DQ a_k] - \rho \tilde{\delta} \mathbb{E}[a_k^* TD(\mathbb{E}_j Q) DQ a_k]$$

$$+ \psi_j a_k^* TDT a_k \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 + \psi_j \sum_{\ell=1}^{j} \frac{\rho \tilde{t}_{\ell\ell} \tilde{d}_\ell a_k^* T a_\ell a_\ell^* T a_k}{1 + \tilde{d}_\ell \delta}$$

$$+ a_k^* TDT a_k \varphi_j + \mathcal{O}(n^{-1/2}).$$

Plugging this and Eq. (5.6) into (5.4), and noticing that $\rho \tilde{t}_{\ell\ell} (a_\ell \mathcal{T}_\ell a_\ell (1 + \tilde{d}_\ell \delta)^{-1} + 1) = (1 + \tilde{d}_\ell \delta)^{-1}$, we obtain:

$$\zeta_{kj} = a_k^* TDT a_k + \psi_j \left(\sum_{\ell=1}^{j} \frac{a_k^* T a_\ell \tilde{d}_\ell a_\ell^* T a_k}{(1 + \tilde{d}_\ell \delta)^2} + a_k^* TDT a_k \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 \right)$$

$$+ a_k^* TDT a_k \varphi_j + \mathcal{O}(n^{-1/2}). \tag{5.7}$$

5.2. Step 2: Expression of $\psi_j = n^{-1} \operatorname{Tr} \mathbb{E}[(\mathbb{E}_j Q) DQD]$

Using Identity (5.2), we obtain:

$$\psi_j = \frac{1}{n} \operatorname{Tr} \mathbb{E}[TDQD] + \frac{\rho \tilde{\delta}}{n} \operatorname{Tr} \mathbb{E}[TD(\mathbb{E}_j Q) DQD]$$

$$+ \frac{1}{n} \operatorname{Tr} \mathbb{E}[TA(I + \delta \tilde{D})^{-1} A^* (\mathbb{E}_j Q) DQD] - \frac{1}{n} \operatorname{Tr} \mathbb{E}[T(\mathbb{E}_j \Sigma \Sigma^* Q) DQD] \tag{5.8}$$

$$= \frac{1}{n} \operatorname{Tr} DTDT + \frac{\rho \tilde{\delta}}{n} \operatorname{Tr} \mathbb{E}[TD(\mathbb{E}_j Q) DQD] + X + Z + \varepsilon, \tag{5.9}$$

where X and Z are the last two terms of the right-hand side of (5.8), and where $\varepsilon = \mathcal{O}(n^{-1})$ by Theorem 3.1(2). Due to the presence of the multiplying factor n^{-1}, the treatment of X and Z is simpler here than the treatment of their analogues for ζ_{kj}. We skip hereafter the details related to the bounds over the ε's. The term X satisfies

$$X = \frac{1}{n} \sum_{\ell=1}^{n} \frac{\mathbb{E}[a_\ell^* (\mathbb{E}_j Q) DQDT a_\ell]}{1 + \delta \tilde{d}_\ell}$$

$$= \frac{1}{n} \sum_{\ell=1}^{n} \frac{\mathbb{E}[a_\ell^* (\mathbb{E}_j Q_\ell) DQDT a_\ell]}{1 + \delta \tilde{d}_\ell} - \frac{1}{n} \sum_{\ell=1}^{n} \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} (\mathbb{E}_j a_\ell^* Q_\ell \eta_\ell \eta_\ell^* Q_\ell) DQDT a_\ell]}{1 + \delta \tilde{d}_\ell} + \varepsilon$$

$$= \frac{1}{n} \sum_{\ell=1}^{n} \frac{\mathbb{E}[a_\ell^* (\mathbb{E}_j Q_\ell) DQDT a_\ell]}{1 + \delta \tilde{d}_\ell} - \frac{1}{n} \sum_{\ell=1}^{n} \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_\ell^* \mathcal{T}_\ell a_\ell a_\ell^* (\mathbb{E}_j Q_\ell) DQDT a_\ell]}{1 + \delta \tilde{d}_\ell}$$

$$- \frac{1}{n} \sum_{\ell=1}^{j} \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_\ell^* \mathcal{T}_\ell a_\ell y_\ell^* (\mathbb{E}_j Q_\ell) DQDT a_\ell]}{1 + \delta \tilde{d}_\ell} + \varepsilon',$$

where $\max(|\varepsilon|, |\varepsilon'|) = \mathcal{O}(n^{-1/2})$. As $1 - \rho\tilde{t}_{\ell\ell}a_\ell^* \mathcal{T}_\ell a_\ell = \rho\tilde{t}_{\ell\ell}(1 + \tilde{d}_\ell\delta)$,

$$
\begin{aligned}
X &= \frac{1}{n}\sum_{\ell=1}^{n}\mathbb{E}[\rho\tilde{t}_{\ell\ell}a_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell] \\
&\quad - \frac{1}{n}\sum_{\ell=1}^{j}\frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell}a_\ell^* \mathcal{T}_\ell a_\ell y_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell]}{1+\delta\tilde{d}_\ell} + \mathcal{O}(n^{-1/2}) \\
&= \frac{1}{n}\sum_{\ell=1}^{n}\mathbb{E}[\rho\tilde{t}_{\ell\ell}a_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell] \\
&\quad + \frac{1}{n}\sum_{\ell=1}^{j}\frac{\mathbb{E}[\rho^2\tilde{t}_{\ell\ell}^2 a_\ell^* \mathcal{T}_\ell a_\ell y_\ell^*(\mathbb{E}_j Q_\ell)DQ_\ell\eta_\ell\eta_\ell^* Q_\ell DTa_\ell]}{1+\delta\tilde{d}_\ell} + \mathcal{O}(n^{-1/2}) \\
&= \frac{1}{n}\sum_{\ell=1}^{n}\mathbb{E}[\rho\tilde{t}_{\ell\ell}a_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell] + \frac{\psi_j}{n}\sum_{\ell=1}^{j}\frac{\tilde{d}_\ell a_\ell^* Ta_\ell a_\ell^* TDTa_\ell}{(1+\delta\tilde{d}_\ell)^3} + \mathcal{O}(n^{-1/2}),
\end{aligned}
\tag{5.10}
$$

where (3.11) is used to obtain the last equation. The term Z can be expressed as:

$$
\begin{aligned}
Z &= -\frac{1}{n}\sum_{\ell=1}^{n}\operatorname{Tr}\mathbb{E}[\rho\tilde{t}_{\ell\ell}T(\mathbb{E}_j\eta_\ell\eta_\ell^* Q_\ell)DQD] + \mathcal{O}(n^{-1/2}) \\
&= -\frac{1}{n}\sum_{\ell=1}^{n}\mathbb{E}[\rho\tilde{t}_{\ell\ell}a_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell] \\
&\quad - \left(\frac{1}{n}\sum_{\ell=1}^{j}\mathbb{E}[\rho\tilde{t}_{\ell\ell}y_\ell^*(\mathbb{E}_j Q_\ell)DQDTy_\ell] + \frac{1}{n}\sum_{\ell=j+1}^{n}\frac{1}{n}\operatorname{Tr}\mathbb{E}[\rho\tilde{d}_\ell\tilde{t}_{\ell\ell}TD(\mathbb{E}_j Q_\ell)DQD]\right) \\
&\quad - \frac{1}{n}\sum_{\ell=1}^{j}\mathbb{E}[\rho\tilde{t}_{\ell\ell}y_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell] - \frac{1}{n}\sum_{\ell=1}^{j}\mathbb{E}[\rho\tilde{t}_{\ell\ell}a_\ell^*(\mathbb{E}_j Q_\ell)DQDTy_\ell] \\
&\quad + \mathcal{O}(n^{-1/2}) \\
&\stackrel{\triangle}{=} Z_1 + Z_2 + Z_3 + Z_4 + \mathcal{O}(n^{-1/2}).
\end{aligned}
$$

The term Z_1 cancels with the first term in the right-hand side of X's decomposition (5.10). The terms Z_2, Z_3 and Z_4 satisfy:

$$
\begin{aligned}
Z_2 &= -\frac{\rho\tilde{\delta}}{n}\operatorname{Tr}\mathbb{E}[TD(\mathbb{E}_j Q)DQD] \\
&\quad + \frac{1}{n}\sum_{\ell=1}^{j}\mathbb{E}[\rho^2\tilde{t}_{\ell\ell}^2 y_\ell^*(\mathbb{E}_j Q_\ell)DQ_\ell\eta_\ell\eta_\ell^* Q_\ell DTy_\ell] + \mathcal{O}(n^{-1/2}) \\
&= -\frac{\rho\tilde{\delta}}{n}\operatorname{Tr}\mathbb{E}[TD(\mathbb{E}_j Q)DQD] + \psi_j\frac{1}{n}\operatorname{Tr}DTDT\frac{1}{n}\sum_{\ell=1}^{j}\rho^2\tilde{d}_\ell^2\tilde{t}_{\ell\ell}^2 + \mathcal{O}(n^{-1/2}),
\end{aligned}
$$

$$Z_3 = \frac{1}{n} \sum_{\ell=1}^{j} \mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2 y_\ell^* (\mathbb{E}_j Q_\ell) DQ_\ell \eta_\ell \eta_\ell^* Q_\ell DT a_\ell] + \mathcal{O}(n^{-1/2})$$

$$= \psi_j \frac{1}{n} \sum_{\ell=1}^{j} \rho \tilde{d}_\ell \tilde{t}_{\ell\ell} \frac{a_\ell^* T D T a_\ell}{1 + \tilde{d}_\ell \delta} + \mathcal{O}(n^{-1/2}) \quad \text{(see Theorem 3.1(4) and (3.11))},$$

$$Z_4 = \frac{1}{n} \sum_{\ell=1}^{j} \mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2 a_\ell^* (\mathbb{E}_j Q_\ell) DQ_\ell \eta_\ell \eta_\ell^* Q_\ell DT y_\ell] + \mathcal{O}(n^{-1/2})$$

$$= \frac{1}{n} \operatorname{Tr} DTDT \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{d}_\ell \tilde{t}_{\ell\ell}^2 \mathbb{E}[a_\ell^* (\mathbb{E}_j Q_\ell) DQ_\ell a_\ell] + \mathcal{O}(n^{-1/2}).$$

Plugging these terms in (5.9), we obtain:

$$\psi_j = \gamma + \psi_j \left(\frac{1}{n} \sum_{\ell=1}^{j} a_\ell^* T D T a_\ell \left(\frac{\tilde{d}_\ell a_\ell^* T a_\ell}{(1 + \delta \tilde{d}_\ell)^3} + \frac{\rho \tilde{d}_\ell \tilde{t}_{\ell\ell}}{1 + \tilde{d}_\ell \delta} \right) + \frac{\gamma}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{d}_\ell^2 \tilde{t}_{\ell\ell}^2 \right)$$

$$+ \gamma \varphi_j + \mathcal{O}(n^{-1/2})$$

$$= \gamma + \psi_j \left(\frac{1}{n} \sum_{\ell=1}^{j} \frac{\tilde{d}_\ell a_\ell^* T D T a_\ell}{(1 + \delta \tilde{d}_\ell)^2} + \frac{\gamma}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{d}_\ell^2 \tilde{t}_{\ell\ell}^2 \right) + \gamma \varphi_j + \mathcal{O}(n^{-1/2}), \quad (5.11)$$

using (3.10) and (3.11).

