FLUCTUATIONS OF THE SNR AT THE WIENER FILTER OUTPUT FOR LARGE DIMENSIONAL SIGNALS

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ABSTRACT

Consider the linear Wiener receiver for multidimensional signals. Such a receiver is frequently encountered in wireless communications and in array processing, and the Signal to noise ratio (SNR) at its output is a popular performance index. The SNR can be modeled as a random quadratic form and in order to study this quadratic form, one can rely on well-know results in Random Matrix Theory, if one assumes that the dimension of the received and transmitted signals go to infinity, their ratio remaining constant. In this paper, we study the asymptotic behavior of the SNR for a large class of multidimensional signals (MIMO, CDMA, MC-CDMA transmissions). More precisely, we provide a deterministic approximation of the SNR, that depends on the system parameters; furthermore, the fluctuations of the SNR around this deterministic approximation are shown to be Gaussian, with variance decreasing as $1/K$, where K is the dimension of the transmitted signal.

***Index Terms*—** Antenna Arrays, CDMA, Central Limit Theorem, MC-CDMA, Random Matrix Theory, Wiener Filtering.

1. INTRODUCTION

The model. Consider the N dimensional received signal

$$\mathbf{r} = \mathbf{\Sigma s} + \mathbf{n}$$

where $\mathbf{s} = [s_0, s_1, \ldots, s_K]^{\mathrm{T}}$ is the transmitted complex vector signal with size $K+1$ satisfying $\mathbb{E}\mathbf{ss}^* = \mathbf{I}_{K+1}$, matrix $\mathbf{\Sigma}$ represents the channel in the wide sense and $\mathbf{n}$ is the independent additive white Gaussian noise (AWGN) with covariance matrix $\mathbb{E}\mathbf{nn}^* = \rho \mathbf{I}_N > \mathbf{0}$. In this article, we are interested in the performance of the linear Wiener estimate (also called LMMSE for Linear Minimum Mean Squared Error estimate) of signal s_0. Among the various performance indexes, we shall focus on the Signal to Noise Ratio (SNR) which can be expressed as follows: Partition the channel matrix as $\mathbf{\Sigma} = [\mathbf{y}\ \mathbf{Y}]$, then the Wiener estimate $\hat{s}_0$ of s_0 writes $\hat{s}_0 = \mathbf{y}^* \left(\mathbf{\Sigma\Sigma}^* + \rho \mathbf{I}_N\right)^{-1} \mathbf{r}$ and the associated SNR β_K is given by:

$$\beta_K = \mathbf{y}^* \left(\mathbf{YY}^* + \rho \mathbf{I}_N\right)^{-1} \mathbf{y} \, .$$

This expression rarely provides a clear insight on the impact of the channel model parameters (such as the load factor K/N, the power distribution of the transmission data streams, the correlation structure of the channel paths in the context of multiantenna transmissions, etc.) on the performances of the LMMSE estimator and matrix $\mathbf{\Sigma}$ is often modeled as a random matrix. This assumption is justified by the fluctuating nature of the channel paths in the context of MIMO communications, the pseudo-random nature of the spreading sequences in spread spectrum applications, etc.

In the sequel, we shall assume that $\mathbf{\Sigma}$ is random. In this case, the SNR β_K is random and it is of interest to get a deterministic approximation of it based on the system parameters, and to study its fluctuations around this approximation. The theory of Large Random Matrices is a popular tool to address this problem, widely used in multidimensional signal processing and in communication engineering, since the seminal papers of Telatar [1] and Tse *et al.* [2, 3]. Let $N \to \infty$ with $K/N \to \alpha > 0$ (denoted in the sequel by "$K \to \infty$" for short). As amply shown in the literature, there are many statistical models related to $\mathbf{\Sigma}$ for which there exists a deterministic sequence $\bar{\beta}_K$ such that $\beta_K - \bar{\beta}_K \to 0$ almost surely (a.s.). Beyond the convergence of the SNR, a natural practical and theoretical problem concerns the study of its fluctuations (to evaluate for example the outage probability, etc.). Despite its interest, there are very few related articles in the literature [3, 4]. In this paper, we provide a Central Limit Theorem (CLT) for β_K as $K \to \infty$ for a general model of matrix $\mathbf{\Sigma}$ described as follows. Assume that the $N \times (K+1)$ matrix $\mathbf{\Sigma} = (\mathbf{\Sigma}_{nk})$ $(1 \le n \le N, 0 \le k \le K)$ is given by:

