OUTAGE PROBABILITY APPROXIMATION FOR THE WIENER FILTER SINR IN MIMO SYSTEMS

Abla Kammoun[(1)]*, Malika Kharouf*[(2)]*, Walid Hachem*[(3)] *and Jamal Najim*[(3)]

[(1)] ENST (France), [(2)]Casablanca University (Morocco), [(3)] CNRS/ ENST (France).
(abla.kammoun, malika.kharouf, walid.hachem, jamal.najim@enst.fr)

ABSTRACT

This paper studies the fluctuations of the post-processing SNR at the output of the linear MMSE receiver in (receive) correlated multiple input multiple output (MIMO) systems. Although it is known that asymptotically, the SNR behaves like a Gaussian random variable, this approximation may yield to inaccurate estimates for small dimension. In order to circumvent this, we use Gamma and Generalized Gamma distributions to approximate the probability distribution of the SINR. The first three asymptotic moments of the SNR are computed and are used to adjust Gamma and Generalized Gamma distributions and to accurately approximate the Bit Error Rate (BER) and its outage probability. We provide simulations which strongly support the Gamma approximation, even for a small number of emitting/receive antennas.

Index Terms— Large Random Matrices, Bit Error Rate, outage probability, MIMO systems.

1. INTRODUCTION

We consider an uplink transmission system, in which a base station equipped with N correlated antennas detects the symbols of a given user of interest in the presence of K interfering users. The N dimensional received signal can be written:

$$\mathbf{r} = \mathbf{\Sigma s} + \mathbf{n}\ ,$$

where $\mathbf{s} = [s_0, \cdots, s_K]^{\mathrm{T}}$ is the transmitted complex vector signal satisfying $\mathbb{E}\mathbf{s}\mathbf{s}^* = \mathbf{I}_{K+1}$, and $\mathbf{\Sigma}$ is the $N \times (K+1)$ channel matrix. Considering an exponentially decaying profile, Σ writes:

$$\mathbf{\Sigma} = \frac{1}{\sqrt{K}} \mathbf{\Psi}^{\frac{1}{2}} \mathbf{W} \sqrt{\mathbf{P}}\ ,$$

where $\Psi(i,j) = a^{|i-j|}$ $(1 \leq i,j \leq N)$ with $0 < a < 1$, $\mathbf{P} = \mathrm{diag}\,(p_0, \cdots, p_K)$ is the deterministic matrix of the powers given to the different users and $\mathbf{W} = (W_{ij};\ 1 \leq i \leq N, 0 \leq j \leq K)$ is a complex $(K+1) \times (K+1)$ Gaussian matrix whose entries are independent and identically distributed (i.i.d.) with a variance of $\frac{K}{N}$; we shall denote $\mathbf{W} = [W_0, \cdots, W_K]$, the W_j's being $\mathbf{W}$'s columns. Let $\tilde{\mathbf{P}} = \mathrm{diag}\,(p_1, \cdots, p_K)$, $\tilde{\mathbf{W}} = [W_1, \cdots, W_K]$. The associated SNR β_K is given by:

$$\beta_K = y^* (\mathbf{Y}\mathbf{Y}^* + \rho \mathbf{I}_N)^{-1} y$$

where $y = \sqrt{\frac{p_0}{K}} \mathbf{\Psi}^{\frac{1}{2}} W_0$ and $\mathbf{Y} = \frac{1}{\sqrt{K}} \mathbf{\Psi}^{\frac{1}{2}} \tilde{\mathbf{W}} \tilde{\mathbf{P}}^{\frac{1}{2}}$. Considering a spectral decomposition of $\mathbf{\Psi} = \mathbf{U} \mathbf{\Lambda} \mathbf{U}^*$, the SINR β_K is given by:

$$\beta_K = \frac{p_0}{K} \mathbf{z}^* \mathbf{D}^{\frac{1}{2}} \left(\frac{1}{K} \mathbf{D}^{\frac{1}{2}} \mathbf{Z} \tilde{\mathbf{D}} \mathbf{Z}^* \mathbf{D}^{\frac{1}{2}} + \rho \mathbf{I} \right)^{-1} \mathbf{D}^{\frac{1}{2}} \mathbf{z} \qquad (1)$$

where:

$$\mathbf{D} = \frac{K}{N} \mathbf{\Lambda}, \ \text{ and } \ \tilde{\mathbf{D}} = \tilde{\mathbf{P}}$$

and $\mathbf{z}$ is a $N \times 1$ standard Gaussian vector, $\mathbf{Z} = (Z_{ij})$ is a $N \times K$ standard Gaussian matrix, both being independent.