5.3. Step 3: Relation between ζ_{kj} and θ_{kj} for $k \leq j$

The term ζ_{kj} can be written as

$$\zeta_{kj} = \mathbb{E}[a_k^* \mathbb{E}_j (Q_k - \rho \tilde{q}_{kk} Q_k \eta_k \eta_k^* Q_k) D(Q_k - \rho \tilde{q}_{kk} Q_k \eta_k \eta_k^* Q_k) a_k]$$

$$= \theta_{kj} - \rho \tilde{t}_{kk} \mathbb{E}[a_k^* \mathbb{E}_j (Q_k \eta_k \eta_k^* Q_k) DQ_k a_k] - \rho \tilde{t}_{kk} \mathbb{E}[a_k^* \mathbb{E}_j (Q_k) DQ_k \eta_k \eta_k^* Q_k a_k]$$

$$+ \rho^2 \tilde{t}_{kk}^2 \mathbb{E}[a_k^* \mathbb{E}_j (Q_k \eta_k \eta_k^* Q_k) DQ_k \eta_k \eta_k^* Q_k a_k] + \mathcal{O}(n^{-1/2})$$

$$\stackrel{\triangle}{=} \theta_{kj} + X_1 + X_2 + X_3 + \mathcal{O}(n^{-1/2}).$$

Using similar arguments as those developed previously, we get:

$$X_1 = -\rho \tilde{t}_{kk} a_k^* \mathcal{T}_k a_k \mathbb{E}[a_k^* \mathbb{E}_j (Q_k) DQ_k a_k] + \mathcal{O}(n^{-1/2})$$

$$= -\rho \tilde{t}_{kk} a_k^* \mathcal{T}_k a_k \theta_{kj} + \mathcal{O}(n^{-1/2}),$$

$$X_2 = -\rho \tilde{t}_{kk} a_k^* \mathcal{T}_k a_k \theta_{kj} + \mathcal{O}(n^{-1/2}).$$

As $k \leq j$,

$$X_3 = \rho^2 \tilde{t}_{kk}^2 (a_k^* \mathcal{T}_k a_k)^2 \mathbb{E}[\eta_k^* \mathbb{E}_j (Q_k) DQ_k \eta_k] + \mathcal{O}(n^{-1/2})$$

$$= \rho^2 \tilde{t}_{kk}^2 (a_k^* \mathcal{T}_k a_k)^2 (\theta_{kj} + \tilde{d}_k \psi_j) + \mathcal{O}(n^{-1/2}).$$

Using (3.10) and (3.11), we finally obtain:

$$\zeta_{kj} = \rho^2 \tilde{t}_{kk}^2 (1 + \tilde{d}_k \delta)^2 \theta_{kj} + \tilde{d}_k \left(\frac{a_k^* T a_k}{1 + \tilde{d}_k \delta} \right)^2 \psi_j + \mathcal{O}(n^{-1/2}). \tag{5.12}$$

5.4. *Step 4: A system of perturbed linear equations in (ψ_j, φ_j). Proof of (5.1)*

Combining (5.12) with (5.7), we obtain

$$\rho^2 \tilde{d}_k \tilde{t}_{kk}^2 \theta_{kj} = \tilde{d}_k \frac{a_k^* T D T a_k}{(1 + \tilde{d}_k \delta)^2} + \left(\sum_{\ell=1}^{j} \frac{\tilde{d}_k a_k^* T a_\ell \tilde{d}_\ell a_\ell^* T a_k}{(1 + \tilde{d}_k \delta)^2 (1 + \tilde{d}_\ell \delta)^2} \right.$$

$$+ \frac{\tilde{d}_k a_k^* T D T a_k}{(1 + \tilde{d}_k \delta)^2} \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 - \tilde{d}_k^2 \frac{(a_k^* T a_k)^2}{(1 + \tilde{d}_k \delta)^4} \left. \right) \psi_j$$

$$+ \frac{\tilde{d}_k a_k^* T D T a_k}{(1 + \tilde{d}_k \delta)^2} \varphi_j + \mathcal{O}(n^{-1/2}) \tag{5.13}$$

which implies that $\varphi_j = \frac{1}{n} \sum_{k=1}^{j} \rho^2 \tilde{d}_k \tilde{t}_{kk}^2 \theta_{kj}$ satisfies

$$(1 - F_j) \varphi_j - (G_j + F_j M_j) \psi_j = F_j + \mathcal{O}(n^{-1/2})$$

where

$$F_j = \frac{1}{n} \sum_{k=1}^{j} \frac{a_k^* T D T a_k \tilde{d}_k}{(1 + \tilde{d}_k \delta)^2},$$

$$M_j = \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2, \tag{5.14}$$

$$G_j = \frac{1}{n} \sum_{k=1}^{j} \sum_{\substack{\ell=1 \\ \ell \neq k}}^{j} \frac{\tilde{d}_k \tilde{d}_\ell |a_k^* T a_\ell|^2}{(1 + \tilde{d}_k \delta)^2 (1 + \tilde{d}_\ell \delta)^2}.$$

With these new notations, Eq. (5.11) is rewritten

$$-\gamma \varphi_j + (1 - F_j - \gamma M_j) \psi_j = \gamma + \mathcal{O}(n^{-1/2}),$$

and we end up with a system of two perturbed linear equations in (φ_j, ψ_j):

$$\begin{cases} (1 - F_j) \varphi_j - (G_j + F_j M_j) \psi_j = F_j + \mathcal{O}(n^{-1/2}), \\ -\gamma \varphi_j + (1 - F_j - \gamma M_j) \psi_j = \gamma + \mathcal{O}(n^{-1/2}). \end{cases} \tag{5.15}$$

The determinant of this system is $\boldsymbol{\Delta}_j = (1 - F_j)^2 - \gamma M_j - \gamma G_j$. The following lemma establishes the link between the $\boldsymbol{\Delta}_j$'s and Δ_n as defined in Theorem 2.2.

Lemma 5.3. *Recall the definition of Δ_n:*

$$\Delta_n = \left(1 - \frac{1}{n}\operatorname{Tr} D^{\frac{1}{2}}TA(I + \delta\tilde{D})^{-2}\tilde{D}A^*TD^{\frac{1}{2}}\right)^2 - \rho^2\gamma\tilde{\gamma}.$$

The determinants $\boldsymbol{\Delta}_j$ decrease as j goes from 1 to n; moreover, $\boldsymbol{\Delta}_n$ coincides with Δ_n.

Proof of Lemma 5.3 is postponed to Appendix B.3.

Solving this system of equations and using the lemma in conjunction with the fact $\liminf \Delta_n > 0$, established in Lemma 3.4, we obtain:

$$\begin{bmatrix} \varphi_j \\ \psi_j \end{bmatrix} = \frac{1}{\boldsymbol{\Delta}_j}\begin{bmatrix} F_j(1 - F_j) + \gamma G_j \\ \gamma \end{bmatrix} + \varepsilon_j,$$

where $\|\varepsilon_j\| = \mathcal{O}(n^{-1/2})$. Replacing into (5.13), we obtain

$$\frac{2\rho^2\tilde{d}_j\tilde{t}_{jj}^2\theta_{jj}}{n} = 2(F_j - F_{j-1}) + (G_j - G_{j-1} + 2M_j(F_j - F_{j-1}))\psi_j$$

$$+ 2(F_j - F_{j-1})\varphi_j + \mathcal{O}(n^{-3/2})$$

$$= 2(F_j - F_{j-1}) + \frac{\gamma(G_j - G_{j-1}) + 2\gamma M_j(F_j - F_{j-1})}{\boldsymbol{\Delta}_j}$$

$$+ \frac{2(F_j - F_{j-1})(F_j(1 - F_j) + \gamma G_j)}{\boldsymbol{\Delta}_j} + \mathcal{O}(n^{-3/2})$$

which leads to

$$\frac{1}{n}\sum_{j=1}^{n}(\rho^2\tilde{t}_{jj}^2\tilde{d}_j^2\psi_j + 2\rho^2\tilde{d}_j\tilde{t}_{jj}^2\theta_{jj})$$

$$= \sum_{j=1}^{n}\frac{2(F_j - F_{j-1})(1 - F_j) + \gamma(M_j - M_{j-1}) + \gamma(G_j - G_{j-1})}{\boldsymbol{\Delta}_j} + \mathcal{O}(n^{-1/2}).$$

On the other hand, $\boldsymbol{\Delta}_{j-1} - \boldsymbol{\Delta}_j = 2(F_j - F_{j-1})(1 - F_j) + \gamma(M_j - M_{j-1}) + \gamma(G_j - G_{j-1}) + \mathcal{O}(n^{-2})$, hence, due to Lemma 5.3 and to $\liminf \Delta_n > 0$,

$$\frac{1}{n}\sum_{j=1}^{n}(\rho^2\tilde{t}_{jj}^2\tilde{d}_j^2\psi_j + 2\rho^2\tilde{d}_j\tilde{t}_{jj}^2\theta_{jj}) = \sum_{j=1}^{n}\frac{\boldsymbol{\Delta}_{j-1} - \boldsymbol{\Delta}_j}{\boldsymbol{\Delta}_j} + \mathcal{O}(n^{-1/2})$$

$$= \sum_{j=1}^{n}\log\left(1 + \frac{\boldsymbol{\Delta}_{j-1} - \boldsymbol{\Delta}_j}{\boldsymbol{\Delta}_j}\right) + \mathcal{O}(n^{-1/2})$$

$$= \sum_{j=1}^{n} \log \frac{\boldsymbol{\Delta}_{j-1}}{\boldsymbol{\Delta}_j} + \mathcal{O}(n^{-1/2})$$

$$= -\log(\Delta_n) + \mathcal{O}(n^{-1/2})$$

which proves (5.1). Lemma 4.3 is proven.

6. Proof of Theorem 2.2 (Part III)

In this section, we complete the proof of Theorem 2.2. Proof of Lemma 4.4 is very close to the proof of Lemma 4.3; we therefore only provide its main landmarks. We finally establish the main estimates over (Θ_n).

6.1. *Elements of proof for Lemma 4.4*

Proof of Lemma 4.4 relies on the following counterpart of Lemma 3.2:

Lemma 6.1. *Assume that the setting of Lemma 3.2 holds true; and let $\mathbb{E}x^2 = \vartheta$. Then for any $p \geq 2$,*

$$\mathbb{E}|\boldsymbol{x}^T M \boldsymbol{x} - \vartheta \operatorname{Tr} M|^p \leq K_p((\mathbb{E}|x_1|^4 \operatorname{Tr} MM^*)^{p/2} + \mathbb{E}|x_1|^{2p} \operatorname{Tr}(MM^*)^{p/2}).$$

Proof. The result is obtained upon noticing that

$$\boldsymbol{x}^T M \boldsymbol{x} = \frac{1}{4} \sum_{k=0}^{3} \mathbf{i}^k (\mathbf{i}^k \bar{\boldsymbol{x}} + \boldsymbol{x})^* M (\mathbf{i}^k \bar{\boldsymbol{x}} + \boldsymbol{x})$$

and using Lemma 3.2. □

Here are the main steps of the proof. Introducing the notations

$$\underline{\psi}_j = \frac{1}{n} \operatorname{Tr} \mathbb{E}[(\mathbb{E}_j Q) D \bar{Q} D],$$

$$\underline{\varsigma}_{kj} = \mathbb{E}[a_k^*(\mathbb{E}_j Q) D \bar{Q} \bar{a}_k],$$

$$\underline{\theta}_{kj} = \mathbb{E}[a_k^*(\mathbb{E}_j Q_k) D \bar{Q}_k \bar{a}_k],$$

$$\underline{\varphi}_j = \frac{1}{n} \sum_{k=1}^{j} \rho^2 \tilde{d}_k \tilde{t}_{kk}^2 \underline{\theta}_{kj},$$

and adapting Lemma 5.1, we only need to prove that:

$$\frac{1}{n} \sum_{j=1}^{n} (\rho^2 \tilde{t}_{jj}^2 \tilde{d}_j^2 |\vartheta|^2 \underline{\psi}_j + 2\rho^2 \tilde{d}_j \tilde{t}_{jj}^2 \operatorname{Re}(\vartheta \underline{\theta}_{jj})) + \log \underline{\Delta}_n \xrightarrow[n\to\infty]{} 0.$$

Similar derivations as those performed in Steps 1–3 in Sec. 5 yield the perturbed system:

$$\begin{cases} (1 - \vartheta \underline{F}_j)\underline{\varphi}_j - (\bar{\vartheta}\underline{G}_j + |\vartheta|^2 \underline{F}_j M_j)\underline{\psi}_j = \underline{F}_j + \mathcal{O}(n^{-1/2}), \\ -\vartheta \gamma \underline{\varphi}_j + (1 - \bar{\vartheta}\underline{F}_j - \gamma|\vartheta|^2 M_j)\underline{\psi}_j = \gamma + \mathcal{O}(n^{-1/2}), \end{cases}$$

where

$$\underline{F}_j = \frac{1}{n}\sum_{k=1}^{j}\frac{a_k^* T D \bar{T} \bar{a}_k \tilde{d}_k}{(1+\tilde{d}_k\delta)^2} \in \mathbb{C}, \quad \underline{G}_j = \frac{1}{n}\sum_{k=1}^{j}\sum_{\substack{\ell=1\\ \ell\neq k}}^{j}\frac{\tilde{d}_k\tilde{d}_\ell(a_k^* T a_\ell)^2}{(1+\tilde{d}_k\delta)^2(1+\tilde{d}_\ell\delta)^2} \in \mathbb{R},$$

$$M_j = \frac{1}{n}\sum_{\ell=1}^{j}\rho^2\tilde{t}_{\ell\ell}^2\tilde{d}_\ell^2.$$

The determinant of this system is:

$$\underline{\boldsymbol{\Delta}}_j = |1-\vartheta\underline{F}_j|^2 - |\vartheta|^2\underline{\gamma}(M_j+\underline{G}_j).$$

By (3.13), $0 \leq \underline{\gamma} \leq \gamma$; furthermore, $|\vartheta| \leq 1$, $|\underline{F}_j| \leq F_j$, and $|\underline{G}_j| \leq G_j$. As a result, $\underline{\boldsymbol{\Delta}}_j \geq \boldsymbol{\Delta}_j$. Hence, by Lemma 5.3, the perturbation remains of order $\mathcal{O}(n^{-1/2})$ after solving the system. Performing the same derivations as in Step 4 in Sec. 5, it can be established that $\underline{\boldsymbol{\Delta}}_n = \underline{\Delta}_n$. We finally end up with:

$$\frac{1}{n}\sum_{j=1}^{n}(\rho^2\tilde{t}_{jj}^2\tilde{d}_j^2|\vartheta|^2\underline{\psi}_j + 2\rho^2\tilde{d}_j\tilde{t}_{jj}^2\operatorname{Re}(\vartheta\underline{\theta}_{jj})) = \sum_{j=1}^{n}\frac{\underline{\boldsymbol{\Delta}}_{j-1}-\underline{\boldsymbol{\Delta}}_j}{\underline{\boldsymbol{\Delta}}_j} + \mathcal{O}(n^{-1/2}),$$

$$= -\log(\underline{\Delta}_n) + \mathcal{O}(n^{-1/2}),$$

which is the desired result.

