

# A CLT FOR INFORMATION-THEORETIC STATISTICS OF NON-CENTERED GRAM RANDOM MATRICES

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ABSTRACT. In this article, we study the fluctuations of the random variable:

$$\mathcal{I}_n(\rho) = \frac{1}{N} \log \det (\Sigma_n \Sigma_n^* + \rho I_N), \quad (\rho > 0)$$

where  $\Sigma_n = n^{-1/2} D_n^{1/2} X_n \tilde{D}_n^{1/2} + A_n$ , as the dimensions of the matrices go to infinity at the same pace. Matrices  $X_n$  and  $A_n$  are respectively random and deterministic  $N \times n$  matrices; matrices  $D_n$  and  $\tilde{D}_n$  are deterministic and diagonal, with respective dimensions  $N \times N$  and  $n \times n$ ; matrix  $X_n = (X_{ij})$  has centered, independent and identically distributed entries with unit variance, either real or complex.

We prove that when centered and properly rescaled, the random variable  $\mathcal{I}_n(\rho)$  satisfies a Central Limit Theorem and has a Gaussian limit. The variance of  $\mathcal{I}_n(\rho)$  depends on the moment  $\mathbb{E}X_{ij}^2$  of the variables  $X_{ij}$  and also on its fourth cumulant  $\kappa = \mathbb{E}|X_{ij}|^4 - 2 - |\mathbb{E}X_{ij}^2|^2$ .

The main motivation comes from the field of wireless communications, where  $\mathcal{I}_n(\rho)$  represents the mutual information of a multiple antenna radio channel. This article closely follows the companion article "A CLT for Information-theoretic statistics of Gram random matrices with a given variance profile", *Ann. Appl. Probab. (2008)* by Hachem et al., however the study of the fluctuations associated to non-centered large random matrices raises specific issues, which are addressed here.

**Key words and phrases:** Random Matrix, Spectral measure, Stieltjes Transform, Central Limit Theorem.

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## 1. INTRODUCTION

**The model, the statistics, and the literature.** Consider a  $N \times n$  random matrix  $\Sigma_n = (\xi_{ij}^n)$  which has the expression

$$\Sigma_n = \frac{1}{\sqrt{n}} D_n^{\frac{1}{2}} X_n \tilde{D}_n^{\frac{1}{2}} + A_n, \quad (1.1)$$

where  $A_n = (a_{ij}^n)$  is a deterministic  $N \times n$  matrix,  $D_n$  and  $\tilde{D}_n$  are diagonal deterministic matrices with nonnegative entries, with respective dimensions  $N \times N$  and  $n \times n$ ;  $X_n = (X_{ij})$  is a  $N \times n$  matrix with the entries  $X_{ij}$ 's being centered, independent and identically distributed (i.i.d.) random variables with unit variance  $\mathbb{E}|X_{ij}|^2 = 1$  and finite 16<sup>th</sup> moment.

Consider the following linear statistics of the eigenvalues:

$$\mathcal{I}_n(\rho) = \frac{1}{N} \log \det (\Sigma_n \Sigma_n^* + \rho I_N) = \frac{1}{N} \sum_{i=1}^N \log(\lambda_i + \rho),$$

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where  $I_N$  is the  $N \times N$  identity matrix,  $\rho > 0$  is a given parameter and the  $\lambda_i$ 's are the eigenvalues of matrix  $\Sigma_n \Sigma_n^*$  ( $\Sigma_n^*$  stands for the Hermitian adjoint of  $\Sigma_n$ ). This functional, known as the mutual information for multiple antenna radio channels, is fundamental in wireless communication as it characterizes the performance of a (coherent) communication over a wireless Multiple-Input Multiple-Output (MIMO) channel with gain matrix  $\Sigma_n$ . When  $\Sigma_n$  follows the model described by (1.1), the deterministic matrix  $A_n$  accounts for the so-called specular component, while  $D_n$  and  $\tilde{D}_n$  account for the correlations in certain bases at the receiving and emitting sides, respectively.

Since the seminal work of Telatar [37], the study of the mutual information  $\mathcal{I}_n(\rho)$  of a MIMO channel (and other performance indicators) in the regime where the dimensions of the gain matrix grow to infinity at the same pace has turned to be extremely fruitful. However, non-centered channel matrices have been comparatively less studied from this point of view, as their analysis is more difficult due to the presence of the deterministic matrix  $A_n$ . First order results can be found in Girko [15, 16]; Dozier and Silverstein [11, 12] established convergence results for the spectral measure; and the systematic study of the convergence of  $\mathcal{I}_n(\rho)$  for a correlated Rician channel has been undertaken by Hachem et al. in [20, 13], etc. The fluctuations of  $\mathcal{I}_n$  are important as well, for the computation of the outage probability of a MIMO channel for instance. With the help of the replica method, Taricco [35, 36] provided a closed-form expression for the asymptotic variance of  $\mathcal{I}_n$  when the elements of  $X_n$  are Gaussian.

The purpose of this article is to establish a Central Limit Theorem (CLT) for  $\mathcal{I}_n(\rho)$  in the following regime

$$n \rightarrow \infty \quad \text{and} \quad 0 < \liminf \frac{N}{n} \leq \limsup \frac{N}{n} < \infty ,$$

(simply denoted by  $n \rightarrow \infty$  in the sequel) under mild assumptions for matrices  $X_n$ ,  $A_n$ ,  $D_n$  and  $\tilde{D}_n$ .

The contributions of this article are twofold. From a wireless communication perspective, the fluctuations of  $\mathcal{I}_n$  are established, regardless of the Gaussianity of the entries and the CLT conjectured by Tarrico is fully proven. Also, this article concludes a series of studies devoted to Rician MIMO channels, initiated in [20] where a deterministic equivalent of the mutual information was provided, and continued in [13] where the computation of the ergodic capacity was addressed and an iterative algorithm proposed.

From a mathematical point of view, the study of the fluctuations of  $\mathcal{I}_n$  is the first attempt (up to our knowledge) to establish a CLT for a linear statistics of the eigenvalues of a Gram non-centered matrix (so-called signal plus noise model in [11, 12]). It complements (but does not supersede) the CLT established in [21] for a centered Gram matrix with a given variance profile. The fact that matrix  $\Sigma_n$  is non-centered ( $\mathbb{E} \Sigma_n = A_n$ ) raises specific issues, from a different nature than those addressed in close-by results [1, 4, 21], etc. These issues arise from the presence in the computations of bilinear forms  $u_n^* Q_n(z) v_n$  where at least one of the vectors  $u_n$  or  $v_n$  is deterministic. Often, the deterministic vector is related to the columns of matrix  $A_n$ , and has to be dealt with in such a way that the assumption over the spectral norm of  $A_n$  is exploited.

Another important contribution of this paper is to establish the CLT regardless of specific assumptions on the real or complex nature of the underlying random variables. It is in particular *not* assumed that the random variables are Gaussian, neither that whenever the

random variables  $X_{ij}$  are complex, their second moment  $\mathbb{E}X_{ij}^2$  is zero; nor is assumed that the random variables are circular<sup>1</sup>. As we shall see, all these assumptions, if assumed, would have resulted in substantial simplifications. As a reward however, we obtain a variance expression which smoothly depends upon  $\mathbb{E}X_{ij}^2$ , whose value is 1 in the real case, and zero in the complex case where the real and imaginary parts are not correlated.

Interestingly, the mutual information  $\mathcal{I}_n$  has a strong relationship with the Stieltjes transform  $f_n(z) = \frac{1}{N} \text{Trace}(\Sigma_n \Sigma_n^* - zI_N)^{-1}$  of the spectral measure of  $\Sigma_n \Sigma_n^*$ :

$$\mathcal{I}_n(\rho) = \log \rho + \int_{\rho}^{\infty} \left( \frac{1}{w} - f_n(-w) \right) \mathbf{d}w .$$

Accordingly, the study of the fluctuations of  $\mathcal{I}_n$  is also an important step toward the study of general linear statistics of  $\Sigma_n \Sigma_n^*$ 's eigenvalues which can be expressed via the Stieltjes transform:

$$\frac{1}{N} \text{Trace} h(\Sigma_n \Sigma_n) = \frac{1}{N} \sum_{i=1}^N h(\lambda_i) = -\frac{1}{2i\pi} \oint_{\mathcal{C}} h(z) f_n(z) \mathbf{d}z ,$$

for some well-chosen contour  $\mathcal{C}$  (see for instance [4]).

Fluctuations for particular linear statistics (and general classes of linear statistics) of large random matrices have been widely studied: CLTs for Wigner matrices can be traced back to Girko [14] (see also [17]). Results for this class of matrices have also been obtained by Khorunzhy et al. [27], Boutet de Monvel and Khorunzhy [7], Johansson [24], Sinai and Sochnikov [33], Soshnikov [34], Cabanal-Duvillard [8], Guionnet [18], Anderson and Zeitouni [1], Mingo and Speicher [29], Chatterjee [9], Lytova and Pastur [28], etc. The case of Gram matrices has been studied in Arharov [2], Jonsson [25], Bai and Silverstein [4], Hachem et al. [21], and also in [28, 29, 9]. Fluctuation results dedicated to wireless communication applications have been developed in the centered case ( $A_n = 0$ ) by Debbah and Müller [10] and Tulino and Verdù [38] (based on Bai and Silverstein [4]), Hachem et al. [19] (for Gaussian entries) and [21]. Other fluctuation results either based on the replica method or on saddle-point analysis have been developed by Moustakas, Sengupta and coauthors [30, 31], and Tarrico [35, 36].

**Presentation of the results.** We first introduce the fundamental equations needed to express the deterministic approximation of the mutual information and the variance in the CLT.

*Fundamental equations, deterministic equivalents.* We collect here results from [20]. The following system of equations

$$\begin{cases} \delta_n(z) &= \frac{1}{n} \text{Tr} D_n \left( -z(I_N + \tilde{\delta}_n(z)D_n) + A_n(I_n + \delta_n(z)\tilde{D}_n)^{-1}A_n^* \right)^{-1} \\ \tilde{\delta}_n(z) &= \frac{1}{n} \text{Tr} \tilde{D}_n \left( -z(I_n + \delta_n(z)\tilde{D}_n) + A_n^*(I_N + \tilde{\delta}_n(z)D_n)^{-1}A_n \right)^{-1} \end{cases}, \quad z \in \mathbb{C} - \mathbb{R}^+ \quad (1.2)$$

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<sup>1</sup>A random variable  $X \in \mathbb{C}$  is circular if the distribution of  $X$  is equal to the distribution of  $\rho X$  for every  $\rho \in \mathbb{C}$ ,  $|\rho| = 1$ . This assumption is very often relevant in wireless communication and has an important consequence; it implies that all the cross moments  $\mathbb{E}|X|^k X^\ell$  ( $\ell \geq 1$ ) are zero.

admits a unique solution  $(\delta_n, \tilde{\delta}_n)$  in the class of Stieltjes transforms of nonnegative measures<sup>2</sup> with support in  $\mathbb{R}^+$ . Matrices  $T_n(z)$  and  $\tilde{T}_n(z)$  defined by

$$\begin{cases} T_n(z) &= \left( -z(I_N + \tilde{\delta}_n(z)D_n) + A_n(I_n + \delta_n\tilde{D}_n)^{-1}A_n^* \right)^{-1} \\ \tilde{T}_n(z) &= \left( -z(I_n + \delta_n(z)\tilde{D}_n) + A_n^*(I_N + \tilde{\delta}_n D_n)^{-1}A_n \right)^{-1} \end{cases} \quad (1.3)$$

are approximations of the resolvent  $Q_n(z) = (\Sigma_n \Sigma_n^* - zI_N)^{-1}$  and the co-resolvent  $\tilde{Q}_n(z) = (\Sigma_n^* \Sigma_n - zI_N)^{-1}$  in the sense that  $(\xrightarrow{a.s.})$  stands for almost sure convergence):

$$\frac{1}{n} \text{Tr} (Q_n(z) - T_n(z)) \xrightarrow[n \rightarrow \infty]{a.s.} 0, \quad z \in \mathbb{C} - \mathbb{R}^+$$

which readily gives a deterministic approximation of the Stieltjes transform  $N^{-1} \text{Tr} Q_n(z)$  of the spectral measure of  $\Sigma_n \Sigma_n^*$  in terms of  $T_n$  (and similarly for  $\tilde{Q}_n$  and  $\tilde{T}_n$ ). Also proven in [22] is the convergence of bilinear forms

$$u_n^*(Q_n(z) - T_n(z))v_n \xrightarrow[n \rightarrow \infty]{a.s.} 0, \quad z \in \mathbb{C} - \mathbb{R}^+ \quad (1.4)$$

where  $(u_n)$  and  $(v_n)$  are sequences of  $N \times 1$  deterministic vectors with bounded Euclidean norms, which complements the picture of  $T_n$  approximating  $Q_n$ .

Matrices  $T_n = (t_{ij}; 1 \leq i, j \leq N)$  and  $\tilde{T}_n = (\tilde{t}_{ij}; 1 \leq i, j \leq n)$  will play a fundamental role in the sequel and enable us to express a deterministic equivalent to  $\mathbb{E}\mathcal{I}_n(\rho)$ . Define  $V_n(\rho)$  by:

$$\begin{aligned} V_n(\rho) &= \frac{1}{N} \log \det \left( \rho(I_N + \tilde{\delta}_n D_n) + A_n(I_n + \delta_n \tilde{D}_n)^{-1} A_n^* \right) \\ &\quad + \frac{1}{N} \log \det(I_n + \delta_n \tilde{D}_n) - \frac{\rho n}{N} \delta_n \tilde{\delta}_n, \end{aligned} \quad (1.5)$$

where  $\delta_n$  and  $\tilde{\delta}_n$  are evaluated at  $z = -\rho$ . Then the difference  $\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho)$  goes to zero as  $n \rightarrow \infty$ .

In order to study the fluctuations  $N(\mathcal{I}_n(\rho) - V_n(\rho))$  and to establish a CLT, we study separately the quantity  $N(\mathcal{I}_n(\rho) - \mathbb{E}\mathcal{I}_n(\rho))$  from which the fluctuations arise and the quantity  $N(\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho))$  which yields a bias.

*The fluctuations.* In every case where the fluctuations of the mutual information have been studied, the variance of  $N(\mathcal{I}_n(\rho) - V_n(\rho))$  always proved to take a remarkably simple closed-form expression (see for instance [30, 36, 38] and in a more mathematical flavour [19, 21]). The same phenomenon again occurs for the matrix model  $\Sigma_n$  under consideration. Drop the subscripts  $N, n$  and let

$$\gamma = \frac{1}{n} \text{Tr} D T D T, \quad \tilde{\gamma} = \frac{1}{n} \text{Tr} \tilde{D} \tilde{T} \tilde{D} \tilde{T}, \quad \underline{\gamma} = \frac{1}{n} \text{Tr} D T D \bar{T}, \quad \tilde{\underline{\gamma}} = \frac{1}{n} \text{Tr} \tilde{D} \tilde{T} \tilde{D} \bar{\tilde{T}}, \quad (1.6)$$

where  $\bar{M}$  stands for the (elementwise) conjugate of matrix  $M$ . Let

$$\vartheta = \mathbb{E}(X_{ij})^2 \quad \text{and} \quad \kappa = \mathbb{E}|X_{ij}|^4 - 2 - |\vartheta|^2.$$

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<sup>2</sup>In fact,  $\delta_n$  is the Stieltjes transform of a measure with total mass equal to  $n^{-1} \text{Tr} D_n$  while  $\tilde{\delta}_n$  is the Stieltjes transform of a measure with total mass equal to  $n^{-1} \text{Tr} \tilde{D}_n$ .

Let

$$\begin{aligned} \Theta_n = & -\log \left( \left( 1 - \frac{1}{n} \text{Tr} D^{\frac{1}{2}} T A (I + \delta \tilde{D})^{-1} \tilde{D} (I + \delta \tilde{D})^{-1} A^* T D^{\frac{1}{2}} \right)^2 - \rho^2 \gamma \tilde{\gamma} \right) \\ & - \log \left( \left| 1 - \vartheta \frac{1}{n} \text{Tr} D^{\frac{1}{2}} \bar{T} \bar{A} (I + \delta \tilde{D})^{-1} \tilde{D} (I + \delta \tilde{D})^{-1} A^* T D^{\frac{1}{2}} \right|^2 - |\vartheta|^2 \rho^2 \underline{\gamma} \underline{\tilde{\gamma}} \right) \\ & + \kappa \frac{\rho^2}{n^2} \sum_i d_i^2 t_{ii}^2 \sum_j \tilde{d}_j^2 \tilde{t}_{jj}^2, \end{aligned}$$

where  $d_i = [D_n]_{ii}$ ,  $\tilde{d}_j = [\tilde{D}_n]_{jj}$ , and all the needed quantities are evaluated at  $z = -\rho$ . The CLT can then be expressed as:

$$\frac{N}{\sqrt{\Theta_n}} (\mathcal{I}_n - \mathbb{E}\mathcal{I}_n) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1),$$

where  $\xrightarrow{\mathcal{D}}$  stands for convergence in distribution. Although complicated at first sight, variance  $\Theta_n$  encompasses the case of standard real random variables ( $\vartheta = 1$ ), standard complex random variables ( $\vartheta = 0$ ) and all the intermediate cases  $0 < |\vartheta| < 1$ . Moreover,  $\Theta_n$  often takes simpler forms if the variables are Gaussian, real, etc. (see for instance Remark 2.2).

*The bias.* When the entries of  $X_n$  are complex Gaussian with independent and identically distributed real and imaginary parts,  $\kappa = \vartheta = 0$ , and it has already been proven in [13] that  $\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho) = \mathcal{O}(n^{-2})$ . When any of  $\kappa$  or  $\vartheta$  is non zero, a bias term  $\mathcal{B}_n(\rho) \neq 0$  appears in the sense that

$$N (\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho)) - \mathcal{B}_n(\rho) \xrightarrow[n \rightarrow \infty]{} 0.$$

We establish the existence of this bias and provide its expression in the case where  $A = 0$ .

**Outline of the article.** In Section 2, we provide the main assumptions and state the main results of the paper: Definition of the variance  $\Theta_n$  and asymptotic fluctuations of  $N (\mathcal{I}_n(\rho) - \mathbb{E}\mathcal{I}_n(\rho))$  (Theorem 2.2), asymptotic bias of  $N (\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho))$  (Proposition 2.3). Notations, important estimates and classical results are provided in Section 3. Sections 4, 5 and 6 are devoted to the proof of Theorem 2.2. In Section 4, the general framework of the proof is exposed; in Section 5, the central part of the CLT and of the identification of the variance are established; remaining proofs are provided in Section 6. Finally, proof of Proposition 2.3 (bias) is provided in Section 7.

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## 2. THE CENTRAL LIMIT THEOREM FOR $\mathcal{I}_n(\rho)$

**2.1. Notations, assumptions and first-order results.** Let  $\mathbf{i} = \sqrt{-1}$ . As usual,  $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ . Denote by  $\xrightarrow{\mathcal{P}}$  the convergence in probability of random variables and by  $\xrightarrow{\mathcal{D}}$  the convergence in distribution of probability measures. Denote by  $\text{diag}(a_i; 1 \leq i \leq k)$  the  $k \times k$  diagonal matrix whose diagonal entries are the  $a_i$ 's. Element  $(i, j)$  of matrix  $M$  will be either denoted  $m_{ij}$  or  $[M]_{ij}$  depending on the notational context. If  $M$  is a  $n \times n$  square matrix,  $\text{diag}(M) = \text{diag}(m_{ii}; 1 \leq i \leq n)$ . Denote by  $M^T$  the matrix transpose of  $M$ ,

by  $M^*$  its Hermitian adjoint, by  $\bar{M}$  the (elementwise) conjugate of matrix  $M$ , by  $\text{Tr}(M)$  its trace and  $\det(M)$  its determinant (if  $M$  is square). When dealing with vectors,  $\|\cdot\|$  will refer to the Euclidean norm. In the case of matrices,  $\|\cdot\|$  will refer to the spectral norm. We shall denote by  $K$  a generic constant that does not depend on  $n$  and that might change from a line to another. If  $(u_n)$  is a sequence of real numbers, then  $u_n = \mathcal{O}(v_n)$  stands for  $|u_n| \leq K|v_n|$  where constant  $K$  does not depend on  $n$ .

Recall that

$$\Sigma_n = \frac{1}{\sqrt{n}} D_n^{1/2} X_n \tilde{D}_n^{1/2} + A_n, \quad (2.1)$$

denote  $D_n = \text{diag}(d_i, 1 \leq i \leq N)$  and  $\tilde{D}_n = \text{diag}(\tilde{d}_j, 1 \leq j \leq n)$ . When no confusion can occur, we shall often drop subscripts and superscripts  $n$  for readability. Recall also that the asymptotic regime of interest is:

$$n \rightarrow \infty \quad \text{and} \quad 0 < \liminf \frac{N}{n} \leq \limsup \frac{N}{n} < \infty,$$

and will be simply denoted by  $n \rightarrow \infty$  in the sequel. We can assume without loss of generality that there exist nonnegative real numbers  $\ell^-$  and  $\ell^+$  such that:

$$0 < \ell^- \leq \frac{N}{n} \leq \ell^+ < \infty \quad \text{as} \quad n \rightarrow \infty. \quad (2.2)$$

**Assumption A-1.** *The random variables  $(X_{ij}^n; 1 \leq i \leq N, 1 \leq j \leq n, n \geq 1)$  are complex, independent and identically distributed. They satisfy*

$$\mathbb{E}X_{11}^n = 0, \quad \mathbb{E}|X_{11}^n|^2 = 1 \quad \text{and} \quad \mathbb{E}|X_{11}^n|^{16} < \infty.$$

*Remark 2.1.* (Gaussian distributions) If  $X_{11}$  is a standard complex or real Gaussian random variable, then  $\kappa = 0$ . More precisely, in the complex case,  $\text{Re}(X_{11})$  and  $\text{Im}(X_{11})$  are independent real Gaussian random variables, then  $\vartheta = \kappa = 0$ ; in the real case, then  $\vartheta = 1$  while  $\kappa = 0$ .

**Assumption A-2.** *The family of deterministic  $N \times n$  complex matrices  $(A_n, n \geq 1)$  is bounded for the spectral norm:*

$$\mathbf{a}_{\max} = \sup_{n \geq 1} \|A_n\| < \infty.$$

**Assumption A-3.** *The families of real deterministic  $N \times N$  and  $n \times n$  matrices  $(D_n)$  and  $(\tilde{D}_n)$  are diagonal with non-negative diagonal elements, and are bounded for the spectral norm as  $n \rightarrow \infty$ :*

$$\mathbf{d}_{\max} = \sup_{n \geq 1} \|D_n\| < \infty \quad \text{and} \quad \tilde{\mathbf{d}}_{\max} = \sup_{n \geq 1} \|\tilde{D}_n\| < \infty.$$

Moreover,

$$\mathbf{d}_{\min} = \inf_{n \geq 1} \frac{1}{n} \text{Tr} D_n > 0 \quad \text{and} \quad \tilde{\mathbf{d}}_{\min} = \inf_{n \geq 1} \frac{1}{n} \text{Tr} \tilde{D}_n > 0.$$

**Theorem 2.1** (First order results - [20, 13]). *Consider the  $N \times n$  matrix  $\Sigma_n$  given by (2.1) and assume that A-1, A-2 and A-3 hold true. Then, the system (1.2) admits a unique solution  $(\delta_n, \tilde{\delta}_n)$  in the class of Stieltjes transforms of nonnegative measures. Moreover,*

$$\frac{1}{n} \text{Tr} (Q_n(z) - T_n(z)) \xrightarrow[n \rightarrow \infty]{a.s.} 0 \quad \text{and} \quad \frac{1}{n} \text{Tr} (\tilde{Q}_n(z) - \tilde{T}_n(z)) \xrightarrow[n \rightarrow \infty]{a.s.} 0 \quad \text{for any } z \in \mathbb{C} - \mathbb{R}^+.$$

**2.2. The Central Limit Theorem.** In this section, we state the CLT then provide the asymptotic bias in some particular cases.