The asymptotic behavior of β_K together with its fluctuations is of major interest in digital communication and there are many statistical models related to $\mathbf{\Sigma}$ for which a deterministic sequence $\bar{\beta}_K$ is shown to satisfy $\beta_K - \bar{\beta}_K \ \to\ 0$ almost surely (a.s.); this approximation is generally defined as the solution of an implicit equation. We can cite for instance the works of [1] and [2] which derived the asymptotic first moment for the SINR for uncorrelated channels. Asymptotic normality of SINR was proved in [3] for the equal power case, and is stated in the companion paper [4] for the general case.

Based on the asymptotic normality, [5] proposed the following BER (denoted by BER_∞) whose formula is given by:

$$\mathrm{BER}_\infty = \frac{1}{\sqrt{2\pi}} \int_{\bar{\beta}_K}^{\infty} e^{-t^2/2} dt$$

where $\bar{\beta}_K$ is the asymptotic first moment. For small dimensions, it is shown in [6] that this expression yields inaccurate estimates for the bit error rate and another technique is proposed, that provides a good accuracy even for very small dimensions. The main idea consists in using other distributions than the Gaussian one whose moments (up to a given order) asymptotically equate the moments of the post-processing SNR. However, for the general correlated case, only upper bounds of the asymptotic moments were provided in [6].

Based on recent work in [7, 8, 9] and also on the companion paper [4], we provide closed-form expressions for the first three moments of the post-processing SNR, generalizing the work of [6] to a correlated context. Using similar techniques, we also derive expressions for the bit error rate and the outage probability.

2. ASYMPTOTIC MOMENTS COMPUTATION

We shall first introduce deterministic quantities that play a central role in the sequel. These quantities allow us to provide deterministic equivalents for the first, second and third moments of the SNR as defined in (1). Proofs are omitted, but we refer to [9, 4] for more details.

Proposition 1. *The following system*

$$\left\{ \begin{array}{lcl} \kappa & = & \frac{1}{K} \mathrm{tr} \left(\mathbf{D} \left(\rho (\mathbf{I}_N + \tilde{\kappa} \mathbf{D}) \right)^{-1} \right) \\ \tilde{\kappa} & = & \frac{1}{K} \mathrm{tr} \left(\widetilde{\mathbf{D}} \left(\rho (\mathbf{I}_K + \kappa \widetilde{\mathbf{D}}) \right)^{-1} \right) \end{array} \right. \qquad (2)$$

admits a unique solution such that $\kappa > 0$, $\tilde{\kappa} > 0$.

The solutions κ and $\tilde{\kappa}$ can be very easily determined using standard iterative algorithms. We define $\mathbf{\Gamma}$ and $\widetilde{\mathbf{\Gamma}}$ as the deterministic diagonal matrices given by

$$\mathbf{\Gamma} = \frac{1}{\rho}(\mathbf{I} + \tilde{\kappa}\mathbf{D})^{-1} \quad \text{and} \quad \widetilde{\mathbf{\Gamma}} = \frac{1}{\rho}(\mathbf{I} + \kappa\widetilde{\mathbf{D}})^{-1} .$$

Note that κ and $\tilde{\kappa}$ write $\kappa = \frac{1}{K}\mathrm{tr}\mathbf{D}\mathbf{\Gamma}$ and $\tilde{\kappa} = \frac{1}{K}\mathrm{tr}\tilde{\mathbf{D}}\tilde{\mathbf{\Gamma}}$. We are now in position to provide the deterministic equivalents. Define ζ and $\tilde{\zeta}$ as:

$$\zeta = \frac{1}{K}\mathrm{tr}\mathbf{D}^2\mathbf{\Gamma}^2 \quad \text{and} \quad \tilde{\zeta} = \frac{1}{K}\mathrm{tr}\widetilde{\mathbf{D}}^2\widetilde{\mathbf{\Gamma}}^2 .$$

and introduce the following quantities

1. (second asymptotic moment)

$$\Omega_K^2 = \zeta\left(\left(\mathbb{E}|Z_{11}|^4 - 1\right) + \frac{\rho^2\zeta\tilde{\zeta}}{1-\rho^2\zeta\tilde{\zeta}}\right) ,$$