6.2. *Estimates over* Θ_n

In order to conclude the proof of Theorem 2.2, it remains to prove that $0 < \liminf_n \Theta_n \leq \limsup_n \Theta_n < \infty$.

Consider first the upper bound. By Lemma 3.4, $\sup_n(-\log\Delta_n) < \infty$. As $\underline{\Delta}_n \geq \Delta_n$, $\log\underline{\Delta}_n$ is defined and $\sup_n(-\log\underline{\Delta}_n) < \infty$. By Lemma 3.3, the cumulant term in the expression of Θ_n is bounded, hence $\limsup_n \Theta_n < \infty$.

We now prove that $\liminf \Theta_n > 0$. To this end, write:

$$\Theta_n = \sum_{j=1}^{n}\left(\frac{\boldsymbol{\Delta}_{j-1}-\boldsymbol{\Delta}_j}{\boldsymbol{\Delta}_j} + \frac{\underline{\boldsymbol{\Delta}}_{j-1}-\underline{\boldsymbol{\Delta}}_j}{\underline{\boldsymbol{\Delta}}_j}\right) + \kappa\frac{\rho^2}{n^2}\sum_{i=1}^{N}d_i^2t_{ii}^2\sum_{j=1}^{n}\tilde{d}_j^2\tilde{t}_{jj}^2 + \mathcal{O}(n^{-1/2}),$$

$$= \sum_{j=1}^{n}\left(\frac{\gamma(G_j-G_{j-1})}{\boldsymbol{\Delta}_j} + \frac{|\vartheta|^2\underline{\gamma}(\underline{G}_j-\underline{G}_{j-1})}{\underline{\boldsymbol{\Delta}}_j}\right)$$

$$+ \sum_{j=1}^{n}\left(\frac{2(F_j-F_{j-1})(1-F_j)}{\boldsymbol{\Delta}_j} + \frac{2\operatorname{Re}(\vartheta(\underline{F}_j-\underline{F}_{j-1})(1-\bar{\vartheta}\underline{F}_j))}{\underline{\boldsymbol{\Delta}}_j}\right)$$

$$+ \frac{\rho^2}{n}\sum_{j=1}^{n}\tilde{d}_j^2\tilde{t}_{jj}^2\left(\frac{\gamma}{\boldsymbol{\Delta}_j} + |\vartheta|^2\frac{\underline{\gamma}}{\underline{\boldsymbol{\Delta}}_j} + \frac{\kappa}{n}\sum_{i=1}^{N}d_i^2t_{ii}^2\right) + \mathcal{O}(n^{-1/2})$$

$$\stackrel{\triangle}{=} Z_{1,n} + Z_{2,n} + Z_{3,n} + \mathcal{O}(n^{-1/2}).$$

We prove in the sequel that $Z_{1,n} \geq 0$, $Z_{2,n} \geq 0$, and that $\liminf_n Z_{3,n} > 0$. It has already been noticed that $\underline{\Delta}_j \geq \Delta_j$; moreover, it can be proven by direct computation that $|\underline{G}_j - \underline{G}_{j-1}| \leq G_j - G_{j-1}$, hence $Z_{1,n} \geq 0$. As

$$(1 - F_j) - \frac{\gamma(M_j + G_j)}{1 - F_j} \leq |1 - \vartheta \underline{F}_j| - \frac{|\vartheta|^2 \underline{\gamma}(M_j + \underline{G}_j)}{|1 - \vartheta \underline{F}_j|},$$

this implies that $\underline{\Delta}_j^{-1}|1 - \vartheta \underline{F}_j| \leq \Delta_j^{-1}(1 - F_j)$. Noticing in addition that $|\underline{F}_j - \underline{F}_{j-1}| \leq F_j - F_{j-1}$, we get $Z_{2,n} \geq 0$. The cumulant $\kappa = \mathbb{E}|X_{11}|^4 - 2 - |\vartheta|^2$ satisfies $\kappa \geq -1 - |\vartheta|^2$, hence

$$Z_{3,n} \geq \frac{\rho^2}{n^2} \sum_{j=1}^{n} \tilde{d}_j^2 \tilde{t}_{jj}^2 \left(\left(\frac{1}{\Delta_j} - 1 \right) + |\vartheta|^2 \left(\frac{1}{\underline{\Delta}_j} - 1 \right) \right) \sum_{i=1}^{N} d_i^2 t_{ii}^2$$

$$+ \frac{\rho^2}{n} \sum_{j=1}^{n} \tilde{d}_j^2 \tilde{t}_{jj}^2 \left(\frac{1}{n\Delta_j} \sum_{\substack{k,\ell=1 \\ k \neq \ell}}^{N} d_k^2 |t_{kl}|^2 + \frac{1}{n\underline{\Delta}_j} \sum_{\substack{k,\ell=1 \\ k \neq \ell}}^{N} d_k^2 (t_{kl})^2 \right)$$

$$\geq \frac{\rho^2}{n^2} \sum_{j=1}^{n} \tilde{d}_j^2 \tilde{t}_{jj}^2 \left(\left(\frac{1}{\Delta_j} - 1 \right) + |\vartheta|^2 \left(\frac{1}{\underline{\Delta}_j} - 1 \right) \right) \sum_{i=1}^{N} d_i^2 t_{ii}^2$$

$$= \frac{\rho^2}{n^2} \sum_{j=1}^{n} \tilde{d}_j^2 \tilde{t}_{jj}^2 p_j \sum_{i=1}^{N} d_i^2 t_{ii}^2.$$

As the term p_j is linear in $|\vartheta|^2 \in [0,1]$, $p_j \geq \min(\Delta_j^{-1}(1 - \Delta_j), \Delta_j^{-1} + \underline{\Delta}_j^{-1} - 2)$. We have

$$\Delta_j \underline{\Delta}_j \leq ((1 - F_j)^2 - \gamma(M_j + G_j))((1 + F_j)^2 + \gamma(M_j + G_j))$$

$$= (1 - F_j^2)^2 - \gamma^2(M_j + G_j)^2 - 4\gamma F_j(M_j + G_j) \leq 1.$$

Hence $\Delta_j^{-1} + \underline{\Delta}_j^{-1} - 2 \geq \Delta_j^{-1} + \Delta_j - 2 = \Delta_j^{-1}(1 - \Delta_j)^2$. As $1 - \Delta_j \geq \gamma M_j$, we get $p_j \geq \gamma^2 M_j^2$, which implies that

$$Z_{3,n} \geq \frac{\rho^2 \gamma^2}{n^2} \sum_{j=1}^{n} \tilde{d}_j^2 \tilde{t}_{jj}^2 M_j^2 \sum_{i=1}^{N} d_i^2 t_{ii}^2 = \gamma^2 \left(\sum_{j=1}^{n} M_j^2 (M_j - M_{j-1}) \right) \frac{1}{n} \sum_{i=1}^{N} d_i^2 t_{ii}^2$$

$$= \frac{\gamma^2}{3} \left(\sum_{j=1}^{n} M_j^3 - M_{j-1}^3 \right) \frac{1}{n} \sum_{i=1}^{N} d_i^2 t_{ii}^2 + \mathcal{O}(n^{-1}) = \frac{\gamma^2 M_n^3}{3} \frac{1}{n} \sum_{i=1}^{N} d_i^2 t_{ii}^2 + \mathcal{O}(n^{-1}),$$

whose liminf is positive by Lemma 3.3.

The estimates over the variance are therefore established. This completes the proof of Theorem 2.2.

7. Proposition 2.1 (Bias): Main Steps of the Proof

Proof of Proposition 2.1(i) can be found in [13, Theorem 2]. Let us prove (ii). The same arguments as in the companion article [21] allow to write the bias term as:

$$N(\mathbb{E}\,\mathcal{I}_n(\rho) - V_n(\rho)) = \int_\rho^\infty \mathrm{Tr}(T(-\omega) - \mathbb{E}Q(-\omega))\mathbf{d}\omega.$$

Recall that in the centered case where $A = 0$, $T(-\omega)$ and $\tilde{T}(-\omega)$ take the simple form $T(-\omega) = [\omega(I_N + \tilde{\delta}(-\omega)D)]^{-1}$ and $\tilde{T}(-\omega) = [\omega(I_N + \delta(-\omega)\tilde{D})]^{-1}$, which implies that $\underline{\gamma} = \gamma$ and $\underline{\tilde{\gamma}} = \tilde{\gamma}$. We introduce the following intermediate quantities:

$$\alpha(-\omega) = \frac{1}{n}\,\mathrm{Tr}\,D\mathbb{E}Q(-\omega), \quad \tilde{\alpha}(-\omega) = \frac{1}{n}\,\mathrm{Tr}\,\tilde{D}\mathbb{E}\tilde{Q}(-\omega),$$

$$C(-\omega) = (\omega(I_N + \tilde{\alpha}(-\omega)D))^{-1}, \quad \tilde{C}(-\omega) = (\omega(I_n + \alpha(-\omega)\tilde{D}))^{-1}.$$

From Theorem 3.1, $n^{-1}\,\mathrm{Tr}\,U(C - T) \to 0$ and $n^{-1}\,\mathrm{Tr}\,\tilde{U}(\tilde{C} - \tilde{T}) \to 0$ for any sequences of deterministic matrices U and $\tilde{U}$ with bounded spectral norms.

The proof consists of two steps:

7.1. *Step 1*

Given $\omega > 0$, let

$$\beta_n(\omega) = \left(\kappa + \frac{|\vartheta|^2}{1 - \omega^2|\vartheta|^2\gamma\tilde{\gamma}}\right) R_n(\omega)$$

where

$$R_n(\omega) = \frac{\frac{\omega^3}{n}\,\mathrm{Tr}\,DT^2}{1 - \omega^2\gamma\tilde{\gamma}}\left(\frac{\gamma}{n}\,\mathrm{Tr}(\tilde{D}\tilde{T})^3 - \frac{\omega\tilde{\gamma}^2}{n}\,\mathrm{Tr}(DT)^3\right) - \frac{\omega^2\tilde{\gamma}}{n}\,\mathrm{Tr}\,D^2T^3.$$

The purpose of this step is to show that $\int_\rho^\infty |\beta_n(\omega)|\mathbf{d}(\omega) < \infty$ and that

$$N(\mathbb{E}\,\mathcal{I}_n(\rho) - V_n(\rho)) - \int_\rho^\infty \beta_n(\omega)\mathbf{d}(\omega) \xrightarrow[n\to\infty]{} 0. \tag{7.1}$$

By inspecting the expression of $\beta_n(\omega)$, by using Lemmas 3.3 and 3.4 and by recalling that $1 - \omega^2\gamma\tilde{\gamma} = \Delta_n$ taken at $z = -\omega$, we obtain after a small derivation that $|\beta_n(\omega)| \leq K/\omega^3$ on $[\rho, \infty)$ where K does not depend on n nor on ω. This proves the integrability of $|\beta_n(\omega)|$. By taking up the poof of [21, Inequality (7.10)] with minor modifications, we also show that

$$|\mathrm{Tr}(T(-\omega) - \mathbb{E}Q(-\omega))| \leq \frac{K}{\omega^2}$$

hence, showing

$$\mathrm{Tr}(T(-\omega) - \mathbb{E}Q(-\omega)) - \beta(\omega) \xrightarrow[n\to\infty]{} 0 \tag{7.2}$$

and applying the Dominated Convergence Theorem leads to (7.1). In order to show (7.2), we start by writing for $z = -\omega$

$$\operatorname{Tr}(T - \mathbb{E}Q) = \operatorname{Tr}(T - C) + \operatorname{Tr}(C - \mathbb{E}Q).$$

Using the decomposition $\operatorname{Tr}(T - C) = \operatorname{Tr} C(C^{-1} - T^{-1})T = \omega(\tilde{\alpha} - \tilde{\delta}) \operatorname{Tr} DCT$, we obtain

$$\operatorname{Tr}(T - \mathbb{E}Q) = \omega n(\tilde{\alpha} - \tilde{\delta})\frac{1}{n} \operatorname{Tr} DCT + \operatorname{Tr}(C - \mathbb{E}Q). \tag{7.3}$$

