POSITIVE SOLUTIONS FOR LARGE RANDOM LINEAR SYSTEMS

Pierre Bizeul(1), *Maxime Clenet*(2), *Jamal Najim*(2)

(1) Institut de Mathematiques de Jussieu, Sorbonne Université, France
(2) Centre National de la Recherche Scientifique and Université Paris-Est Marne-La-Vallée, France

ABSTRACT

Consider a large linear system with random underlying matrix:

$$\mathbf{x}_n = \mathbf{1}_n + \frac{1}{\alpha_n\sqrt{\beta_n}} M_n \mathbf{x}_n \ ,$$

where $\mathbf{x}_n$ is the unknown, $\mathbf{1}_n$ is a vector of ones, M_n is a random matrix and α_n, β_n are scaling parameters to be specified. We investigate the componentwise positivity of the solution $\mathbf{x}_n$ depending on the scaling factors, as the dimensions of the system grow to infinity.

We consider 2 models of interest: The case where matrix M_n has independent and identically distributed standard Gaussian random variables, and a sparse case with a growing number of vanishing entries.

In each case, there exists a phase transition for the scaling parameters below which there is no positive solution to the system with growing probability and above which there is a positive solution with growing probability.

These questions arise from feasibility and stability issues for large biological communities with interactions.

***Index Terms*—** Linear equation, Large Random Matrices, Extreme values, Lotka-Volterra equations, feasibility and stability in foodwebs.

1. INTRODUCTION

Consider a large linear system with random matrix M_n:

$$\mathbf{x}_n = \mathbf{1}_n + \frac{1}{\alpha_n\sqrt{\beta_n}} M_n \mathbf{x}_n \ , \tag{1}$$

where $[n] = \{1, \cdots, n\}$, $\mathbf{x}_n = (x_k)_{k\in[n]}$ is a $n \times 1$ unknown vector, $\mathbf{1}_n$ is a $n \times 1$ vector of ones, $M_n = A_n \odot X_n$ is a $n \times n$ random matrix, where A_n represents a deterministic adjacency matrix of a given graph, accounting for the sparsity of M_n, and X_n is a $n \times n$ matrix of independent and identically distributed (i.i.d.) standard Gaussian $\mathcal{N}(0,1)$ random variables. The Hadamard product $M_n = A_n \odot X_n$ accounts for the entrywise product $M_{ij} = A_{ij}X_{ij}$, hence A_n acts as a deterministic sparsity pattern over the random matrix X_n.

The sequences α_n and β_n are two deterministic positive sequences going to infinity with different roles: β_n is such that the spectral norm of matrix $\beta_n^{-1/2} M_n$ is of order 1, while the parameter α_n represents the extra normalization needed to obtain a positive solution $\mathbf{x}_n$.

In the following, we investigate the componentwise positivity of the solution $\mathbf{x}_n$ for two specific models: the full matrix model (FMM), where

$$A_n = \mathbf{1}_n \mathbf{1}_n^T, \quad M_n = X_n \quad \text{and} \quad \beta_n = n \, . \tag{2}$$

For this model, we will state a theorem established in [1].

We also consider a sparse matrix model (SMM) where A_n is the adjacency matrix of a d-regular graph. For this model, we present a conjecture and some simulations.

The positivity of the x_k's is a key issue in the study of Large Lotka-Volterra systems, widely used in mathematical biology and ecology to model populations with interactions.

Consider for instance a given foodweb and denote by $\mathbf{x}_n(t) = (x_k(t))_{k\in[n]}$ the vector of abundances of the various species within the foodweb at time t. A standard way to connect these abundances is via a Lotka-Volterra (LV) system of equations that writes

$$\frac{dx_k(t)}{dt} = x_k(t) \left(r_k - \theta x_k(t) + \frac{1}{\alpha_n\sqrt{\beta_n}} \sum_{\ell\in[n]} M_{k\ell} x_\ell(t) \right) \tag{3}$$

for $k \in [n]$. In this equation, r_k represents the intrinsic growth rate of species k, θ is a coefficient reflecting intraspecific competition, and $M_{k\ell}$ is the per capita effect of species ℓ on species k. In the absence of any prior information, the interactions $M_{k\ell}$ can be modelled as random.

Remark 1. *Notice that without interactions ($M_n = 0$), equation* (3) *is simply a logistic differential equation.*

In the following, we will focus on the idealized model where $r_k = \theta = 1$.

At the equilibrium $\frac{d\mathbf{x}_n}{dt} = 0$, the abundance vector $\mathbf{x}_n$ is solution of (1) and a key issue is the existence of a *feasible* solution, that is a solution $\mathbf{x}_n$ with positive components x_k.

A major motivation for the present study comes from the paper [2] where it is established that for the full matrix case and under the standard normalization $\alpha_n = \alpha$ fixed and $\beta_n = n$, there are no feasible solutions.