2. (third asymptotic moment)

$$\nu = \frac{\mathbb{E}(|Z_{11}|^2-1)^3}{(1-\rho^2\zeta\tilde{\zeta})^3}\left(\frac{1}{K}\mathrm{Tr}\,\mathbf{D}^3\mathbf{\Gamma}^3 - \frac{\rho^3\zeta^3}{K}\mathrm{Tr}\,\tilde{\mathbf{D}}^3\tilde{\mathbf{\Gamma}}^3\right).$$

Theorem 1. *Recall that β_K is given by* (1). *Then, under mild technical conditions involving $\mathbf{D}$ and $\tilde{\mathbf{D}}$, the following convergences hold true:*

1. *(First moment approximation)*

$$\mathbb{E}\left(\frac{\beta_K}{p_0}\right) - \kappa_K \xrightarrow[K\to\infty]{} 0 ,$$

where $\kappa_K = \kappa$ is given by (2).

2. *(Second moment approximation)*

$$\mathbb{E}\left(\frac{\beta_K}{p_0} - \mathbb{E}\left(\frac{\beta_K}{p_0}\right)\right)^2 - \frac{\Omega_K^2}{K} \xrightarrow[K\to\infty]{} 0 ,$$

3. *(Third moment approximation)*

$$\mathbb{E}\left(\frac{\beta_K}{p_0} - \mathbb{E}\left(\frac{\beta_K}{p_0}\right)\right)^3 - \frac{\nu}{K^2} \xrightarrow[K\to\infty]{} 0 .$$

Remark 1. *The proof of the two first moments approximation is stated in [4] and does not depend on the Gaussiannity of the entries. The proof of the third moment approximation relies on Gaussian computations as developed at length in [10].*

3. BIT ERROR RATE AND OUTAGE PROBABILITY EXPRESSIONS

3.1. Gamma and generalized Gamma approximation

In [6], it was shown that Gaussian approximation is not accurate for small dimensions and is not suitable for computing error probabilites: One can notice that the SNR is always positive and has a strictly positive central third moment while a normal Gaussian variable has a zero third moment and may take negative values.

To overcome these difficulties, Gamma distribution and generalized Gamma distribution were proposed and were shown to be accurate even for very small dimensions (e.g. 2 transmit antennas).

Gamma Approximation. Recall that if a random variable X is Gamma-distributed $G(\alpha, b)$, then:

$$\mathbb{E}X = \alpha b, \quad \mathrm{var}(X) = \alpha b^2, \quad \mathbb{E}(X - \mathbb{E}X)^3 = 2\alpha b^3 .$$

We approximate the SNR by a Gamma random variable $G(\alpha, b)$ whose parameters are determined by solving:

$$\mathbb{E}\left(\frac{\beta_K}{p_0}\right) \simeq \kappa_K = \alpha b \quad \text{and} \quad \mathrm{var}\left(\frac{\beta_K}{p_0}\right) \simeq \frac{\Omega_K^2}{K} = \alpha b^2 .$$

We then approximate β_K by:

$$\frac{\beta_K}{p_0} \sim G\left(\frac{K\kappa_K^2}{\Omega_K^2}, \frac{\Omega_K^2}{K\kappa_K}\right) .$$

Generalized Gamma Approximation. Recall that if a random variable X follows a generalized Gamma distribution $G(\alpha, b, \xi)$, then:

$$\mathbb{E}X = \alpha b, \quad \mathrm{var}(X) = \alpha b^2, \quad \mathbb{E}(X - \mathbb{E}X)^3 = (\xi+1)\alpha b^3 .$$

Assuming a generalized Gamma distribution, the first three moments of the SNR can be written as:

$$\begin{aligned} \mathbb{E}\left(\frac{\beta_K}{p_0}\right) \simeq \kappa_K &= \alpha b , \\ \mathrm{var}\left(\frac{\beta_K}{p_0}\right) \simeq \frac{\Omega_K^2}{K} &= \alpha b^2 , \\ \mathbb{E}\left(\frac{\beta_K}{p_0} - \mathbb{E}\left(\frac{\beta_K}{p_0}\right)\right)^3 \simeq \frac{\nu}{K^2} &= (\xi+1)\alpha b^3 . \end{aligned}$$