On the other hand, writing $n(\alpha - \delta) = \operatorname{Tr} D(\mathbb{E}Q - C) + \operatorname{Tr} D(C - T) = \operatorname{Tr} D(\mathbb{E}Q - C) - \omega(\tilde{\alpha} - \tilde{\delta}) \operatorname{Tr} D^2CT$ and similarly for $n(\tilde{\alpha} - \tilde{\delta})$, we obtain the system

$$\begin{bmatrix} 1 & \omega n^{-1} \operatorname{Tr} D^2CT \\ \omega n^{-1} \operatorname{Tr} \tilde{D}^2\tilde{C}\tilde{T} & 1 \end{bmatrix} \begin{bmatrix} n(\alpha - \delta) \\ n(\tilde{\alpha} - \tilde{\delta}) \end{bmatrix} = \begin{bmatrix} \operatorname{Tr} D(\mathbb{E}Q - C) \\ \operatorname{Tr} \tilde{D}(\mathbb{E}\tilde{Q} - \tilde{C}) \end{bmatrix}. \tag{7.4}$$

Consequently, in order to show (7.2), we need to look for approximations of $\operatorname{Tr} U(\mathbb{E}Q - C)$ and $\operatorname{Tr} \tilde{U}(\mathbb{E}\tilde{Q} - \tilde{C})$ for deterministic matrices U and $\tilde{U}$ with bounded spectral norms:

Lemma 7.1. *Assume that the setting of Proposition* 2.1 *holds true. Fix* $z = -\omega < 0$ *and let* $(U_n)_n$ *(respectively,* $(\tilde{U}_n)_n$*) be a sequence of* $N \times N$ *(respectively,* $n \times n$*) diagonal deterministic matrices such that* $\sup_n \max(\|U_n\|, \|\tilde{U}_n\|) < \infty$*. Then,*

$$\operatorname{Tr} U_n(C - \mathbb{E}Q) + \kappa\frac{\omega^2\tilde{\gamma}}{n} \operatorname{Tr} UD^2T^3 + |\vartheta|^2 \frac{\omega^2\tilde{\gamma}}{1 - \omega^2|\vartheta|^2\gamma\tilde{\gamma}}\frac{1}{n} \operatorname{Tr} UD^2T^3 \xrightarrow[n\to\infty]{} 0, \tag{7.5}$$

$$\operatorname{Tr} \tilde{U}_n(\tilde{C} - \mathbb{E}\tilde{Q}) + \kappa\frac{\omega^2\gamma}{n} \operatorname{Tr} \tilde{U}\tilde{D}^2\tilde{T}^3 + |\vartheta|^2 \frac{\omega^2\gamma}{1 - \omega^2|\vartheta|^2\gamma\tilde{\gamma}}\frac{1}{n} \operatorname{Tr} \tilde{U}\tilde{D}^2\tilde{T}^3 \xrightarrow[n\to\infty]{} 0. \tag{7.6}$$

Recalling that $n^{-1} \operatorname{Tr} U(C - T) \to 0$ and taking $U = D^2T$, we have $n^{-1} \operatorname{Tr} D^2CT - \gamma \to 0$. Similarly, $n^{-1} \operatorname{Tr} \tilde{D}^2\tilde{C}\tilde{T} - \tilde{\gamma} \to 0$. Solving system (7.4) and using this lemma with $U = D$ and $\tilde{U} = \tilde{D}$, we obtain

$$n(\tilde{\alpha} - \tilde{\delta}) = \frac{1}{1 - \omega^2\gamma\tilde{\gamma}}(-\omega\tilde{\gamma} \operatorname{Tr} D(\mathbb{E}Q - C) + \operatorname{Tr} \tilde{D}(\mathbb{E}\tilde{Q} - \tilde{C})) + \varepsilon$$

$$= \frac{\kappa}{1 - \omega^2\gamma\tilde{\gamma}}\left(\frac{\omega^2\gamma}{n} \operatorname{Tr}(\tilde{D}\tilde{T})^3 - \frac{\omega^3\tilde{\gamma}^2}{n} \operatorname{Tr}(DT)^3\right)$$

$$+ \frac{|\vartheta|^2}{(1 - \omega^2\gamma\tilde{\gamma})(1 - \omega^2|\vartheta|^2\gamma\tilde{\gamma})}\left(\frac{\omega^2\gamma}{n} \operatorname{Tr}(\tilde{D}\tilde{T})^3 - \frac{\omega^3\tilde{\gamma}^2}{n} \operatorname{Tr}(DT)^3\right) + \varepsilon'$$

where $\varepsilon, \varepsilon' \to 0$. Using Lemma 7.1 again in (7.3) with $U = I$, we obtain (7.2).

The remainder of this paragraph is devoted to the proof of Lemma **7.1**. Recall the following notations:

$$\tilde{b}_j(-\omega) = \frac{1}{\omega(1 + \frac{\tilde{d}_j}{n} \operatorname{Tr} DQ_j(-\omega))} \quad \text{and}$$

$$e_j(-\omega) = \eta_j^* Q_j(-\omega)\eta_j - \frac{\tilde{d}_j}{n} \operatorname{Tr} DQ_j(-\omega).$$

Starting with

$$\operatorname{Tr} U(\mathbb{E}Q - C) = \operatorname{Tr} \mathbb{E}[UC(C^{-1} - Q^{-1})Q]$$
$$= -\operatorname{Tr} \mathbb{E}[UC\Sigma\Sigma^* Q] + \omega\tilde{\alpha} \operatorname{Tr} \mathbb{E}[UCDQ]$$

and using (3.8) and (3.7), we obtain $\operatorname{Tr} U(\mathbb{E}Q - C) = Z_1 + Z_2 + Z_3$ where

$$Z_1 = \sum_{j=1}^{n} \mathbb{E}[\omega^2 \tilde{b}_j^2 e_j \eta_j^* Q_j UC\eta_j],$$

$$Z_2 = -\sum_{j=1}^{n} \mathbb{E}[\omega^3 \tilde{q}_{jj} \tilde{b}_j^2 e_j^2 \eta_j^* Q_j UC\eta_j],$$

$$Z_3 = \omega\tilde{\alpha}\mathbb{E} \operatorname{Tr} UCDQ - \frac{\omega}{n} \sum_{j=1}^{n} \mathbb{E}\tilde{b}_j \tilde{d}_j \operatorname{Tr} DQ_j UC.$$

In the remainder, we omit the study of the negligible terms to focus on the deterministic equivalent formulas; in this spirit, we shall denote by ε a negligible term whose value might change from line to line.

The term Z_1 is

$$Z_1 = \sum_{j=1}^{n} \mathbb{E}\left[\omega^2 \tilde{b}_j^2 e_j \left(\eta_j^* Q_j UC\eta_j - \tilde{d}_j \frac{\operatorname{Tr} DQ_j UC}{n}\right)\right].$$

Using Identity (3.20) with $M = Q_j$, $P = Q_j UC$ and $\boldsymbol{u} = 0$, we obtain:

$$Z_1 = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}\left[\omega^2 \tilde{b}_j^2 \left(\frac{\tilde{d}_j^2}{n} \operatorname{Tr} Q_j DQ_j UCD + \frac{|\vartheta|^2}{n} \tilde{d}_j^2 \operatorname{Tr} Q_j DCU\bar{Q}_j D\right.\right.$$

$$\left.\left. + \frac{\kappa}{n} \sum_{i=1}^{N} \tilde{d}_j^2 d_i^2 [Q_j]_{ii}[Q_j UC]_{ii}\right)\right] + \varepsilon.$$

It is not difficult to check that

$$Z_2 = -\sum_{j=1}^{n} \mathbb{E}\left[\omega^3 \tilde{b}_j^3 \tilde{d}_j \frac{\operatorname{Tr} DQ_j UC}{n} e_j^2\right] + \varepsilon.$$

Turning to Z_3, we have

$$Z_3 = \frac{\omega}{n} \sum_{j=1}^{n} \mathbb{E}[\tilde{d}_j \mathbb{E}\tilde{q}_{jj} \operatorname{Tr} UCDQ - \tilde{d}_j \tilde{b}_j \operatorname{Tr} UCDQ_j]$$

$$= \frac{\omega}{n} \sum_{j=1}^{n} \tilde{d}_j \mathbb{E}[\mathbb{E}\tilde{q}_{jj}(\operatorname{Tr} UCDQ_j - \omega \tilde{q}_{jj} \operatorname{Tr} UCDQ_j \eta_j \eta_j^* Q_j) - \tilde{b}_j \operatorname{Tr} UCDQ_j]$$

$$= \frac{\omega}{n} \sum_{j=1}^{n} \tilde{d}_j \mathbb{E}[(\mathbb{E}\tilde{q}_{jj} - \tilde{b}_j) \operatorname{Tr} UCDQ_j] - \frac{\omega^2}{n} \sum_{j=1}^{n} \tilde{d}_j \mathbb{E}\tilde{q}_{jj} \mathbb{E}[\tilde{b}_j \eta_j^* Q_j UCDQ_j \eta_j] + \varepsilon.$$

Replacing $\tilde{b}_j$ by $\tilde{q}_{jj} + \omega \tilde{b}_j^2 e_j - \omega^2 \tilde{q}_{jj} \tilde{b}_j^2 e_j^2$, we have

$$\mathbb{E}[(\mathbb{E}\tilde{q}_{jj} - \tilde{b}_j) \operatorname{Tr} UCDQ_j] = \mathbb{E}[(\mathbb{E}\tilde{q}_{jj} - \tilde{q}_{jj}) \operatorname{Tr} UCDQ_j]$$
$$+ \mathbb{E}[\omega^2 \tilde{q}_{jj} \tilde{b}_j^2 e_j^2 \operatorname{Tr} UCDQ_j].$$

Since

$$\tilde{q}_{jj} - \mathbb{E}\tilde{q}_{jj} = \frac{\mathbb{E}(\eta_j^* Q_j \eta_j) - \eta_j^* Q_j \eta_j}{\omega(1 + \eta_j^* Q_j \eta_j)(1 + \mathbb{E}(\eta_j^* Q_j \eta_j))}$$

$$+ \mathbb{E}\left[\frac{\eta_j^* Q_j \eta_j - \mathbb{E}(\eta_j^* Q_j \eta_j)}{\omega(1 + \eta_j^* Q_j \eta_j)(1 + \mathbb{E}(\eta_j^* Q_j \eta_j))}\right]$$

we have $\mathbb{E}(\tilde{q}_{jj} - \mathbb{E}\tilde{q}_{jj})^2 = \mathcal{O}(1/n)$. Hence

$$\mathbb{E}[(\mathbb{E}\tilde{q}_{jj} - \tilde{q}_{jj}) \operatorname{Tr} UCDQ_j] = \mathbb{E}[(\mathbb{E}\tilde{q}_{jj} - \tilde{q}_{jj})(\operatorname{Tr} UCDQ_j - \mathbb{E} \operatorname{Tr} UCDQ_j)]$$
$$= \mathcal{O}(n^{-1/2})$$

by Theorem 3.1(5). It follows that $\mathbb{E}[(\mathbb{E}\tilde{q}_{jj} - \tilde{b}_j) \operatorname{Tr} UCDQ_j] = \mathbb{E}[\omega^2 \tilde{b}_j^3 e_j^2 \operatorname{Tr} UCDQ_j] + \varepsilon$, hence

$$Z_2 + Z_3 = -\frac{1}{n} \sum_{j=1}^{n} \mathbb{E}\left[\omega^2 \tilde{b}_j^2 \tilde{d}_j^2 \frac{1}{n} \operatorname{Tr} Q_j DQ_j UCD\right] + \varepsilon.$$

Taking the sum $Z_1 + Z_2 + Z_3$, the terms that do not depend on ϑ nor on κ cancel out, and we are left with

$$\operatorname{Tr} U(\mathbb{E}Q - C) = |\vartheta|^2 \omega^2 \tilde{\gamma} \frac{1}{n} \mathbb{E}[\operatorname{Tr} QDCU\bar{Q}D] + \kappa \omega^2 \tilde{\gamma} \frac{1}{n} \operatorname{Tr} UD^2 T^3 + \varepsilon,$$

where we relied on the usual approximations for the diagonal entries of the resolvent (see Lemma 4.1) to obtain the term in κ. We now briefly characterize the asymptotic

behavior of $n^{-1} \operatorname{Tr} \mathbb{E} Q D C U \bar{Q} D$. Starting with $Q = T + \omega \tilde{\delta} T D Q - T \Sigma \Sigma^* Q$, we have

$$\frac{1}{n} \operatorname{Tr} \mathbb{E} Q D C U \bar{Q} D = \frac{1}{n} \operatorname{Tr} \mathbb{E} T D C U \bar{Q} D + \frac{\omega \tilde{\delta}}{n} \operatorname{Tr} \mathbb{E} T D Q D C U \bar{Q} D$$

$$- \frac{1}{n} \operatorname{Tr} \mathbb{E} T \Sigma \Sigma^* Q D C U \bar{Q} D,$$

and

$$-\frac{1}{n} \operatorname{Tr} \mathbb{E} T \Sigma \Sigma^* Q D C U \bar{Q} D = -\frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \omega \tilde{q}_{jj} \eta_j^* Q_j D C U \bar{Q} D T \eta_j$$

$$= -\frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \omega \tilde{q}_{jj} \eta_j^* Q_j D C U \bar{Q}_j D T \eta_j$$

$$+ \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \omega^2 \tilde{q}_{jj}^2 \eta_j^* Q_j D C U \bar{Q}_j \bar{\eta}_j \eta_j^T D T \eta_j$$

$$= -\frac{\omega \tilde{\delta}}{n} \operatorname{Tr} \mathbb{E} T D Q D C U \bar{Q} D$$

$$+ |\vartheta|^2 \omega^2 \gamma \tilde{\gamma} \frac{\operatorname{Tr} \mathbb{E} Q D C U \bar{Q} D}{n} + \varepsilon.$$

We therefore get

$$\frac{1}{n} \operatorname{Tr} \mathbb{E} Q D C U \bar{Q} D = \frac{\frac{1}{n} \operatorname{Tr} U D^2 T^3}{1 - \omega^2 |\vartheta|^2 \gamma \tilde{\gamma}} + \varepsilon$$

and Convergence (7.5) of Lemma 7.1 is shown. Convergence (7.6) is proven similarly. Lemma 7.1 is proven, and Step 1 of the proof of Proposition 2.1(ii) is established.