$$\mathbf{\Sigma}_{nk} = \frac{\sigma_{nk}}{\sqrt{K}} W_{nk} \tag{1}$$

where (σ^2_{nk}) is a sequence of real numbers called a variance profile and where the complex random variables W_{nk} are independent and identically distributed (i.i.d.) such that $\mathbb{E}W_{nk} = 0$, $\mathbb{E}W^2_{nk} = 0$, and $\mathbb{E}|W_{nk}|^2 = 1$. In this case, the quadratic form β_K is given by:

$$\beta_K = \frac{1}{K} \mathbf{w}_0^* \mathbf{D}_0^{1/2} \left(\mathbf{YY}^* + \rho \mathbf{I}_N\right)^{-1} \mathbf{D}_0^{1/2} \mathbf{w}_0 \tag{2}$$

where $\mathbf{w}_0 = [W_{10}, W_{20}, \ldots, W_{N0}]^{\mathrm{T}}$ and $\mathbf{D}_0$ is the $N \times N$ diagonal nonnegative matrix $\mathbf{D}_0 = \mathrm{diag}(\sigma^2_{n0};\ 1 \le n \le N)$. An important special case that we shall describe carefully in the sequel is when the variance profile is *separable*, i.e $\sigma^2_{nk} = d_n \tilde{d}_k$.

Application to large dimensional signals. There are many applications of the general model (1) to the study of large dimensional signals. We mention a few below.

Multiple antenna transmissions with $K+1$ *antennas at the transmission side and* N *antennas at the reception side*. Consider the transmission model $\mathbf{r} = \mathbf{\Xi s} + \mathbf{n}$ where $\mathbf{\Xi} = \frac{1}{\sqrt{K}} \mathbf{HP}^{1/2}$, matrix $\mathbf{H}$ is a $N \times (K+1)$ random matrix with complex Gaussian elements representing the radio channel and $\mathbf{P} = \mathrm{diag}(p_k; 0 \le k \le K)$ is the (deterministic) matrix of the powers given to the different users. Write $\mathbf{H} = [\mathbf{h}_0 \cdots \mathbf{h}_K]$, and assume the columns $\mathbf{h}_k$ are independent, which is realistic when the transmitters are distant one from another. Let $\mathbf{C}_k$ be the covariance matrix $\mathbf{C}_k = \mathbb{E}\mathbf{h}_k \mathbf{h}_k^*$ and

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let $\mathbf{C}_k = \mathbf{U}_k \mathbf{\Lambda}_k \mathbf{U}_k$ be a spectral decomposition of $\mathbf{C}_k$ where $\mathbf{\Lambda}_k = \mathrm{diag}(\lambda_{nk};\ 1 \le n \le N)$ is the matrix of eigenvalues. If the eigenvector matrices $\mathbf{U}_k$ $(0 \le k \le K)$ are all equal (note that sometimes they are all identified with the Fourier $N \times N$ matrix [5]), then one can show (see for instance [6]) that matrix $\mathbf{\Xi}$ introduced above can be replaced with matrix $\mathbf{\Sigma}$ of Model (1) where the W_{nk} are standard Gaussian i.i.d. and $\sigma^2_{nk} = \lambda_{nk} p_k$.

In the so-called Kronecker model with correlations at reception, it is furthermore assumed that $\mathbf{\Lambda}_k = \mathrm{diag}(\lambda_n;\ 1 \le n \le N)$ for $0 \le k \le K$. This model is accounted for by the separable variance profile case $\sigma^2_{nk} = \lambda_n p_k$.

CDMA transmissions on flat fading channels. Here N is the spreading factor, $K+1$ is the number of users, and

$$\mathbf{\Sigma} = \mathbf{W}\mathbf{P}^{1/2} \tag{3}$$

where $\mathbf{W}$ is the $N \times (K+1)$ signature matrix assumed here to have random i.i.d. elements with mean zero, variance $1/N$ and where $\mathbf{P} = \mathrm{diag}(p_k;\ 0 \le k \le K)$ is the users powers matrix. In this case, the variance profile $\sigma^2_{nk} = d_n \tilde{d}_k$ is separable with $d_n = 1$ and $\tilde{d}_k = \frac{K}{N} p_k$.