Equating these moments with the asymptotic moments of the SNR, we will get the same α and b as for the Gamma approximation. As in [6], we define RS to be the ratio of the third central moment of the approximated Gamma distribution to the asymptotic third central moment. The parameter ξ writes then as:

$$\xi = \frac{2}{RS} - 1 .$$

The generalized Gamma distribution does not have a closed form but its Moment Generating Function (MGF) has the following expression:

$$MGF(s) = \begin{cases} \exp\left(\frac{\alpha}{\xi-1}\left(1 - (1-b\xi s)^{\frac{\xi-1}{\xi}}\right)\right) & \text{if } \xi > 1 \\ \exp\left(\frac{\alpha}{1-\xi}\left((1-b\xi s)^{\frac{\xi-1}{\xi}} - 1\right)\right) & \text{if } \xi < 1 \end{cases} . \tag{3}$$

3.2. Bit error rate approximation

Gamma approximation. As in [6], we use Gamma approximation. The bit error rate writes as:

$$\begin{aligned} \mathrm{BER} &= \frac{1}{2\Gamma(\alpha)\sqrt{2\pi}}\frac{\Gamma(1/2+\alpha)(1/b)^{\alpha}}{\alpha(1/2+1/b)^{1/2+\alpha}} \\ &\quad \times {}_2F_1\left(1, 1/2+\alpha, \alpha+1, \frac{1/b}{1/2+1/b}\right) , \end{aligned}$$

where $\Gamma(y) = \int_0^\infty t^{y-1}e^{-t}\,dt$ and ${}_2F_1$ is the hypergeometric function defined by:

$${}_2F_1\left(1, 1/2+\alpha, \alpha+1, \frac{1/b}{1/2+1/b}\right) = \sum_{n=0}^{\infty} \frac{\frac{\Gamma(1/2+\alpha+n)}{\Gamma(1/2+\alpha)}}{\frac{\Gamma(1+\alpha+n)}{\Gamma(1+\alpha)}} \frac{\left(\frac{1/b}{1/2+1/b}\right)^n}{n!} .$$

Generalized Gamma approximation. According to [11], under QPSK constellation and using the generalized Gamma approximation, the BER has the following expression:

$$\text{BER} = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} MGF(-\frac{1}{2\sin^2\phi})d\phi,$$

where MGF is given by (3).

4. OUTAGE PROBABILITY

Recall that the outage probability is the probability that the post processing SNR falls below a threshold.

Gamma approximation. Using the Gamma approximation, the outage probability writes:

$$P_{out}(y) = \frac{1}{\Gamma(\alpha)}\Gamma_{in}\left(\alpha, \frac{y}{b}\right) \tag{4}$$

where Γ has been introduced previously and the incomplete Gamma function Γ_{in} is defined as: $\Gamma_{in}(\alpha, y) = \int_0^y t^{\alpha-1}e^{-t}\,dt$.

Generalized Gamma approximation. Only the MGF of the generalized Gamma distribution has a closed form expression. Knowing the MGF, one can compute the cumulative distribution function, by applying the technique of saddle point approximation [12]. Let $K(y) = \log(MGF(y))$ denote the cumulative generating function. Let y denote the threshold SNR. First, we compute t_y the solution of $K'(t_y) = y$. Let w_0 and u_0 given by:

$$\begin{aligned} w_0 &= \operatorname{sign}(t_y)\left(2(t_y y - K(t_y)\right)^{\frac{1}{2}}, \\ u_0 &= t_y\sqrt{K''(t_y)}\,. \end{aligned}$$

The saddle point approximation of the outage probability writes as:

$$P_{out}(y) = \Phi(w_0) + \phi(w_0)\left(\frac{1}{w_0} - \frac{1}{u_0}\right), \tag{5}$$

where Φ and ϕ denote respectively the standard normal cumulative distribution function and probability distribution function.