**Theorem 2.2** (The CLT). *Consider the  $N \times n$  matrix  $\Sigma_n$  given by (2.1) and assume that **A-1**, **A-2** and **A-3** hold true. Recall the definitions of  $\delta$  and  $\tilde{\delta}$  given by (1.2),  $T$  and  $\tilde{T}$  given by (1.3),  $\gamma$ ,  $\tilde{\gamma}$ ,  $\underline{\gamma}$  and  $\underline{\tilde{\gamma}}$  given by (1.6). Let  $\rho > 0$ . All the considered quantities are evaluated at  $z = -\rho$ . Define  $\Delta_n$  and  $\underline{\Delta}_n$  as*

$$\Delta_n = \left( 1 - \frac{1}{n} \operatorname{Tr} D^{\frac{1}{2}} T A (I + \delta \tilde{D})^{-2} \tilde{D} A^* T D^{\frac{1}{2}} \right)^2 - \rho^2 \gamma \tilde{\gamma}$$

and

$$\underline{\Delta}_n = \left| 1 - \vartheta \frac{1}{n} \operatorname{Tr} D^{\frac{1}{2}} \tilde{T} \bar{A} (I + \delta \tilde{D})^{-2} \tilde{D} A^* T D^{\frac{1}{2}} \right|^2 - |\vartheta|^2 \rho^2 \underline{\gamma} \underline{\tilde{\gamma}}.$$

Then the real numbers

$$\Theta_n = -\log \Delta_n - \log \underline{\Delta}_n + \kappa \frac{\rho^2}{n^2} \sum_{i=1}^N d_i^2 t_{ii}^2 \sum_{j=1}^n \tilde{d}_j^2 \tilde{t}_{jj}^2 \quad (2.3)$$

are well-defined and satisfy:

$$0 < \liminf_n \Theta_n \leq \limsup_n \Theta_n < \infty \quad (2.4)$$

as  $n \rightarrow \infty$ . Let

$$\mathcal{I}_n(\rho) = \frac{1}{N} \log \det (\Sigma_n \Sigma_n^* + \rho I_N) ,$$

then the following convergence holds true:

$$\frac{N}{\sqrt{\Theta_n}} (\mathcal{I}_n(\rho) - \mathbb{E} \mathcal{I}_n(\rho)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1) .$$

*Remark 2.2.* (Simpler forms for the variance) We consider here special cases where the variance  $\Theta_n$  takes a simpler form.

- (1) The standard complex Gaussian case. Assume that the  $X_{ij}$ 's are standard complex Gaussian random variables, i.e. that both the real and imaginary parts of  $X_{ij}$  are independent real Gaussian random variables, each with variance  $1/2$ . In this case,  $\vartheta = \kappa = 0$  and  $\Theta_n$  is equal to  $-\log \Delta_n$ , and we in particular recover the variance formula given in [36].
- (2) The standard real case. Assume that the  $X_{ij}$ 's are standard real random variables, assume also that  $A$  has real entries. Then  $\Delta_n$  and  $\underline{\Delta}_n$  are equal.
- (3) The 'signal plus noise' model. In this case,  $D_n = I_N$  and  $\tilde{D}_n = I_n$ , which already yields simplifications in the variance expression. In the case where  $\vartheta = 0$ , the variance is:

$$\Theta_n = -\log \left( \left( 1 - n^{-1} (1 + \delta)^{-2} \operatorname{Tr} T A A^* T \right)^2 - \rho^2 \gamma \tilde{\gamma} \right) + \frac{\kappa \rho^2}{n^2} \sum_i d_i^2 t_{ii}^2 \sum_j \tilde{d}_j^2 \tilde{t}_{jj}^2 .$$

As one may easily check, the first term of the variance only depends upon the spectrum of  $AA^*$ . The second term however also depends on the eigenvectors of  $AA^*$  (see for instance [26]).

A full study of the asymptotic bias turns out to be extremely involved and would have substantially increased the volume of this paper. In the following proposition, we restrict our study to two important particular cases:

**Proposition 2.3** (The bias - particular cases). *Assume that the setting of Theorem 2.2 holds true.*

- (i) *If the random variables  $(X_{ij}^n; i, j, n)$  are complex with  $\text{Re}(X_{ij}^n)$  and  $\text{Im}(X_{ij}^n)$  independent, both with distribution  $\mathcal{N}(0, 1/2)$ , then:*

$$N(\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho)) = \mathcal{O}\left(\frac{1}{N}\right).$$

- (ii) *If  $A_n = 0$ , let the quantities  $\gamma$  and  $\tilde{\gamma}$  be evaluated at  $z = -\rho$  and consider*

$$\mathcal{B}_n = -\frac{\kappa}{2}\rho^2\gamma\tilde{\gamma} + \frac{1}{2}\log(1 - |\vartheta|^2\rho^2\gamma\tilde{\gamma}). \quad (2.5)$$

Then

$$N(\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho)) - \mathcal{B}_n \xrightarrow[n \rightarrow \infty]{} 0.$$

*Remark 2.3.* Observe that  $T(z) = [-z(I + \tilde{\delta}(z)D)]^{-1}$  and  $\tilde{T}(z) = [-z(I + \delta(z)\tilde{D})]^{-1}$  when  $A = 0$ . It is interesting to notice that  $\mathcal{B}_n$  coincides in that case with  $-0.5 \times$  the sum of the two last terms at the right hand side (r.h.s.) of (2.3).

Proof of Proposition 2.3 is deferred to Section 7.

### 3. NOTATIONS AND CLASSICAL RESULTS

**3.1. Further notations.** We denote by  $Y$  the  $N \times n$  matrix  $n^{-1/2}D^{1/2}X\tilde{D}^{1/2}$ ; by  $(\eta_j)$ ,  $(a_j)$  and  $(y_j)$  the columns of matrices  $\Sigma$ ,  $A$  and  $Y$ . Denote by  $\Sigma_j$ ,  $A_j$ , and  $Y_j$  the matrices  $\Sigma$ ,  $A$ , and  $Y$  where column  $j$  has been removed. The associated resolvent is  $Q_j(z) = (\Sigma_j\Sigma_j^* - zI_N)^{-1}$ . We shall often write  $Q$ ,  $Q_j$ ,  $T$  for  $Q(z)$ ,  $Q_j(z)$ ,  $T(z)$ , etc. We denote by  $\tilde{D}_j$  matrix  $\tilde{D}$  where row and column  $j$  have been removed. We also denote by  $A_{1:j}$  and  $\Sigma_{1:j}$  the  $N \times j$  matrices  $A_{1:j} = [a_1, \dots, a_j]$  and  $\Sigma_{1:j} = [\eta_1, \dots, \eta_j]$ . Denote by  $\mathbb{E}_j$  the conditional expectation with respect to the  $\sigma$ -field  $\mathcal{F}_j$  generated by the vectors  $(y_\ell, 1 \leq \ell \leq j)$ . By convention,  $\mathbb{E}_0 = \mathbb{E}$ .

We introduce here intermediate quantities of constant use in the rest of the paper. For  $1 \leq j \leq n$ , let:

$$\begin{aligned} \tilde{b}_j(z) &= \frac{-1}{z\left(1 + a_j^*Q_j(z)a_j + \frac{\tilde{d}_j}{n}\text{Tr}DQ_j(z)\right)}, \\ e_j(z) &= \eta_j^*Q_j(z)\eta_j - \left(\frac{\tilde{d}_j}{n}\text{Tr}DQ_j(z) + a_j^*Q_j(z)a_j\right) \\ &= y_j^*Q_j(z)y_j - \frac{\tilde{d}_j}{n}\text{Tr}DQ_j(z) + a_j^*Q_j(z)y_j + y_j^*Q_j(z)a_j. \end{aligned} \quad (3.1)$$

**3.2. Important identities.** Recall the following classical identity for the inverse of a perturbed matrix (see [23, Section 0.7.4]):

$$(A + XRY)^{-1} = A^{-1} - A^{-1}X(R^{-1} + YA^{-1}X)^{-1}YA^{-1}. \quad (3.2)$$



*Identities involving the resolvents.* The following identity expresses the diagonal elements  $\tilde{q}_{jj}(z) = [\tilde{Q}(z)]_{jj}$  of the co-resolvent; the two following ones are obtained from (3.2).

$$\tilde{q}_{jj}(z) = \frac{-1}{z(1 + \eta_j^* Q_j(z) \eta_j)}, \quad (3.3)$$

$$\begin{aligned} Q(z) &= Q_j(z) - \frac{Q_j(z) \eta_j \eta_j^* Q_j(z)}{1 + \eta_j^* Q_j(z) \eta_j} \\ &= Q_j(z) + z \tilde{q}_{jj}(z) Q_j(z) \eta_j \eta_j^* Q_j(z) \end{aligned} \quad (3.4)$$

$$Q_j(z) = Q(z) + \frac{Q(z) \eta_j \eta_j^* Q(z)}{1 - \eta_j^* Q(z) \eta_j} \quad (3.5)$$

$$1 + \eta_j^* Q_j(z) \eta_j = \frac{1}{1 - \eta_j^* Q(z) \eta_j} \quad (3.6)$$

Notice that

$$\tilde{q}_{jj}(z) = \tilde{b}_j(z) + z \tilde{q}_{jj}(z) \tilde{b}_j(z) e_j(z). \quad (3.7)$$

and that  $0 < \tilde{b}_j(-\rho), \tilde{q}_{jj}(-\rho) < \rho^{-1}$ . These facts will be repeatedly used in the remainder. A useful consequence of (3.4) is:

$$\eta_j^* Q(z) = \frac{\eta_j^* Q_j(z)}{1 + \eta_j^* Q_j(z) \eta_j} = -z \tilde{q}_{jj}(z) \eta_j^* Q_j(z). \quad (3.8)$$

*Identities involving the deterministic equivalents  $T$  and  $\tilde{T}$ .* Define the  $N \times N$  matrix  $\mathcal{T}_j(z)$  as

$$\mathcal{T}_j(z) = \left( -z(I_N + \tilde{\delta}(z)D) + A_j(I_{n-1} + \delta(z)\tilde{D}_j)^{-1}A_j^* \right)^{-1}, \quad (3.9)$$

where  $\delta$  and  $\tilde{\delta}$  are defined in (1.2). Notice that matrix  $\mathcal{T}_j$  is not obtained in general by solving the analogue of system (1.3) where  $A$  is replaced with  $A_j$  and when  $\tilde{D}$  is truncated accordingly. This matrix naturally pops up when expressing the diagonal elements  $\tilde{t}_{jj}$  of  $\tilde{T}$ . Indeed, we obtain (see Appendix A.1):

$$\tilde{t}_{jj}(z) = \frac{-1}{z \left( 1 + a_j^* \mathcal{T}_j(z) a_j + \tilde{d}_j \delta(z) \right)}. \quad (3.10)$$

Let  $b$  be a given  $N \times 1$  vector. The following identity is also shown in Appendix A.1:

$$-z \tilde{t}_{\ell\ell}(z) a_\ell^* \mathcal{T}_\ell(z) b = \frac{a_\ell^* T(z) b}{1 + \tilde{d}_\ell \delta(z)}. \quad (3.11)$$

Thanks to (3.2), we also have

$$\tilde{T}(z) = -z^{-1}(I + \delta(z)\tilde{D})^{-1} + z^{-1}(I + \delta(z)\tilde{D})^{-1}A^*T(z)A(I + \delta(z)\tilde{D})^{-1}. \quad (3.12)$$

**3.3. Important estimates.** We gather in this section matrix estimates which will be of constant use in the sequel. In all the remainder,  $z$  will belong to the open negative real axis, and will be fixed to  $z = -\rho$  until Section 7.

Let  $A$  and  $B$  be two square matrices. Then

$$|\mathrm{Tr}(AB)| \leq \sqrt{\mathrm{Tr}(AA^*)} \sqrt{\mathrm{Tr}(BB^*)} \quad (3.13)$$

When  $B$  is Hermitian non negative, then a consequence of Von Neumann's trace theorem is

$$|\operatorname{Tr}(AB)| \leq \|A\| \operatorname{Tr} B. \quad (3.14)$$

The following lemma gives an estimate for a rank-one perturbation of the resolvent ([21, Lemma 6.3] and [32, Lemma 2.6]):

**Lemma 3.1.** *The resolvents  $Q$  and the perturbed resolvent  $Q_j$  satisfy for  $z = -\rho$ :*

$$|\operatorname{Tr} A(Q - Q_j)| \leq \frac{\|A\|}{\rho}$$

for any  $N \times N$  matrix  $A$ .

The following results describe the asymptotic behaviour of quadratic forms based on the resolvent.

**Lemma 3.2** (Bai and Silverstein, Lemma 2.7 in [3]). *Let  $\mathbf{x} = (x_1, \dots, x_n)$  be a  $n \times 1$  vector where the  $x_i$  are centered i.i.d. complex random variables with unit variance. Let  $M$  be a  $n \times n$  deterministic complex matrix. Then for any  $p \geq 2$ , there exists a constant  $K_p$  for which*

$$\mathbb{E}|\mathbf{x}^* M \mathbf{x} - \operatorname{Tr} M|^p \leq K_p \left( (\mathbb{E}|x_1|^4 \operatorname{Tr} M M^*)^{p/2} + \mathbb{E}|x_1|^{2p} \operatorname{Tr}(M M^*)^{p/2} \right)$$

*Remark 3.1.* There are some important consequences of the previous lemma. Let  $(M_n)$  be a sequence of  $n \times n$  deterministic matrices with bounded spectral norm and  $(\mathbf{x}_n)$  be a sequence of random vectors as in the statement of Lemma 3.2. Then for any  $p \in [2; 8]$ ,

$$\max \left( \mathbb{E} \left| \frac{\mathbf{x}_n^* M_n \mathbf{x}_n}{n} - \frac{\operatorname{Tr} M_n}{n} \right|^p, \mathbb{E}|e_j|^p \right) \leq \frac{K}{n^{p/2}} \quad (3.15)$$

where  $e_j$  is given by (3.1) (the estimate  $\mathbb{E}|e_j|^p = \mathcal{O}(n^{-p/2})$  is proven in Appendix A.2).

*Remark 3.2.* By replacing  $\mathbb{E}|x_1|^4$  with  $\max_i \mathbb{E}|x_i|^4$  and  $\mathbb{E}|x_1|^{2p}$  with  $\max_i \mathbb{E}|x_i|^{2p}$ , Lemma 3.2 can be extended to the case where elements of vector  $\mathbf{x}$  are independent but not necessarily identically distributed [5, Lemma B.26]. Accordingly, the results of this paper remain true when the  $X_{ij}$  are independent but not necessarily identically distributed, provided  $\mathbb{E}|X_{ij}|^2 = 1$ ,  $\mathbb{E}X_{ij}^2 = \vartheta$ , and  $\mathbb{E}|X_{ij}|^4 - 2 - |\vartheta|^2 = \kappa$  for all  $i, j$ , and  $\sup_n \max_{i,j} \mathbb{E}|X_{ij}|^{16} < \infty$ .

The following theorem is proven in Appendix A.3:

**Theorem 3.3.** *Assume that the setting of Theorem 2.2 holds true. Let  $(u_n)$  and  $(v_n)$  be two sequences of deterministic complex  $N \times 1$  vectors bounded in the Euclidean norm:*

$$\sup_{n \geq 1} \max(\|u_n\|, \|v_n\|) < \infty,$$

and let  $(U_n)$  be a sequence of deterministic  $N \times N$  matrices with bounded spectral norms:

$$\sup_{n \geq 1} \|U_n\| < \infty.$$

Then,

- (1) *There exists a constant  $K$  for which*

$$\sum_{j=1}^n \mathbb{E}|u_n^* Q_j a_j|^2 \leq K.$$

(2) *The following holds true:*

$$\left| \frac{1}{n} \operatorname{Tr} U(T - \mathbb{E}Q) \right| \leq \frac{K}{n} .$$

(3) *For every  $p \in [1, 2]$ , there exists a constant  $K_p$  such that:*

$$\mathbb{E} |u_n^*(Q - T)v_n|^{2p} \leq \frac{K_p}{n^p} .$$

(4) *For every  $p \in [1, 2]$ , there exists a constant  $K_p$  such that:*

$$\mathbb{E} |u_n^*(Q_j - \mathcal{T}_j)v_n|^{2p} \leq \frac{K_p}{n^p} .$$

(5) *There exists a constant  $K$  such that*

$$\mathbb{E} |\operatorname{Tr} U(Q - \mathbb{E}Q)|^2 < K .$$

The following results stem from Lemma 3.2 and Theorem 3.3 and will be of constant use in the sequel. Recalling (3.7) and (3.15) along with the bounds on  $\tilde{q}_{jj}$  and  $\tilde{b}_j$ , we have for any  $p \in [2, 8]$

$$\mathbb{E} |\tilde{q}_{jj} - \tilde{b}_j|^p \leq \frac{K}{n^{p/2}} . \quad (3.16)$$

Of course, the counterpart of Theorem 3.3 for the co-resolvent  $\tilde{Q}$  and matrix  $\tilde{T}$  holds true. In particular, taking the vectors  $u_n$  and  $v_n$  as the  $j$ th canonical vector of  $\mathbb{C}^n$  yields the following estimate for any  $p \in [2, 4]$ :

$$\mathbb{E} |\tilde{q}_{jj} - \tilde{t}_{jj}|^p \leq \frac{K}{n^{p/2}} . \quad (3.17)$$

The following two lemmas, proven in Appendices A.4 and A.5, provide some important bounds:

**Lemma 3.4.** *Assume that the setting of Theorem 2.2 holds true. Then, the following quantities satisfy:*

$$\begin{aligned} \frac{d_{\min}}{\rho + d_{\max} \tilde{d}_{\max} + \alpha_{\max}^2} &\leq \delta_n \leq \frac{\ell^+ d_{\max}}{\rho} , & \frac{\tilde{d}_{\min}}{\rho + \ell^+ d_{\max} \tilde{d}_{\max} + \alpha_{\max}^2} &\leq \tilde{\delta}_n \leq \frac{\tilde{d}_{\max}}{\rho} , \\ \frac{d_{\min}}{(\rho + d_{\max} \tilde{d}_{\max} + \alpha_{\max}^2)^2} &\leq \frac{1}{n} \operatorname{Tr} DT^2 \leq \frac{\ell^+ d_{\max}}{\rho^2} , & \frac{\tilde{d}_{\min}}{(\rho + \ell^+ d_{\max} \tilde{d}_{\max} + \alpha_{\max}^2)^2} &\leq \frac{1}{n} \operatorname{Tr} \tilde{D}\tilde{T}^2 \leq \frac{\tilde{d}_{\max}}{\rho^2} , \\ \frac{d_{\min}^2}{\ell^+ (\rho + d_{\max} \tilde{d}_{\max} + \alpha_{\max}^2)^2} &\leq \gamma_n \leq \frac{\ell^+ d_{\max}^2}{\rho^2} , & \frac{\tilde{d}_{\min}^2}{(\rho + \ell^+ d_{\max} \tilde{d}_{\max} + \alpha_{\max}^2)^2} &\leq \tilde{\gamma}_n \leq \frac{\tilde{d}_{\max}^2}{\rho^2} , \\ \frac{d_{\min}^2}{\ell^+ (\rho + d_{\max} \tilde{d}_{\max} + \alpha_{\max}^2)^2} &\leq \frac{1}{n} \sum_{i=1}^N d_i^2 t_{ii}^2 \leq \frac{\ell^+ d_{\max}^2}{\rho^2} , & \frac{\tilde{d}_{\min}^2}{(\rho + \ell^+ d_{\max} \tilde{d}_{\max} + \alpha_{\max}^2)^2} &\leq \frac{1}{n} \sum_{j=1}^n \tilde{d}_j^2 \tilde{t}_{jj}^2 \leq \frac{\tilde{d}_{\max}^2}{\rho^2} . \end{aligned}$$

**Lemma 3.5.** *Assume that the setting of Theorem 2.2 holds true. Then*

$$\sup_n \frac{1}{n} \operatorname{Tr} D^{1/2} T A (I + \delta \tilde{D})^{-2} \tilde{D} A^* T D^{1/2} < 1 .$$

Moreover, the sequence  $(\Delta_n)$  as defined in Theorem 2.2 satisfies

$$\Delta_n \geq \frac{\rho}{n\delta} \operatorname{Tr} DT^2 - \frac{\rho}{n\delta} \operatorname{Tr} \tilde{D}\tilde{T}^2$$

and

$$\liminf_n \Delta_n > 0 .$$

**3.4. Other important results.** The main result we shall rely on to establish the Central Limit Theorem is the following CLT for martingales:

**Theorem 3.6** (CLT for martingales, Th. 35.12 in [6]). *Let  $\gamma_1^{(n)}, \gamma_2^{(n)}, \dots, \gamma_n^{(n)}$  be a martingale difference sequence with respect to the increasing filtration  $\mathcal{F}_1^{(n)}, \dots, \mathcal{F}_n^{(n)}$ . Assume that there exists a sequence of real positive numbers  $\Upsilon_n^2$  such that*

$$\frac{1}{\Upsilon_n^2} \sum_{j=1}^n \mathbb{E}_{j-1} \gamma_j^{(n)2} \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 1. \quad (3.18)$$

Assume further that the Lyapounov condition ([6, Section 27]) holds true:

$$\exists \delta > 0, \quad \frac{1}{\Upsilon_n^{2(1+\delta)}} \sum_{j=1}^n \mathbb{E} \left| \gamma_j^{(n)} \right|^{2+\delta} \xrightarrow[n \rightarrow \infty]{} 0.$$

Then  $\Upsilon_n^{-1} \sum_{j=1}^n \gamma_j^{(n)}$  converges in distribution to  $\mathcal{N}(0, 1)$ .

*Remark 3.3.* Note that if moreover  $\liminf_n \Upsilon_n^2 > 0$ , it is sufficient to prove:

$$\sum_{j=1}^n \mathbb{E}_{j-1} \gamma_j^{(n)2} - \Upsilon_n^2 \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0, \quad (3.19)$$

instead of (3.18).

We now state a covariance identity (the proof of which is straightforward and therefore omitted) for quadratic forms based on non-centered vectors. This identity explains to some extent the various terms obtained in the variance.

Let  $\mathbf{x} = (x_1, \dots, x_N)^T$  be a  $N \times 1$  vector where the  $x_i$  are centered i.i.d. complex random variables with unit variance. Let  $\mathbf{y} = N^{-1/2} D^{1/2} \mathbf{x}$  where  $D$  is a  $N \times N$  diagonal nonnegative deterministic matrix. Let  $M = (m_{ij})$  and  $P = (p_{ij})$  be  $N \times N$  deterministic complex matrices and let  $\mathbf{u}$  be a  $N \times 1$  deterministic vector.

If  $M$  is an  $N \times N$  matrix,  $\text{vdiag}(M)$  stands for the  $N \times 1$  vector  $[M_{11}, \dots, M_{NN}]^T$ .