MC-CDMA transmissions on frequency selective channels. In the uplink, the matrix $\mathbf{\Sigma}$ writes $\mathbf{\Sigma} = [\mathbf{H}_0 \mathbf{w}_0 \cdots \mathbf{H}_K \mathbf{w}_K]$ where $\mathbf{H}_k = \mathrm{diag}(h_k [\exp(2\imath\pi(n-1)/N)];\ 1 \le n \le N)$ is the radio channel matrix of user k in the discrete Fourier domain and $\mathbf{W} = [\mathbf{w}_0, \cdots, \mathbf{w}_K]$ is the $N \times (K+1)$ signature matrix with i.i.d. elements as in the CDMA case above. Modeling the channels transfer functions as deterministic functions yields a non-separable variance profile: $\sigma^2_{nk} = \frac{K}{N} |h_k(\exp(2\imath\pi(n-1)/N))|^2$.

In the downlink, we have $\mathbf{\Sigma} = \mathbf{H}\mathbf{W}\mathbf{P}^{1/2}$ where $\mathbf{H} = \mathrm{diag}(h[\exp(2\imath\pi(n-1)/N)];\ 1 \le n \le N)$ is the radio channel matrix in the discrete Fourier domain, the $N \times (K+1)$ signature matrix $\mathbf{W}$ is as above, and $\mathbf{P} = \mathrm{diag}(p_k;\ 0 \le k \le K)$ is the matrix of the powers given to the different users. This model yields a separable variance profile $\sigma^2_{nk} = d_n \tilde{d}_k$ with $d_n = \frac{K}{N} |h(\exp(2\imath\pi(n-1)/N))|^2$ and $d_k = p_k$.

About the Literature. In the communication engineering literature, the CLT for the quadratic forms has probably been considered for the first time in [3], where the authors consider a matrix $\mathbf{\Sigma}$ with i.i.d. entries. Recently, [4] considered the more general CDMA Model (3). The model considered in this paper includes the models of [3] and [4] as special cases. The approach used here to establish the CLT is powerful yet simple. It is based on the representation of β_K as the sum of a martingale difference sequence and the use of the CLT for martingales [7].

Outline of the paper. This paper is organized as follows. In Section 2, we introduce various deterministic quantities needed to express the limiting behaviour of β_K and the variance in the CLT; we then apply general first order results and describe the limiting behaviour of β_K. The CLT for β_K is stated in Section 3. A sketch of proof for the CLT is presented in Section 4. Finally, we provide simulations in Section 5.

2. DETERMINISTIC APPROXIMATION OF THE SNR

Recall that $\beta_K = \mathbf{y}^* \left(\mathbf{Y}\mathbf{Y}^* + \rho \mathbf{I}_N\right)^{-1} \mathbf{y}$ where $\mathbf{\Sigma} = [\mathbf{y}\ \mathbf{Y}]$. The SNR β_K is a random number which becomes closer and closer to its expectation as $K \to \infty$. However, the expectation (whose computation would rely on massive monte-carlo simulations) does not say much about the dependence of the SNR to the channel model parameters. In order to circumvent this issue, we introduce a deterministic equivalent $\bar{\beta}_K$ to the SNR which depends explicitly on the channel parametres and whose computation can be performed with many straightforward numerical routines.

The deterministic approximation $\bar{\beta}_K$ plays a central role in the expression of the CLT. Indeed, we shall study the fluctuations of $\beta_K - \bar{\beta}_K$ in the next section.

Denote by $\mathbb{C}_+ = \{z \in \mathbb{C} :\ \mathrm{im}(z) > 0\}$. We say that a complex function $t(z)$ belongs to class $\mathcal{S}$ (Stieltjes) if

- $t(z)$ is analytical in $\mathbb{C} - [0, \infty)$,
- $t(z) \in \mathbb{C}_+$ for all $z \in \mathbb{C}_+$,
- $\mathrm{im}(z)|t(z)|$ is bounded over $\mathbb{C}_+$, that is:

$$\sup_{z \in \mathbb{C}_+} \mathrm{im}(z)|t(z)| < K < \infty.$$

We introduce the diagonal matrices

$$\begin{aligned} \mathbf{D}_k &= \mathrm{diag}(\sigma^2_{nk};\ 1 \le n \le N) \quad \text{for } 1 \le k \le K\ , \\ \widetilde{\mathbf{D}}_n &= \mathrm{diag}(\sigma^2_{nk};\ 1 \le k \le K) \quad \text{for } 1 \le n \le N\ . \end{aligned}$$