5. SIMULATION RESULTS

In the sequel, we consider a MIMO transmission in the uplink link. The base station is equipped by N antennas and detects the symbols of a given user in the presence of K interfering users. The power of the user of interest is set to $P = 1$. Let $\mathbf{P_u}$ denote the vector containing the interfering users' powers. We set $\mathbf{P_u}$ (up to a permutation) to:

$$\mathbf{P_u} = \begin{cases} [4P\ \ 5P] & \text{if } K = 2 \\ [P\ \ P\ \ 2P\ \ 4P] & \text{if } K = 4 \end{cases}$$

if $K = 2^p(p \geq 3)$, we assume that the interfering users are arranged into 5 classes according to their powers. The power of each class as well as the proportion of users within this class are given in table1.

5.1. Asymptotic moments.

Figures 1, 2 and 3 plot the theoretical and empirical moments of the SNR for 2 simulation scenarios which correspond respectively to $N = K = 4$ and $N = 2K = 8$. The correlation coefficient is set to $a = 0.5$.

Table 1. Power and proportion of each user class

class	1	2	3	4	5
Power	P	$2P$	$4P$	$8P$	$16P$
Proportion	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{4}$

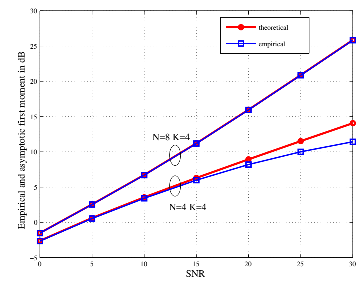

Fig. 1. Theoretical and Empirical First Moment of the SNR

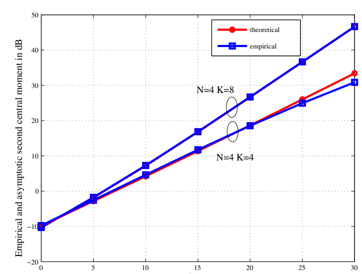

Fig. 2. Theoretical and Empirical Second Moment of the SNR

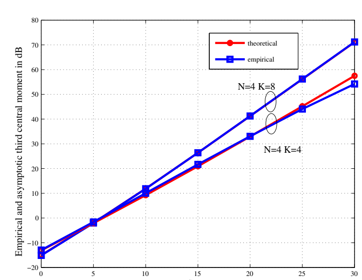

Fig. 3. Theoretical and Empirical Third Moment of the SNR

5.2. Bit error rate

Figure 4 plots the theoretical as well as the empirical bit error rate. The system dimensions N and K are set respectively to 4 and 2. The correlation coefficient a is set to 0.1. We note that a good accuracy is achieved when using Gamma approximation or generalized Gamma approximation.

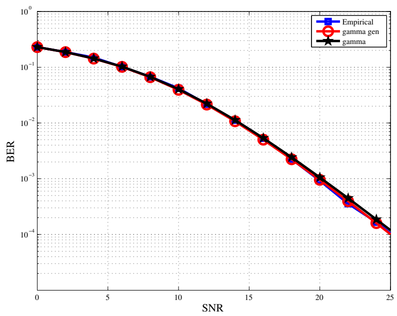

Fig. 4. Bit error rate probability

5.3. Outage Probability

Figure 5 plots the outage probability with respect to the threshold SNR when the correlation coefficient is set to $a = 0.9$ and the system dimensions are set to $N = 2K = 4$. The SNR $\triangleq \frac{1}{\rho}$ is set to 0dB. In the legend, 'Gamma' and 'generalized Gamma' stands for the expressions of the outage probability obtained when using respectively (4) and (5), while 'empirical' refers to the curve obtained by simulations. We note that a better accuracy in the estimation of outage probability is achieved when using generalized Gamma distribution.

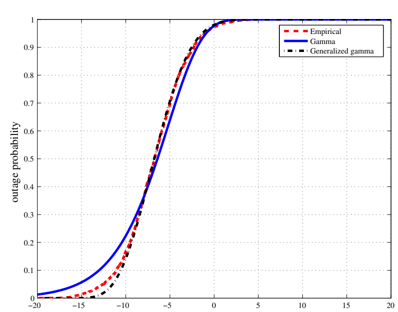

Fig. 5. Outage probability

6. SKETCH OF PROOF FOR THE MAIN THEOREM

We focus on the deterministic equivalent for the third moment. The first moment has already been established in [9], while the second moment is stated in [4].