Denote by  $\Upsilon(M)$  the random variable:

$$\Upsilon(M) = (\mathbf{y} + \mathbf{u})^* M (\mathbf{y} + \mathbf{u}).$$

Then  $\mathbb{E}\Upsilon(M) = \frac{1}{N} \text{Tr} DM + \mathbf{u}^* M \mathbf{u}$  and the covariance between  $\Upsilon(M)$  and  $\Upsilon(P)$  is:

$$\begin{aligned} & \mathbb{E} [(\Upsilon(M) - \mathbb{E}\Upsilon(M)) (\Upsilon(P) - \mathbb{E}\Upsilon(P))] \\ &= \frac{1}{N^2} \text{Tr}(MDPD) + \frac{1}{N} (\mathbf{u}^* MDP\mathbf{u} + \mathbf{u}^* PDM\mathbf{u}) \\ & \quad + \frac{\mathbb{E}[x_1^2]}{N^2} \text{Tr}(MDP^T D) + \frac{\mathbb{E}[x_1^2]}{N} \mathbf{u}^* PDM^T \bar{\mathbf{u}} + \frac{\mathbb{E}[\bar{x}_1^2]}{N} \mathbf{u}^T M^T DP\mathbf{u} \\ & \quad + \frac{\mathbb{E}[|x_1|^2 x_1]}{N^{3/2}} \left( \mathbf{u}^* PD^{3/2} \text{vdiag}(M) + \mathbf{u}^* MD^{3/2} \text{vdiag}(P) \right) \\ & \quad + \frac{\mathbb{E}[|x_1|^2 \bar{x}_1]}{N^{3/2}} \left( \text{vdiag}(P)^T D^{3/2} M\mathbf{u} + \text{vdiag}(M)^T D^{3/2} P\mathbf{u} \right) \\ & \quad + \frac{\kappa}{N^2} \sum_{i=1}^N d_{ii}^2 m_{ii} p_{ii}, \end{aligned} \quad (3.20)$$

where  $\kappa = \mathbb{E}|x_1|^4 - 2 - |\mathbb{E}x_1^2|^2$ .

*Remark 3.4.* Identity (3.20) is the cornerstone for the proof of the CLT; it is the counterpart of Identity (1.15) in [4]. The complexity of Identity (3.20) with respect to [4, Identity (1.15)] lies in 8 extra terms and stems from two elements:

- (1) The fact that matrix  $\Sigma$  is non-centered.
- (2) The fact that the random variables  $X_{ij}$ 's are either real and complex with no particular assumption on their second moment (in particular,  $\mathbb{E}X_{ij}^2$  can be non zero in the complex case).

It is this identity which induces to a large extent all the computations in the present article.

#### 4. PROOF OF THEOREM 2.2 (PART I)

*Decomposition of  $\mathcal{I}_n - \mathbb{E}\mathcal{I}_n$ , Cumulant and cross-moments terms in the variance*

**4.1. Decomposition of  $\mathcal{I}_n - \mathbb{E}\mathcal{I}_n$  as a sum of martingale differences.** Denote by

$$\Gamma_j = \frac{\eta_j^* Q_j \eta_j - \left( \frac{\tilde{d}_j}{n} \operatorname{Tr} DQ_j + a_j^* Q_j a_j \right)}{1 + \frac{\tilde{d}_j}{n} \operatorname{Tr} DQ_j + a_j^* Q_j a_j}.$$

With this notation at hand, the decomposition of  $\mathcal{I}_n - \mathbb{E}\mathcal{I}_n$  as

$$\mathcal{I}_n - \mathbb{E}\mathcal{I}_n = \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) \log(1 + \Gamma_j) \quad (4.1)$$

follows verbatim from [21, Section 6.2]. Moreover, it is a matter of bookkeeping to establish the following (cf. [21, Section 6.4]):

$$\sum_{j=1}^n \mathbb{E}_{j-1} ((\mathbb{E}_j - \mathbb{E}_{j-1}) \log(1 + \Gamma_j))^2 - \sum_{j=1}^n \mathbb{E}_{j-1} (\mathbb{E}_j \Gamma_j)^2 \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0. \quad (4.2)$$

Hence, the details are omitted. In view of Theorem 3.6, Eq. (3.19), (4.1) and (4.2), the CLT will be established if one proves the following 3 results:

- (1) (Lyapounov condition)

$$\exists \delta > 0, \quad \sum_{j=1}^n \mathbb{E} |\mathbb{E}_j \Gamma_j|^{2+\delta} \xrightarrow[n \rightarrow \infty]{} 0,$$

- (2) (Martingale increments and variance)

$$\sum_{j=1}^n \mathbb{E}_{j-1} (\mathbb{E}_j \Gamma_j)^2 - \Theta_n \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0.$$

- (3) (estimates over the variance)

$$0 < \liminf_n \Theta_n \leq \limsup_n \Theta_n < \infty$$

It is straightforward (and hence omitted) to verify Lyapounov condition. The convergence toward the variance is the cornerstone of the proof of the CLT: The rest of this section together with much of Section 5 are devoted to establish it. The estimates over the variance  $\Theta_n$ , also central to apply Theorem 3.6, are established in Section 6.2.

Notice that  $\mathbb{E}_{j-1}(\mathbb{E}_j \Gamma_j)^2 = \mathbb{E}_{j-1}(\mathbb{E}_j \rho \tilde{b}_j e_j)^2$ . We prove hereafter that

$$\sum_{j=1}^n \mathbb{E}_{j-1}(\mathbb{E}_j \rho \tilde{b}_j e_j)^2 - \sum_{j=1}^n \rho^2 \tilde{t}_{jj}^2 \mathbb{E}_{j-1}(\mathbb{E}_j e_j)^2 \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0. \quad (4.3)$$

The inequality  $\mathbb{E}|\tilde{b}_j - \tilde{t}_{jj}|^2 \leq 2\mathbb{E}|\tilde{b}_j - \tilde{q}_{jj}|^2 + 2\mathbb{E}|\tilde{q}_{jj} - \tilde{t}_{jj}|^2$  in conjunction with Estimates (3.16) and (3.17) yield  $\mathbb{E}|\tilde{b}_j - \tilde{t}_{jj}|^2 = \mathcal{O}(n^{-1})$ . Moreover,

$$\begin{aligned} & \mathbb{E} \left| \mathbb{E}_{j-1}(\mathbb{E}_j \rho \tilde{b}_j e_j)^2 - \mathbb{E}_{j-1}(\rho \tilde{t}_{jj} \mathbb{E}_j e_j)^2 \right| \leq \mathbb{E} \left| \left( \mathbb{E}_j \rho \tilde{b}_j e_j \right)^2 - \left( \mathbb{E}_j \rho \tilde{t}_{jj} e_j \right)^2 \right| \\ & = \mathbb{E} \left| \left( \mathbb{E}_j (\rho \tilde{b}_j - \rho \tilde{t}_{jj}) e_j \right) \left( \mathbb{E}_j (\rho \tilde{b}_j + \rho \tilde{t}_{jj}) e_j \right) \right| \leq \mathbb{E} \left| (\rho \tilde{b}_j - \rho \tilde{t}_{jj}) e_j \left( \mathbb{E}_j (\rho \tilde{b}_j + \rho \tilde{t}_{jj}) e_j \right) \right| \\ & \leq K n^{-3/2} \end{aligned}$$

using Cauchy-Schwarz inequality and (3.15). This implies (4.3). Let  $\varsigma = \mathbb{E}(|X_{11}^2| | X_{11})$ . Using Identity (3.20), we develop the quantity  $\mathbb{E}_{j-1}(\mathbb{E}_j e_j)^2$ :

$$\begin{aligned} & \sum_{j=1}^n \rho^2 \tilde{t}_{jj}^2 \mathbb{E}_{j-1}(\mathbb{E}_j e_j)^2 \\ & = \frac{\kappa}{n^2} \sum_{j=1}^n \rho^2 \tilde{d}_j^2 \tilde{t}_{jj}^2 \sum_{i=1}^N d_i^2 [\mathbb{E}_j Q_j]_{ii}^2 \\ & \quad + \frac{4}{n} \sum_{j=1}^n \rho^2 \tilde{d}_j^{3/2} \tilde{t}_{jj}^2 \operatorname{Re} \left( \varsigma \frac{a_j^* (\mathbb{E}_j Q_j) D^{3/2} \operatorname{vdiag}(\mathbb{E}_j Q_j)}{\sqrt{n}} \right) \\ & \quad + \frac{1}{n} \sum_{j=1}^n \rho^2 \tilde{t}_{jj}^2 \left( \frac{\tilde{d}_j^2}{n} \operatorname{Tr}(\mathbb{E}_j Q_j) D(\mathbb{E}_j Q_j) D + 2 \tilde{d}_j a_j^* (\mathbb{E}_j Q_j) D(\mathbb{E}_j Q_j) a_j \right) \\ & \quad + \frac{1}{n} \sum_{j=1}^n \rho^2 \tilde{t}_{jj}^2 \left( |\vartheta|^2 \frac{\tilde{d}_j^2}{n} \operatorname{Tr}(\mathbb{E}_j Q_j) D(\mathbb{E}_j \bar{Q}_j) D + 2 \operatorname{Re} \left( \vartheta \tilde{d}_j a_j^* (\mathbb{E}_j Q_j) D(\mathbb{E}_j \bar{Q}_j) \bar{a}_j \right) \right) \\ & \triangleq \sum_{j=1}^n \chi_{1j} + \sum_{j=1}^n \chi_{2j} + \sum_{j=1}^n \chi_{3j} + \sum_{j=1}^n \chi_{4j}. \end{aligned}$$

**4.2. Key lemmas for the identification of the variance.** The remainder of the proof of Theorem 2.2 is devoted to find deterministic equivalents for the terms  $\sum_{j=1}^n \chi_{\ell j}$  for  $\ell = 1, 2, 3, 4$ .

**Lemma 4.1.** *Assume that the setting of Theorem 2.2 holds true, then:*

$$\sum_{j=1}^n \chi_{1j} - \frac{\kappa \rho^2}{n^2} \sum_{i=1}^N \sum_{j=1}^n d_i^2 t_{ii}^2 \tilde{d}_j^2 \tilde{t}_{jj}^2 \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0.$$

*Proof.* Write

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^N d_i^2 [\mathbb{E}_j Q_j]_{ii}^2 - \frac{1}{n} \sum_{i=1}^N d_i^2 [\mathbb{E}_j Q_j]_{ii} t_{ii} &= \\ \frac{1}{n} \sum_{i=1}^N d_i^2 [\mathbb{E}_j Q_j]_{ii} \mathbb{E}_j ([Q_j]_{ii} - [Q]_{ii}) + \frac{1}{n} \sum_{i=1}^N d_i^2 [\mathbb{E}_j Q_j]_{ii} ([\mathbb{E}_j Q]_{ii} - t_{ii}) &= \varepsilon_{1,j} + \varepsilon_{2,j}. \end{aligned}$$

The term  $|\varepsilon_{1,j}| = n^{-1} |\mathbb{E}_j [\text{Tr } D^2 \text{diag}(\mathbb{E}_j Q_j)(Q_j - Q)]|$  is of order  $\mathcal{O}(n^{-1})$  thanks to Lemma 3.1. Moreover,  $\mathbb{E}|\varepsilon_{2,j}| = \mathcal{O}(n^{-1/2})$  by the analogue of (3.17) for the diagonal elements of the resolvent. Hence,

$$\sum_{j=1}^n \chi_{1j} - \frac{\kappa \rho^2}{n} \sum_{j=1}^n \tilde{d}_j^2 \tilde{t}_{jj} \left( \frac{1}{n} \sum_{i=1}^N d_i^2 [\mathbb{E}_j Q_j]_{ii} t_{ii} \right) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0.$$

Iterating the same arguments, we can replace the remaining term  $\mathbb{E}_j [Q_j]_{ii}$  by  $t_{ii}$  to obtain the desired result.  $\square$

**Lemma 4.2.** *Assume that the setting of Theorem 2.2 holds true. Then:*

$$\sum_{j=1}^n \chi_{2j} \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0.$$

*Proof.* We have

$$\begin{aligned} \mathbb{E} \left| \sum_{j=1}^n \chi_{2j} \right| &\leq \frac{K}{n} \sum_{j=1}^n \mathbb{E} \left| a_j^* (\mathbb{E}_j Q_j) D^{3/2} \frac{\text{vdiag}(Q_j)}{\sqrt{n}} \right| \\ &\leq \frac{K}{n} \sum_{j=1}^n \mathbb{E} \left| a_j^* Q_j D^{3/2} \frac{\text{vdiag}(T)}{\sqrt{n}} \right| + \frac{K}{n} \sum_{j=1}^n \mathbb{E} \left| a_j^* (\mathbb{E}_j Q_j) D^{3/2} \frac{\text{vdiag}(Q - T)}{\sqrt{n}} \right| \\ &\quad + \frac{K}{n} \sum_{j=1}^n \mathbb{E} \left| a_j^* (\mathbb{E}_j Q_j) D^{3/2} \frac{\text{vdiag}(Q_j - Q)}{\sqrt{n}} \right|. \end{aligned} \quad (4.4)$$

The first term satisfies

$$\sum_{j=1}^n \mathbb{E} \left| a_j^* Q_j D^{3/2} \frac{\text{vdiag}(T)}{\sqrt{n}} \right| \leq \sqrt{n} \left( \sum_{j=1}^n \mathbb{E} \left| a_j^* Q_j D^{3/2} \frac{\text{vdiag}(T)}{\sqrt{n}} \right|^2 \right)^{1/2}.$$

As  $\|n^{-1/2} D^{3/2} \text{vdiag}(T)\| = (n^{-1} \sum_{i=1}^N d_i^3 t_{ii}^2)^{1/2} \leq K$ , Theorem 3.3-(1) can be applied, and the first term at the r.h.s. of (4.4) is of order  $n^{-1/2}$ . We now deal with the second term at the r.h.s.

$$\mathbb{E} \left| a_j^* (\mathbb{E}_j Q_j) D^{3/2} \frac{\text{vdiag}(Q - T)}{\sqrt{n}} \right| \leq K \mathbb{E} \left\| \frac{\text{vdiag}(Q - T)}{\sqrt{n}} \right\| \leq \left( \frac{K}{n} \sum_{i=1}^N \mathbb{E} |q_{ii} - t_{ii}|^2 \right)^{1/2} \leq \frac{K}{\sqrt{n}}$$

by (3.17). We now consider the third term. Since  $\|a_j^*(\mathbb{E}_j Q_j) D^{3/2}\|$  is uniformly bounded,

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n \mathbb{E} \left| a_j^*(\mathbb{E}_j Q_j) D^{3/2} \frac{\text{vdiag}(Q_j - Q)}{\sqrt{n}} \right| \\ = \frac{1}{n^{3/2}} \sum_{j=1}^n \mathbb{E} \left| \text{Tr} \left( \text{diag}(a_j^*(\mathbb{E}_j Q_j) D^{3/2})(Q_j - Q) \right) \right| \leq \frac{K}{\sqrt{n}} \end{aligned}$$

by Lemma 3.1.  $\square$

**Lemma 4.3.** *Assume that the setting of Theorem 2.2 holds true, then:*

$$\sum_{j=1}^n \chi_{3j} + \log \left( \left( 1 - \frac{1}{n} \text{Tr} D^{\frac{1}{2}} T A (I + \delta \tilde{D})^{-2} \tilde{D} A^* T D^{\frac{1}{2}} \right)^2 - \rho^2 \gamma \tilde{\gamma} \right) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0.$$

**Lemma 4.4.** *Assume that the setting of Theorem 2.2 holds true, then:*

$$\sum_{j=1}^n \chi_{4j} + \log \left( \left| 1 - \vartheta \frac{1}{n} \text{Tr} D^{\frac{1}{2}} \bar{T} \bar{A} (I + \delta \tilde{D})^{-2} \tilde{D} A^* T D^{\frac{1}{2}} \right|^2 - |\vartheta|^2 \rho^2 \underline{\gamma} \tilde{\gamma} \right) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0.$$

The core of the paper is devoted to the proof of Lemma 4.3. This proof is provided in Section 5. The proof of Lemma 4.4 follows the same canvas with minor differences. Elements of this proof are given in Section 6.

## 5. PROOF OF THEOREM 2.2 (PART II)

This section is devoted to the proof of Lemma 4.3. We begin with the following lemma which implies that  $\sum_{j=1}^n \chi_{3j}$  can be replaced by its expectation.

**Lemma 5.1.** *For any  $N \times 1$  vector  $a$  with bounded Euclidean norm, we have,*

$$\max_j \text{var}(a^*(\mathbb{E}_j Q) D (\mathbb{E}_j Q) a) = \mathcal{O}(n^{-1}) \quad \text{and} \quad \max_j \text{var}(\text{Tr}(\mathbb{E}_j Q) D (\mathbb{E}_j Q) D) = \mathcal{O}(1).$$

Proof of Lemma 5.1 is postponed to Appendix B.1. Observe that:

$$\begin{aligned} \sum_{i=1}^j \chi_{3j} &= \frac{1}{n} \sum_{j=1}^n \rho^2 \tilde{t}_{jj}^2 \left( \frac{\tilde{d}_j^2}{n} \text{Tr}(\mathbb{E}_j Q_j) D (\mathbb{E}_j Q_j) D + 2\tilde{d}_j a_j^*(\mathbb{E}_j Q_j) D (\mathbb{E}_j Q_j) a_j \right) \\ &= \frac{1}{n} \sum_{j=1}^n \rho^2 \tilde{t}_{jj}^2 \left( \frac{\tilde{d}_j^2}{n} \text{Tr}(\mathbb{E}_j Q) D (\mathbb{E}_j Q) D + 2\tilde{d}_j a_j^*(\mathbb{E}_j Q_j) D (\mathbb{E}_j Q_j) a_j \right) + \mathcal{O}(n^{-1}), \end{aligned}$$

due to Lemma 3.1. Consider the following notations:

$$\begin{aligned} \psi_j &= \frac{1}{n} \text{Tr} \mathbb{E} [(\mathbb{E}_j Q) D (\mathbb{E}_j Q) D] = \frac{1}{n} \text{Tr} \mathbb{E} [(\mathbb{E}_j Q) D Q D], \\ \zeta_{kj} &= \mathbb{E} [a_k^*(\mathbb{E}_j Q) D (\mathbb{E}_j Q) a_k] = \mathbb{E} [a_k^*(\mathbb{E}_j Q) D Q a_k], \\ \theta_{kj} &= \mathbb{E} [a_k^*(\mathbb{E}_j Q_k) D (\mathbb{E}_j Q_k) a_k] = \mathbb{E} [a_k^*(\mathbb{E}_j Q_k) D Q_k a_k], \\ \varphi_j &= \frac{1}{n} \sum_{k=1}^j \rho^2 \tilde{d}_k \tilde{t}_{kk}^2 \theta_{kj}. \end{aligned}$$



Thanks to Lemma 5.1, we only need to show that

$$\frac{1}{n} \sum_{j=1}^n \left( \rho^2 \tilde{t}_{jj}^2 \tilde{d}_j^2 \psi_j + 2\rho^2 \tilde{d}_j \tilde{t}_{jj}^2 \theta_{jj} \right) + \log \Delta_n \xrightarrow[n \rightarrow \infty]{} 0. \quad (5.1)$$

There are structural links between the various quantities  $\psi_j$ ,  $\zeta_{kj}$ ,  $\theta_{kj}$  and  $\varphi_j$ . The idea behind the proof is to establish the equations between these quantities. Solving these equations will yield explicit expressions which will enable to identify  $\frac{1}{n} \sum_{j=1}^n \left( \rho^2 \tilde{t}_{jj}^2 \tilde{d}_j^2 \psi_j + 2\rho^2 \tilde{d}_j \tilde{t}_{jj}^2 \theta_{jj} \right)$  as the deterministic quantity  $-\log \Delta_n$  up to a vanishing error term.

Proof of (5.1) is broken down into four steps. In the first step, we establish an equation between  $\zeta_{kj}$ ,  $\psi_j$  and  $\varphi_j$  (up to  $\mathcal{O}(n^{-1/2})$ ): Eq. (5.7). In the second step, we establish an equation between  $\psi_j$  and  $\varphi_j$ : Eq. (5.11). In the third step, we establish an equation between  $\zeta_{kj}$ ,  $\psi_j$  and  $\theta_{kj}$ : Eq. (5.12). Gathering these results, we obtain a  $2 \times 2$  linear system (5.15) whose solutions are  $\psi_j$  and  $\varphi_j$ . In the fourth step, we solve this system and finally establish (5.1).

**5.1. Step 1: Expression of  $\zeta_{kj} = \mathbb{E}[a_k^*(\mathbb{E}_j Q) D Q a_k]$ .** Writing

$$Q = T + T(T^{-1} - Q^{-1})Q = T + T \left( \rho \tilde{\delta} D + A(I + \delta \tilde{D})^{-1} A^* - \Sigma \Sigma^* \right) Q, \quad (5.2)$$

we have:

$$\begin{aligned} \zeta_{kj} &= \mathbb{E} \left[ a_k^* \mathbb{E}_j \left[ T + T \left( \rho \tilde{\delta} D + A(I + \delta \tilde{D})^{-1} A^* - \Sigma \Sigma^* \right) Q \right] D Q a_k \right], \\ &= \mathbb{E}[a_k^* T D Q a_k] + \rho \tilde{\delta} \mathbb{E}[a_k^* T D (\mathbb{E}_j Q) D Q a_k] \\ &\quad + \mathbb{E}[a_k^* T A (I + \delta \tilde{D})^{-1} A^* (\mathbb{E}_j Q) D Q a_k] - \mathbb{E}[a_k^* T (\mathbb{E}_j \Sigma \Sigma^* Q) D Q a_k], \end{aligned} \quad (5.3)$$

$$\triangleq a_k^* T D T a_k + \rho \tilde{\delta} \mathbb{E}[a_k^* T D (\mathbb{E}_j Q) D Q a_k] + X + Z + \varepsilon, \quad (5.4)$$

where  $X$  and  $Z$  are the last two terms at the r.h.s. of (5.3) and where  $|\varepsilon| = \mathcal{O}(n^{-1/2})$  by Theorem 3.3-(3). Beginning with  $X$ , we have

$$\begin{aligned} X &= \sum_{\ell=1}^n \frac{\mathbb{E}[a_k^* T a_\ell a_\ell^* (\mathbb{E}_j Q) D Q a_k]}{1 + \delta \tilde{d}_\ell} \\ &= \sum_{\ell=1}^n \frac{\mathbb{E}[a_k^* T a_\ell a_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell} - \sum_{\ell=1}^n \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_k^* T a_\ell a_\ell^* (\mathbb{E}_j Q_\ell \eta_\ell^* Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell} + \varepsilon_1, \\ &= \sum_{\ell=1}^n \frac{\mathbb{E}[a_k^* T a_\ell a_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell} - \sum_{\ell=1}^n \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_k^* T a_\ell a_\ell^* (\mathbb{E}_j Q_\ell a_\ell \eta_\ell^* Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell} + \varepsilon_1 + \varepsilon_2, \\ &= \sum_{\ell=1}^n \frac{\mathbb{E}[a_k^* T a_\ell a_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell} - \sum_{\ell=1}^n \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell (\mathbb{E}_j \eta_\ell^* Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell} + \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \\ &\triangleq X_1 + X_2 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \end{aligned}$$

where

$$\begin{aligned}\varepsilon_1 &= -\sum_{\ell=1}^n \frac{\mathbb{E}[a_k^* T a_\ell (\mathbb{E}_j (\rho \tilde{q}_{\ell\ell} - \rho \tilde{t}_{\ell\ell}) a_\ell^* Q_\ell \eta_\ell \eta_\ell^* Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell}, \\ \varepsilon_2 &= -\sum_{\ell=1}^n \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_k^* T a_\ell a_\ell^* (\mathbb{E}_j Q_\ell y_\ell \eta_\ell^* Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell}, \\ \varepsilon_3 &= -\sum_{\ell=1}^n \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_k^* T a_\ell (\mathbb{E}_j a_\ell (Q_\ell - \mathcal{T}_\ell) a_\ell \eta_\ell^* Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell}.\end{aligned}\quad (5.5)$$