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2. THE FULL MATRIX MODEL

In the FMM (2), the convergence of the spectral norm

$$\|n^{-1/2}X_n\| \xrightarrow[n\to\infty]{a.s.} 2 \tag{4}$$

is well-known (see [3]) hence the normalization $\beta_n = n$. The following phase transition phenomenon occurs:

Theorem 1. *Let $\alpha_n \to \infty$ and denote by $\alpha_n^* = \sqrt{2\log(n)}$. Consider the solution*

$$\mathbf{x}_n = \mathbf{1}_n + \frac{M_n}{\alpha_n\sqrt{n}}\mathbf{x}_n \quad \Leftrightarrow \quad \mathbf{x}_n = \left(I_n - \frac{M_n}{\alpha_n\sqrt{n}}\right)^{-1}\mathbf{1}_n\,,$$

where $M_n = X_n$.

- *If there exists $\varepsilon > 0$ such that $\alpha_n \leq (1-\varepsilon)\alpha_n^*$ eventually, then*

$$\mathbb{P}\left\{\min_{k\in[n]} x_k > 0\right\} \xrightarrow[n\to\infty]{} 0\,. \tag{5}$$

- *If there exists $\varepsilon > 0$ such that $\alpha_n \geq (1+\varepsilon)\alpha_n^*$ eventually, then*

$$\mathbb{P}\left\{\min_{k\in[n]} x_k > 0\right\} \xrightarrow[n\to\infty]{} 1\,. \tag{6}$$

Theorem 1 has been established in [1].

Remark 2. *Notice that if $\alpha_n = \alpha$ is fixed, the solution of the system* (1) *has already been studied by Hwang and Geman [4] for non-Gaussian i.i.d. standardized entries. A major conclusion of this work is the asymptotic independence and Gaussian fluctuations of any finite number of $\mathbf{x}_n$'s components:*

$$(x_1,\cdots,x_M)^T \quad \xrightarrow[n\to\infty]{\mathcal{D}} \quad \mathcal{N}_M\left(\mathbf{1}_M, \sigma_\alpha^2 I_M\right)\ , \tag{7}$$

where M is fixed, $\mathcal{N}_M$ represents a M-valued Gaussian vector and $\sigma_\alpha^2 > 0$. An easy consequence of (7) *yields* (5)*. This result has been exploited in [2] to state the absence of feasible solution to* (3) *if $\alpha_n = \alpha > 0$ is fixed.*

Elements of proof. The system (1) writes

$$\left(I_n - \frac{M_n}{\alpha_n\sqrt{n}}\right)\mathbf{x}_n = \mathbf{1}_n\,.$$

By (4), the spectral norm of $\alpha_n^{-1}n^{-1/2}M_n$ goes to zero and one can safely invert the previous equation, and unfold the resolvent $\left(I_n - \frac{M_n}{\alpha_n\sqrt{n}}\right)^{-1}$ as a matrix infinite series:

$$\begin{aligned}\mathbf{x}_n &= \left(I_n - \frac{M_n}{\alpha_n\sqrt{n}}\right)^{-1}\mathbf{1}_n\,,\\ &= \mathbf{1}_n + \frac{M_n}{\alpha_n\sqrt{n}}\mathbf{1}_n + \sum_{\ell=2}^{\infty}\left(\frac{M_n}{\alpha_n\sqrt{n}}\right)^{\ell}\mathbf{1}_n\,.\end{aligned}$$

Denote by $\mathbf{e}_k$ the k-th canonical vector and keep the first two terms in the previous expansion, then x_k writes

$$\begin{aligned}x_k &= \mathbf{e}_k^T\mathbf{x}_n = 1 + \mathbf{e}_k^T\frac{M_n}{\alpha_n\sqrt{n}}\mathbf{1}_n + \cdots\\ &= 1 + \frac{1}{\alpha_n}\frac{\sum_{j=1}^n X_{kj}}{\sqrt{n}} + \cdots\end{aligned}$$

Notice that $Z_k = n^{-1/2}\sum_{j=1}^n X_{kj}$ is exactly $\mathcal{N}(0,1)$-distributed and that the Z_k's are independent. In particular,

$$\min_{k\in[n]} x_k \approx 1 + \frac{\min_{k\in[n]} Z_k}{\alpha_n} \approx 1 - \frac{\sqrt{2\log(n)}}{\alpha_n}$$

by standard extreme value theory[1]. This immediatly yields the conclusions of the theorem by comparing the relative positions of $\alpha_n^* = \sqrt{2\log(n)}$ and α_n. $\square$

The main input of [1] is to establish that the remaining term

$$R_k = \mathbf{e}_k^T\sum_{\ell=2}^{\infty}\left(\frac{M_n}{\alpha_n\sqrt{n}}\right)^{\ell}\mathbf{1}_n$$

has no effect on the positivity of $\mathbf{x}_n$ and can be neglected.

3. THE SPARSE MATRIX MODEL

We focus on the following SMM: consider a deterministic $n\times n$ adjacency matrix A_n of a d-regular (directed) graph, that is a matrix whose entry A_{ij} equals 1 if the edge (ij) belongs to the graph of order n, and zero else, and where each vertex $1,\cdots,n$ has exactly d neighbours. This in particular implies that there are exactly d non-null entries in each row and each column of A_n, and the total number of non-null entries of matrix A_n is nd.