Let $\mathbf{Q}(\rho) = (\mathbf{Y}\mathbf{Y}^* + \rho\mathbf{I}_N)^{-1}$, then the post-processing SNR writes: $\beta_K = \frac{1}{K}\mathbf{z}^*\mathbf{D}^{\frac{1}{2}}\mathbf{Q}\mathbf{D}^{\frac{1}{2}}\mathbf{z}$. Consider the eigenvalue decomposition of $\mathbf{D}^{\frac{1}{2}}\mathbf{Q}\mathbf{D}^{\frac{1}{2}} = \mathbf{U}\Upsilon\mathbf{U}^*$, then:

$$\mathbb{E}(\beta_K - \mathbb{E}(\beta_K))^3 = \mathbb{E}\left(\frac{1}{K}\omega^*\Upsilon\omega - \frac{1}{K}\mathbb{E}\mathrm{Tr}(\Upsilon)\right)^3 ,$$

where ω is a standard $N \times 1$ Gaussian vector. Using concentration inequalities as in [6, eq. (86)-(87)], one can prove that:

$$K^2\mathbb{E}(\beta_K - \mathbb{E}(\beta_K))^3 = \frac{1}{K}\mathbb{E}\mathrm{Tr}(\Upsilon^3)\mathbb{E}\left(|\omega_1|^2 - 1\right)^3 + O\left(\frac{1}{K}\right) , \quad (6)$$

where ω_1 is a standard circular Gaussian random variable. As

$$\mathrm{Tr}(\Upsilon^3) = \mathrm{Tr}(\mathbf{DQDQDQ}) ,$$

it is sufficient to derive an asymptotic equivalent for

$$\frac{1}{K}\mathrm{Tr}(\mathbf{DQDQDQ})$$

in order to get an equivalent for the third centered moment of β_K. In the sequel, we shall heavily rely on the results and techniques developed in [10]. Let us first introduce slightly different notations to comply with those in [10]. Denote by

$$\mathbf{H}(t) = \left(\frac{t}{K}\mathbf{X}\mathbf{X}^* + \mathbf{I}\right)^{-1} \quad \text{where} \quad \mathbf{X} = \sqrt{K}\mathbf{Y} ,$$

so that $\mathbf{Q}(\rho) = \frac{1}{\rho}\mathbf{H}\left(\frac{1}{\rho}\right)$. Denote:

$$\mathbf{T}(t) = \frac{1}{t}\mathbf{\Gamma}\left(\frac{1}{t}\right) \quad \text{and} \quad \widetilde{\mathbf{T}}(t) = \frac{1}{t}\widetilde{\mathbf{\Gamma}}\left(\frac{1}{t}\right) .$$

We need to introduce a few more intermediate quantities. Let

$$\widetilde{\mathbf{R}}(t) = \left(\mathbf{I} + t\left(\frac{1}{K}\mathrm{Tr}\mathbf{D}\mathbb{E}H(t)\right)\tilde{\mathbf{D}}\right)^{-1}$$

and

$$\mathbf{R}(t) = \left(\mathbf{I} + t\left(\frac{1}{K}\mathrm{Tr}\widetilde{\mathbf{D}}\widetilde{\mathbf{R}}(t)\right)\mathbf{D}\right)^{-1} ,$$

and finally $\gamma(t) = \frac{1}{K}\mathrm{Tr}\mathbf{D}^2\mathbf{T}^2(t)$ and $\tilde{\gamma}(t) = \frac{1}{K}\mathrm{Tr}\tilde{\mathbf{D}}^2\widetilde{\mathbf{T}}^2(t)$. We are now in position to proceed into the computations. First note that

$$\mathbb{E}\mathrm{Tr}(\mathbf{DHDHDH}) = \sum_{p=1}^{N}\mathbb{E}\left[\mathbf{DHDHDH}\right]_{pp}$$

(we drop the dependence in t). Note that $\mathbf{H} = \mathbf{I} - \frac{t}{K}\mathbf{HXX}^*$, thus:

$$\left[\mathbf{HDHDH}\right]_{pp} = \left[\mathbf{HDHD}\right]_{pp} - t\left[\mathbf{HDHDH}\frac{\mathbf{XX}^*}{K}\right]_{pp} .$$