7.2. *Step 2*

The purpose of this step is to show that

$$R(\omega) = \frac{1}{2} \frac{\mathbf{d}}{\mathbf{d}\omega} (\omega^2 \gamma(-\omega) \tilde{\gamma}(-\omega)).$$

Plugging into the expression of $\beta_n(\omega)$, it is straightforward to show that $\int_\rho^\infty \beta_n(\omega) \mathbf{d}(\omega)$ coincides with $\mathcal{B}_n$ given by (2.5).

Recall that

$$R(\omega) = \frac{\omega^3 \gamma}{1 - \omega^2 \gamma \tilde{\gamma}} \frac{\operatorname{Tr} D T^2}{n} \frac{\operatorname{Tr}(\tilde{D}\tilde{T})^3}{n} - \frac{\omega^4 \tilde{\gamma}^2}{1 - \omega^2 \gamma \tilde{\gamma}} \frac{\operatorname{Tr} D T^2}{n} \frac{\operatorname{Tr}(D T)^3}{n} - \omega^2 \tilde{\gamma} \frac{\operatorname{Tr} D^2 T^3}{n}$$

$$= R_1 + R_2 + R_3.$$

Our method is similar to [19, §V.B]. We start by showing that the derivatives of $\tilde{\gamma}(-\omega)$ and $\gamma(-\omega)$ that we denote respectively as $\tilde{\gamma}'$ and γ' are

$$
\begin{aligned}
\tilde{\gamma}' &= \frac{\mathbf{d}\tilde{\gamma}(-\omega)}{\mathbf{d}\omega} = -\frac{2}{\omega}\tilde{\gamma} + \frac{2\omega}{1-\omega^2\gamma\tilde{\gamma}}\frac{\operatorname{Tr}(\tilde{D}\tilde{T})^3}{n}\frac{\operatorname{Tr} DT^2}{n}, \\
\gamma' &= \frac{\mathbf{d}\gamma(-\omega)}{\mathbf{d}\omega} = -\frac{2}{\omega}\gamma + \frac{2\omega}{1-\omega^2\gamma\tilde{\gamma}}\frac{\operatorname{Tr}(DT)^3}{n}\frac{\operatorname{Tr}\tilde{D}\tilde{T}^2}{n}.
\end{aligned}
\tag{7.7}
$$

We have

$$
\tilde{\gamma}' = \frac{1}{n}\sum_{j=1}^{n}\tilde{d}_j^2\frac{\mathbf{d}}{\mathbf{d}\omega}\left(\frac{1}{\omega^2(1+\tilde{d}_j\delta(-\omega))^2}\right) = -\frac{2}{\omega}\tilde{\gamma} - 2\omega\delta'\frac{\operatorname{Tr}(\tilde{D}\tilde{T})^3}{n}
\tag{7.8}
$$

where we put $\delta' = \mathbf{d}\delta(-\omega)/\mathbf{d}\omega$. This derivative can be expressed as

$$
\delta' = \frac{1}{n}\sum_{i=1}^{N}d_i\frac{\mathbf{d}}{\mathbf{d}\omega}\left(\frac{1}{\omega(1+d_i\tilde{\delta}(-\omega))}\right) = -\frac{\delta}{\omega} - \omega\gamma\tilde{\delta}'
$$

where $\tilde{\delta}' = \mathbf{d}\tilde{\delta}(-\omega)/\mathbf{d}\omega$. Similarly, $\tilde{\delta}' = -\omega^{-1}\tilde{\delta} - \omega\tilde{\gamma}\delta'$. Combining the two equations, we obtain

$$
\delta' = \frac{\gamma\tilde{\delta} - \omega^{-1}\delta}{1-\omega^2\gamma\tilde{\gamma}}.
\tag{7.9}
$$

Since $T = [\omega(I + \tilde{\delta}D)]^{-1}$ and $\tilde{T} = [\omega(I + \delta\tilde{D})]^{-1}$, we have

$$
T = \omega^{-1}I_N - \tilde{\delta}DT, \quad \tilde{T} = \omega^{-1}I_n - \delta\tilde{D}\tilde{T}.
\tag{7.10}
$$

This leads to

$$
\begin{aligned}
\frac{1}{n}\operatorname{Tr} DT^2 &= \frac{1}{n}\operatorname{Tr} DT((\omega^{-1}I - \tilde{\delta}DT) = \omega^{-1}\delta - \gamma\tilde{\delta}, \quad \text{and} \\
\frac{1}{n}\operatorname{Tr}\tilde{D}\tilde{T}^2 &= \omega^{-1}\tilde{\delta} - \tilde{\gamma}\delta.
\end{aligned}
\tag{7.11}
$$

Combining with (7.9) and (7.8), we obtain the first equation of (7.7), the second being obtained similarly. Using the first equation, the term R_1 can be expressed as

$$
R_1 = \frac{1}{2}(\omega^2\gamma\tilde{\gamma}' + 2\omega\gamma\tilde{\gamma}).
$$

Turning to R_2, we have

$$
\begin{aligned}
R_2 &= -\frac{\omega^4\tilde{\gamma}^2}{1-\omega^2\gamma\tilde{\gamma}}\frac{\operatorname{Tr}(DT)^3}{n}\frac{\operatorname{Tr} DT(\omega^{-1}I - \tilde{\delta}DT)}{n} \\
&= -\frac{\omega^3\delta\tilde{\gamma}^2}{1-\omega^2\gamma\tilde{\gamma}}\frac{\operatorname{Tr}(DT)^3}{n} + \frac{\omega^4\tilde{\delta}\gamma\tilde{\gamma}^2}{1-\omega^2\gamma\tilde{\gamma}}\frac{\operatorname{Tr}(DT)^3}{n} \\
&\overset{(a)}{=} \frac{\omega^3\tilde{\gamma}}{1-\omega^2\gamma\tilde{\gamma}}\frac{\operatorname{Tr}(DT)^3}{n}\frac{\operatorname{Tr}\tilde{D}\tilde{T}^2}{n} - \omega^2\tilde{\gamma}\tilde{\delta}\frac{\operatorname{Tr}(DT)^3}{n} \\
&\overset{(b)}{=} \frac{1}{2}(\omega^2\gamma'\tilde{\gamma} + 2\omega\gamma\tilde{\gamma}) - \omega^2\tilde{\gamma}\tilde{\delta}\frac{\operatorname{Tr}(DT)^3}{n}
\end{aligned}
$$

where (a) is due to $\tilde{\gamma}\delta = \omega^{-1}\tilde{\delta} - n^{-1}\operatorname{Tr}\tilde{D}\tilde{T}^2$, see (7.11), and (b) is due to (7.7). Considering R_3, we have by (7.10),

$$-\omega^2\tilde{\gamma}\tilde{\delta}\frac{\operatorname{Tr}(DT)^3}{n} - \omega^2\tilde{\gamma}\frac{\operatorname{Tr}D^2T^3}{n} = -\omega^2\tilde{\gamma}\frac{\operatorname{Tr}D^2T^2(T+\tilde{\delta}DT)}{n} = -\omega\gamma\tilde{\gamma}.$$

We therefore have $R(\omega) = 0.5(\omega^2\gamma'\tilde{\gamma} + \omega^2\gamma\tilde{\gamma}' + 2\omega\gamma\tilde{\gamma}) = 0.5(\omega^2\gamma\tilde{\gamma})'$. Proposition 2.1 is proven.

Acknowledgments

Appendix A. Proofs for Sec. 3

A.1. *Proofs of Eqs. (3.10) and (3.11)*

Proof of Eq. (3.10) mainly relies on matrix identity (3.2) and on the following identity for the inverse of a partitioned matrix (see, for instance [23, Sec. 0.7.3]):

$$\text{If } A = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \text{then } (A^{-1})_{11} = (a_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}. \qquad \text{(A.1)}$$

To lighten the computations, let us introduce the following notations:

$$\mathcal{I} = (I_n + \tilde{\delta}D)^{-1}, \quad \tilde{\mathcal{I}} = (I_{N-1} + \delta\tilde{D}_1)^{-1}.$$

In order to express a diagonal element of $\tilde{T}$, say $\tilde{t}_{11}$ (without loss of generality), let us first write:

$$\tilde{T} = \begin{bmatrix} -z(1+\delta\tilde{d}_1) + a_1^*\mathcal{I}a_1 & a_1^*\mathcal{I}A_1 \\ A_1^*\mathcal{I}a_1 & -z\tilde{\mathcal{I}}^{-1} + A_1^*\mathcal{I}A_1 \end{bmatrix}^{-1}.$$

Hence, according to (A.1):

$$\frac{1}{\tilde{t}_{11}} = -z(1+\delta\tilde{d}_1) + a_1^*\mathcal{I}a_1 - a_1^*\mathcal{I}A_1[-z\tilde{\mathcal{I}}^{-1} + A_1^*\mathcal{I}A_1]^{-1}A_1^*\mathcal{I}a_1$$

$$\overset{(a)}{=} -z(1+\delta\tilde{d}_1) + a_1^*\mathcal{I}a_1 - a_1^*\mathcal{I}A_1\left[-\frac{1}{z}\tilde{\mathcal{I}} + \frac{1}{z}\tilde{\mathcal{I}}A_1^*\mathcal{T}_1A_1\tilde{\mathcal{I}}\right]A_1^*\mathcal{I}a_1$$

$$\overset{(b)}{=} -z(1+\delta\tilde{d}_1) + a_1^*\mathcal{I}a_1 + \frac{1}{z}a_1^*\mathcal{I}(\mathcal{T}_1^{-1} + z\mathcal{I}^{-1})\mathcal{I}a_1$$

$$\quad - \frac{1}{z}a_1^*\mathcal{I}A_1\tilde{\mathcal{I}}A_1^*[I_N + z\mathcal{T}_1\mathcal{I}^{-1}]\mathcal{I}a_1$$

$$\overset{(b)}{=} -z(1 + \delta \tilde{d}_1) + 2a_1^* \mathcal{I} a_1 + \frac{1}{z} a_1^* \mathcal{I} \mathcal{T}_1^{-1} \mathcal{I} a_1$$

$$- \frac{1}{z} a_1^* \mathcal{I} [\mathcal{T}_1^{-1} + z\mathcal{I}^{-1}][I_N + z\mathcal{T}_1 \mathcal{I}^{-1}] \mathcal{I} a_1$$

$$= -z(1 + \delta \tilde{d}_1) - z a_1^* \mathcal{T}_1 a_1,$$

where (a) follows from (3.2), (b) from equalities

$$\mathcal{T}_1 A_1 \tilde{\mathcal{I}} A_1^* = I_N + z\mathcal{T}_1 \mathcal{I}^{-1} \quad \text{and} \quad A_1 \tilde{\mathcal{I}} A_1^* = \mathcal{T}_1^{-1} + z\mathcal{I}^{-1}$$

which follow from the mere definition of $\mathcal{T}_1$. Finally, (3.10) is established.

Let us now turn to the proof of (3.11). Notice first that T can be expressed as

$$T = \left[-z(I_N + \tilde{\delta} D) + A_1 (I_{n-1} + \tilde{\delta} \tilde{D}_1)^{-1} A_1^* + \frac{a_1 a_1^*}{1 + \delta \tilde{d}_1} \right]^{-1}.$$

Applying (3.2) readily yields:

$$T = \mathcal{T}_1 - \mathcal{T}_1 a_1 (1 + \delta \tilde{d}_1 + a_1^* \mathcal{T}_1 a_1) a_1^* \mathcal{T}_1.$$

It remains to multiply by a_1^* (left), b (right) and to use (3.10) to establish (3.11).