Using (3.3) and (3.8),  $\varepsilon_1$  can be written as:

$$\varepsilon_1 = \mathbb{E} \left[ \mathbb{E}_j \left( a_k^* T A \operatorname{diag}(\xi_\ell) (I + \delta \tilde{D})^{-1} \Sigma^* Q \right) D Q a_k \right],$$

where  $\xi_\ell = \rho(\tilde{q}_{\ell\ell} - \tilde{t}_{\ell\ell})(1 + \eta_\ell^* Q_\ell \eta_\ell) a_\ell^* Q_\ell \eta_\ell$ . Recalling that  $\|\Sigma^* Q\|$  is bounded, we obtain  $|\varepsilon_1| \leq K \mathbb{E} \|a_k^* T A \operatorname{diag}(\xi_\ell)\| \leq K (\sum_{\ell=1}^n \|a_k^* T A\|_\ell^2 \mathbb{E} \xi_\ell^2)^{1/2} \leq K/\sqrt{n}$  by (3.17) and the boundedness of  $\mathbb{E}|X_{11}|^{16}$  (Assumption **A1**). We show similarly that  $\varepsilon_2$  and  $\varepsilon_3$  (with the help of Theorem 3.3-(4)) are of order  $\mathcal{O}(n^{-1/2})$ . We now develop  $X_2$  as:

$$\begin{aligned}X_2 &= -\sum_{\ell=1}^n \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell (\mathbb{E}_j Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell} - \sum_{\ell=1}^j \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k]}{1 + \delta \tilde{d}_\ell}, \\ &\triangleq U_1 + U_2.\end{aligned}$$

The term  $U_2$  can be expressed as:

$$\begin{aligned}U_2 &= \sum_{\ell=1}^j \frac{\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell} \tilde{q}_{\ell\ell} a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell \eta_\ell^* Q_\ell a_k]}{1 + \delta \tilde{d}_\ell}, \\ &= \sum_{\ell=1}^j \frac{\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2 a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell \eta_\ell^* Q_\ell a_k]}{1 + \delta \tilde{d}_\ell} + \mathcal{O}(n^{-1/2}).\end{aligned}$$

Write  $\eta_\ell \eta_\ell^* = a_\ell a_\ell^* + a_\ell y_\ell^* + y_\ell y_\ell^* + y_\ell a_\ell^*$ . The term in  $a_\ell a_\ell^*$  is zero. Turning to the term in  $a_\ell y_\ell^*$ , we have  $\mathbb{E}|y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell a_\ell y_\ell^* Q_\ell a_k| = \mathcal{O}(n^{-1})$ , hence

$$\sum_{\ell=1}^j \frac{|\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2 a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell a_\ell y_\ell^* Q_\ell a_k]|}{1 + \delta \tilde{d}_\ell} \leq \frac{K}{n} \sum_{\ell=1}^j |a_k^* T a_\ell| \leq \frac{K}{\sqrt{n}}.$$

Moreover,

$$\begin{aligned}&\sum_{\ell=1}^j \frac{|\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2 a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell y_\ell^* Q_\ell a_k]|}{1 + \delta \tilde{d}_\ell} \\ &= \sum_{\ell=1}^j \frac{|\mathbb{E} \left[ \rho^2 \tilde{t}_{\ell\ell}^2 a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell \left( y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell - \tilde{d}_\ell n^{-1} \operatorname{Tr} D(\mathbb{E}_j Q_\ell) D Q_\ell \right) y_\ell^* Q_\ell a_k \right]|}{1 + \delta \tilde{d}_\ell} \\ &\leq \frac{K}{n} \sum_{\ell=1}^j |a_k^* T a_\ell| = \mathcal{O}(n^{-1/2}).\end{aligned}$$

The term in  $y_\ell a_\ell^*$  is written as

$$\sum_{\ell=1}^j \frac{\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2 a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell a_\ell^* Q_\ell a_k]}{1 + \delta \tilde{d}_\ell} = \psi_j \sum_{\ell=1}^j \frac{\mathbb{E}[\rho^2 \tilde{d}_{\ell\ell}^2 a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell a_\ell^* Q_\ell a_k]}{1 + \delta \tilde{d}_\ell} + \varepsilon$$

where  $\varepsilon = \mathcal{O}(n^{-1})$  by Lemmas 5.1 and 3.1. The remaining term in the r.h.s. can be handled by the following lemma which is proven in appendix B.2:

**Lemma 5.2.** *Let  $(u) = (u_n)_{n \in \mathbb{N}}$  be a sequence of vectors with bounded Euclidean norms. Let  $(\alpha_\ell)_{1 \leq \ell \leq n} = (\alpha_{\ell,n})_{1 \leq \ell \leq n}$  be an array of bounded real numbers. Then:*

$$\sum_{\ell=1}^j \alpha_\ell u^* T a_\ell \mathbb{E}[a_\ell Q_\ell u] = \sum_{\ell=1}^j \frac{\alpha_\ell u^* T a_\ell a_\ell^* T u}{\rho \tilde{t}_{\ell\ell} (1 + \tilde{d}_\ell \delta)} + \mathcal{O}(n^{-1/2}).$$

Applying this lemma with  $u = a_k$  and  $\alpha_\ell = \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell (1 + \tilde{d}_\ell)^{-1} a_\ell \mathcal{T}_\ell a_\ell$ , we obtain

$$U_2 = \psi_j \sum_{\ell=1}^j \frac{\rho \tilde{d}_{\ell\ell} \tilde{t}_{\ell\ell} a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell a_\ell^* T a_k}{(1 + \delta \tilde{d}_\ell)^2} + \mathcal{O}(n^{-1/2}).$$

Gathering these results, and using the identity  $(1 - \rho \tilde{t}_{\ell\ell} a_\ell^* \mathcal{T}_\ell a_\ell) = \rho \tilde{t}_{\ell\ell} (1 + \tilde{d}_\ell \delta)$  (see (3.10)), we obtain

$$X = \sum_{\ell=1}^n \rho \tilde{t}_{\ell\ell} a_k^* T a_\ell \mathbb{E}[a_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k] + \psi_j \sum_{\ell=1}^j \frac{\rho \tilde{d}_{\ell\ell} \tilde{t}_{\ell\ell} a_k^* T a_\ell a_\ell^* \mathcal{T}_\ell a_\ell a_\ell^* T a_k}{(1 + \tilde{d}_\ell \delta)^2} + \mathcal{O}(n^{-1/2}). \quad (5.6)$$

We now turn to the term  $Z$  in (5.4).

$$Z = - \sum_{\ell=1}^n \mathbb{E}[a_k^* T (\mathbb{E}_j \rho \tilde{q}_{\ell\ell} \eta_\ell \eta_\ell^* Q_\ell) D Q a_k] = - \sum_{\ell=1}^n \rho \tilde{t}_{\ell\ell} \mathbb{E}[a_k^* T (\mathbb{E}_j \eta_\ell \eta_\ell^* Q_\ell) D Q a_k] + \varepsilon,$$

where

$$\varepsilon = \sum_{\ell=1}^n \mathbb{E}[a_k^* T (\mathbb{E}_j \rho (\tilde{q}_{\ell\ell} - \tilde{t}_{\ell\ell}) \eta_\ell \eta_\ell^* Q_\ell) D Q a_k]$$

satisfies  $\varepsilon = \mathcal{O}(n^{-1/2})$  (same arguments as for  $\varepsilon_1$  in (5.5)). Writing  $\eta_\ell \eta_\ell^* = a_\ell a_\ell^* + y_\ell y_\ell^* + a_\ell y_\ell^* + y_\ell a_\ell^*$ , we obtain:

$$\begin{aligned} Z &= - \sum_{\ell=1}^n \rho \tilde{t}_{\ell\ell} a_k^* T a_\ell \mathbb{E}[a_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k] \\ &\quad - \left( \sum_{\ell=1}^j \rho \tilde{t}_{\ell\ell} \mathbb{E}[a_k^* T y_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k] + \frac{1}{n} \sum_{\ell=j+1}^n \rho \tilde{t}_{\ell\ell} \tilde{d}_\ell \mathbb{E}[a_k^* T D (\mathbb{E}_j Q_\ell) D Q a_k] \right) \\ &\quad - \sum_{\ell=1}^j \rho \tilde{t}_{\ell\ell} a_k^* T a_\ell \mathbb{E}[y_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k] - \sum_{\ell=1}^j \rho \tilde{t}_{\ell\ell} \mathbb{E}[a_k^* T y_\ell a_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k] + \mathcal{O}(n^{-1/2}) \\ &\triangleq Z_1 + Z_2 + Z_3 + Z_4 + \mathcal{O}(n^{-1/2}). \end{aligned}$$

The term  $Z_1$  cancels with the first term in the decomposition of  $X$  (first term at the r.h.s. of (5.6)). The term  $Z_2$  can be written as:

$$\begin{aligned} Z_2 &= - \left( \sum_{\ell=1}^j \rho \tilde{t}_{\ell\ell} \mathbb{E}[a_k^* T y_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell a_k] + \frac{1}{n} \sum_{\ell=j+1}^n \rho \tilde{t}_{\ell\ell} \tilde{d}_\ell \mathbb{E}[a_k^* T D(\mathbb{E}_j Q_\ell) D Q a_k] \right) \\ &\quad + \sum_{\ell=1}^j \rho^2 \tilde{t}_{\ell\ell}^2 \mathbb{E}[a_k^* T y_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell \eta_\ell^* Q_\ell a_k] + \varepsilon \\ &\triangleq W_1 + W_2 + \varepsilon, \end{aligned}$$

where  $\varepsilon$  follows from the substitution of  $\rho \tilde{q}_{\ell\ell}$  with  $\rho \tilde{t}_{\ell\ell}$  and satisfies  $\varepsilon = \mathcal{O}(n^{-1/2})$  as in (5.5). Consider first  $W_1$ :

$$W_1 = - \left( \frac{1}{n} \sum_{\ell=1}^j \rho \tilde{t}_{\ell\ell} \tilde{d}_\ell \mathbb{E}[a_k^* T D(\mathbb{E}_j Q_\ell) D Q_\ell a_k] + \frac{1}{n} \sum_{\ell=j+1}^n \rho \tilde{t}_{\ell\ell} \tilde{d}_\ell \mathbb{E}[a_k^* T D(\mathbb{E}_j Q_\ell) D Q a_k] \right).$$

Write:

$$(\mathbb{E}_j Q_\ell) D Q_\ell - (\mathbb{E}_j Q) D Q = (\mathbb{E}_j Q_\ell) D(Q_\ell - Q) + (\mathbb{E}_j Q_\ell - \mathbb{E}_j Q) D Q.$$

Using (3.5) and (3.6),

$$\begin{aligned} \frac{1}{n} \sum_{\ell=1}^j \rho \tilde{t}_{\ell\ell} \tilde{d}_\ell |\mathbb{E}[a_k^* T D(\mathbb{E}_j Q_\ell) D(Q_\ell - Q) a_k]| &\leq \frac{K}{n} \sum_{\ell=1}^j (\mathbb{E}|(1 + \eta_\ell^* Q_\ell \eta_\ell)|^2)^{1/2} (\mathbb{E}|\eta_\ell^* Q a_k|^2)^{1/2} \\ &\leq \frac{K}{\sqrt{n}} (\mathbb{E} a_k^* Q \Sigma_{1:j} \Sigma_{1:j}^* Q a_k)^{1/2} = \mathcal{O}(n^{-1/2}), \end{aligned}$$

and the same arguments apply to the term  $(\mathbb{E}_j Q_\ell - \mathbb{E}_j Q) D Q$ . Hence,

$$W_1 = -\rho \tilde{\delta} \mathbb{E}[a_k^* T D(\mathbb{E}_j Q) D Q a_k] + \mathcal{O}(n^{-1/2}).$$

Turning to  $W_2$ , we have:

$$\begin{aligned} &\sum_{\ell=1}^j \rho^2 \tilde{t}_{\ell\ell}^2 \mathbb{E}[a_k^* T y_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell a_\ell^* Q_\ell a_k] \\ &= \sum_{\ell=1}^j \rho^2 \tilde{t}_{\ell\ell}^2 \mathbb{E} \left[ a_k^* T y_\ell \left( y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell - \frac{\tilde{d}_\ell}{n} \text{Tr} D(\mathbb{E}_j Q_\ell) D Q_\ell \right) a_\ell^* Q_\ell a_k \right] \end{aligned}$$

whose modulus is of order  $\mathcal{O}(n^{-1/2})$ . The term

$$\sum_{\ell=1}^j \rho^2 \tilde{t}_{\ell\ell}^2 \mathbb{E}[a_k^* T y_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell a_\ell y_\ell^* Q_\ell a_k]$$

can be handled similarly.

The term

$$\sum_{\ell=1}^j \rho^2 \tilde{t}_{\ell\ell}^2 \mathbb{E}[a_k^* T y_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell a_\ell a_\ell^* Q_\ell a_k] = \frac{1}{n} \sum_{\ell=1}^j \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell \mathbb{E}[a_k^* T D(\mathbb{E}_j Q_\ell) D Q_\ell a_\ell a_\ell^* Q_\ell a_k]$$

is bounded by  $Kn^{-1/2}$ . Finally,

$$\begin{aligned} W_2 &= \sum_{\ell=1}^j \rho^2 \tilde{t}_{\ell\ell}^2 \mathbb{E}[y_\ell^* Q_\ell a_k a_k^* T y_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell] + \mathcal{O}(n^{-1/2}), \\ &\stackrel{(a)}{=} \psi_j a_k^* T D T a_k \frac{1}{n} \sum_{\ell=1}^j \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 + \mathcal{O}(n^{-1/2}), \end{aligned}$$

where (a) follows by standard arguments as those already developed.

The term  $Z_3$  satisfies

$$Z_3 = \sum_{\ell=1}^j \rho^2 \tilde{t}_{\ell\ell}^2 a_k^* T a_\ell \mathbb{E}[y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell \eta_\ell^* Q_\ell a_k] + \mathcal{O}(n^{-1/2}).$$

Writing  $\eta_\ell \eta_\ell^* = y_\ell y_\ell^* + a_\ell y_\ell^* + a_\ell a_\ell^* + y_\ell a_\ell^*$  and relying arguments as those already developed, one can check that the only non-negligible contribution stems from the term containing  $y_\ell a_\ell^*$ . Hence,

$$\begin{aligned} Z_3 &= \psi_j \sum_{\ell=1}^j \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell a_k^* T a_\ell \mathbb{E}[a_\ell^* Q_\ell a_k] + \mathcal{O}(n^{-1/2}), \\ &= \psi_j \sum_{\ell=1}^j \frac{\rho \tilde{t}_{\ell\ell} \tilde{d}_\ell a_k^* T a_\ell a_\ell^* T a_k}{1 + \tilde{d}_\ell \delta} + \mathcal{O}(n^{-1/2}), \end{aligned}$$

by Lemma (5.2). Similarly,

$$\begin{aligned} Z_4 &= \sum_{\ell=1}^j \rho^2 \tilde{t}_{\ell\ell}^2 \mathbb{E}[a_k^* T y_\ell a_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell \eta_\ell^* Q_\ell a_k] + \mathcal{O}(n^{-1/2}), \\ &= a_k^* T D T a_k \frac{1}{n} \sum_{\ell=1}^j \rho^2 \tilde{d}_\ell \tilde{t}_{\ell\ell}^2 \mathbb{E}[a_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell a_\ell] + \mathcal{O}(n^{-1/2}), \\ &= a_k^* T D T a_k \varphi_j + \mathcal{O}(n^{-1/2}). \end{aligned}$$

Gathering these results, we obtain

$$\begin{aligned} Z &= - \sum_{\ell=1}^n \rho \tilde{t}_{\ell\ell} a_k^* T a_\ell \mathbb{E}[a_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k] - \rho \tilde{\delta} \mathbb{E}[a_k^* T D (\mathbb{E}_j Q) D Q a_k] \\ &\quad + \psi_j a_k^* T D T a_k \frac{1}{n} \sum_{\ell=1}^j \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 + \psi_j \sum_{\ell=1}^j \frac{\rho \tilde{t}_{\ell\ell} \tilde{d}_\ell a_k^* T a_\ell a_\ell^* T a_k}{1 + \tilde{d}_\ell \delta} + a_k^* T D T a_k \varphi_j + \mathcal{O}(n^{-1/2}). \end{aligned}$$

Plugging this and Eq. (5.6) into (5.4), and noticing that  $\rho \tilde{t}_{\ell\ell} (a_\ell T_\ell a_\ell (1 + \tilde{d}_\ell \delta)^{-1} + 1) = (1 + \tilde{d}_\ell \delta)^{-1}$ , we obtain:

$$\begin{aligned} \zeta_{kj} &= a_k^* T D T a_k + \psi_j \left( \sum_{\ell=1}^j \frac{a_k^* T a_\ell \tilde{d}_\ell a_\ell^* T a_k}{(1 + \tilde{d}_\ell \delta)^2} + a_k^* T D T a_k \frac{1}{n} \sum_{\ell=1}^j \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 \right) \\ &\quad + a_k^* T D T a_k \varphi_j + \mathcal{O}(n^{-1/2}). \quad (5.7) \end{aligned}$$

5.2. **Step 2: Expression of  $\psi_j = n^{-1} \text{Tr} \mathbb{E}[(\mathbb{E}_j Q) D Q D]$ .** Using Identity (5.2), we obtain:

$$\begin{aligned} \psi_j &= \frac{1}{n} \text{Tr} \mathbb{E}[T D Q D] + \frac{\rho \tilde{\delta}}{n} \text{Tr} \mathbb{E}[T D (\mathbb{E}_j Q) D Q D] \\ &\quad + \frac{1}{n} \text{Tr} \mathbb{E}[T A (I + \delta \tilde{D})^{-1} A^* (\mathbb{E}_j Q) D Q D] - \frac{1}{n} \text{Tr} \mathbb{E}[T (\mathbb{E}_j \Sigma \Sigma^* Q) D Q D], \end{aligned} \quad (5.8)$$

$$= \frac{1}{n} \text{Tr} D T D T + \frac{\rho \tilde{\delta}}{n} \text{Tr} \mathbb{E}[T D (\mathbb{E}_j Q) D Q D] + X + Z + \varepsilon, \quad (5.9)$$

where  $X$  and  $Z$  are the last two terms of the r.h.s. of (5.8), and where  $\varepsilon = \mathcal{O}(n^{-1})$  by Theorem 3.3-(2). Due to the presence of the multiplying factor  $n^{-1}$ , the treatment of  $X$  and  $Z$  is simpler here than the treatment of their analogues for  $\zeta_{kj}$ . We skip hereafter the details related to the bounds over the  $\varepsilon$ 's. The term  $X$  satisfies

$$\begin{aligned} X &= \frac{1}{n} \sum_{\ell=1}^n \frac{\mathbb{E}[a_\ell^* (\mathbb{E}_j Q) D Q D T a_\ell]}{1 + \delta \tilde{d}_\ell}, \\ &= \frac{1}{n} \sum_{\ell=1}^n \frac{\mathbb{E}[a_\ell^* (\mathbb{E}_j Q_\ell) D Q D T a_\ell]}{1 + \delta \tilde{d}_\ell} - \frac{1}{n} \sum_{\ell=1}^n \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} (\mathbb{E}_j a_\ell^* Q_\ell \eta_\ell \eta_\ell^* Q_\ell) D Q D T a_\ell]}{1 + \delta \tilde{d}_\ell} + \varepsilon, \\ &= \frac{1}{n} \sum_{\ell=1}^n \frac{\mathbb{E}[a_\ell^* (\mathbb{E}_j Q_\ell) D Q D T a_\ell]}{1 + \delta \tilde{d}_\ell} - \frac{1}{n} \sum_{\ell=1}^n \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_\ell^* \mathcal{T}_\ell a_\ell a_\ell^* (\mathbb{E}_j Q_\ell) D Q D T a_\ell]}{1 + \delta \tilde{d}_\ell} \\ &\quad - \frac{1}{n} \sum_{\ell=1}^j \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_\ell^* \mathcal{T}_\ell a_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q D T a_\ell]}{1 + \delta \tilde{d}_\ell} + \varepsilon', \end{aligned}$$

where  $\max(|\varepsilon|, |\varepsilon'|) = \mathcal{O}(n^{-1/2})$ . As  $1 - \rho \tilde{t}_{\ell\ell} a_\ell^* \mathcal{T}_\ell a_\ell = \rho \tilde{t}_{\ell\ell} (1 + \tilde{d}_\ell \delta)$ ,

$$\begin{aligned} X &= \frac{1}{n} \sum_{\ell=1}^n \mathbb{E}[\rho \tilde{t}_{\ell\ell} a_\ell^* (\mathbb{E}_j Q_\ell) D Q D T a_\ell] - \frac{1}{n} \sum_{\ell=1}^j \frac{\mathbb{E}[\rho \tilde{t}_{\ell\ell} a_\ell^* \mathcal{T}_\ell a_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q D T a_\ell]}{1 + \delta \tilde{d}_\ell} + \mathcal{O}(n^{-1/2}), \\ &= \frac{1}{n} \sum_{\ell=1}^n \mathbb{E}[\rho \tilde{t}_{\ell\ell} a_\ell^* (\mathbb{E}_j Q_\ell) D Q D T a_\ell] + \frac{1}{n} \sum_{\ell=1}^j \frac{\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2 a_\ell^* \mathcal{T}_\ell a_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell \eta_\ell^* Q_\ell D T a_\ell]}{1 + \delta \tilde{d}_\ell} + \mathcal{O}(n^{-1/2}), \\ &= \frac{1}{n} \sum_{\ell=1}^n \mathbb{E}[\rho \tilde{t}_{\ell\ell} a_\ell^* (\mathbb{E}_j Q_\ell) D Q D T a_\ell] + \frac{\psi_j}{n} \sum_{\ell=1}^j \frac{\tilde{d}_\ell a_\ell^* \mathcal{T}_\ell a_\ell a_\ell^* T D T a_\ell}{(1 + \delta \tilde{d}_\ell)^3} + \mathcal{O}(n^{-1/2}), \end{aligned} \quad (5.10)$$

where (3.11) is used to obtain the last equation. The term  $Z$  can be expressed as:

$$\begin{aligned}
Z &= -\frac{1}{n} \sum_{\ell=1}^n \text{Tr} \mathbb{E}[\rho \tilde{t}_{\ell\ell} T (\mathbb{E}_j \eta_\ell \eta_\ell^* Q_\ell) D Q D] + \mathcal{O}(n^{-1/2}), \\
&= -\frac{1}{n} \sum_{\ell=1}^n \mathbb{E}[\rho \tilde{t}_{\ell\ell} a_\ell^* (\mathbb{E}_j Q_\ell) D Q D T a_\ell] \\
&\quad - \left( \frac{1}{n} \sum_{\ell=1}^j \mathbb{E}[\rho \tilde{t}_{\ell\ell} y_\ell^* (\mathbb{E}_j Q_\ell) D Q D T y_\ell] + \frac{1}{n} \sum_{\ell=j+1}^n \frac{1}{n} \text{Tr} \mathbb{E}[\rho \tilde{d}_\ell \tilde{t}_{\ell\ell} T D (\mathbb{E}_j Q_\ell) D Q D] \right) \\
&\quad - \frac{1}{n} \sum_{\ell=1}^j \mathbb{E}[\rho \tilde{t}_{\ell\ell} y_\ell^* (\mathbb{E}_j Q_\ell) D Q D T a_\ell] - \frac{1}{n} \sum_{\ell=1}^j \mathbb{E}[\rho \tilde{t}_{\ell\ell} a_\ell^* (\mathbb{E}_j Q_\ell) D Q D T y_\ell] + \mathcal{O}(n^{-1/2}), \\
&\triangleq Z_1 + Z_2 + Z_3 + Z_4 + \mathcal{O}(n^{-1/2}).
\end{aligned}$$

The term  $Z_1$  cancels with the first term in the r.h.s. of  $X$ 's decomposition (5.10). The terms  $Z_2$ ,  $Z_3$  and  $Z_4$  satisfy:

$$\begin{aligned}
Z_2 &= -\frac{\rho \tilde{\delta}}{n} \text{Tr} \mathbb{E}[T D (\mathbb{E}_j Q) D Q D] + \frac{1}{n} \sum_{\ell=1}^j \mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2 y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell \eta_\ell^* Q_\ell D T y_\ell] + \mathcal{O}(n^{-1/2}) \\
&= -\frac{\rho \tilde{\delta}}{n} \text{Tr} \mathbb{E}[T D (\mathbb{E}_j Q) D Q D] + \psi_j \frac{1}{n} \text{Tr} D T D T \frac{1}{n} \sum_{\ell=1}^j \rho^2 \tilde{d}_\ell^2 \tilde{t}_{\ell\ell}^2 + \mathcal{O}(n^{-1/2}), \\
Z_3 &= \frac{1}{n} \sum_{\ell=1}^j \mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2 y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell \eta_\ell^* Q_\ell D T a_\ell] + \mathcal{O}(n^{-1/2}) \\
&= \psi_j \frac{1}{n} \sum_{\ell=1}^j \rho \tilde{d}_\ell \tilde{t}_{\ell\ell} \frac{a_\ell^* T D T a_\ell}{1 + \tilde{d}_\ell \delta} + \mathcal{O}(n^{-1/2}) \quad (\text{see Th.3.3-(4) and (3.11)}), \\
Z_4 &= \frac{1}{n} \sum_{\ell=1}^j \mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2 a_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell \eta_\ell^* Q_\ell D T y_\ell] + \mathcal{O}(n^{-1/2}) \\
&= \frac{1}{n} \text{Tr} D T D T \frac{1}{n} \sum_{\ell=1}^j \rho^2 \tilde{d}_\ell \tilde{t}_{\ell\ell}^2 \mathbb{E}[a_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell a_\ell] + \mathcal{O}(n^{-1/2}).
\end{aligned}$$

Plugging these terms in (5.9), we obtain:

$$\begin{aligned}
\psi_j &= \gamma + \psi_j \left( \frac{1}{n} \sum_{\ell=1}^j a_\ell^* T D T a_\ell \left( \frac{\tilde{d}_\ell a_\ell^* T a_\ell}{(1 + \delta \tilde{d}_\ell)^3} + \frac{\rho \tilde{d}_\ell \tilde{t}_{\ell\ell}}{1 + \tilde{d}_\ell \delta} \right) + \frac{\gamma}{n} \sum_{\ell=1}^j \rho^2 \tilde{d}_\ell^2 \tilde{t}_{\ell\ell}^2 \right) + \gamma \varphi_j + \mathcal{O}(n^{-1/2}), \\
&= \gamma + \psi_j \left( \frac{1}{n} \sum_{\ell=1}^j \frac{\tilde{d}_\ell a_\ell^* T D T a_\ell}{(1 + \delta \tilde{d}_\ell)^2} + \frac{\gamma}{n} \sum_{\ell=1}^j \rho^2 \tilde{d}_\ell^2 \tilde{t}_{\ell\ell}^2 \right) + \gamma \varphi_j + \mathcal{O}(n^{-1/2}), \tag{5.11}
\end{aligned}$$

using (3.10) and (3.11).