Proposition 1 *([8, 9]) The system of $N+K$ functional equations*

$$\begin{cases} t_n(z) = \dfrac{-1}{z\left(1 + \frac{1}{K}\mathrm{tr}(\widetilde{\mathbf{D}}_n \widetilde{\mathbf{T}}(z))\right)}, & 1 \le n \le N \\ \tilde{t}_k(z) = \dfrac{-1}{z\left(1 + \frac{1}{K}\mathrm{tr}(\mathbf{D}_k \mathbf{T}(z))\right)}, & 1 \le k \le K \end{cases}$$

where

$$\begin{aligned} \mathbf{T}(z) &= \mathrm{diag}(t_1(z), \ldots, t_N(z)) \\ \widetilde{\mathbf{T}}(z) &= \mathrm{diag}(\tilde{t}_1(z), \ldots, \tilde{t}_K(z)) \end{aligned}$$

has a unique solution $(\mathbf{T}, \widetilde{\mathbf{T}})$ among the diagonal matrices for which the t_n and the $\tilde{t}_k$ belong to class $\mathcal{S}$.

The asymptotic behaviour of β_N is characterized by the following theorem:

Theorem 1 *([8, 10, 9]) Let $\bar{\beta}_K = \dfrac{1}{K}\mathrm{tr}\mathbf{D}_0\mathbf{T}(-\rho)$ where $\mathbf{T}$ is given by Proposition 1. Then*

$$\beta_K - \bar{\beta}_K \xrightarrow[K \to \infty]{} 0 \quad \textit{almost surely.}$$

Remark 1 *In matrix model* (1)*, one sometimes assumes that the variance profile σ^2_{nk} is obtained from the samples of a continuous nonnegative function $\pi(x, y)$ defined on $[0, 1]^2$ at points $\left(\frac{n}{N}, \frac{k}{K+1}\right)$, i.e. $\sigma^2_{nk} = \pi\left(\frac{n}{N}, \frac{k}{K+1}\right)$. In this case, the sequence $\bar{\beta}_K$ and δ_K defined in Theorem 1 above converge to limits that are solutions of integral equations (see for instance [10, 11]). The same holds true for δ_K defined in Corollary 1 below.*

Of particular importance is the separable case where $\sigma^2_{nk} = d_n \tilde{d}_k$. In this case, $\mathbf{D}_k$ writes: $\mathbf{D}_k = \tilde{d}_k \mathbf{D}$ and $\widetilde{\mathbf{D}}_n = d_n \widetilde{\mathbf{D}}$ where $\mathbf{D} = \mathrm{diag}(d_n;\ 1 \le n \le N)$ and $\widetilde{\mathbf{D}} = \mathrm{diag}(\tilde{d}_k;\ 1 \le k \le K)$, and the system of $N+K$ functional equations reduces to two equations:

Proposition 2 *The system of two functional equations*

$$\begin{cases} \delta(z) &= \frac{1}{K}\mathrm{tr}\left(\mathbf{D}\left(-z(\mathbf{I}_N+\tilde{\delta}(z)\mathbf{D})\right)^{-1}\right) \\ \tilde{\delta}(z) &= \frac{1}{K}\mathrm{tr}\left(\widetilde{\mathbf{D}}\left(-z(\mathbf{I}_K+\delta(z)\widetilde{\mathbf{D}})\right)^{-1}\right) \end{cases} \tag{4}$$

admits a unique solution $(\delta,\tilde{\delta}) \in \mathcal{S}^2$. Moreover, letting $z=-\rho\in(-\infty,0)$, we have $\delta(-\rho)>0$, $\tilde{\delta}(-\rho)>0$.