A.2. *Proof of inequality (3.15)*

We provide here some elements to establish that $\mathbb{E} |e_j|^p = \mathcal{O}(n^{-p/2})$. Recall the definition (3.1) of e_j and write:

$$\mathbb{E} |e_j|^p \leq K \left\{ \mathbb{E} \left| y_j^* Q_j y_j - \frac{\tilde{d}_j}{n} \operatorname{Tr} DQ_j \right|^p + \mathbb{E} |a_j^* Q_j y_j|^p + \mathbb{E} |y_j^* Q_j a_j|^p \right\}.$$

The first term of the right-hand side can be directly estimated with the help of Lemma 3.2. The two remaining terms are similar and can be estimated in the following way:

$$\mathbb{E} |a_j^* Q_j y_j|^p = \mathbb{E} (y_j^* Q_j^* a_j a_j^* Q_j y_j)^{p/2}$$

$$\leq K \left(\mathbb{E} \left| y_j^* Q_j^* a_j a_j^* Q_j y_j - \frac{d_j^2}{n} \operatorname{Tr} Q_j^* a_j a_j^* Q_j \right|^{p/2} + \mathbb{E} \left| \frac{1}{n} \operatorname{Tr} Q_j^* a_j a_j^* Q_j \right|^{p/2} \right).$$

The first term of the right-hand side can be handled with the help of Lemma 3.2 (notice that $Q_j^* a_j a_j^* Q_j$ is of rank one and has a bounded spectral norm), and the second term is directly of the right order.

A.3. *Proof of Theorem 3.1*

Items (1)–(3) of Theorem 3.1 are shown in [22]. Let us show Theorem 3.1(4). Denote by $(\delta_j, \tilde{\delta}_j)$ the solution of System (1.2) when A and $\tilde{D}$ are replaced with A_j and $\tilde{D}_j$ respectively. Let T_j and $\tilde{T}_j$ be the matrices associated to $(\delta_j, \tilde{\delta}_j)$ as in Eq. (1.3).

Then $\mathbb{E}|u^*(Q_j - T_j)v|^{2p} \leq K_p n^{-p}$ by Item (3), and we only need to show that $|u^*(\mathcal{T}_j - T_j)v| \leq K/\sqrt{n}$. We have

$$|\delta - \delta_j| = \frac{1}{n}|\mathrm{Tr}\, D(T - T_j)|$$

$$\leq \frac{1}{n}|\mathrm{Tr}\,\mathbb{E}D(T - Q)| + \frac{1}{n}|\mathrm{Tr}\,\mathbb{E}D(Q - Q_j)| + \frac{1}{n}|\mathrm{Tr}\,\mathbb{E}D(Q_j - T_j)|$$

$$= \mathcal{O}(n^{-1})$$

by Item (2) and Lemma 3.1. Moreover,

$$|\tilde{\delta} - \tilde{\delta}_j| \leq \frac{1}{n}|\mathrm{Tr}\,\mathbb{E}\tilde{D}(\tilde{T} - \tilde{Q})| + \frac{1}{n}|\mathbb{E}(\mathrm{Tr}\,\tilde{D}\tilde{Q} - \mathrm{Tr}\,\tilde{D}_j\tilde{Q}_j)| + \frac{1}{n}|\mathrm{Tr}\,\mathbb{E}\tilde{D}_j(\tilde{Q}_j - \tilde{T}_j)|.$$

In order to deal with the middle term at the right-hand side, assume without generality loss that $j = 1$. Using the identity in [23, Sec. 0.7.3] for the inverse of a partitioned matrix, we obtain

$$\tilde{Q} = \begin{bmatrix} \tilde{q}_{11} & \times \\ \times & \tilde{Q}_1 + \tilde{q}_{11}\tilde{Q}_1\Sigma_1^*\eta_1\eta_1^*\Sigma_1\tilde{Q}_1 \end{bmatrix}$$

hence $\mathbb{E}(\mathrm{Tr}\,\tilde{D}\tilde{Q} - \mathrm{Tr}\,\tilde{D}_1\tilde{Q}_1) = \mathcal{O}(1)$, which shows that $|\tilde{\delta} - \tilde{\delta}_j| = \mathcal{O}(n^{-1})$. We now have

$$u^*(\mathcal{T}_j - T_j)v = u^*\mathcal{T}_j(T_j^{-1} - \mathcal{T}_j^{-1})T_jv$$

$$= u^*\mathcal{T}_j(\rho(\tilde{\delta}_j - \tilde{\delta})D + (\delta - \delta_j)A_j(I + \delta_j\tilde{D}_j)^{-1}\tilde{D}_j(I + \delta\tilde{D}_j)^{-1}A_j^*)T_jv$$

$$= \mathcal{O}(n^{-1})$$

which proves Item (4). In order to prove Item (5), we develop $\mathrm{Tr}\,U(Q - \mathbb{E}Q)$ as a sum of martingale differences:

$$\mathrm{Tr}\,U(Q - \mathbb{E}Q) = \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1})\,\mathrm{Tr}\,UQ$$

$$= \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1})\,\mathrm{Tr}\,U(Q - Q_j)$$

$$= -\sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1})(\rho\tilde{q}_{jj}\eta_j^*Q_jUQ_j\eta_j)$$

by (3.4), hence $\mathbb{E}|\mathrm{Tr}\,U(Q - \mathbb{E}Q)|^2 = \sum_{j=1}^n \mathbb{E}|(\mathbb{E}_j - \mathbb{E}_{j-1})(\rho\tilde{q}_{jj}\eta_j^*Q_jUQ_j\eta_j)|^2$. We now use (3.7). We have

$$\mathbb{E}|(\mathbb{E}_j - \mathbb{E}_{j-1})(\rho\tilde{b}_j\eta_j^*Q_jUQ_j\eta_j)|^2$$

$$= \mathbb{E}\left|\mathbb{E}_j\left(\rho\tilde{b}_j(\eta_j^*Q_jUQ_j\eta_j - \tilde{d}_j\frac{\mathrm{Tr}\,DQ_jUQ_j}{n} - a_j^*Q_jUQ_ja_j)\right)\right|^2$$

$$= \mathcal{O}(n^{-1})$$

by Lemma 3.2, and furthermore,

$$\mathbb{E}|(\mathbb{E}_j - \mathbb{E}_{j-1})(\rho^2 \tilde{q}_{jj} \tilde{b}_j e_j \eta_j^* Q_j U Q_j \eta_j)|^2 \leq K(\mathbb{E}e_j^4 \mathbb{E}|\eta_j^* Q_j U Q_j \eta_j|^4)^{1/2}$$
$$= \mathcal{O}(n^{-1})$$

by Lemma 3.2 and (3.15). This shows Theorem 3.1(5).

A.4. *Proof of Lemma 3.3*

The two first upper bounds are easy to obtain, given that δ_n and $\tilde{\delta}_n$ are Stieltjes transforms of non-negative measures with respective total mass $n^{-1}\operatorname{Tr} D$ and $n^{-1}\operatorname{Tr}\tilde{D}$. Now $\operatorname{Tr} DT^2 \leq \boldsymbol{d}_{\max}\operatorname{Tr} T^2$ by Inequality (3.14), which in turn is smaller than $N\boldsymbol{d}_{\max}\rho^{-2}$, hence the third upper bound, and the other upper bounds can be proven similarly. Let us now prove the first lower bound.

$$\operatorname{Tr} D = \operatorname{Tr}(T^{\frac{1}{2}} D T^{\frac{1}{2}} T^{-1}) \overset{(a)}{\leq} \operatorname{Tr}(DT) \times \|T^{-1}\|$$
$$\leq \operatorname{Tr}(DT) \times (\rho(1 + \tilde{\delta}_n \boldsymbol{d}_{\max}) + \boldsymbol{a}_{\max}^2 \|(I + \delta_n \tilde{D})^{-1}\|)$$
$$\overset{(b)}{\leq} \operatorname{Tr}(DT) \times (\rho + \boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max} + \boldsymbol{a}_{\max}^2),$$

where (a) follows from (3.14) and (b) from the upper bound on $\tilde{\delta}_n$. This readily yields δ_n's lower bound and $\tilde{\delta}_n$'s lower bound which can be proven similarly. Writing $\operatorname{Tr} D \leq (\operatorname{Tr} DT^2)\|T^{-1}\|^2$, we obtain the lower bounds on $n^{-1}\operatorname{Tr} DT^2$ and $n^{-1}\operatorname{Tr}\tilde{D}\tilde{T}^2$ similarly. The lower bound for γ_n follows from the same ideas:

$$\left(\frac{1}{N}\operatorname{Tr} D\right)^2 \leq \frac{1}{N}\operatorname{Tr} D^2 = \frac{1}{N}\operatorname{Tr}(T^{\frac{1}{2}} D^2 T^{\frac{1}{2}} T^{-1})$$

$$\leq \frac{1}{N}\operatorname{Tr}(T^{\frac{1}{2}} D^2 T^{\frac{1}{2}}) \times \|T^{-1}\| = \frac{1}{N}\operatorname{Tr}(T^{\frac{1}{2}} D T^{\frac{1}{2}} T^{-1} T^{\frac{1}{2}} D T^{\frac{1}{2}}) \times \|T^{-1}\|$$

$$\leq \frac{1}{N}\operatorname{Tr}(TDTD) \times \|T^{-1}\|^2,$$

and one readily obtains γ_n's lower bound (and similarly $\tilde{\gamma}_n$'s lower bound) using Assumption 2.3 and the upper estimate previously obtained for $\|T^{-1}\|$.

The two last series of inequalities related to $n^{-1}\sum_{i=1}^N d_i^2 t_{ii}^2$ and $n^{-1}\sum_{j=1}^n \tilde{d}_j^2 \tilde{t}_{jj}^2$ can be proven with similar arguments (lower bounds are in fact easier to obtain as one can directly get lower bounds for t_{ii} and $\tilde{t}_{jj}$ — using (3.10) for instance).

A.5. *Proof of Lemma 3.4*

From (1.3), $TA(I + \delta\tilde{D})^{-1}A^* = I - \rho T(I + \tilde{\delta}D)$. Moreover, $(I + \delta\tilde{D})^{-1}\tilde{D} = \delta^{-1}I - \delta^{-1}(I + \delta\tilde{D})^{-1}$. Hence

$$\frac{1}{n}\operatorname{Tr} D^{1/2} TA(I + \delta\tilde{D})^{-2}\tilde{D}A^* T D^{\frac{1}{2}} \leq \frac{1}{n\delta}\operatorname{Tr} DTA(I + \delta\tilde{D})^{-1}A^* T$$

$$= 1 - \frac{\rho}{n\delta}\operatorname{Tr} DT^2 - \rho\frac{\tilde{\delta}}{\delta}\gamma \qquad (A.2)$$

which proves the first assertion with the help of the results of Lemma 3.3. Similarly,

$$\frac{1}{n} \operatorname{Tr} \tilde{D}^{1/2} \tilde{T} A^* (I + \tilde{\delta} D)^{-2} D A \tilde{T} \tilde{D}^{\frac{1}{2}} \leq 1 - \frac{\rho}{n\tilde{\delta}} \operatorname{Tr} \tilde{D} \tilde{T}^2 - \rho \frac{\delta}{\tilde{\delta}} \tilde{\gamma}. \qquad (A.3)$$

We now show that the left hand sides (l.h.s.) of (A.2) and (A.3) are equal. Using the well known matrix identity $(I + UV)^{-1} U = U(I + VU)^{-1}$,

$$
\begin{aligned}
TA&(I + \delta \tilde{D})^{-1} \\
&= \rho^{-1}(I + \tilde{\delta} D)^{-1}(I + \rho^{-1}A(I + \delta \tilde{D})^{-1}A^*(I + \tilde{\delta} D)^{-1})^{-1}A(I + \delta \tilde{D})^{-1} \\
&= \rho^{-1}(I + \tilde{\delta} D)^{-1}A(I + \delta \tilde{D})^{-1}(I + \rho^{-1}A^*(I + \tilde{\delta} D)^{-1}A(I + \delta \tilde{D})^{-1})^{-1} \\
&= (I + \tilde{\delta} D)^{-1}A\tilde{T}.
\end{aligned}
$$

Plugging this identity in the left-hand side of (A.2), and identifying with the left-hand side of (A.3), we obtain the result. As a consequence, we have

$$
\left(1 - \frac{1}{n} \operatorname{Tr} D^{1/2} TA(I + \delta \tilde{D})^{-2} \tilde{D} A^* T D^{\frac{1}{2}} \right)^2
$$

$$
\geq \left(\frac{\rho}{n\delta} \operatorname{Tr} DT^2 + \rho \frac{\tilde{\delta}}{\delta} \gamma \right) \left(\frac{\rho}{n\tilde{\delta}} \operatorname{Tr} \tilde{D} \tilde{T}^2 + \rho \frac{\delta}{\tilde{\delta}} \tilde{\gamma} \right)
$$

$$
\geq \rho^2 \gamma \tilde{\gamma} + \frac{\rho}{n\delta} \operatorname{Tr} DT^2 \frac{\rho}{n\tilde{\delta}} \operatorname{Tr} \tilde{D} \tilde{T}^2
$$

which is the second assertion. By Lemma 3.3, this leads to $\liminf \Delta_n > 0$. Lemma 3.4 is proven.