**5.3. Step 3: Relation between  $\zeta_{kj}$  and  $\theta_{kj}$  for  $k \leq j$ .** The term  $\zeta_{kj}$  can be written as

$$\begin{aligned}\zeta_{kj} &= \mathbb{E}[a_k^* \mathbb{E}_j(Q_k - \rho \tilde{q}_{kk} Q_k \eta_k \eta_k^* Q_k) D(Q_k - \rho \tilde{q}_{kk} Q_k \eta_k \eta_k^* Q_k) a_k] \\ &= \theta_{kj} - \rho \tilde{t}_{kk} \mathbb{E}[a_k^* \mathbb{E}_j(Q_k \eta_k \eta_k^* Q_k) D Q_k a_k] - \rho \tilde{t}_{kk} \mathbb{E}[a_k^* \mathbb{E}_j(Q_k) D Q_k \eta_k \eta_k^* Q_k a_k] \\ &\quad + \rho^2 \tilde{t}_{kk}^2 \mathbb{E}[a_k^* \mathbb{E}_j(Q_k \eta_k \eta_k^* Q_k) D Q_k \eta_k \eta_k^* Q_k a_k] + \mathcal{O}(n^{-1/2}) \\ &\triangleq \theta_{kj} + X_1 + X_2 + X_3 + \mathcal{O}(n^{-1/2}).\end{aligned}$$

Using similar arguments as those developed previously, we get:

$$\begin{aligned}X_1 &= -\rho \tilde{t}_{kk} a_k^* \mathcal{T}_k a_k \mathbb{E}[a_k^* \mathbb{E}_j(Q_k) D Q_k a_k] + \mathcal{O}(n^{-1/2}) = -\rho \tilde{t}_{kk} a_k^* \mathcal{T}_k a_k \theta_{kj} + \mathcal{O}(n^{-1/2}), \\ X_2 &= -\rho \tilde{t}_{kk} a_k^* \mathcal{T}_k a_k \theta_{kj} + \mathcal{O}(n^{-1/2}).\end{aligned}$$

As  $k \leq j$ ,

$$\begin{aligned}X_3 &= \rho^2 \tilde{t}_{kk}^2 (a_k^* \mathcal{T}_k a_k)^2 \mathbb{E}[\eta_k^* \mathbb{E}_j(Q_k) D Q_k \eta_k] + \mathcal{O}(n^{-1/2}), \\ &= \rho^2 \tilde{t}_{kk}^2 (a_k^* \mathcal{T}_k a_k)^2 (\theta_{kj} + \tilde{d}_k \psi_j) + \mathcal{O}(n^{-1/2}).\end{aligned}$$

Using (3.10) and (3.11), we finally obtain:

$$\zeta_{kj} = \rho^2 \tilde{t}_{kk}^2 (1 + \tilde{d}_k \delta)^2 \theta_{kj} + \tilde{d}_k \left( \frac{a_k^* \mathcal{T}_k a_k}{1 + \tilde{d}_k \delta} \right)^2 \psi_j + \mathcal{O}(n^{-1/2}). \quad (5.12)$$

**5.4. Step 4: A system of perturbed linear equations in  $(\psi_j, \varphi_j)$ . Proof of (5.1).**

Combining (5.12) with (5.7), we obtain

$$\begin{aligned}\rho^2 \tilde{d}_k \tilde{t}_{kk}^2 \theta_{kj} &= \tilde{d}_k \frac{a_k^* \mathcal{T}_k a_k}{(1 + \tilde{d}_k \delta)^2} \\ &+ \left( \sum_{\ell=1}^j \frac{\tilde{d}_k a_k^* \mathcal{T}_\ell a_\ell \tilde{d}_\ell a_\ell^* \mathcal{T}_k a_k}{(1 + \tilde{d}_k \delta)^2 (1 + \tilde{d}_\ell \delta)^2} + \frac{\tilde{d}_k a_k^* \mathcal{T}_k a_k}{(1 + \tilde{d}_k \delta)^2} \frac{1}{n} \sum_{\ell=1}^j \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 - \tilde{d}_k^2 \frac{(a_k^* \mathcal{T}_k a_k)^2}{(1 + \tilde{d}_k \delta)^4} \right) \psi_j \\ &\quad + \frac{\tilde{d}_k a_k^* \mathcal{T}_k a_k}{(1 + \tilde{d}_k \delta)^2} \varphi_j + \mathcal{O}(n^{-1/2})\end{aligned} \quad (5.13)$$

which implies that  $\varphi_j = \frac{1}{n} \sum_{k=1}^j \rho^2 \tilde{d}_k \tilde{t}_{kk}^2 \theta_{kj}$  satisfies

$$(1 - F_j) \varphi_j - (G_j + F_j M_j) \psi_j = F_j + \mathcal{O}(n^{-1/2})$$

where

$$\begin{aligned}F_j &= \frac{1}{n} \sum_{k=1}^j \frac{a_k^* \mathcal{T}_k a_k \tilde{d}_k}{(1 + \tilde{d}_k \delta)^2}, \\ M_j &= \frac{1}{n} \sum_{\ell=1}^j \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2, \\ G_j &= \frac{1}{n} \sum_{k=1}^j \sum_{\substack{\ell=1 \\ \ell \neq k}}^j \frac{\tilde{d}_k \tilde{d}_\ell |a_k^* \mathcal{T}_\ell a_\ell|^2}{(1 + \tilde{d}_k \delta)^2 (1 + \tilde{d}_\ell \delta)^2}.\end{aligned} \quad (5.14)$$

With these new notations, equation (5.11) is rewritten

$$-\gamma \varphi_j + (1 - F_j - \gamma M_j) \psi_j = \gamma + \mathcal{O}(n^{-1/2}),$$



and we end up with a system of two perturbed linear equations in  $(\varphi_j, \psi_j)$ :

$$\begin{cases} (1 - F_j)\varphi_j - (G_j + F_j M_j)\psi_j &= F_j + \mathcal{O}(n^{-1/2}) \\ -\gamma\varphi_j + (1 - F_j - \gamma M_j)\psi_j &= \gamma + \mathcal{O}(n^{-1/2}) \end{cases}. \quad (5.15)$$

The determinant of this system is  $\Delta_j = (1 - F_j)^2 - \gamma M_j - \gamma G_j$ . The following lemma establishes the link between the  $\Delta_j$ 's and  $\Delta_n$  as defined in Theorem 2.2.

**Lemma 5.3.** *Recall the definition of  $\Delta_n$  :*

$$\Delta_n = \left(1 - \frac{1}{n} \operatorname{Tr} D^{\frac{1}{2}} T A (I + \delta \tilde{D})^{-2} \tilde{D} A^* T D^{\frac{1}{2}}\right)^2 - \rho^2 \gamma \tilde{\gamma}.$$

*The determinants  $\Delta_j$  decrease as  $j$  goes from 1 to  $n$ ; moreover,  $\Delta_n$  coincides with  $\Delta_n$ .*

Proof of Lemma 5.3 is postponed to Appendix B.3.

Solving this system of equations and using the lemma in conjunction with the fact  $\liminf \Delta_n > 0$ , established in Lemma 3.5, we obtain:

$$\begin{bmatrix} \varphi_j \\ \psi_j \end{bmatrix} = \frac{1}{\Delta_j} \begin{bmatrix} F_j(1 - F_j) + \gamma G_j \\ \gamma \end{bmatrix} + \varepsilon_j,$$

where  $\|\varepsilon_j\| = \mathcal{O}(n^{-1/2})$ . Replacing into (5.13), we obtain

$$\begin{aligned} \frac{2\rho^2 \tilde{d}_j \tilde{t}_{jj}^2 \theta_{jj}}{n} &= 2(F_j - F_{j-1}) + (G_j - G_{j-1} + 2M_j(F_j - F_{j-1}))\psi_j + 2(F_j - F_{j-1})\varphi_j + \mathcal{O}(n^{-3/2}) \\ &= 2(F_j - F_{j-1}) + \frac{\gamma(G_j - G_{j-1}) + 2\gamma M_j(F_j - F_{j-1})}{\Delta_j} \\ &\quad + \frac{2(F_j - F_{j-1})(F_j(1 - F_j) + \gamma G_j)}{\Delta_j} + \mathcal{O}(n^{-3/2}) \end{aligned}$$

which leads to

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n \left( \rho^2 \tilde{t}_{jj}^2 \tilde{d}_j^2 \psi_j + 2\rho^2 \tilde{d}_j \tilde{t}_{jj}^2 \theta_{jj} \right) \\ = \sum_{j=1}^n \frac{2(F_j - F_{j-1})(1 - F_j) + \gamma(M_j - M_{j-1}) + \gamma(G_j - G_{j-1})}{\Delta_j} + \mathcal{O}(n^{-1/2}). \end{aligned}$$

On the other hand,  $\Delta_{j-1} - \Delta_j = 2(F_j - F_{j-1})(1 - F_j) + \gamma(M_j - M_{j-1}) + \gamma(G_j - G_{j-1}) + \mathcal{O}(n^{-2})$ , hence, due to Lemma 5.3 and to  $\liminf \Delta_n > 0$ ,

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n \left( \rho^2 \tilde{t}_{jj}^2 \tilde{d}_j^2 \psi_j + 2\rho^2 \tilde{d}_j \tilde{t}_{jj}^2 \theta_{jj} \right) &= \sum_{j=1}^n \frac{\Delta_{j-1} - \Delta_j}{\Delta_j} + \mathcal{O}(n^{-1/2}) \\ &= \sum_{j=1}^n \log \left( 1 + \frac{\Delta_{j-1} - \Delta_j}{\Delta_j} \right) + \mathcal{O}(n^{-1/2}) \\ &= \sum_{j=1}^n \log \frac{\Delta_{j-1}}{\Delta_j} + \mathcal{O}(n^{-1/2}) = -\log(\Delta_n) + \mathcal{O}(n^{-1/2}) \end{aligned}$$

which proves (5.1). Lemma 4.3 is proven.

## 6. PROOF OF THEOREM 2.2 (PART III)

In this section, we complete the proof of Theorem 2.2. Proof of Lemma 4.4 is very close to the proof of Lemma 4.3; we therefore only provide its main landmarks. We finally establish the main estimates over  $(\Theta_n)$ .

**6.1. Elements of proof for Lemma 4.4.** Proof of Lemma 4.4 relies on the following counterpart of Lemma 3.2:

**Lemma 6.1.** *Assume that the setting of Lemma 3.2 holds true; and let  $\mathbb{E}x^2 = \vartheta$ . Then for any  $p \geq 2$ ,*

$$\mathbb{E}|\mathbf{x}^T M \mathbf{x} - \vartheta \operatorname{Tr} M|^p \leq K_p \left( (\mathbb{E}|x_1|^4 \operatorname{Tr} M M^*)^{p/2} + \mathbb{E}|x_1|^{2p} \operatorname{Tr}(M M^*)^{p/2} \right).$$

*Proof.* The result is obtained upon noticing that

$$\mathbf{x}^T M \mathbf{x} = \frac{1}{4} \sum_{k=0}^3 \mathbf{i}^k (\mathbf{i}^k \bar{\mathbf{x}} + \mathbf{x})^* M (\mathbf{i}^k \bar{\mathbf{x}} + \mathbf{x})$$

and using Lemma 3.2. □

Here are the main steps of the proof. Introducing the notations

$$\begin{aligned} \underline{\psi}_j &= \frac{1}{n} \operatorname{Tr} \mathbb{E} [(\mathbb{E}_j Q) D \bar{Q} D] , \\ \underline{\zeta}_{kj} &= \mathbb{E} [a_k^* (\mathbb{E}_j Q) D \bar{Q} \bar{a}_k] , \\ \underline{\theta}_{kj} &= \mathbb{E} [a_k^* (\mathbb{E}_j Q_k) D \bar{Q}_k \bar{a}_k] , \\ \underline{\varphi}_j &= \frac{1}{n} \sum_{k=1}^j \rho^2 \tilde{d}_k \tilde{t}_{kk}^2 \underline{\theta}_{kj} , \end{aligned}$$

and adapting Lemma 5.1, we only need to prove that:

$$\frac{1}{n} \sum_{j=1}^n \left( \rho^2 \tilde{t}_{jj}^2 \tilde{d}_j^2 |\vartheta|^2 \underline{\psi}_j + 2\rho^2 \tilde{d}_j \tilde{t}_{jj}^2 \operatorname{Re}(\vartheta \underline{\theta}_{jj}) \right) + \log \underline{\Delta}_n \xrightarrow{n \rightarrow \infty} 0 .$$

Similar derivations as those performed in Steps 1-3 in Section 5 yield the perturbed system:

$$\begin{cases} (1 - \vartheta \underline{F}_j) \underline{\varphi}_j - (\vartheta \underline{G}_j + |\vartheta|^2 \underline{F}_j M_j) \underline{\psi}_j &= \underline{F}_j + \mathcal{O}(n^{-1/2}) \\ -\vartheta \underline{\gamma} \underline{\varphi}_j + (1 - \vartheta \underline{F}_j - \underline{\gamma} |\vartheta|^2 M_j) \underline{\psi}_j &= \underline{\gamma} + \mathcal{O}(n^{-1/2}) \end{cases} ,$$

where

$$\begin{aligned} \underline{F}_j &= \frac{1}{n} \sum_{k=1}^j \frac{a_k^* T D \bar{T} \bar{a}_k \tilde{d}_k}{(1 + \tilde{d}_k \delta)^2} \in \mathbb{C}, & \underline{G}_j &= \frac{1}{n} \sum_{k=1}^j \sum_{\substack{\ell=1 \\ \ell \neq k}}^j \frac{\tilde{d}_k \tilde{d}_\ell (a_k^* T a_\ell)^2}{(1 + \tilde{d}_k \delta)^2 (1 + \tilde{d}_\ell \delta)^2} \in \mathbb{R}, \\ M_j &= \frac{1}{n} \sum_{\ell=1}^j \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 . \end{aligned}$$

The determinant of this system is:

$$\underline{\Delta}_j = |1 - \vartheta \underline{F}_j|^2 - |\vartheta|^2 \underline{\gamma} (M_j + \underline{G}_j) .$$

By (3.13),  $0 \leq \underline{\gamma} \leq \gamma$ ; furthermore,  $|\vartheta| \leq 1$ ,  $|\underline{E}_j| \leq F_j$ , and  $|\underline{G}_j| \leq G_j$ . As a result,  $\underline{\Delta}_j \geq \Delta_j$ . Hence, by Lemma 5.3, the perturbation remains of order  $\mathcal{O}(n^{-1/2})$  after solving the system. Performing the same derivations as in Step 4 in Section 5, it can be established that  $\underline{\Delta}_n = \Delta_n$ . We finally end up with:

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n \left( \rho^2 \tilde{t}_{jj}^2 \tilde{d}_j^2 |\vartheta|^2 \underline{\psi}_j + 2\rho^2 \tilde{d}_j \tilde{t}_{jj}^2 \operatorname{Re}(\vartheta \underline{\theta}_{jj}) \right) &= \sum_{j=1}^n \frac{\underline{\Delta}_{j-1} - \underline{\Delta}_j}{\underline{\Delta}_j} + \mathcal{O}(n^{-1/2}), \\ &= -\log(\underline{\Delta}_n) + \mathcal{O}(n^{-1/2}), \end{aligned}$$

which is the desired result.

**6.2. Estimates over  $\Theta_n$ .** In order to conclude the proof of Theorem 2.2, it remains to prove that  $0 < \liminf_n \Theta_n \leq \limsup_n \Theta_n < \infty$ .

Consider first the upper bound. By Lemma 3.5,  $\sup_n(-\log \Delta_n) < \infty$ . As  $\underline{\Delta}_n \geq \Delta_n$ ,  $\log \underline{\Delta}_n$  is defined and  $\sup_n(-\log \underline{\Delta}_n) < \infty$ . By Lemma 3.4, the cumulant term in the expression of  $\Theta_n$  is bounded, hence  $\limsup_n \Theta_n < \infty$ .

We now prove that  $\liminf \Theta_n > 0$ . To this end, write:

$$\begin{aligned} \Theta_n &= \sum_{j=1}^n \left( \frac{\Delta_{j-1} - \Delta_j}{\Delta_j} + \frac{\underline{\Delta}_{j-1} - \underline{\Delta}_j}{\underline{\Delta}_j} \right) + \kappa \frac{\rho^2}{n^2} \sum_{i=1}^N d_i^2 t_{ii}^2 \sum_{j=1}^n \tilde{d}_j^2 \tilde{t}_{jj}^2 + \mathcal{O}(n^{-1/2}), \\ &= \sum_{j=1}^n \left( \frac{\gamma(G_j - G_{j-1})}{\Delta_j} + \frac{|\vartheta|^2 \underline{\gamma}(\underline{G}_j - \underline{G}_{j-1})}{\underline{\Delta}_j} \right) \\ &\quad + \sum_{j=1}^n \left( \frac{2(F_j - F_{j-1})(1 - F_j)}{\Delta_j} + \frac{2 \operatorname{Re}(\vartheta(\underline{E}_j - \underline{E}_{j-1})(1 - \bar{\vartheta} \bar{\underline{E}}_j))}{\underline{\Delta}_j} \right) \\ &\quad + \frac{\rho^2}{n} \sum_{j=1}^n \tilde{d}_j^2 \tilde{t}_{jj}^2 \left( \frac{\gamma}{\Delta_j} + |\vartheta|^2 \frac{\underline{\gamma}}{\underline{\Delta}_j} + \frac{\kappa}{n} \sum_{i=1}^N d_i^2 t_{ii}^2 \right) + \mathcal{O}(n^{-1/2}), \\ &\stackrel{\Delta}{=} Z_{1,n} + Z_{2,n} + Z_{3,n} + \mathcal{O}(n^{-1/2}). \end{aligned}$$

We prove in the sequel that  $Z_{1,n} \geq 0$ ,  $Z_{2,n} \geq 0$ , and that  $\liminf_n Z_{3,n} > 0$ . It has already been noticed that  $\underline{\Delta}_j \geq \Delta_j$ ; moreover, it can be proven by direct computation that  $|\underline{G}_j - \underline{G}_{j-1}| \leq G_j - G_{j-1}$ , hence  $Z_{1,n} \geq 0$ . As

$$(1 - F_j) - \frac{\gamma(M_j + G_j)}{1 - F_j} \leq |1 - \vartheta \underline{E}_j| - \frac{|\vartheta|^2 \underline{\gamma}(M_j + \underline{G}_j)}{|1 - \vartheta \underline{E}_j|},$$

this implies that  $\underline{\Delta}_j^{-1} |1 - \vartheta \underline{E}_j| \leq \Delta_j^{-1} (1 - F_j)$ . Noticing in addition that  $|\underline{E}_j - \underline{E}_{j-1}| \leq F_j - F_{j-1}$ , we get  $Z_{2,n} \geq 0$ . The cumulant  $\kappa = \mathbb{E}|X_{11}|^4 - 2 - |\vartheta|^2$  satisfies  $\kappa \geq -1 - |\vartheta|^2$ ,

hence

$$\begin{aligned}
Z_{3,n} &\geq \frac{\rho^2}{n^2} \sum_{j=1}^n \tilde{d}_j^2 \tilde{t}_{jj}^2 \left( \left( \frac{1}{\underline{\Delta}_j} - 1 \right) + |\vartheta|^2 \left( \frac{1}{\underline{\Delta}_j} - 1 \right) \right) \sum_{i=1}^N d_i^2 t_{ii}^2 \\
&\quad + \frac{\rho^2}{n} \sum_{j=1}^n \tilde{d}_j^2 \tilde{t}_{jj}^2 \left( \frac{1}{n \underline{\Delta}_j} \sum_{\substack{k,\ell=1 \\ k \neq \ell}}^N d_k^2 |t_{k\ell}|^2 + \frac{1}{n \underline{\Delta}_j} \sum_{\substack{k,\ell=1 \\ k \neq \ell}}^N d_k^2 (t_{k\ell})^2 \right), \\
&\geq \frac{\rho^2}{n^2} \sum_{j=1}^n \tilde{d}_j^2 \tilde{t}_{jj}^2 \left( \left( \frac{1}{\underline{\Delta}_j} - 1 \right) + |\vartheta|^2 \left( \frac{1}{\underline{\Delta}_j} - 1 \right) \right) \sum_{i=1}^N d_i^2 t_{ii}^2 = \frac{\rho^2}{n^2} \sum_{j=1}^n \tilde{d}_j^2 \tilde{t}_{jj}^2 p_j \sum_{i=1}^N d_i^2 t_{ii}^2.
\end{aligned}$$

As the term  $p_j$  is linear in  $|\vartheta|^2 \in [0, 1]$ ,  $p_j \geq \min(\underline{\Delta}_j^{-1}(1 - \underline{\Delta}_j), \underline{\Delta}_j^{-1} + \underline{\Delta}_j^{-1} - 2)$ . We have

$$\begin{aligned}
\underline{\Delta}_j \underline{\Delta}_j &\leq ((1 - F_j)^2 - \gamma(M_j + G_j)) ((1 + F_j)^2 + \gamma(M_j + G_j)), \\
&= (1 - F_j^2)^2 - \gamma^2(M_j + G_j)^2 - 4\gamma F_j(M_j + G_j) \leq 1.
\end{aligned}$$

Hence  $\underline{\Delta}_j^{-1} + \underline{\Delta}_j^{-1} - 2 \geq \underline{\Delta}_j^{-1} + \underline{\Delta}_j - 2 = \underline{\Delta}_j^{-1}(1 - \underline{\Delta}_j)^2$ . As  $1 - \underline{\Delta}_j \geq \gamma M_j$ , we get  $p_j \geq \gamma^2 M_j^2$ , which implies that

$$\begin{aligned}
Z_{3,n} &\geq \frac{\rho^2 \gamma^2}{n^2} \sum_{j=1}^n \tilde{d}_j^2 \tilde{t}_{jj}^2 M_j^2 \sum_{i=1}^N d_i^2 t_{ii}^2 = \gamma^2 \left( \sum_{j=1}^n M_j^2 (M_j - M_{j-1}) \right) \frac{1}{n} \sum_{i=1}^N d_i^2 t_{ii}^2 \\
&= \frac{\gamma^2}{3} \left( \sum_{j=1}^n M_j^3 - M_{j-1}^3 \right) \frac{1}{n} \sum_{i=1}^N d_i^2 t_{ii}^2 + \mathcal{O}(n^{-1}) = \frac{\gamma^2 M_n^3}{3} \frac{1}{n} \sum_{i=1}^N d_i^2 t_{ii}^2 + \mathcal{O}(n^{-1}),
\end{aligned}$$

whose liminf is positive by Lemma 3.4.