In this particular case, $\mathbf{D}_0=\tilde{d}_0\mathbf{D}$, the matrix functions $\mathbf{T}$ and $\widetilde{\mathbf{T}}$ defined by Proposition 1 are given by $\mathbf{T}=-\frac{1}{z}(\mathbf{I}+\tilde{\delta}\mathbf{D})^{-1}$ and $\widetilde{\mathbf{T}}=-\frac{1}{z}(\mathbf{I}+\delta\widetilde{\mathbf{D}})^{-1}$; hence we have:

Corollary 1 *([10, 11]) Assume the separable case $\sigma_{nk}^2=d_n\tilde{d}_k$. Then*

$$\frac{\beta_K}{\tilde{d}_0}-\delta_K \xrightarrow[K\to\infty]{} 0 \qquad a.s.$$

where $\delta_K=\delta$ with $(\delta,\tilde{\delta})$ being the solution of System (4) at $z=-\rho$.

Remark 2 *In the separable case, $\beta_K/\tilde{d}_0$ often represents the SNR of user 0 normalized to this user's power. Therefore, we can naturally interpret the approximation δ_K as an asymptotic normalized SNR. This approximation, as well as the asymptotic variance of the normalized SNR $\beta_K/\tilde{d}_0$ defined in Corollary 2 is the same for all users.*

3. FLUCTUATIONS FOR THE SNR: THE CLT

In the sequel, we shall use Landau notation:

$$U_K=\mathcal{O}(V_K) \quad\Leftrightarrow\quad |U_K|\le C|V_K|,\ K\in\mathbb{N},$$

for some constant C.

In order to express the CLT, and especially the variance that appears in the CLT, we build upon what has been introduced in the previous section (matrices $\mathbf{T}$ and $\tilde{\mathbf{T}}$) and introduce the following quantities: Let $\mathbf{A}$ and $\boldsymbol{\Delta}$ be the $K\times K$ matrices

$$\begin{aligned} \mathbf{A} &= \left[\frac{1}{K}\frac{\frac{1}{K}\mathrm{tr}\mathbf{D}_\ell\mathbf{D}_m\mathbf{T}(-\rho)^2}{\left(1+\frac{1}{K}\mathrm{tr}\mathbf{D}_\ell\mathbf{T}(-\rho)\right)^2}\right]_{\ell,m=1}^{K} \quad \text{and} \\ \boldsymbol{\Delta} &= \mathrm{diag}\left(\left(1+\frac{1}{K}\mathrm{tr}\mathbf{D}_\ell\mathbf{T}(-\rho)\right)^2;\ 1\le\ell\le K\right) \end{aligned}$$

where $\mathbf{T}$ is defined by Proposition 1. Let $\mathbf{g}$ be the $K\times 1$ vector

$$\mathbf{g}=\left[\frac{1}{K}\mathrm{tr}\mathbf{D}_0\mathbf{D}_1\mathbf{T}(-\rho)^2,\cdots,\frac{1}{K}\mathrm{tr}\mathbf{D}_0\mathbf{D}_K\mathbf{T}(-\rho)^2\right]^{\mathrm{T}}.$$

We are now in position to express the CLT (which holds under slight technical assumptions):

Theorem 2 *With the notations introduces above, the following hold true:*

1) *The sequence of real numbers*

$$\Theta_K^2=(\mathbb{E}|W_{10}|^4-1)\frac{1}{K}\mathrm{tr}\mathbf{D}_0^2\mathbf{T}^2 + \frac{1}{K}\mathbf{g}^{\mathrm{T}}(\mathbf{I}_K-\mathbf{A})^{-1}\boldsymbol{\Delta}^{-1}\mathbf{g} \tag{5}$$

is well defined and furthermore

$$0<\liminf_K \Theta_K^2 \le \limsup_K \Theta_K^2 <\infty\,.$$

2) *The sequence β_K satisfies*

$$\sqrt{K}\left(\frac{\beta_K-\bar{\beta}_K}{\Theta_K}\right)\xrightarrow[K\to\infty]{}\mathcal{N}(0,1)$$

in distribution where $\bar{\beta}_K$ is defined in Theorem 1.

Remark 3 *In the context of MIMO channels, the mutual information per transmit antenna*

$$\mathcal{I}_K=\frac{1}{K}\log\det\left(\rho\mathbf{I}+\Sigma\boldsymbol{\Sigma}^*\right)$$

is another popular performance index whose fluctuations have been studied in details [12, 13] and whose speed of convergence is $\sqrt{K}$-faster than the fluctuations of the SINR. Indeed, if $\overline{\mathcal{I}}_K$ stands for the deterministic equivalent of the mutual information $\mathcal{I}_K$, the CLT expresses as:

$$K\left(\frac{\mathcal{I}_K-\overline{\mathcal{I}}_K}{\Theta_K^{MI}}\right)\xrightarrow[n\to\infty]{}\mathcal{N}(0,1)$$

in distribution, where $\Theta_K^{MI}=\mathcal{O}(1)$ approximates the standard deviation of the mutual information $K\mathcal{I}_K$. As a practical consequence, $\overline{\mathcal{I}}_K$ remains a good approximation of $\mathcal{I}_K$ even for small values of K, while this is not the case for $\bar{\beta}_K$.