Appendix B. Additional Proofs for Sec. 5

B.1. *Proof of Lemma 5.1*

Let us show that $\max_j \operatorname{var}(a^* \mathbb{E}_j QDE_j Qa) = \mathcal{O}(n^{-1})$. We have

$$
a^* \mathbb{E}_j QD\mathbb{E}_j Qa - \mathbb{E}a^* \mathbb{E}_j QD\mathbb{E}_j Qa = \sum_{i=1}^{j} (\mathbb{E}_i - \mathbb{E}_{i-1})(a^* \mathbb{E}_j QD\mathbb{E}_j Qa)
$$

$$
= \sum_{i=1}^{j} (\mathbb{E}_i - \mathbb{E}_{i-1}) \| D^{1/2} \mathbb{E}_j (Q_i - \rho \tilde{q}_{ii} Q_i \eta_i \eta_i^* Q_i) a \|^2
$$

$$
= \sum_{i=1}^{j} (\mathbb{E}_i - \mathbb{E}_{i-1})[-2\rho \operatorname{Re}(a^* (\mathbb{E}_j Q_i) D(\mathbb{E}_j \tilde{q}_{ii} Q_i \eta_i \eta_i^* Q_i) a)
$$

$$
+ \| \mathbb{E}_j (\rho \tilde{q}_{ii} \eta_i^* Q_i a D^{1/2} Q_i \eta_i) \|^2]
$$

$$
\stackrel{\triangle}{=} 2 \operatorname{Re}(X) + Z, \qquad (B.1)
$$

and the variance of $a^* \mathbb{E}_j QD\mathbb{E}_j Qa$ is the sum of the variances of these martingale increments. Consider the term X. Recalling that $\tilde{q}_{ii} = \tilde{b}_i - \rho \tilde{q}_{ii} \tilde{b}_i e_i$,

$$X = -\rho \sum_{i=1}^{j} (\mathbb{E}_i - \mathbb{E}_{i-1})(\tilde{b}_i \eta_i^* Q_i aa^* (\mathbb{E}_j Q_i) DQ_i \eta_i)$$

$$+ \rho^2 \sum_{i=1}^{j} (\mathbb{E}_i - \mathbb{E}_{i-1})(\tilde{b}_i \tilde{q}_{ii} e_i \eta_i^* Q_i aa^* (\mathbb{E}_j Q_i) DQ_i \eta_i)$$

$$\triangleq X_1 + X_2.$$

Let $M_i = Q_i aa^* (\mathbb{E}_j Q_i) DQ_i$. The term X_1 satisfies

$$\mathbb{E}|X_1|^2 = \rho^2 \sum_{i=1}^{j} \mathbb{E} \left| \mathbb{E}_i \tilde{b}_i \left(y_i^* M_i y_i - \frac{\tilde{d}_i}{n} \operatorname{Tr} DM_i + y_i^* M_i a_i + a_i^* M_i y_i \right) \right|^2. \quad \text{(B.2)}$$

Since M_i is a rank one matrix, $\sum_{i=1}^{j} \mathbb{E}|y_i^* M_i y_i - \tilde{d}_i \operatorname{Tr} DM_i/n|^2 \le K/n$. Moreover,

$$\sum_{i=1}^{j} \mathbb{E}|y_i^* M_i a_i|^2 = \frac{1}{n} \sum_{i=1}^{j} \tilde{d}_i \mathbb{E}(a^* Q_i DQ_i a|a^* (\mathbb{E}_j Q_i) DQ_i a_i|^2)$$

$$\le \frac{K}{n} \sum_{i=1}^{j} \mathbb{E}|a^* (\mathbb{E}_j Q_i) DQ_i a_i|^2.$$

The summand at the right-hand side of the inequality satisfies:

$$|a^* (\mathbb{E}_j Q_i) DQ_i a_i|^2 \le 4(|a^* (\mathbb{E}_j Q) DQ a_i|^2 + |a^* (\mathbb{E}_j (Q_i - Q)) D(Q_i - Q)a_i|^2$$

$$+ |a^* (\mathbb{E}_j Q) D(Q_i - Q)a_i|^2 + |a^* (\mathbb{E}_j (Q_i - Q)) DQ a_i|^2)$$

$$\triangleq 4(W_{i,1} + W_{i,2} + W_{i,3} + W_{i,4}).$$

Recalling that $A_{1:j} = [a_1, \ldots, a_j]$, we have

$$\sum_{i=1}^{j} W_{i,1} = a^* (\mathbb{E}_j Q) DQ A_{1:j} A_{1:j}^* QD(\mathbb{E}_j Q)a \le K.$$

Recalling (3.5) and (3.6), and writing $\xi_i = 1 + \eta_i^* Q_i \eta_i$, we have:

$$W_{i,2} \le \frac{a_i^* Q \eta_i \eta_i^* Q a_i}{1 - \eta_i^* Q \eta_i} \times a^* \mathbb{E}_j (\xi_i Q \eta_i \eta_i^* Q) D \frac{Q \eta_i \eta_i^* Q}{1 - \eta_i^* Q \eta_i} D\mathbb{E}_j (\xi_i Q \eta_i \eta_i^* Q)a.$$

As $\|(1 - \eta_i^* Q \eta_i)^{-1} Q \eta_i \eta_i^* Q\| = \|Q - Q_i\| \le K$ and $\|Q\Sigma\| \le K$, we have $\sum_{i=1}^{j} \mathbb{E} W_{i,2} \le \sum_{i=1}^{j} \mathbb{E}[|a^* Q \eta_i|^2 |\xi_i|^2]$. Writing $\xi_i = (\rho \tilde{b}_i)^{-1} + e_i$, and noticing that $(\rho \tilde{b}_i)^{-1}$ is bounded, we obtain:

$$\sum_{i=1}^{j} \mathbb{E} W_{i,2} \le 2\mathbb{E} a^* Q \Sigma_{1:j} \operatorname{diag}((\rho \tilde{b}_1)^{-1}, \ldots, (\rho \tilde{b}_j)^{-1}) \Sigma_{1:j}^* Qa + K \sum_{i=1}^{j} \mathbb{E}|e_i|^2 \le K.$$

The terms $W_{i,3}$ and $W_{i,4}$ can be handled by similar derivations.

We get that $\sum_{i=1}^{j} \mathbb{E}|y_i^* M_i a_i|^2 \le K/n$. The terms $a_i^* M_i y_i$ on the right-hand side of (B.2) satisfy $\sum_{i=1}^{j} \mathbb{E}|a_i^* M_i y_i|^2 \le K n^{-1} \sum_{i=1}^{j} \mathbb{E}|a_i^* Q_i a|^2 \le K/n$, which proves that $\mathbb{E}|X_1|^2 \le K/n$.

We now consider X_2, which satisfies $\mathbb{E}|X_2|^2 \le 2\rho^4 \sum_{i=1}^{j} \mathbb{E}|\tilde{b}_i \tilde{q}_{ii} e_i \eta_i^* M_i \eta_i|^2$. We have

$$\sum_{i=1}^{j} \mathbb{E}|\tilde{b}_i \tilde{q}_{ii} e_i a_i^* M_i a_i|^2 \le K \sum_{i=1}^{j} \mathbb{E}\mathbb{E}^{(i)}|e_i a_i^* M_i a_i|^2$$

$$\le \frac{K}{n} \sum_{i=1}^{j} \mathbb{E}|a_i^* M_i a_i|^2 \le \frac{K}{n} \sum_{i=1}^{j} \mathbb{E}|a_i^* Q_i a|^2 \le \frac{K}{n},$$

where $\mathbb{E}^{(i)} = \mathbb{E}[\cdot|y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n]$. Moreover,

$$\sum_{i=1}^{j} \mathbb{E}|\tilde{b}_i \tilde{q}_{ii} e_i y_i^* M_i a_i|^2 \le K \sum_{i=1}^{j} (\mathbb{E}|e_i|^4)^{1/2} (\mathbb{E}|y_i^* M_i a_i|^4)^{1/2} \le \frac{K}{n}$$

and similarly for the terms in $a_i^* M_i y_i$ and in $y_i^* M_i y_i$. We get that $\mathbb{E}|X_2|^2 = \mathcal{O}(n^{-1})$. We now turn to the term Z of Eq. (B.1). To control the variance of Z, we only need to control the variances of the terms:

$$Z_1 = \sum_{i=1}^{j} (\mathbb{E}_i - \mathbb{E}_{i-1})(\mathbb{E}_j \tilde{q}_{ii} a^* Q_i y_i \eta_i^* Q_i) D (\mathbb{E}_j \tilde{q}_{ii} Q_i \eta_i a_i^* Q_i a),$$

$$Z_2 = \sum_{i=1}^{j} (\mathbb{E}_i - \mathbb{E}_{i-1}) \|\mathbb{E}_j (\tilde{q}_{ii} y_i^* Q_i a D^{1/2} Q_i \eta_i)\|^2,$$

$$Z_3 = \sum_{i=1}^{j} (\mathbb{E}_i - \mathbb{E}_{i-1}) \|\mathbb{E}_j (\tilde{q}_{ii} a_i^* Q_i a D^{1/2} Q_i \eta_i)\|^2.$$

The first term satisfies

$$\mathbb{E}|Z_1|^2 \le 2 \sum_{i=1}^{j} \mathbb{E}|(\mathbb{E}_j \tilde{q}_{ii} a^* Q_i y_i \eta_i^* Q_i) D (\mathbb{E}_j \tilde{q}_{ii} Q_i \eta_i a_i^* Q_i a)|^2$$

$$\le 2 \sum_{i=1}^{j} \mathbb{E}|(\mathbb{E}_j a^* Q_i y_i \eta_i^* Q_i) D Q_i \eta_i a_i^* Q_i a|^2$$

$$= 2 \sum_{i=1}^{j} \mathbb{E}[|a_i^* Q_i a|^2 \mathbb{E}^{(i)}|(\mathbb{E}_j a^* Q_i y_i \eta_i^* Q_i) D Q_i \eta_i|^2]$$

$$\le \frac{K}{n} \sum_{i=1}^{j} \mathbb{E}|a_i^* Q_i a|^2$$

$$= \mathcal{O}(n^{-1}),$$

where the second inequality comes from $\mathbb{E}|\mathbb{E}_j(X)\mathbb{E}_j(Y)|^2 = \mathbb{E}|\mathbb{E}_j(X\mathbb{E}_j(Y))|^2 \leq \mathbb{E}|X\mathbb{E}_j(Y)|^2$. The terms Z_2 and Z_3 can be handled similarly; details are omitted. The result is $\mathbb{E}|Z|^2 \leq K/n$.

Hence, $\mathrm{var}(a^*(\mathbb{E}_jQ)^2a) = \mathcal{O}(n^{-1})$. The estimate $\mathrm{var}(\mathrm{Tr}(\mathbb{E}_jQ)D(\mathbb{E}_jQ)) = \mathcal{O}(1)$ can be established similarly.