Performing the same derivations as in [10], one can prove that:

$$\frac{1}{K}\mathbb{E}\left[\mathrm{Tr}\left(\mathbf{DHDHDH}\right)\right] = \chi_1 + \chi_2 + \chi_3 + \mathcal{O}(K^{-2}) \qquad (7)$$

where

$$\begin{aligned}
\chi_1 &= \frac{1}{K}\mathbb{E}\left[\mathrm{Tr}\left(\mathbf{D^2RHDH}\right)\right] \ , \\
\chi_2 &= \frac{t^2}{K}\mathbb{E}\left[\mathrm{Tr}\left(\mathbf{DHDHDH}\right)\frac{1}{K}\mathrm{Tr}\left[\mathbf{DRH}\frac{\mathbf{X\tilde{D}\tilde{R}X}^*}{K}\right]\right] \ , \\
\chi_3 &= t^2\mathbb{E}\left[\mathrm{Tr}\frac{1}{K}\left(\mathbf{DHDH}\right)\frac{1}{K}\mathrm{Tr}\left(\mathbf{DRHDH}\frac{\mathbf{X\tilde{D}\tilde{R}X}^*}{K}\right)\right] \ .
\end{aligned}$$

We now need to work out the expressions involved in χ_1, χ_2 and χ_3 and remove the expectation by introducing deterministic equivalents. For χ_1, straightforward computations as in [10] yield:

$$\chi_1 = \frac{1}{K(1-t^2\gamma\tilde{\gamma})}\mathrm{Tr}(\mathbf{D}^3\mathbf{T}^3) + \mathcal{O}(K^{-2}) \ . \qquad (8)$$

For χ_2 and χ_3, deterministic equivalents have been provided in [10] (see for instance Proposition 4 in [10]) and yield:

$$\begin{aligned}
\chi_2 &= \frac{t^2}{K}\mathbb{E}\left[\mathrm{Tr}\left(\mathbf{DHDHDH}\right)\right]\gamma\tilde{\gamma} + O(K^{-2}) \ , & (9) \\
\chi_3 &= \frac{t^2\gamma}{(1-t^2\gamma\tilde{\gamma})^2}\left[\frac{\tilde{\gamma}}{K}\mathrm{Tr}(\mathbf{D}^3\mathbf{T}^3)\right. \\
&\quad \left. -\frac{t\gamma^2}{K}\mathrm{Tr}(\tilde{\mathbf{D}}^3\tilde{\mathbf{T}}^3)\right] + O(K^{-2}) \ . & (10)
\end{aligned}$$

Substituting (8), (9) and (10) into (7), we get:

$$\begin{aligned}
\frac{1}{K}\mathbb{E}\mathrm{Tr}\mathbf{DHDHDH} &= \frac{1}{K(1-t^2\gamma\tilde{\gamma})^3}\mathrm{Tr}\mathbf{T}^3\mathbf{D}^3 \\
&\quad - \frac{t^3\gamma^3}{K(1-t^2\gamma\tilde{\gamma})^3}\mathrm{Tr}\widetilde{\mathbf{T}}^3\widetilde{\mathbf{D}}^3 + \mathcal{O}\left(\frac{1}{K^2}\right) \ .
\end{aligned}$$

We now go back to the original notations. Using the equivalences $\mathbf{H}(\rho^{-1}) = \rho\mathbf{Q}(\rho)$, $\mathbf{T}(\rho^{-1}) = \rho\mathbf{\Gamma}(\rho)$, $\gamma = \rho^2\zeta$, and their counterparts for the tilded quantities, we get:

$$\begin{aligned}
\frac{1}{K}\mathbb{E}\mathrm{Tr}\mathbf{DQDQDQ} &= \frac{1}{K\rho^3}\mathbb{E}\mathrm{Tr}\mathbf{DHDHDH} \\
&= \frac{1}{K(1-\rho^2\zeta\tilde{\zeta})^3}\mathrm{Tr}\mathbf{\Gamma}^3\mathbf{D}^3 - \frac{\rho^3\zeta^3}{K(1-\rho^2\zeta\tilde{\zeta})^3}\mathrm{Tr}\widetilde{\mathbf{\Gamma}}^3\widetilde{\mathbf{D}}^3 \\
&\quad + \mathcal{O}\left(\frac{1}{K^2}\right) \ .
\end{aligned}$$

It remains to plug this expression into (6) to get the desired result.

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