Corollary 2 *In the separable case $\sigma_{nk}^2=d_n\tilde{d}_k$, denote by:*

$$\begin{aligned} \gamma &= \frac{1}{K}\mathrm{tr}\mathbf{D}^2\mathbf{T}^2\,, \quad \tilde{\gamma} = \frac{1}{K}\mathrm{tr}\widetilde{\mathbf{D}}^2\widetilde{\mathbf{T}}^2, \\ \Omega_K^2 &= \gamma\left(\left(\mathbb{E}|W_{10}|^4-1\right)+\frac{\rho^2\gamma\tilde{\gamma}}{1-\rho^2\gamma\tilde{\gamma}}\right)\,. \end{aligned}$$

Then, $\Omega_K^2=\frac{\Theta_K^2}{\tilde{d}_0^2}$, where Θ_K^2 is defined in Theorem 2. In particular,

1) *The sequence (Ω_K^2) satisfies:*

$$0<\liminf_K \Omega_K^2\le\limsup_K\Omega_K^2<\infty.$$

(2) *The following convergence holds true:*

$$\sqrt{K}\left(\frac{\beta_K/\tilde{d}_0-\delta_K}{\Omega_K}\right)\xrightarrow[K\to\infty]{}\mathcal{N}(0,1)$$

in distribution.

Remark 4 *These results show in particular that the asymptotic variance Θ_K^2 is minimum with respect to the distribution of the W_{nk} when $|W_{nk}|=1$ with probability one. In the context of CDMA and MC-CDMA, this is the case when the signature matrix elements have their values in a PSK constellation.*

4. SKETCH OF PROOF

Let $\mathbf{Q}$ be the $N\times N$ matrix $\mathbf{Q}=(\mathbf{Y}\mathbf{Y}^*+\rho\mathbf{I}_N)^{-1}$. Recall that the deterministic approximation of β_K is $\bar{\beta}_K=\frac{1}{K}\mathrm{tr}\mathbf{D}_0\mathbf{T}$. Getting back to Equation (2), we can write

$$\begin{aligned} \sqrt{K}(\beta_K-\bar{\beta}_K) &= \frac{1}{\sqrt{K}}\left(\mathbf{w}_0^*\mathbf{D}_0^{1/2}\mathbf{Q}\mathbf{D}_0^{1/2}\mathbf{w}_0-\mathrm{tr}\mathbf{D}_0\mathbf{Q}\right) \\ &\quad +\frac{1}{\sqrt{K}}\mathrm{tr}\mathbf{D}_0\left(\mathbf{Q}-\mathbf{T}\right) \\ &\stackrel{\mathrm{def}}{=} \xi_K+\chi_K \end{aligned}$$

It can be shown [13] that $\mathbb{E}\chi_K^2 = \mathcal{O}(K^{-1})$. On the other hand, by using the independence of $\mathbf{w}_0$ and $\mathbf{Q}$, the fact that the spectral norm of $\mathbf{Q}$ is uniformly bounded and the fact that the elements of $\mathbf{w}_0$ are i.i.d., one can easily show that $\mathbb{E}\xi_K^2 = \mathcal{O}(1)$ as $K \to \infty$ by using the well-known identity

$$\mathbb{E}\left(\boldsymbol{x}^* M \boldsymbol{x} - \mathrm{tr} M\right)^2 = \mathrm{tr} M^2 + \kappa \sum_{i=1}^{K} m_{ii}^2 \ ,$$

where M is a deterministic matrix and $\boldsymbol{x} = (x_1, \cdots, x_K)$ is a $K \times 1$ vector with unit variance centered i.i.d. complex random variables and $\kappa = \mathbb{E}|x_1|^4 - 2$.