B.2. *Proof of Lemma 5.2*

Recalling the expression (3.10) of $\tilde{t}_{\ell\ell}$, we notice that $(1 - \rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell) = \rho\tilde{t}_{\ell\ell}(1 + \tilde{d}_\ell\delta)$ is bounded below. It follows from Theorem 3.1(3) that

$$\sum_{\ell=1}^{j} \frac{\alpha_\ell u^*Ta_\ell\mathbb{E}[a_\ell^*Qu]}{1 - \rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell} = \sum_{\ell=1}^{j} \frac{\alpha_\ell u^*Ta_\ell a_\ell^*Tu}{\rho\tilde{t}_{\ell\ell}(1 + \tilde{d}_\ell\delta)} + \mathcal{O}(n^{-1/2}).$$

Moreover,

$$\sum_{\ell=1}^{j} \alpha_\ell u^*Ta_\ell \left(\frac{\mathbb{E}[a_\ell^*Qu]}{1 - \rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell} - \mathbb{E}[a_\ell^*Q_\ell u] \right)$$

$$= \sum_{\ell=1}^{j} \alpha_\ell u^*Ta_\ell\mathbb{E}\left[a_\ell^*Q_\ell u\left(\frac{1 - \rho\tilde{q}_{\ell\ell}a_\ell^*Q_\ell\eta_\ell}{1 - \rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell} - 1 \right) \right]$$

$$- \sum_{\ell=1}^{j} \alpha_\ell u^*Ta_\ell \frac{\mathbb{E}[\rho\tilde{q}_{\ell\ell}a_\ell^*Q_\ell\eta_\ell y_\ell^*Q_\ell u]}{1 - \rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell} = \varepsilon_1 + \varepsilon_2.$$

We have $\varepsilon_1 = \sum_{\ell=1}^{j} \alpha_\ell u^*Ta_\ell\mathbb{E}[a_\ell^*Q_\ell u\xi_\ell]$ where $\mathbb{E}|\xi_\ell|^p \leq Kn^{-p/2}$ for $p \geq 2$. It follows that $|\varepsilon_1| \leq (\sum_{\ell=1}^{j} \alpha_\ell^2|u^*Ta_\ell|^2\mathbb{E}\xi_\ell^2)^{1/2}(\sum_{\ell=1}^{j} \mathbb{E}|a_\ell^*Q_\ell u|^2)^{1/2} \leq K/\sqrt{n}$ by Theorem 3.1(1). By writing

$$\varepsilon_2 = -\sum_{\ell=1}^{j} \alpha_\ell u^*Ta_\ell \frac{\mathbb{E}[(\rho\tilde{q}_{\ell\ell}a_\ell^*Q_\ell\eta_\ell - \mathbb{E}[\rho\tilde{q}_{\ell\ell}a_\ell^*Q_\ell\eta_\ell])y_\ell^*Q_\ell u]}{1 - \rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell}$$

and proceeding similarly to ε_1, we obtain $|\varepsilon_2| \leq K/\sqrt{n}$, which completes the proof of Lemma 5.2.

B.3. *Proof of Lemma 5.3*

The F_j increase to $F_n = n^{-1}\mathrm{Tr}\, D^{1/2}TA(1 + \delta\tilde{D})^{-2}\tilde{D}A^*TD^{1/2} < 1$ by Lemma 3.4. As $\gamma > 0$ and M_j and G_j are increasing, $\mathbf{\Delta}_j$ is decreasing. In order to show that $\mathbf{\Delta}_n = \Delta_n$, we only need to show that $M_n + G_n = \rho^2\tilde{\gamma}$. We have

$$G_n = \frac{1}{n}\mathrm{Tr}\,\tilde{D}(I + \delta\tilde{D})^{-2}A^*TA\tilde{D}(I + \delta\tilde{D})^{-2}A^*TA - \frac{1}{n}\sum_{k=1}^{n}\left(\frac{\tilde{d}_k a_k^*Ta_k}{(1 + \delta\tilde{d}_k)^2} \right)^2.$$

Recall from (3.10) and (3.11) that $(1 + \delta\tilde{d}_k)^{-2} a_k^* T a_k = (1 + \delta\tilde{d}_k)^{-1} - \rho\tilde{t}_{kk}$. Hence

$$\frac{1}{n} \sum_{k=1}^{n} \left(\frac{\tilde{d}_k a_k^* T a_k}{(1 + \delta\tilde{d}_k)^2} \right)^2 = \frac{1}{n} \sum_{k=1}^{n} \rho^2 \tilde{d}_k^2 \tilde{t}_{kk}^2 - \frac{1}{n} \sum_{k=1}^{n} \frac{\rho\tilde{d}_k^2 \tilde{t}_{kk}}{(1 + \delta\tilde{d}_k)} + \frac{1}{n} \sum_{k=1}^{n} \frac{\tilde{d}_k^2 a_k^* T a_k}{(1 + \delta\tilde{d}_k)^3}$$

which results in

$$M_n + G_n = \frac{\rho}{n} \operatorname{Tr} \tilde{D} \tilde{T} \tilde{D} (I + \delta\tilde{D})^{-1} - \frac{1}{n} \operatorname{Tr} \tilde{D}^2 (I + \delta\tilde{D})^{-3} A^* T A$$

$$+ \frac{1}{n} \operatorname{Tr} \tilde{D} (I + \delta\tilde{D})^{-2} A^* T A \tilde{D} (I + \delta\tilde{D})^{-2} A^* T A.$$

Now, one can check with the help of (3.12) that $\rho^2 \tilde{\gamma} = \rho^2 n^{-1} \operatorname{Tr} \tilde{D} \tilde{T} \tilde{D} \tilde{T}$ is equal to the right-hand side of this equation. Lemma 5.3 is proven.

References

[1] G. W. Anderson and O. Zeitouni, A CLT for a band matrix model, *Probab. Theory Related Fields* **134**(2) (2006) 283–338.

[2] L. V. Arharov, Limit theorems for the characteristic roots of a sample covariance matrix, *Dokl. Akad. Nauk SSSR* **199** (1971) 994–997. MR MR0309171 (46 #8281).

[3] Z. D. Bai and J. W. Silverstein, No eigenvalues outside the support of the limiting spectral distribution of large-dimensional sample covariance matrices, *Ann. Probab.* **26**(1) (1998) 316–345. MR 99b:60041.

[4] Z. D. Bai and J. W. Silverstein, CLT for linear spectral statistics of large-dimensional sample covariance matrices, *Ann. Probab.* **32**(1A) (2004) 553–605. MR MR2040792 (2005b:60046).

[5] Z. Bai and J. W. Silverstein, *Spectral Analysis of Large Dimensional Random Matrices*, 2nd edn., Springer Series in Statistics (Springer, New York, 2010). MR 2567175 (2011d:60014).

[6] P. Billingsley, *Probability and Measure*, Wiley Series in Probability and Mathematical Statistics (John Wiley & Sons Inc., New York, 1995). MR MR1324786 (95k:60001).

[7] A. Boutet de Monvel and A. Khorunzhy, Limit theorems for random matrices, *Markov Process. Related Fields* **4**(2) (1998) 175–197. MR MR1641617 (99k:60100).

[8] T. Cabanal-Duvillard, Fluctuations de la loi empirique de grandes matrices aléatoires, *Ann. Inst. H. Poincaré Probab. Statist.* **37**(3) (2001) 373–402. MR MR1831988 (2002c:15040).

[9] S. Chatterjee, Fluctuations of eigenvalues and second order Poincaré inequalities, *Probab. Theory Related Fields* **143**(1–2) (2009) 1–40. MR MR2449121 (2009j:60044).

[10] M. Debbah and R. Müller, On the limiting behavior of directional MIMO models, in *Proceeding of the Sixth Baiona Workshop On Signal Processing in Communication* (Spain, September 2003).

[11] R. Brent Dozier and J. W. Silverstein, Analysis of the limiting spectral distribution of large dimensional information-plus-noise type matrices, *J. Multivariate Anal.* **98**(6) (2007) 1099–1122. MR MR2326242 (2008h:62056).

[12] R. Brent Dozier and J. W. Silverstein, On the empirical distribution of eigenvalues of large dimensional information-plus-noise-type matrices, *J. Multivariate Anal.* **98**(4) (2007) 678–694. MR MR2322123 (2009d:60078).

[13] J. Dumont, W. Hachem, S. Lasaulce, P. Loubaton and J. Najim, On the capacity achieving covariance matrix for rician MIMO channels: An asymptotic approach, *Information Theory, IEEE Transactions on* **56**(3) (2010) 1048–1069.

[14] V. L. Girko, Sluchainye matritsy (russian) [random matrices], *Izdat. Ob″ed. "Višča Škola" pri Kiev. Gosudarstv. Univ., Kiev* (1975). MR MR0394812 (52 #15611).

[15] V. L. Girko, *Theory of Stochastic Canonical Equations*. Vol. I, Mathematics and Its Applications, Vol. 535 (Kluwer Academic Publishers, Dordrecht, 2001). MR 2003f:60048.

[16] V. L. Girko, *Theory of Stochastic Canonical Equations*. Vol. II, Mathematics and Its Applications, Vol. 535 (Kluwer Academic Publishers, Dordrecht, 2001). MR MR1887676 (2003e:15027).

[17] V. L. Girko, Thirty years of the central resolvent law and three laws on the $1/n$ expansion for resolvent of random matrices, *Random Oper. Stochastic Equations* **11**(2) (2003) 167–212. MR MR2001000 (2004h:60100).

[18] A. Guionnet, Large deviations upper bounds and central limit theorems for non-commutative functionals of Gaussian large random matrices, *Ann. Inst. H. Poincaré Probab. Statist.* **38**(3) (2002) 341–384. MR MR1899457 (2003a:60042).

[19] W. Hachem, O. Khorunzhy, P. Loubaton, J. Najim and L. Pastur, A new approach for mutual information analysis of large dimensional multi-antenna channels, *IEEE Trans. Inform. Theory* **54**(9) (2008) 3987–4004. MR MR2450782.

[20] W. Hachem, P. Loubaton and J. Najim, Deterministic equivalents for certain functionals of large random matrices, *Ann. Appl. Probab.* **17**(3) (2007) 875–930.

[21] W. Hachem, P. Loubaton and J. Najim, A CLT for information-theoretic statistics of gram random matrices with a given variance profile, *Ann. Appl. Probab.* **18**(6) (2008) 2071–2130. MR MR2473651 (2009j:15084).

[22] W. Hachem, P. Loubaton, J. Najim and P. Vallet, On bilinear forms based on the resolvent of large random matrices, Tech. report, Tlcom Paristech and Universit Paris-Est (may 2010), arXiv:1004.3848, available at http://front.math.ucdavis.edu/1004.3848.

[23] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, 1990), corrected reprint of the 1985 original. MR 1084815 (91i:15001).

[24] K. Johansson, On fluctuations of eigenvalues of random Hermitian matrices, *Duke Math. J.* **91**(1) (1998) 151–204. MR MR1487983 (2000m:82026),

[25] D. Jonsson, Some limit theorems for the eigenvalues of a sample covariance matrix, *J. Multivariate Anal.* **12**(1) (1982) 1–38. MR MR650926 (83m:62085).

[26] A. Kammoun, M. Kharouf, W. Hachem, J. Najim and A. El Kharroubi, On the fluctuations of the mutual information for non centered MIMO channels: The non gaussian case, *Signal Processing Advances in Wireless Communications (SPAWC), 2010 IEEE Eleventh International Workshop* (June 2010), pp. 1–5.

[27] A. Khorunzhy, B. Khoruzhenko and L. Pastur, Asymptotic properties of large random matrices with independent entries, *J. Math. Phys.* **37**(10) (1996) 5033–5060. MR 97j:82082.

[28] A. Lytova and L. Pastur, Central limit theorem for linear eigenvalue statistics of the Wigner and the sample covariance random matrices, *Metrika* **69**(2–3) (2009) 153–172. MR MR2481919.

[29] J. A. Mingo and R. Speicher, Second order freeness and fluctuations of random matrices. I. Gaussian and Wishart matrices and cyclic Fock spaces, *J. Funct. Anal.* **235**(1) (2006) 226–270. MR MR2216446 (2007h:46080).

[30] A. L. Moustakas, S. H. Simon and A. M. Sengupta, MIMO capacity through correlated channels in the presence of correlated interferers and noise: a (not so) large N analysis, *IEEE Trans. Inform. Theory* **49**(10) (2003) 2545–2561. MR MR2028000 (2004k:94041).

[31] A. M. Sengupta and P. P. Mitra, Capacity of multivariate channels with multiplicative noise: I. random matrix techniques and large-n expansions for full transfer matrices, preprint (2010), available at http://arxiv.org/abs/physics/0010081.

[32] J. W. Silverstein and Z. D. Bai, On the empirical distribution of eigenvalues of a class of large-dimensional random matrices, *J. Multivariate Anal.* **54**(2) (1995) 175–192. MR 97b:60053.

[33] Ya. Sinai and A. Soshnikov, Central limit theorem for traces of large random symmetric matrices with independent matrix elements, *Bol. Soc. Brasil. Mat. (N.S.)* **29**(1) (1998) 1–24. MR MR1620151 (99f:60053).

[34] A. Soshnikov, The central limit theorem for local linear statistics in classical compact groups and related combinatorial identities, *Ann. Probab.* **28**(3) (2000) 1353–1370. MR MR1797877 (2002f:15035).

[35] G. Taricco, On the capacity of separately-correlated MIMO Rician fading channels, in *Proc. 49th Annual IEEE Globecom Conference, 2006 (San Francisco (CA), USA)* (November 2006).

[36] G. Taricco, Asymptotic mutual information statistics of separately correlated Rician fading MIMO channels, *IEEE Trans. Inform. Theory* **54**(8) (2008) 3490–3504. MR MR2451009 (2009i:94076).

[37] I. E. Telatar, *Capacity of Multi-Antenna Gaussian Channel*, European Transactions on Telecommunications (1999).

[38] A. Tulino and S. Verdú, Random Matrix Theory and Wireless Communications, *Foundations and Trends in Communications and Information Theory*, Vol. 1 (Now Publishers, June 2004), pp. 1–182.