The estimates over the variance are therefore established. This completes the proof of Theorem 2.2.

## 7. PROPOSITION 2.3 (BIAS): MAIN STEPS OF THE PROOF

Proof of Proposition 2.3-(i) can be found in [13, Theorem 2]. Let us prove (ii). The same arguments as in the companion article [21] allow to write the bias term as:

$$N(\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho)) = \int_{\rho}^{\infty} \text{Tr}(T(-\omega) - \mathbb{E}Q(-\omega)) \mathbf{d}\omega,$$

Recall that in the centered case where  $A = 0$ ,  $T(-\omega)$  and  $\tilde{T}(-\omega)$  take the simple form  $T(-\omega) = [\omega(I_N + \tilde{\delta}(-\omega)D)]^{-1}$  and  $\tilde{T}(-\omega) = [\omega(I_N + \delta(-\omega)\tilde{D})]^{-1}$ , which implies that  $\underline{\gamma} = \gamma$  and  $\tilde{\underline{\gamma}} = \tilde{\gamma}$ . We introduce the following intermediate quantities:

$$\begin{aligned}
\alpha(-\omega) &= \frac{1}{n} \text{Tr} D \mathbb{E}Q(-\omega), & \tilde{\alpha}(-\omega) &= \frac{1}{n} \text{Tr} \tilde{D} \mathbb{E}\tilde{Q}(-\omega), \\
C(-\omega) &= \left( \omega(I_N + \tilde{\alpha}(-\omega)D) \right)^{-1}, & \tilde{C}(-\omega) &= \left( \omega(I_N + \alpha(-\omega)\tilde{D}) \right)^{-1}.
\end{aligned}$$

From Theorem 3.3,  $n^{-1} \text{Tr} U(C - T) \rightarrow 0$  and  $n^{-1} \text{Tr} \tilde{U}(\tilde{C} - \tilde{T}) \rightarrow 0$  for any sequences of deterministic matrices  $U$  and  $\tilde{U}$  with bounded spectral norms.

The proof consists of two steps:

7.1. **Step 1.** Given  $\omega > 0$ , let

$$\beta_n(\omega) = \left( \kappa + \frac{|\vartheta|^2}{1 - \omega^2 |\vartheta|^2 \gamma \tilde{\gamma}} \right) R_n(\omega)$$

where

$$R_n(\omega) = \frac{\omega^3 \operatorname{Tr} DT^2}{1 - \omega^2 \gamma \tilde{\gamma}} \left( \frac{\gamma}{n} \operatorname{Tr}(\tilde{D}\tilde{T})^3 - \frac{\omega \tilde{\gamma}^2}{n} \operatorname{Tr}(DT)^3 \right) - \frac{\omega^2 \tilde{\gamma}}{n} \operatorname{Tr} D^2 T^3.$$

The purpose of this step is to show that  $\int_{\rho}^{\infty} |\beta_n(\omega)| \mathbf{d}(\omega) < \infty$  and that

$$N(\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho)) - \int_{\rho}^{\infty} \beta_n(\omega) \mathbf{d}(\omega) \xrightarrow[n \rightarrow \infty]{} 0. \quad (7.1)$$

By inspecting the expression of  $\beta_n(\omega)$ , by using Lemmas 3.4 and 3.5 and by recalling that  $1 - \omega^2 \gamma \tilde{\gamma} = \Delta_n$  taken at  $z = -\omega$ , we obtain after a small derivation that  $|\beta_n(\omega)| \leq K/\omega^3$  on  $[\rho, \infty)$  where  $K$  does not depend on  $n$  nor on  $\omega$ . This proves the integrability of  $|\beta_n(\omega)|$ . By taking up the poof of [21, Inequality (7.10)] with minor modifications, we also show that

$$|\operatorname{Tr}(T(-\omega) - \mathbb{E}Q(-\omega))| \leq \frac{K}{\omega^2}$$

hence, showing

$$\operatorname{Tr}(T(-\omega) - \mathbb{E}Q(-\omega)) - \beta(\omega) \xrightarrow[n \rightarrow \infty]{} 0 \quad (7.2)$$

and applying the Dominated Convergence Theorem leads to (7.1). In order to show (7.2), we start by writing for  $z = -\omega$

$$\operatorname{Tr}(T - \mathbb{E}Q) = \operatorname{Tr}(T - C) + \operatorname{Tr}(C - \mathbb{E}Q).$$

Using the decomposition  $\operatorname{Tr}(T - C) = \operatorname{Tr} C(C^{-1} - T^{-1})T = \omega(\tilde{\alpha} - \tilde{\delta}) \operatorname{Tr} DCT$ , we obtain

$$\operatorname{Tr}(T - \mathbb{E}Q) = \omega n(\tilde{\alpha} - \tilde{\delta}) \frac{1}{n} \operatorname{Tr} DCT + \operatorname{Tr}(C - \mathbb{E}Q). \quad (7.3)$$

On the other hand, writing  $n(\alpha - \delta) = \operatorname{Tr} D(\mathbb{E}Q - C) + \operatorname{Tr} D(C - T) = \operatorname{Tr} D(\mathbb{E}Q - C) - \omega(\tilde{\alpha} - \tilde{\delta}) \operatorname{Tr} D^2 CT$  and similarly for  $n(\tilde{\alpha} - \tilde{\delta})$ , we obtain the system

$$\begin{bmatrix} 1 & \omega n^{-1} \operatorname{Tr} D^2 CT \\ \omega n^{-1} \operatorname{Tr} \tilde{D}^2 \tilde{C} \tilde{T} & 1 \end{bmatrix} \begin{bmatrix} n(\alpha - \delta) \\ n(\tilde{\alpha} - \tilde{\delta}) \end{bmatrix} = \begin{bmatrix} \operatorname{Tr} D(\mathbb{E}Q - C) \\ \operatorname{Tr} \tilde{D}(\mathbb{E}\tilde{Q} - \tilde{C}) \end{bmatrix}. \quad (7.4)$$

Consequently, in order to show (7.2), we need to look for approximations of  $\operatorname{Tr} U(\mathbb{E}Q - C)$  and  $\operatorname{Tr} \tilde{U}(\mathbb{E}\tilde{Q} - \tilde{C})$  for deterministic matrices  $U$  and  $\tilde{U}$  with bounded spectral norms:

**Lemma 7.1.** *Assume that the setting of Proposition 2.3 holds true. Fix  $z = -\omega < 0$  and let  $(U_n)_n$  (resp.  $(\tilde{U}_n)_n$ ) be a sequence of  $N \times N$  (resp.  $n \times n$ ) diagonal deterministic matrices such that  $\sup_n \max(\|U_n\|, \|\tilde{U}_n\|) < \infty$ . Then,*

$$\operatorname{Tr} U_n(C - \mathbb{E}Q) + \kappa \frac{\omega^2 \tilde{\gamma}}{n} \operatorname{Tr} U D^2 T^3 + |\vartheta|^2 \frac{\omega^2 \tilde{\gamma}}{1 - \omega^2 |\vartheta|^2 \gamma \tilde{\gamma}} \frac{1}{n} \operatorname{Tr} U D^2 T^3 \xrightarrow[n \rightarrow \infty]{} 0, \quad (7.5)$$

$$\operatorname{Tr} \tilde{U}_n(\tilde{C} - \mathbb{E}\tilde{Q}) + \kappa \frac{\omega^2 \gamma}{n} \operatorname{Tr} \tilde{U} \tilde{D}^2 \tilde{T}^3 + |\vartheta|^2 \frac{\omega^2 \gamma}{1 - \omega^2 |\vartheta|^2 \gamma \tilde{\gamma}} \frac{1}{n} \operatorname{Tr} \tilde{U} \tilde{D}^2 \tilde{T}^3 \xrightarrow[n \rightarrow \infty]{} 0. \quad (7.6)$$

Recalling that  $n^{-1} \text{Tr} U(C - T) \rightarrow 0$  and taking  $U = D^2T$ , we have  $n^{-1} \text{Tr} D^2CT - \gamma \rightarrow 0$ . Similarly,  $n^{-1} \text{Tr} \tilde{D}^2\tilde{C}\tilde{T} - \tilde{\gamma} \rightarrow 0$ . Solving system (7.4) and using this lemma with  $U = D$  and  $\tilde{U} = \tilde{D}$ , we obtain

$$\begin{aligned} n(\tilde{\alpha} - \tilde{\delta}) &= \frac{1}{1 - \omega^2\gamma\tilde{\gamma}} \left( -\omega\tilde{\gamma} \text{Tr} D(\mathbb{E}Q - C) + \text{Tr} \tilde{D}(\mathbb{E}\tilde{Q} - \tilde{C}) \right) + \varepsilon \\ &= \frac{\kappa}{1 - \omega^2\gamma\tilde{\gamma}} \left( \frac{\omega^2\gamma}{n} \text{Tr}(\tilde{D}\tilde{T})^3 - \frac{\omega^3\tilde{\gamma}^2}{n} \text{Tr}(DT)^3 \right) \\ &\quad + \frac{|\vartheta|^2}{(1 - \omega^2\gamma\tilde{\gamma})(1 - \omega^2|\vartheta|^2\gamma\tilde{\gamma})} \left( \frac{\omega^2\gamma}{n} \text{Tr}(\tilde{D}\tilde{T})^3 - \frac{\omega^3\tilde{\gamma}^2}{n} \text{Tr}(DT)^3 \right) + \varepsilon' \end{aligned}$$

where  $\varepsilon, \varepsilon' \rightarrow 0$ . Using Lemma 7.1 again in (7.3) with  $U = I$ , we obtain (7.2).

The remainder of this paragraph is devoted to the proof of Lemma 7.1.

Recall the following notations:

$$\tilde{b}_j(-\omega) = \frac{1}{\omega \left( 1 + \frac{\tilde{d}_j}{n} \text{Tr} DQ_j(-\omega) \right)} \quad \text{and} \quad e_j(-\omega) = \eta_j^* Q_j(-\omega) \eta_j - \frac{\tilde{d}_j}{n} \text{Tr} DQ_j(-\omega).$$

Starting with

$$\text{Tr} U(\mathbb{E}Q - C) = \text{Tr} \mathbb{E}[UC(C^{-1} - Q^{-1})Q] = -\text{Tr} \mathbb{E}[UC\Sigma\Sigma^*Q] + \omega\tilde{\alpha} \text{Tr} \mathbb{E}[UCDQ]$$

and using (3.8) and (3.7), we obtain  $\text{Tr} U(\mathbb{E}Q - C) = Z_1 + Z_2 + Z_3$  where

$$\begin{aligned} Z_1 &= \sum_{j=1}^n \mathbb{E} \left[ \omega^2 \tilde{b}_j^2 e_j \eta_j^* Q_j UC \eta_j \right], \\ Z_2 &= - \sum_{j=1}^n \mathbb{E} \left[ \omega^3 \tilde{q}_{jj} \tilde{b}_j^2 e_j^2 \eta_j^* Q_j UC \eta_j \right], \\ Z_3 &= \omega \tilde{\alpha} \mathbb{E} \text{Tr} UCDQ - \frac{\omega}{n} \sum_{j=1}^n \mathbb{E} \tilde{b}_j \tilde{d}_j \text{Tr} DQ_j UC. \end{aligned}$$

In the remainder, we omit the study of the negligible terms to focus on the deterministic equivalent formulas; in this spirit, we shall denote by  $\varepsilon$  a negligible term whose value might change from line to line.

The term  $Z_1$  is

$$Z_1 = \sum_{j=1}^n \mathbb{E} \left[ \omega^2 \tilde{b}_j^2 e_j \left( \eta_j^* Q_j UC \eta_j - \tilde{d}_j \frac{\text{Tr} DQ_j UC}{n} \right) \right].$$

Using Identity (3.20) with  $M = Q_j$ ,  $P = Q_j UC$  and  $\mathbf{u} = 0$ , we obtain:

$$\begin{aligned} Z_1 &= \frac{1}{n} \sum_{j=1}^n \mathbb{E} \left[ \omega^2 \tilde{b}_j^2 \left( \frac{\tilde{d}_j^2}{n} \text{Tr} Q_j DQ_j UC D + \frac{|\vartheta|^2}{n} \tilde{d}_j^2 \text{Tr} Q_j DCU \bar{Q}_j D \right. \right. \\ &\quad \left. \left. + \frac{\kappa}{n} \sum_{i=1}^N \tilde{d}_j^2 d_i^2 [Q_j]_{ii} [Q_j UC]_{ii} \right) \right] + \varepsilon. \end{aligned}$$

It is not difficult to check that

$$Z_2 = - \sum_{j=1}^n \mathbb{E} \left[ \omega^3 \tilde{b}_j^3 \tilde{d}_j \frac{\text{Tr} DQ_j UC}{n} e_j^2 \right] + \varepsilon.$$

Turning to  $Z_3$ , we have

$$\begin{aligned} Z_3 &= \frac{\omega}{n} \sum_{j=1}^n \mathbb{E} \left[ \tilde{d}_j \mathbb{E} \tilde{q}_{jj} \text{Tr} UC DQ - \tilde{d}_j \tilde{b}_j \text{Tr} UC DQ_j \right] \\ &= \frac{\omega}{n} \sum_{j=1}^n \tilde{d}_j \mathbb{E} \left[ \mathbb{E} \tilde{q}_{jj} (\text{Tr} UC DQ_j - \omega \tilde{q}_{jj} \text{Tr} UC DQ_j \eta_j \eta_j^* Q_j) - \tilde{b}_j \text{Tr} UC DQ_j \right] \\ &= \frac{\omega}{n} \sum_{j=1}^n \tilde{d}_j \mathbb{E} \left[ (\mathbb{E} \tilde{q}_{jj} - \tilde{b}_j) \text{Tr} UC DQ_j \right] - \frac{\omega^2}{n} \sum_{j=1}^n \tilde{d}_j \mathbb{E} \tilde{q}_{jj} \mathbb{E} \left[ \tilde{b}_j \eta_j^* Q_j UC DQ_j \eta_j \right] + \varepsilon. \end{aligned}$$

Replacing  $\tilde{b}_j$  by  $\tilde{q}_{jj} + \omega \tilde{b}_j^2 e_j - \omega^2 \tilde{q}_{jj} \tilde{b}_j^2 e_j^2$ , we have

$$\mathbb{E} \left[ (\mathbb{E} \tilde{q}_{jj} - \tilde{b}_j) \text{Tr} UC DQ_j \right] = \mathbb{E} [(\mathbb{E} \tilde{q}_{jj} - \tilde{q}_{jj}) \text{Tr} UC DQ_j] + \mathbb{E} \left[ \omega^2 \tilde{q}_{jj} \tilde{b}_j^2 e_j^2 \text{Tr} UC DQ_j \right].$$

Since

$$\tilde{q}_{jj} - \mathbb{E} \tilde{q}_{jj} = \frac{\mathbb{E}(\eta_j^* Q_j \eta_j) - \eta_j^* Q_j \eta_j}{\omega(1 + \eta_j^* Q_j \eta_j)(1 + \mathbb{E}(\eta_j^* Q_j \eta_j))} + \mathbb{E} \left[ \frac{\eta_j^* Q_j \eta_j - \mathbb{E}(\eta_j^* Q_j \eta_j)}{\omega(1 + \eta_j^* Q_j \eta_j)(1 + \mathbb{E}(\eta_j^* Q_j \eta_j))} \right]$$

we have  $\mathbb{E}(\tilde{q}_{jj} - \mathbb{E} \tilde{q}_{jj})^2 = \mathcal{O}(1/n)$ . Hence

$$\mathbb{E} [(\mathbb{E} \tilde{q}_{jj} - \tilde{q}_{jj}) \text{Tr} UC DQ_j] = \mathbb{E} [(\mathbb{E} \tilde{q}_{jj} - \tilde{q}_{jj})(\text{Tr} UC DQ_j - \mathbb{E} \text{Tr} UC DQ_j)] = \mathcal{O}(n^{-1/2})$$

by Theorem 3.3-(5). It follows that  $\mathbb{E} \left[ (\mathbb{E} \tilde{q}_{jj} - \tilde{b}_j) \text{Tr} UC DQ_j \right] = \mathbb{E} \left[ \omega^2 \tilde{b}_j^3 e_j^2 \text{Tr} UC DQ_j \right] + \varepsilon$ , hence

$$Z_2 + Z_3 = - \frac{1}{n} \sum_{j=1}^n \mathbb{E} \left[ \omega^2 \tilde{b}_j^3 \tilde{d}_j^2 \frac{1}{n} \text{Tr} Q_j DQ_j UC D \right] + \varepsilon.$$

Taking the sum  $Z_1 + Z_2 + Z_3$ , the terms that do not depend on  $\vartheta$  nor on  $\kappa$  cancel out, and we are left with

$$\text{Tr} U(\mathbb{E}Q - C) = |\vartheta|^2 \omega^2 \tilde{\gamma} \frac{1}{n} \mathbb{E} [\text{Tr} QDCU\bar{Q}D] + \kappa \omega^2 \tilde{\gamma} \frac{1}{n} \text{Tr} UD^2 T^3 + \varepsilon.$$

where we relied on the usual approximations for the diagonal entries of the resolvent (see Lemma 4.1) to obtain the term in  $\kappa$ . We now briefly characterize the asymptotic behavior of  $n^{-1} \text{Tr} \mathbb{E} QDCU\bar{Q}D$ . Starting with  $Q = T + \omega \tilde{\delta} TDQ - T\Sigma\Sigma^*Q$ , we have

$$\frac{1}{n} \text{Tr} \mathbb{E} QDCU\bar{Q}D = \frac{1}{n} \text{Tr} \mathbb{E} TDCU\bar{Q}D + \frac{\omega \tilde{\delta}}{n} \text{Tr} \mathbb{E} TDQDCU\bar{Q}D - \frac{1}{n} \text{Tr} \mathbb{E} T\Sigma\Sigma^*QDCU\bar{Q}D,$$

and

$$\begin{aligned}
-\frac{1}{n} \operatorname{Tr} \mathbb{E} T \Sigma \Sigma^* Q D C U \bar{Q} D &= -\frac{1}{n} \sum_{j=1}^n \mathbb{E} \omega \tilde{q}_{jj} \eta_j^* Q_j D C U \bar{Q} D T \eta_j \\
&= -\frac{1}{n} \sum_{j=1}^n \mathbb{E} \omega \tilde{q}_{jj} \eta_j^* Q_j D C U \bar{Q}_j D T \eta_j \\
&\quad + \frac{1}{n} \sum_{j=1}^n \mathbb{E} \omega^2 \tilde{q}_{jj}^2 \eta_j^* Q_j D C U \bar{Q}_j \bar{\eta}_j \eta_j^T D T \eta_j \\
&= -\frac{\omega \tilde{\delta}}{n} \operatorname{Tr} \mathbb{E} T D Q D C U \bar{Q} D + |\vartheta|^2 \omega^2 \gamma \tilde{\gamma} \frac{\operatorname{Tr} \mathbb{E} Q D C U \bar{Q} D}{n} + \varepsilon.
\end{aligned}$$

We therefore get

$$\frac{1}{n} \operatorname{Tr} \mathbb{E} Q D C U \bar{Q} D = \frac{\frac{1}{n} \operatorname{Tr} U D^2 T^3}{1 - \omega^2 |\vartheta|^2 \gamma \tilde{\gamma}} + \varepsilon$$

and Convergence (7.5) of lemma 7.1 is shown. Convergence (7.6) is proven similarly. Lemma 7.1 is proven, and Step 1 of the proof of Proposition 2.3-(ii) is established.

**7.2. Step 2.** The purpose of this step is to show that

$$R(\omega) = \frac{1}{2} \frac{\mathbf{d}}{\mathbf{d}\omega} (\omega^2 \gamma(-\omega) \tilde{\gamma}(-\omega)).$$

Plugging into the expression of  $\beta_n(\omega)$ , it is straightforward to show that  $\int_{\rho}^{\infty} \beta_n(\omega) \mathbf{d}(\omega)$  coincides with  $\mathcal{B}_n$  given by (2.5).