As a consequence, the asymptotic behaviour of $\sqrt{K}(\beta_K - \bar{\beta}_K)$ is given by ξ_K's behaviour. Denote by $\mathbb{E}_n$ the conditional expectation

$$\mathbb{E}_n[\,\cdot\,] = \mathbb{E}[\,\cdot\,\| W_{n,0}, W_{n+1,0}, \ldots, W_{N,0}, \mathbf{Y}].$$

Put $\mathbb{E}_{N+1}[\,\cdot\,] = \mathbb{E}[\,\cdot\,\|\mathbf{Y}]$ and note that $\mathbb{E}_{N+1}\mathbf{w}_0^* \mathbf{D}_0^{1/2}\mathbf{Q}\mathbf{D}_0^{1/2}\mathbf{w}_0 = \mathrm{tr}\mathbf{D}_0\mathbf{Q}$. With these notations at hand, we have

$$\begin{aligned} \xi_K &= \sum_{n=1}^{N} (\mathbb{E}_n - \mathbb{E}_{n+1}) \frac{\mathbf{w}_0^* \mathbf{D}_0^{1/2}\mathbf{Q}\mathbf{D}_0^{1/2}\mathbf{w}_0}{\sqrt{K}} \\ &\stackrel{\mathrm{def}}{=} \sum_{n=1}^{N} Z_n \ . \end{aligned}$$

The sequence Z_n is readily a martingale difference sequence with respect to the increasing sequence of $\sigma-$fields $\sigma(\mathbf{Y}), \sigma(W_{N,0}, \mathbf{Y}))$, $\ldots, \sigma(W_{1,0}, \ldots, W_{N,0}, \mathbf{Y})$.

Now, the asymptotic behaviour of ξ_K (convergence in distribution toward a Gaussian r.v. and derivation of the variance Θ_K^2) can be characterized with the help of the CLT for martingales [7, Ch. 35]: In order to establish the CLT, it is sufficient to prove that

$$\frac{1}{\Theta_K^2} \sum_{n=1}^{N} \mathbb{E}_{n+1} Z_n^2 \xrightarrow[K\to\infty]{} 1$$

in probability.

5. SIMULATIONS

In this section, the accuracy of the Gaussian approximation is verified by simulation. We consider an MC-CDMA transmission in the uplink direction. The base station detects the symbols of a given user in the presence of K interfering users. We assume that the discrete channel impulse response of each user consists in $L = 5$ i.i.d. Gaussian coefficients with variance L^{-1}. Formally, the results are conditioned to the channel; in praticular, the variance profile is considered as deterministic. All impulse responses are known to the base station.

In this case, $\mathbf{\Sigma}$ is given by:

$$\mathbf{\Sigma} = [\sqrt{p_0}\mathbf{H}_0\mathbf{w}_0 \ \cdots \ \sqrt{p_{K+1}}\mathbf{H}_{K+1}\mathbf{w}_{K+1}]$$

where

- $\mathbf{H}_k = \mathrm{diag}(h_k(\exp(2\imath\pi(n-1)/, N))_{n=1,\ldots,N})$ is the channel matrix of user k in the frequency domain,
- p_k is the amount of power allocated to user k,
- $\boldsymbol{w}_k$ are assumed to belong to QPSK constellation with mean zero and variance $1/N$.

In this case, $\sigma_{n,k}^2$ is given by:

$$\sigma_{n,k}^2 = \frac{K p_k}{N} \left| h_k\left(\exp\left(2i\pi(n-1)/N\right)\right)\right|^2$$

We denote by P the power given to the user of interest. The other users are arranged into 5 classes according to their powers. The power of each class as well as the proportion of users within this class are given in table 1. Figure 1 shows the histogram of $\sqrt{K}(\beta_K - \bar{\beta}_K)$ for $N = 16$ and $N = 64$. We note that as it was predicted by our derived results, the histogram of $\sqrt{K}(\beta_K - \bar{\beta}_K)$ is similar to that of a Gaussian random variable. In Figure 2 the measured second moment of $\beta_K - \bar{\beta}_K$ is compared with Θ_K^2/K. We note that convergence is reached even for $K = 8$.