Recall that

$$R(\omega) = \frac{\omega^3 \gamma}{1 - \omega^2 \gamma \tilde{\gamma}} \frac{\operatorname{Tr} D T^2}{n} \frac{\operatorname{Tr}(\tilde{D} \tilde{T})^3}{n} - \frac{\omega^4 \tilde{\gamma}^2}{1 - \omega^2 \gamma \tilde{\gamma}} \frac{\operatorname{Tr} D T^2}{n} \frac{\operatorname{Tr}(D T)^3}{n} - \omega^2 \tilde{\gamma} \frac{\operatorname{Tr} D^2 T^3}{n} = R_1 + R_2 + R_3$$

Our method is similar to [19, §V.B]. We start by showing that the derivatives of  $\tilde{\gamma}(-\omega)$  and  $\gamma(-\omega)$  that we denote respectively as  $\tilde{\gamma}'$  and  $\gamma'$  are

$$\begin{aligned}
\tilde{\gamma}' &= \frac{\mathbf{d}\tilde{\gamma}(-\omega)}{\mathbf{d}\omega} = -\frac{2}{\omega} \tilde{\gamma} + \frac{2\omega}{1 - \omega^2 \gamma \tilde{\gamma}} \frac{\operatorname{Tr}(\tilde{D} \tilde{T})^3}{n} \frac{\operatorname{Tr} D T^2}{n} \\
\gamma' &= \frac{\mathbf{d}\gamma(-\omega)}{\mathbf{d}\omega} = -\frac{2}{\omega} \gamma + \frac{2\omega}{1 - \omega^2 \gamma \tilde{\gamma}} \frac{\operatorname{Tr}(D T)^3}{n} \frac{\operatorname{Tr} \tilde{D} \tilde{T}^2}{n}.
\end{aligned} \tag{7.7}$$

We have

$$\tilde{\gamma}' = \frac{1}{n} \sum_{j=1}^n \tilde{d}_j^2 \frac{\mathbf{d}}{\mathbf{d}\omega} \left( \frac{1}{\omega^2 (1 + \tilde{d}_j \delta(-\omega))^2} \right) = -\frac{2}{\omega} \tilde{\gamma} - 2\omega \delta' \frac{\operatorname{Tr}(\tilde{D} \tilde{T})^3}{n} \tag{7.8}$$

where we put  $\delta' = \mathbf{d}\delta(-\omega)/\mathbf{d}\omega$ . This derivative can be expressed as

$$\delta' = \frac{1}{n} \sum_{i=1}^N d_i \frac{\mathbf{d}}{\mathbf{d}\omega} \left( \frac{1}{\omega(1 + d_i \delta(-\omega))} \right) = -\frac{\delta}{\omega} - \omega \gamma \tilde{\delta}'$$

where  $\tilde{\delta}' = \mathbf{d}\tilde{\delta}(-\omega)/\mathbf{d}\omega$ . Similarly,  $\tilde{\delta}' = -\omega^{-1} \tilde{\delta} - \omega \tilde{\gamma} \delta'$ . Combining the two equations, we obtain

$$\delta' = \frac{\gamma \tilde{\delta} - \omega^{-1} \delta}{1 - \omega^2 \gamma \tilde{\gamma}}. \tag{7.9}$$



Since  $T = [\omega(I + \tilde{\delta}D)]^{-1}$  and  $\tilde{T} = [\omega(I + \delta\tilde{D})]^{-1}$ , we have

$$T = \omega^{-1}I_N - \tilde{\delta}DT, \quad \tilde{T} = \omega^{-1}I_n - \delta\tilde{D}\tilde{T}. \quad (7.10)$$

This leads to

$$\frac{1}{n} \operatorname{Tr} DT^2 = \frac{1}{n} \operatorname{Tr} DT((\omega^{-1}I - \tilde{\delta}DT) = \omega^{-1}\delta - \gamma\tilde{\delta}, \text{ and } \frac{1}{n} \operatorname{Tr} \tilde{D}\tilde{T}^2 = \omega^{-1}\tilde{\delta} - \tilde{\gamma}\delta. \quad (7.11)$$

Combining with (7.9) and (7.8), we obtain the first equation of (7.7), the second being obtained similarly. Using the first equation, the term  $R_1$  can be expressed as

$$R_1 = \frac{1}{2} (\omega^2\gamma\tilde{\gamma}' + 2\omega\gamma\tilde{\gamma}).$$

Turning to  $R_2$ , we have

$$\begin{aligned} R_2 &= -\frac{\omega^4\tilde{\gamma}^2}{1 - \omega^2\gamma\tilde{\gamma}} \frac{\operatorname{Tr}(DT)^3}{n} \frac{\operatorname{Tr} DT(\omega^{-1}I - \tilde{\delta}DT)}{n} \\ &= -\frac{\omega^3\delta\tilde{\gamma}^2}{1 - \omega^2\gamma\tilde{\gamma}} \frac{\operatorname{Tr}(DT)^3}{n} + \frac{\omega^4\tilde{\delta}\tilde{\gamma}^2}{1 - \omega^2\gamma\tilde{\gamma}} \frac{\operatorname{Tr}(DT)^3}{n} \\ &\stackrel{(a)}{=} \frac{\omega^3\tilde{\gamma}}{1 - \omega^2\gamma\tilde{\gamma}} \frac{\operatorname{Tr}(DT)^3}{n} \frac{\operatorname{Tr} \tilde{D}\tilde{T}^2}{n} - \omega^2\tilde{\gamma}\tilde{\delta} \frac{\operatorname{Tr}(DT)^3}{n} \\ &\stackrel{(b)}{=} \frac{1}{2} (\omega^2\gamma'\tilde{\gamma} + 2\omega\gamma\tilde{\gamma}) - \omega^2\tilde{\gamma}\tilde{\delta} \frac{\operatorname{Tr}(DT)^3}{n} \end{aligned}$$

where (a) is due to  $\tilde{\gamma}\delta = \omega^{-1}\tilde{\delta} - n^{-1} \operatorname{Tr} \tilde{D}\tilde{T}^2$ , see (7.11), and (b) is due to (7.7). Considering  $R_3$ , we have by (7.10),

$$-\omega^2\tilde{\gamma}\tilde{\delta} \frac{\operatorname{Tr}(DT)^3}{n} - \omega^2\tilde{\gamma} \frac{\operatorname{Tr} D^2T^3}{n} = -\omega^2\tilde{\gamma} \frac{\operatorname{Tr} D^2T^2(T + \tilde{\delta}DT)}{n} = -\omega\gamma\tilde{\gamma}.$$

We therefore have  $R(\omega) = 0.5 (\omega^2\gamma'\tilde{\gamma} + \omega^2\gamma\tilde{\gamma}' + 2\omega\gamma\tilde{\gamma}) = 0.5(\omega^2\gamma\tilde{\gamma})'$ . Proposition 2.3 is proven.

## APPENDIX A. PROOFS FOR SECTION 3

**A.1. Proofs of Eq. (3.10) and Eq. (3.11).** Proof of Eq. (3.10) mainly relies on matrix identity (3.2) and on the following identity for the inverse of a partitioned matrix (see for instance [23, Section 0.7.3]):

$$\text{If } A = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \text{then } (A^{-1})_{11} = (a_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}. \quad (\text{A.1})$$

To lighten the computations, let us introduce the following notations:

$$\mathcal{I} = \left( I_n + \tilde{\delta}D \right)^{-1}, \quad \tilde{\mathcal{I}} = \left( I_{N-1} + \delta\tilde{D}_1 \right)^{-1}.$$

In order to express a diagonal element of  $\tilde{T}$ , say  $\tilde{t}_{11}$  (without loss of generality), let us first write:

$$\tilde{T} = \begin{bmatrix} -z(1 + \delta\tilde{d}_1) + a_1^*\mathcal{I}a_1 & a_1^*\mathcal{I}A_1 \\ A_1^*\mathcal{I}a_1 & -z\tilde{\mathcal{I}}^{-1} + A_1^*\mathcal{I}A_1 \end{bmatrix}^{-1}.$$

Hence, according to (A.1):

$$\begin{aligned}
\frac{1}{\tilde{t}_{11}} &= -z(1 + \delta\tilde{d}_1) + a_1^* \mathcal{I} a_1 - a_1^* \mathcal{I} A_1 \left[ -z\tilde{\mathcal{I}}^{-1} + A_1^* \mathcal{I} A_1 \right]^{-1} A_1^* \mathcal{I} a_1 \\
&\stackrel{(a)}{=} -z(1 + \delta\tilde{d}_1) + a_1^* \mathcal{I} a_1 - a_1^* \mathcal{I} A_1 \left[ -\frac{1}{z}\tilde{\mathcal{I}} + \frac{1}{z}\tilde{\mathcal{I}} A_1^* \mathcal{T}_1 A_1 \tilde{\mathcal{I}} \right] A_1^* \mathcal{I} a_1 \\
&\stackrel{(b)}{=} -z(1 + \delta\tilde{d}_1) + a_1^* \mathcal{I} a_1 + \frac{1}{z} a_1^* \mathcal{I} (\mathcal{T}_1^{-1} + z\mathcal{I}^{-1}) \mathcal{I} a_1 \\
&\quad - \frac{1}{z} a_1^* \mathcal{I} A_1 \tilde{\mathcal{I}} A_1^* [I_N + z\mathcal{T}_1 \mathcal{I}^{-1}] \mathcal{I} a_1 \\
&\stackrel{(b)}{=} -z(1 + \delta\tilde{d}_1) + 2a_1^* \mathcal{I} a_1 + \frac{1}{z} a_1^* \mathcal{I} \mathcal{T}_1^{-1} \mathcal{I} a_1 \\
&\quad - \frac{1}{z} a_1^* \mathcal{I} [\mathcal{T}_1^{-1} + z\mathcal{I}^{-1}] [I_N + z\mathcal{T}_1 \mathcal{I}^{-1}] \mathcal{I} a_1 \\
&= -z(1 + \delta\tilde{d}_1) - z a_1^* \mathcal{T}_1 a_1 ,
\end{aligned}$$

where (a) follows from (3.2), (b) from equalities

$$\mathcal{T}_1 A_1 \tilde{\mathcal{I}} A_1^* = I_N + z\mathcal{T}_1 \mathcal{I}^{-1} \quad \text{and} \quad A_1 \tilde{\mathcal{I}} A_1^* = \mathcal{T}_1^{-1} + z\mathcal{I}^{-1}$$

which follow from the mere definition of  $\mathcal{T}_1$ . Finally, (3.10) is established.

Let us now turn to the proof of (3.11). Notice first that  $T$  can be expressed as

$$T = \left[ -z(I_N + \tilde{\delta}D) + A_1(I_{n-1} + \delta\tilde{D}_1)^{-1} A_1^* + \frac{a_1 a_1^*}{1 + \delta\tilde{d}_1} \right]^{-1} .$$

Applying (3.2) readily yields:

$$T = \mathcal{T}_1 - \mathcal{T}_1 a_1 \left( 1 + \delta\tilde{d}_1 + a_1^* \mathcal{T}_1 a_1 \right) a_1^* \mathcal{T}_1 .$$

It remains to multiply by  $a_1^*$  (left),  $b$  (right) and to use (3.10) to establish (3.11).

**A.2. Proof of Inequality (3.15).** We provide here some elements to establish that  $\mathbb{E}|e_j|^p = \mathcal{O}(n^{-p/2})$ . Recall the definition (3.1) of  $e_j$  and write:

$$\mathbb{E}|e_j|^p \leq K \left\{ \mathbb{E} \left| y_j^* Q_j y_j - \frac{\tilde{d}_j}{n} \text{Tr} D Q_j \right|^p + \mathbb{E} |a_j^* Q_j y_j|^p + \mathbb{E} |y_j^* Q_j a_j|^p \right\} .$$

The first term of the r.h.s. can be directly estimated with the help of Lemma 3.2. The two remaining terms are similar and can be estimated in the following way:

$$\begin{aligned}
\mathbb{E} |a_j^* Q_j y_j|^p &= \mathbb{E} (y_j^* Q_j^* a_j a_j^* Q_j y_j)^{p/2} \\
&\leq K \left( \mathbb{E} \left| y_j^* Q_j^* a_j a_j^* Q_j y_j - \frac{d_j^2}{n} \text{Tr} Q_j^* a_j a_j^* Q_j \right|^{p/2} + \mathbb{E} \left| \frac{1}{n} \text{Tr} Q_j^* a_j a_j^* Q_j \right|^{p/2} \right) .
\end{aligned}$$

The first term of the r.h.s. can be handled with the help of Lemma 3.2 (notice that  $Q_j^* a_j a_j^* Q_j$  is of rank one and has a bounded spectral norm), and the second term is directly of the right order.

**A.3. Proof of Theorem 3.3.** Items (1)–(3) of Theorem 3.3 are shown in [22]. Let us show Theorem 3.3-(4). Denote by  $(\delta_j, \tilde{\delta}_j)$  the solution of System (1.2) when  $A$  and  $\tilde{D}$  are replaced with  $A_j$  and  $\tilde{D}_j$  respectively. Let  $T_j$  and  $\tilde{T}_j$  be the matrices associated to  $(\delta_j, \tilde{\delta}_j)$  as in Eq. (1.3). Then  $\mathbb{E}|u^*(Q_j - T_j)v|^{2p} \leq K_p n^{-p}$  by Item (3), and we only need to show that  $|u^*(\mathcal{T}_j - T_j)v| \leq K/\sqrt{n}$ . We have

$$\begin{aligned} |\delta - \delta_j| &= \frac{1}{n} |\text{Tr } D(T - T_j)| \\ &\leq \frac{1}{n} |\text{Tr } \mathbb{E}D(T - Q)| + \frac{1}{n} |\text{Tr } \mathbb{E}D(Q - Q_j)| + \frac{1}{n} |\text{Tr } \mathbb{E}D(Q_j - T_j)| = \mathcal{O}(n^{-1}) \end{aligned}$$

by Item (2) and Lemma 3.1. Moreover,

$$|\tilde{\delta} - \tilde{\delta}_j| \leq \frac{1}{n} \left| \text{Tr } \mathbb{E}\tilde{D}(\tilde{T} - \tilde{Q}) \right| + \frac{1}{n} \left| \mathbb{E}(\text{Tr } \tilde{D}\tilde{Q} - \text{Tr } \tilde{D}_j\tilde{Q}_j) \right| + \frac{1}{n} \left| \text{Tr } \mathbb{E}\tilde{D}_j(\tilde{Q}_j - \tilde{T}_j) \right|.$$

In order to deal with the middle term at the r.h.s., assume without generality loss that  $j = 1$ . Using the identity in [23, Section 0.7.3] for the inverse of a partitioned matrix, we obtain

$$\tilde{Q} = \begin{bmatrix} \tilde{q}_{11} & & \\ \times & \tilde{Q}_1 + \tilde{q}_{11} \tilde{Q}_1 \Sigma_1^* \eta_1 \eta_1^* \Sigma_1 \tilde{Q}_1 & \\ & & \times \end{bmatrix}$$

hence  $\mathbb{E}(\text{Tr } \tilde{D}\tilde{Q} - \text{Tr } \tilde{D}_1\tilde{Q}_1) = \mathcal{O}(1)$ , which shows that  $|\tilde{\delta} - \tilde{\delta}_j| = \mathcal{O}(n^{-1})$ . We now have

$$\begin{aligned} u^*(\mathcal{T}_j - T_j)v &= u^* \mathcal{T}_j (T_j^{-1} - \mathcal{T}_j^{-1}) T_j v \\ &= u^* \mathcal{T}_j \left( \rho(\tilde{\delta}_j - \tilde{\delta})D + (\delta - \delta_j)A_j(I + \delta_j\tilde{D}_j)^{-1}\tilde{D}_j(I + \delta\tilde{D}_j)^{-1}A_j^* \right) T_j v = \mathcal{O}(n^{-1}) \end{aligned}$$

which proves Item (4). In order to prove Item (5), we develop  $\text{Tr } U(Q - \mathbb{E}Q)$  as a sum of martingale differences:

$$\begin{aligned} \text{Tr } U(Q - \mathbb{E}Q) &= \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) \text{Tr } UQ \\ &= \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) \text{Tr } U(Q - Q_j) = - \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) (\rho \tilde{q}_{jj} \eta_j^* Q_j U Q_j \eta_j) \end{aligned}$$

by (3.4), hence  $\mathbb{E} |\text{Tr } U(Q - \mathbb{E}Q)|^2 = \sum_{j=1}^n \mathbb{E} \left| (\mathbb{E}_j - \mathbb{E}_{j-1}) (\rho \tilde{q}_{jj} \eta_j^* Q_j U Q_j \eta_j) \right|^2$ . We now use (3.7). We have

$$\begin{aligned} &\mathbb{E} \left| (\mathbb{E}_j - \mathbb{E}_{j-1}) (\rho \tilde{b}_j \eta_j^* Q_j U Q_j \eta_j) \right|^2 \\ &= \mathbb{E} \left| \mathbb{E}_j \left( \rho \tilde{b}_j \left( \eta_j^* Q_j U Q_j \eta_j - \tilde{d}_j \frac{\text{Tr } D Q_j U Q_j}{n} - a_j^* Q_j U Q_j a_j \right) \right) \right|^2 = \mathcal{O}(n^{-1}) \end{aligned}$$

by Lemma 3.2, and furthermore,

$$\mathbb{E} \left| (\mathbb{E}_j - \mathbb{E}_{j-1}) (\rho^2 \tilde{q}_{jj} \tilde{b}_j e_j \eta_j^* Q_j U Q_j \eta_j) \right|^2 \leq K (\mathbb{E} e_j^4 \mathbb{E} |\eta_j^* Q_j U Q_j \eta_j|^4)^{1/2} = \mathcal{O}(n^{-1})$$

by Lemma 3.2 and (3.15). This shows Th.3.3-(5).

**A.4. Proof of Lemma 3.4.** The two first upper bounds are easy to obtain, given that  $\delta_n$  and  $\tilde{\delta}_n$  are Stieltjes transforms of nonnegative measures with respective total mass  $n^{-1} \operatorname{Tr} D$  and  $n^{-1} \operatorname{Tr} \tilde{D}$ . Now  $\operatorname{Tr} DT^2 \leq \mathbf{d}_{\max} \operatorname{Tr} T^2$  by Inequality (3.14), which in turn is smaller than  $N\mathbf{d}_{\max}\rho^{-2}$ , hence the third upper bound, and the other upper bounds can be proven similarly. Let us now prove the first lower bound.

$$\begin{aligned} \operatorname{Tr} D &= \operatorname{Tr}(T^{\frac{1}{2}}DT^{\frac{1}{2}}T^{-1}) \stackrel{(a)}{\leq} \operatorname{Tr}(DT) \times \|T^{-1}\|, \\ &\leq \operatorname{Tr}(DT) \times \left( \rho(1 + \tilde{\delta}_n \mathbf{d}_{\max}) + \mathbf{a}_{\max}^2 \|(I + \delta_n \tilde{D})^{-1}\| \right), \\ &\stackrel{(b)}{\leq} \operatorname{Tr}(DT) \times \left( \rho + \mathbf{d}_{\max} \tilde{\mathbf{d}}_{\max} + \mathbf{a}_{\max}^2 \right), \end{aligned}$$

where (a) follows from (3.14) and (b) from the upper bound on  $\tilde{\delta}_n$ . This readily yields  $\delta_n$ 's lower bound and  $\tilde{\delta}_n$ 's lower bound which can be proven similarly. Writing  $\operatorname{Tr} D \leq (\operatorname{Tr} DT^2) \|T^{-1}\|^2$ , we obtain the lower bounds on  $n^{-1} \operatorname{Tr} DT^2$  and  $n^{-1} \operatorname{Tr} \tilde{D}\tilde{T}^2$  similarly. The lower bound for  $\gamma_n$  follows from the same ideas:

$$\begin{aligned} \left( \frac{1}{N} \operatorname{Tr} D \right)^2 &\leq \frac{1}{N} \operatorname{Tr} D^2 = \frac{1}{N} \operatorname{Tr}(T^{\frac{1}{2}}D^2T^{\frac{1}{2}}T^{-1}), \\ &\leq \frac{1}{N} \operatorname{Tr}(T^{\frac{1}{2}}D^2T^{\frac{1}{2}}) \times \|T^{-1}\| = \frac{1}{N} \operatorname{Tr}(T^{\frac{1}{2}}DT^{\frac{1}{2}}T^{-1}T^{\frac{1}{2}}DT^{\frac{1}{2}}) \times \|T^{-1}\|, \\ &\leq \frac{1}{N} \operatorname{Tr}(TDTD) \times \|T^{-1}\|^2, \end{aligned}$$

and one readily obtains  $\gamma_n$ 's lower bound (and similarly  $\tilde{\gamma}_n$ 's lower bound) using Assumption **A-3** and the upper estimate previously obtained for  $\|T^{-1}\|$ .

The two last series of inequalities related to  $n^{-1} \sum_{i=1}^N d_i^2 t_{ii}^2$  and  $n^{-1} \sum_{j=1}^n \tilde{d}_j^2 \tilde{t}_{jj}^2$  can be proven with similar arguments (lower bounds are in fact easier to obtain as one can directly get lower bounds for  $t_{ii}$  and  $\tilde{t}_{jj}$  - using (3.10) for instance).

**A.5. Proof of Lemma 3.5.** From (1.3),  $TA(I + \delta\tilde{D})^{-1}A^* = I - \rho T(I + \delta\tilde{D})$ . Moreover,  $(I + \delta\tilde{D})^{-1}\tilde{D} = \delta^{-1}I - \delta^{-1}(I + \delta\tilde{D})^{-1}$ . Hence

$$\begin{aligned} \frac{1}{n} \operatorname{Tr} D^{1/2}TA(I + \delta\tilde{D})^{-2}\tilde{D}A^*TD^{\frac{1}{2}} &\leq \frac{1}{n\delta} \operatorname{Tr} DTA(I + \delta\tilde{D})^{-1}A^*T \\ &= 1 - \frac{\rho}{n\delta} \operatorname{Tr} DT^2 - \rho \frac{\tilde{\delta}}{\delta} \gamma \end{aligned} \quad (\text{A.2})$$

which proves the first assertion with the help of the results of Lemma 3.4. Similarly,

$$\frac{1}{n} \operatorname{Tr} \tilde{D}^{1/2}\tilde{T}A^*(I + \delta\tilde{D})^{-2}D\tilde{A}\tilde{T}\tilde{D}^{\frac{1}{2}} \leq 1 - \frac{\rho}{n\tilde{\delta}} \operatorname{Tr} \tilde{D}\tilde{T}^2 - \rho \frac{\delta}{\tilde{\delta}} \tilde{\gamma} \quad (\text{A.3})$$

We now show that the left hand sides (l.h.s.) of (A.2) and (A.3) are equal. Using the well known matrix identity  $(I + UV)^{-1}U = U(I + VU)^{-1}$ ,

$$\begin{aligned} TA(I + \delta\tilde{D})^{-1} &= \rho^{-1}(I + \delta\tilde{D})^{-1} \left( I + \rho^{-1}A(I + \delta\tilde{D})^{-1}A^*(I + \delta\tilde{D})^{-1} \right)^{-1} A(I + \delta\tilde{D})^{-1} \\ &= \rho^{-1}(I + \delta\tilde{D})^{-1} A(I + \delta\tilde{D})^{-1} \left( I + \rho^{-1}A^*(I + \delta\tilde{D})^{-1}A(I + \delta\tilde{D})^{-1} \right)^{-1} \\ &= (I + \delta\tilde{D})^{-1}A\tilde{T}. \end{aligned}$$

Plugging this identity in the l.h.s. of (A.2), and identifying with the l.h.s. of (A.3), we obtain the result. As a consequence, we have

$$\begin{aligned} \left(1 - \frac{1}{n} \operatorname{Tr} D^{1/2} T A (I + \delta \tilde{D})^{-2} \tilde{D} A^* T D^{1/2}\right)^2 &\geq \left(\frac{\rho}{n\delta} \operatorname{Tr} D T^2 + \rho \frac{\tilde{\delta}}{\delta} \gamma\right) \left(\frac{\rho}{n\tilde{\delta}} \operatorname{Tr} \tilde{D} \tilde{T}^2 + \rho \frac{\delta}{\tilde{\delta}} \tilde{\gamma}\right) \\ &\geq \rho^2 \gamma \tilde{\gamma} + \frac{\rho}{n\delta} \operatorname{Tr} D T^2 \frac{\rho}{n\tilde{\delta}} \operatorname{Tr} \tilde{D} \tilde{T}^2 \end{aligned}$$

which is the second assertion. By Lemma 3.4, this leads to  $\liminf \Delta_n > 0$ . Lemma 3.5 is proven.

## APPENDIX B. ADDITIONAL PROOFS FOR SECTION 5

**B.1. Proof of Lemma 5.1.** Let us show that  $\max_j \operatorname{var}(a^* \mathbb{E}_j Q D \mathbb{E}_j Q a) = \mathcal{O}(n^{-1})$ . We have

$$\begin{aligned} a^* \mathbb{E}_j Q D \mathbb{E}_j Q a - \mathbb{E} a^* \mathbb{E}_j Q D \mathbb{E}_j Q a &= \sum_{i=1}^j (\mathbb{E}_i - \mathbb{E}_{i-1}) (a^* \mathbb{E}_j Q D \mathbb{E}_j Q a) \\ &= \sum_{i=1}^j (\mathbb{E}_i - \mathbb{E}_{i-1}) \left\| D^{1/2} \mathbb{E}_j (Q_i - \rho \tilde{q}_{ii} Q_i \eta_i \eta_i^* Q_i) a \right\|^2 \\ &= \sum_{i=1}^j (\mathbb{E}_i - \mathbb{E}_{i-1}) \left[ -2\rho \operatorname{Re} (a^* (\mathbb{E}_j Q_i) D (\mathbb{E}_j \tilde{q}_{ii} Q_i \eta_i \eta_i^* Q_i) a) \right. \\ &\quad \left. + \|\mathbb{E}_j (\rho \tilde{q}_{ii} \eta_i^* Q_i a D^{1/2} Q_i \eta_i)\|^2 \right] \\ &\triangleq 2 \operatorname{Re}(X) + Z, \end{aligned} \tag{B.1}$$

and the variance of  $a^* \mathbb{E}_j Q D \mathbb{E}_j Q a$  is the sum of the variances of these martingale increments. Consider the term  $X$ . Recalling that  $\tilde{q}_{ii} = \tilde{b}_i - \rho \tilde{q}_{ii} \tilde{b}_i e_i$ ,

$$\begin{aligned} X &= -\rho \sum_{i=1}^j (\mathbb{E}_i - \mathbb{E}_{i-1}) (\tilde{b}_i \eta_i^* Q_i a a^* (\mathbb{E}_j Q_i) D Q_i \eta_i) + \rho^2 \sum_{i=1}^j (\mathbb{E}_i - \mathbb{E}_{i-1}) (\tilde{b}_i \tilde{q}_{ii} e_i \eta_i^* Q_i a a^* (\mathbb{E}_j Q_i) D Q_i \eta_i) \\ &\triangleq X_1 + X_2. \end{aligned}$$

Let  $M_i = Q_i a a^* (\mathbb{E}_j Q_i) D Q_i$ . The term  $X_1$  satisfies

$$\mathbb{E} |X_1|^2 = \rho^2 \sum_{i=1}^j \mathbb{E} \left| \mathbb{E}_i \tilde{b}_i \left( y_i^* M_i y_i - \frac{\tilde{d}_i}{n} \operatorname{Tr} D M_i + y_i^* M_i a_i + a_i^* M_i y_i \right) \right|^2. \tag{B.2}$$

Since  $M_i$  is a rank one matrix,  $\sum_{i=1}^j \mathbb{E} |y_i^* M_i y_i - \tilde{d}_i \operatorname{Tr} D M_i / n|^2 \leq K/n$ . Moreover,

$$\sum_{i=1}^j \mathbb{E} |y_i^* M_i a_i|^2 = \frac{1}{n} \sum_{i=1}^j \tilde{d}_i \mathbb{E} \left( |a^* Q_i D Q_i a| |a^* (\mathbb{E}_j Q_i) D Q_i a_i|^2 \right) \leq \frac{K}{n} \sum_{i=1}^j \mathbb{E} |a^* (\mathbb{E}_j Q_i) D Q_i a_i|^2.$$

The summand at the r.h.s. of the inequality satisfies:

$$\begin{aligned} |a^* (\mathbb{E}_j Q_i) D Q_i a_i|^2 &\leq 4 \left( |a^* (\mathbb{E}_j Q) D Q a_i|^2 + |a^* (\mathbb{E}_j (Q_i - Q)) D (Q_i - Q) a_i|^2 + \right. \\ &\quad \left. |a^* (\mathbb{E}_j Q) D (Q_i - Q) a_i|^2 + |a^* (\mathbb{E}_j (Q_i - Q)) D Q a_i|^2 \right) \\ &\triangleq 4(W_{i,1} + W_{i,2} + W_{i,3} + W_{i,4}). \end{aligned}$$

Recalling that  $A_{1:j} = [a_1, \dots, a_j]$ , we have

$$\sum_{i=1}^j W_{i,1} = a^* (\mathbb{E}_j Q) D Q A_{1:j} A_{1:j}^* Q D (\mathbb{E}_j Q) a \leq K .$$

Recalling (3.5) and (3.6), and writing  $\xi_i = 1 + \eta_i^* Q_i \eta_i$ , we have:

$$W_{i,2} \leq \frac{a_i^* Q \eta_i \eta_i^* Q a_i}{1 - \eta_i^* Q \eta_i} \times a^* \mathbb{E}_j (\xi_i Q \eta_i \eta_i^* Q) D \frac{Q \eta_i \eta_i^* Q}{1 - \eta_i^* Q \eta_i} D \mathbb{E}_j (\xi_i Q \eta_i \eta_i^* Q) a .$$

As  $\|(1 - \eta_i^* Q \eta_i)^{-1} Q \eta_i \eta_i^* Q\| = \|Q - Q_i\| \leq K$  and  $\|Q \Sigma\| \leq K$ , we have  $\sum_{i=1}^j \mathbb{E} W_{i,2} \leq \sum_{i=1}^j \mathbb{E} [|a_i^* Q \eta_i|^2 |\xi_i|^2]$ . Writing  $\xi_i = (\rho \tilde{b}_i)^{-1} + e_i$ , and noticing that  $(\rho \tilde{b}_i)^{-1}$  is bounded, we obtain:

$$\sum_{i=1}^j \mathbb{E} W_{i,2} \leq 2 \mathbb{E} a^* Q \Sigma_{1:j} \text{diag}((\rho \tilde{b}_1)^{-1}, \dots, (\rho \tilde{b}_j)^{-1}) \Sigma_{1:j}^* Q a + K \sum_{i=1}^j \mathbb{E} |e_i|^2 \leq K .$$

The terms  $W_{i,3}$  and  $W_{i,4}$  can be handled by similar derivations.

We get that  $\sum_{i=1}^j \mathbb{E} |y_i^* M_i a_i|^2 \leq K/n$ . The terms  $a_i^* M_i y_i$  on the right hand side of (B.2) satisfy  $\sum_{i=1}^j \mathbb{E} |a_i^* M_i y_i|^2 \leq K n^{-1} \sum_{i=1}^j \mathbb{E} |a_i^* Q_i a|^2 \leq K/n$ , which proves that  $\mathbb{E} |X_1|^2 \leq K/n$ .

We now consider  $X_2$ , which satisfies  $\mathbb{E} |X_2|^2 \leq 2\rho^4 \sum_{i=1}^j \mathbb{E} |\tilde{b}_i \tilde{q}_{ii} e_i \eta_i^* M_i \eta_i|^2$ . We have

$$\begin{aligned} \sum_{i=1}^j \mathbb{E} \left| \tilde{b}_i \tilde{q}_{ii} e_i a_i^* M_i a_i \right|^2 &\leq K \sum_{i=1}^j \mathbb{E} \mathbb{E}^{(i)} |e_i a_i^* M_i a_i|^2 \\ &\leq \frac{K}{n} \sum_{i=1}^j \mathbb{E} |a_i^* M_i a_i|^2 \leq \frac{K}{n} \sum_{i=1}^j \mathbb{E} |a_i^* Q_i a|^2 \leq \frac{K}{n} , \end{aligned}$$

where  $\mathbb{E}^{(i)} = \mathbb{E}[\cdot | y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n]$ . Moreover,

$$\sum_{i=1}^j \mathbb{E} \left| \tilde{b}_i \tilde{q}_{ii} e_i y_i^* M_i a_i \right|^2 \leq K \sum_{i=1}^j (\mathbb{E} |e_i|^4)^{1/2} (\mathbb{E} |y_i^* M_i a_i|^4)^{1/2} \leq \frac{K}{n}$$

and similarly for the terms in  $a_i^* M_i y_i$  and in  $y_i^* M_i y_i$ . We get that  $\mathbb{E} |X_2|^2 = \mathcal{O}(n^{-1})$ . We now turn to the term  $Z$  of equation (B.1). To control the variance of  $Z$ , we only need to control the variances of the terms:

$$\begin{aligned} Z_1 &= \sum_{i=1}^j (\mathbb{E}_i - \mathbb{E}_{i-1}) (\mathbb{E}_j \tilde{q}_{ii} a^* Q_i y_i \eta_i^* Q_i) D (\mathbb{E}_j \tilde{q}_{ii} Q_i \eta_i a_i^* Q_i a), \\ Z_2 &= \sum_{i=1}^j (\mathbb{E}_i - \mathbb{E}_{i-1}) \|\mathbb{E}_j (\tilde{q}_{ii} y_i^* Q_i a D^{1/2} Q_i \eta_i)\|^2, \\ Z_3 &= \sum_{i=1}^j (\mathbb{E}_i - \mathbb{E}_{i-1}) \|\mathbb{E}_j (\tilde{q}_{ii} a_i^* Q_i a D^{1/2} Q_i \eta_i)\|^2. \end{aligned}$$

The first term satisfies

$$\begin{aligned}
\mathbb{E}|Z_1|^2 &\leq 2 \sum_{i=1}^j \mathbb{E} |(\mathbb{E}_j \tilde{q}_{ii} a^* Q_i y_i \eta_i^* Q_i) D(\mathbb{E}_j \tilde{q}_{ii} Q_i \eta_i a_i^* Q_i a)|^2 \\
&\leq 2 \sum_{i=1}^j \mathbb{E} |(\mathbb{E}_j a^* Q_i y_i \eta_i^* Q_i) D Q_i \eta_i a_i^* Q_i a|^2 \\
&= 2 \sum_{i=1}^j \mathbb{E} \left[ |a_i^* Q_i a|^2 \mathbb{E}^{(i)} |(\mathbb{E}_j a^* Q_i y_i \eta_i^* Q_i) D Q_i \eta_i|^2 \right] \\
&\leq \frac{K}{n} \sum_{i=1}^j \mathbb{E} |a_i^* Q_i a|^2 = \mathcal{O}(n^{-1}),
\end{aligned}$$

where the second inequality comes from  $\mathbb{E}|\mathbb{E}_j(X)\mathbb{E}_j(Y)|^2 = \mathbb{E}|\mathbb{E}_j(X\mathbb{E}_j(Y))|^2 \leq \mathbb{E}|X\mathbb{E}_j(Y)|^2$ . The terms  $Z_2$  and  $Z_3$  can be handled similarly; details are omitted. The result is  $\mathbb{E}|Z|^2 \leq K/n$ .

Hence,  $\text{var}(a^*(\mathbb{E}_j Q)^2 a) = \mathcal{O}(n^{-1})$ . The estimate  $\text{var}(\text{Tr}(\mathbb{E}_j Q) D(\mathbb{E}_j Q)) = \mathcal{O}(1)$  can be established similarly.

**B.2. Proof of Lemma 5.2.** Recalling the expression (3.10) of  $\tilde{t}_{\ell\ell}$ , we notice that  $(1 - \rho \tilde{t}_{\ell\ell} a_\ell^* \mathcal{T}_{\ell a_\ell}) = \rho \tilde{t}_{\ell\ell} (1 + \tilde{d}_\ell \delta)$  is bounded below. It follows from Theorem 3.3-(3) that

$$\sum_{\ell=1}^j \frac{\alpha_\ell u^* T a_\ell \mathbb{E}[a_\ell^* Q u]}{1 - \rho \tilde{t}_{\ell\ell} a_\ell^* \mathcal{T}_{\ell a_\ell}} = \sum_{\ell=1}^j \frac{\alpha_\ell u^* T a_\ell a_\ell^* T u}{\rho \tilde{t}_{\ell\ell} (1 + \tilde{d}_\ell \delta)} + \mathcal{O}(n^{-1/2})$$

Moreover,

$$\begin{aligned}
\sum_{\ell=1}^j \alpha_\ell u^* T a_\ell \left( \frac{\mathbb{E}[a_\ell^* Q u]}{1 - \rho \tilde{t}_{\ell\ell} a_\ell^* \mathcal{T}_{\ell a_\ell}} - \mathbb{E}[a_\ell^* Q \ell u] \right) &= \sum_{\ell=1}^j \alpha_\ell u^* T a_\ell \mathbb{E} \left[ a_\ell^* Q \ell u \left( \frac{1 - \rho \tilde{q}_{\ell\ell} a_\ell^* Q \ell \eta_\ell}{1 - \rho \tilde{t}_{\ell\ell} a_\ell^* \mathcal{T}_{\ell a_\ell}} - 1 \right) \right] \\
&\quad - \sum_{\ell=1}^j \alpha_\ell u^* T a_\ell \frac{\mathbb{E}[\rho \tilde{q}_{\ell\ell} a_\ell^* Q \ell \eta_\ell y_\ell^* Q \ell u]}{1 - \rho \tilde{t}_{\ell\ell} a_\ell^* \mathcal{T}_{\ell a_\ell}} = \varepsilon_1 + \varepsilon_2
\end{aligned}$$

We have  $\varepsilon_1 = \sum_{\ell=1}^j \alpha_\ell u^* T a_\ell \mathbb{E}[a_\ell^* Q \ell u \xi_\ell]$  where  $\mathbb{E}|\xi_\ell|^p \leq K n^{-p/2}$  for  $p \geq 2$ . It follows that  $|\varepsilon_1| \leq \left( \sum_{\ell=1}^j \alpha_\ell^2 |u^* T a_\ell|^2 \mathbb{E} \xi_\ell^2 \right)^{1/2} \left( \sum_{\ell=1}^j \mathbb{E} |a_\ell^* Q \ell u|^2 \right)^{1/2} \leq K/\sqrt{n}$  by Theorem 3.3-(1). By writing

$$\varepsilon_2 = - \sum_{\ell=1}^j \alpha_\ell u^* T a_\ell \frac{\mathbb{E}[(\rho \tilde{q}_{\ell\ell} a_\ell^* Q \ell \eta_\ell - \mathbb{E}[\rho \tilde{q}_{\ell\ell} a_\ell^* Q \ell \eta_\ell]) y_\ell^* Q \ell u]}{1 - \rho \tilde{t}_{\ell\ell} a_\ell^* \mathcal{T}_{\ell a_\ell}}$$

and proceeding similarly to  $\varepsilon_1$ , we obtain  $|\varepsilon_2| \leq K/\sqrt{n}$ , which completes the proof of Lemma 5.2.

**B.3. Proof of Lemma 5.3.** The  $F_j$  increase to  $F_n = n^{-1} \text{Tr} D^{1/2} T A (1 + \delta \tilde{D})^{-2} \tilde{D} A^* T D^{1/2} < 1$  by Lemma 3.5. As  $\gamma > 0$  and  $M_j$  and  $G_j$  are increasing,  $\Delta_j$  is decreasing. In order to show that  $\Delta_n = \Delta_n$ , we only need to show that  $M_n + G_n = \rho^2 \tilde{\gamma}$ . We have

$$G_n = \frac{1}{n} \text{Tr} \tilde{D} (I + \delta \tilde{D})^{-2} A^* T A \tilde{D} (I + \delta \tilde{D})^{-2} A^* T A - \frac{1}{n} \sum_{k=1}^n \left( \frac{\tilde{d}_k a_k^* T a_k}{(1 + \delta \tilde{d}_k)^2} \right)^2$$

Recall from (3.10) and (3.11) that  $(1 + \delta \tilde{d}_k)^{-2} a_k^* T a_k = (1 + \delta \tilde{d}_k)^{-1} - \rho \tilde{t}_{kk}$ . Hence

$$\frac{1}{n} \sum_{k=1}^n \left( \frac{\tilde{d}_k a_k^* T a_k}{(1 + \delta \tilde{d}_k)^2} \right)^2 = \frac{1}{n} \sum_{k=1}^n \rho^2 \tilde{d}_k^2 \tilde{t}_{kk}^2 - \frac{1}{n} \sum_{k=1}^n \frac{\rho \tilde{d}_k^2 \tilde{t}_{kk}}{(1 + \delta \tilde{d}_k)} + \frac{1}{n} \sum_{k=1}^n \frac{\tilde{d}_k^2 a_k^* T a_k}{(1 + \delta \tilde{d}_k)^3}$$

which results in

$$\begin{aligned} M_n + G_n &= \frac{\rho}{n} \text{Tr} \tilde{D} \tilde{T} \tilde{D} (I + \delta \tilde{D})^{-1} - \frac{1}{n} \text{Tr} \tilde{D}^2 (I + \delta \tilde{D})^{-3} A^* T A \\ &\quad + \frac{1}{n} \text{Tr} \tilde{D} (I + \delta \tilde{D})^{-2} A^* T A \tilde{D} (I + \delta \tilde{D})^{-2} A^* T A. \end{aligned}$$

Now, one can check with the help of (3.12) that  $\rho^2 \tilde{\gamma} = \rho^2 n^{-1} \text{Tr} \tilde{D} \tilde{T} \tilde{D} \tilde{T}$  is equal to the r.h.s. of this equation. Lemma 5.3 is proven.

#### REFERENCES

- [1] G.W. Anderson and O. Zeitouni. A CLT for a band matrix model. *Probab. Theory Related Fields*, 134(2):283–338, 2006.
- [2] L. V. Arharov. Limit theorems for the characteristic roots of a sample covariance matrix. *Dokl. Akad. Nauk SSSR*, 199:994–997, 1971.
- [3] Z. D. Bai and J. W. Silverstein. No eigenvalues outside the support of the limiting spectral distribution of large-dimensional sample covariance matrices. *Ann. Probab.*, 26(1):316–345, 1998.
- [4] Z. D. Bai and J. W. Silverstein. CLT for linear spectral statistics of large-dimensional sample covariance matrices. *Ann. Probab.*, 32(1A):553–605, 2004.
- [5] Zhidong Bai and Jack W. Silverstein. *Spectral analysis of large dimensional random matrices*. Springer Series in Statistics. Springer, New York, second edition, 2010.
- [6] P. Billingsley. *Probability and measure*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1995.
- [7] A. Boutet de Monvel and A. Khorunzhy. Limit theorems for random matrices. *Markov Process. Related Fields*, 4(2):175–197, 1998.
- [8] T. Cabanal-Duvillard. Fluctuations de la loi empirique de grandes matrices aléatoires. *Ann. Inst. H. Poincaré Probab. Statist.*, 37(3):373–402, 2001.
- [9] S. Chatterjee. Fluctuations of eigenvalues and second order Poincaré inequalities. *Probab. Theory Related Fields*, 143(1-2):1–40, 2009.
- [10] M. Debbah and R. Müller. On the limiting behavior of directional MIMO models. In *Proceeding of the Sixth Baiona Workshop On Signal Processing in Communication*, Spain, September 2003.
- [11] R. Brent Dozier and J. W. Silverstein. Analysis of the limiting spectral distribution of large dimensional information-plus-noise type matrices. *J. Multivariate Anal.*, 98(6):1099–1122, 2007.
- [12] R. Brent Dozier and J. W. Silverstein. On the empirical distribution of eigenvalues of large dimensional information-plus-noise-type matrices. *J. Multivariate Anal.*, 98(4):678–694, 2007.
- [13] J. Dumont, W. Hachem, S. Lasaulce, P. Loubaton, and J. Najim. On the capacity achieving covariance matrix for rician MIMO channels: An asymptotic approach. *Information Theory, IEEE Transactions on*, 56(3):1048–1069, march 2010.
- [14] V. L. Girko. *Sluchainye matritsy (Russian) [Random Matrices]*. Izdat. Ob''ed. “Višča Škola” pri Kiev. Gosudarstv. Univ., Kiev, 1975.
- [15] V. L. Girko. *Theory of stochastic canonical equations. Vol. I*, volume 535 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2001.
- [16] V. L. Girko. *Theory of stochastic canonical equations. Vol. II*, volume 535 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2001.
- [17] V. L. Girko. Thirty years of the central resolvent law and three laws on the  $1/n$  expansion for resolvent of random matrices. *Random Oper. Stochastic Equations*, 11(2):167–212, 2003.
- [18] A. Guionnet. Large deviations upper bounds and central limit theorems for non-commutative functionals of Gaussian large random matrices. *Ann. Inst. H. Poincaré Probab. Statist.*, 38(3):341–384, 2002.
- [19] W. Hachem, O. Khorunzhy, P. Loubaton, J. Najim, and L. Pastur. A new approach for mutual information analysis of large dimensional multi-antenna chennels. *IEEE Trans. Inform. Theory*, 54(9):3987–4004, 2008.



- [20] W. Hachem, P. Loubaton, and J. Najim. Deterministic equivalents for certain functionals of large random matrices. *Ann. Appl. Probab.*, 17(3):875–930, 2007.
- [21] W. Hachem, P. Loubaton, and J. Najim. A CLT for information-theoretic statistics of gram random matrices with a given variance profile. *Ann. Appl. Probab.*, 18(6):2071–2130, 2008.
- [22] W. Hachem, P. Loubaton, J. Najim, and P. Vallet. On bilinear forms based on the resolvent of large random matrices. Technical report, Télécom Paristech and Université Paris-Est, may 2010. arXiv:1004.3848, available at <http://front.math.ucdavis.edu/1004.3848>.
- [23] Roger A. Horn and Charles R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, 1990. Corrected reprint of the 1985 original.
- [24] K. Johansson. On fluctuations of eigenvalues of random Hermitian matrices. *Duke Math. J.*, 91(1):151–204, 1998.
- [25] D. Jonsson. Some limit theorems for the eigenvalues of a sample covariance matrix. *J. Multivariate Anal.*, 12(1):1–38, 1982.
- [26] A. Kammoun, M. Kharouf, W. Hachem, J. Najim, and A. El Kharroubi. On the fluctuations of the mutual information for non centered MIMO channels: The non gaussian case. In *Signal Processing Advances in Wireless Communications (SPAWC), 2010 IEEE Eleventh International Workshop on*, pages 1–5, June 2010.
- [27] A. Khorunzhy, B. Khoruzhenko, and L. Pastur. Asymptotic properties of large random matrices with independent entries. *J. Math. Phys.*, 37(10):5033–5060, 1996.
- [28] A. Lytova and L. Pastur. Central limit theorem for linear eigenvalue statistics of the Wigner and the sample covariance random matrices. *Metrika*, 69(2-3):153–172, 2009.
- [29] J. A. Mingo and R. Speicher. Second order freeness and fluctuations of random matrices. I. Gaussian and Wishart matrices and cyclic Fock spaces. *J. Funct. Anal.*, 235(1):226–270, 2006.
- [30] A. L. Moustakas, S. H. Simon, and A. M. Sengupta. MIMO capacity through correlated channels in the presence of correlated interferers and noise: a (not so) large  $N$  analysis. *IEEE Trans. Inform. Theory*, 49(10):2545–2561, 2003.
- [31] A. M. Sengupta and P.P. Mitra. Capacity of multivariate channels with multiplicative noise: I. random matrix techniques and large- $n$  expansions for full transfer matrices. available at <http://arxiv.org/abs/physics/0010081>, 2000.
- [32] J. W. Silverstein and Z. D. Bai. On the empirical distribution of eigenvalues of a class of large-dimensional random matrices. *J. Multivariate Anal.*, 54(2):175–192, 1995.
- [33] Ya. Sinai and A. Soshnikov. Central limit theorem for traces of large random symmetric matrices with independent matrix elements. *Bol. Soc. Brasil. Mat. (N.S.)*, 29(1):1–24, 1998.
- [34] A. Soshnikov. The central limit theorem for local linear statistics in classical compact groups and related combinatorial identities. *Ann. Probab.*, 28(3):1353–1370, 2000.
- [35] G. Taricco. On the capacity of separately-correlated MIMO Rician fading channels. In *Proc. 49th Annual IEEE Globecom Conference, 2006*, San Francisco (CA), USA, November 2006.
- [36] G. Taricco. Asymptotic mutual information statistics of separately correlated Rician fading MIMO channels. *IEEE Trans. Inform. Theory*, 54(8):3490–3504, 2008.
- [37] I.E. Telatar. Capacity of multi-antenna gaussian channel. *European Transactions on Telecommunications*, 1999.
- [38] A. Tulino and S. Verdú. Random matrix theory and wireless communications. In *Foundations and Trends in Communications and Information Theory*, volume 1, pages 1–182. Now Publishers, June 2004.

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