Table 1. Power and proportion of each user class

class	1	2	3	4	5
Power	P	$2P$	$4P$	$8P$	$16P$
Proportion	1/8	1/4	1/4	1/8	1/4

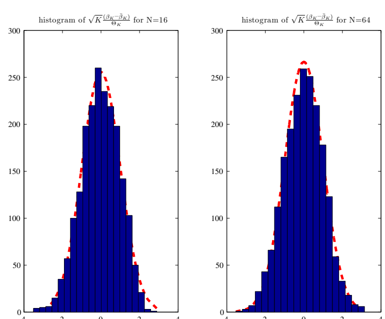

Fig. 1. Histogram of $\frac{\sqrt{K}}{\Theta_K}(\beta_K - \bar{\beta}_K)$ for $N = 16$ and $N = 64$; in red doted line, the Gaussian density

6. CONCLUSION

The Gaussian nature of the SNR at the output of the Wiener receiver for a class of large dimensional signals described by a random transmission model has been established theoretically and verified by simulation.

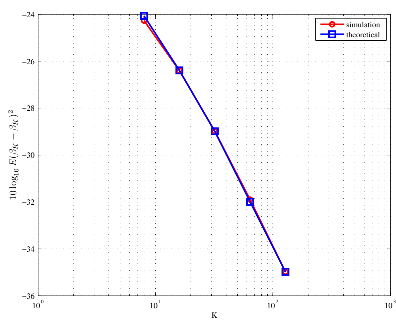

Fig. 2. Second moment of $\beta_K - \bar{\beta}_K$

7. REFERENCES

[1] I.E. Telatar, "Capacity of Multi-Antenna Gaussian Channels," published in *European Transactions on Telecommunications*, vol. 10, no. 6, pp. 585–595, Nov/Dec 1999, Technical Memorandum, Bell Laboratories, Lucent Technologies, October 1995.

[2] D.N.C Tse and S. Hanly, "Linear Multi-User Receiver: Effective Interference, Effective Bandwidth and User Capacity," *IEEE Trans. on IT*, vol. 45, no. 2, pp. 641–657, Mar. 1999.

[3] D.N.C. Tse and O. Zeitouni, "Linear Multiuser Receivers in Random Environments," *IEEE Trans. on IT*, vol. 46, no. 1, pp. 171–188, Jan. 2000.

[4] G.-M. Pan, M.-H Guo, and W. Zhou, "Asymptotic Distributions of the Signal-to-Interference Ratios of LMMSE Detection in Multiuser Communications," *Ann. Appl. Probab.*, vol. 17, no. 1, pp. 181–206, 2007.

[5] A.M. Sayeed, "Deconstructing Multiantenna Fading Channels," *IEEE Trans. on SP*, vol. 50, no. 10, pp. 2563–2579, Oct. 2002.

[6] W. Hachem, P. Loubaton, and J. Najim, "The empirical eigenvalue distribution of a gram matrix: from independence to stationarity," *Markov Process. Related Fields*, vol. 11, no. 4, pp. 629–648, 2005.

[7] P. Billingsley, *Probability and Measure*, John Wiley, 3rd edition, 1995.

[8] V. L. Girko, *Theory of Random Determinants*, Kluwer, Dordrecht, 1990.

[9] W. Hachem, P. Loubaton, and J. Najim, "Deterministic Equivalents for Certain Functionals of Large Random Matrices," *Ann. Appl. Probab.*, vol. 17, no. 3, pp. 875–930, 2007.

[10] L. Li, A.M. Tulino, and S. Verdú, "Design of Reduced-Rank MMSE Multiuser Detectors Using Random Matrix Methods," *IEEE Trans. on IT*, vol. 50, no. 6, pp. 986–1008, June 2004.

[11] J.M. Chaufray, W. Hachem, and Ph. Loubaton, "Asymptotic Analysis of Optimum and Sub-Optimum CDMA Downlink MMSE Receivers," *IEEE Trans. on IT*, vol. 50, no. 11, pp. 2620–2638, Nov. 2004.

[12] A.L. Moustakas, S.H. Simon, and A.M. Sengupta, "MIMO Capacity Through Correlated Channels in the Presence of Correlated Interferers and Noise: A (Not So) Large N Analysis ," *IEEE Trans. on IT*, vol. 49, no. 10, pp. 2545–2561, Oct. 2003.

[13] W. Hachem, Ph. Loubaton, and J. Najim, "A CLT For Information-Theoretic Statistics of Gram Random Matrices with a Given Variance Profile," to be published in *Ann. Appl. Probab.*, `arXiv:0706.0166`.