The spectral radius of M_n. Depending on the magnitude of $d = d_n$, the order of the spectral radius of M_n varies. The following two extreme cases illustrate this fact: consider $A_n^{(1)} = \mathrm{diag}(1)$ and $A_n^{(2)} = \mathbf{1}_n\mathbf{1}_n^T$. In the first case, $d = 1$ and

$$\|M_n^{(1)}\| = \|A_n^{(1)}\odot X_n\| = \max_{i\in[n]}|X_{ii}| \sim \sqrt{2\log(n)}\,.$$

In the second case, $d = d_n = n$ and

$$\|M_n^{(2)}\| = \|A_n^{(2)}\odot X_n\| \sim 2\sqrt{n}\,.$$

This simple example illustrates the fact that the tuning of β_n is non-trivial in the sparse case: if $d = 1$ then $\beta_n = 2\log(n)$ while if $d = n$ then $\beta_n = d_n = n$. In fact, the following phase transition, established by Bandeira and Van Handel in [6], holds:

[1] It is well-known that if the Z_k's are i.i.d. $\mathcal{N}(0,1)$, then $\mathbb{E}\max_{k\in[n]} Z_k = -\mathbb{E}\min_{k\in[n]} Z_k \sim \sqrt{2\log(n)}$, see for instance [5].

- If $d_n \gg \log(n)$ then $\mathbb{E}\|M_n\| \sim \sqrt{d_n}$,
- If $d_n \ll \log(n)$ then $\mathbb{E}\|M_n\| \sim \sqrt{\log(n)}$.

To be more specific, the result by Bandeira and Van Handel [6] writes in our context:

$$\mathbb{E}\|M_n\| \le (1+\varepsilon)\left\{2\sqrt{d_n} + \frac{5}{\sqrt{\log(1+\varepsilon)}}\sqrt{\log(n)}\right\}$$

for any $0 < \varepsilon \le 1/2$ and

$$\mathbb{E}\|M_n\| \ge_K 2\sqrt{d_n} + \sqrt{2\log(d_n n)}\,,$$

where $a_n \ge_K b_n$ means that there exists a constant independent from n such that $a_n \ge K b_n$.

Positivity of the solution $\mathbf{x}_n$. Based on the previous analysis of the spectral norm of $\|M_n\|$, we shall consider the following regime $d_n \gg \log(n)$ where $\|M_n\| \sim \sqrt{d_n}$. We fix $\beta_n = d_n$. Based on simulations (see below), we state the following conjecture:

Conjecture 1. *Let $\alpha_n \to \infty$ and $\alpha_n^* = \sqrt{2\log(n)}$. Let $M_n = A_n \odot X_n$ with A_n the adjacency matrix of a d_n-regular graph, with $d_n \gg \log(n)$. Consider the solution*

$$\mathbf{x}_n = \mathbf{1}_n + \frac{M_n}{\alpha_n\sqrt{d_n}}\mathbf{x}_n \quad \Leftrightarrow \quad \mathbf{x}_n = \left(I_n - \frac{M_n}{\alpha_n\sqrt{d_n}}\right)^{-1}\mathbf{1}_n\,,$$

then

- *If there exists $\varepsilon > 0$ such that $\alpha_n \le (1-\varepsilon)\alpha_n^*$ eventually, then $\mathbb{P}\left\{\min_{k\in[n]} x_k > 0\right\} \xrightarrow[n\to\infty]{} 0$.*
- *If there exists $\varepsilon > 0$ such that $\alpha_n \ge (1+\varepsilon)\alpha_n^*$ eventually, then $\mathbb{P}\left\{\min_{k\in[n]} x_k > 0\right\} \xrightarrow[n\to\infty]{} 1$.*

Remark 3. *The conjecture can be settled in the case where $d \propto n$, i.e. where $dn^{-1} \to c > 0$. It suffices to follow the lines of the proof of Theorem 1 in this regime.*

Arguments. The same argument as Theorem 1 applies when unfolding $\mathbf{x}_n$:

$$x_k = \mathbf{e}_k^T \mathbf{x}_n = 1 + \frac{1}{\alpha_n}\frac{\sum_{j=1}^n A_{kj}X_{kj}}{\sqrt{d_n}} + \cdots$$

Introduce $Z_k = d_n^{-1/2}\sum_{j=1}^n A_{kj}X_{kj}$ and notice that since $\#\{A_{kj} = 1, 1 \le j \le n\} = d_n$, Z_k is $\mathcal{N}(0,1)$-distributed and the Z_k's are independent. Now

$$\min_{1\le k\le n} x_k \approx 1 + \frac{\min_{1\le k\le n} Z_k}{\alpha_n} \approx 1 - \frac{\sqrt{2\log(n)}}{\alpha_n}\,.$$

The conclusion follows as previously. $\square$

Although simulations tend to indicate that the remainder term

$$R_k = \mathbf{e}_k^T \sum_{\ell=2}^{\infty}\left(\frac{M_n}{\alpha_n\sqrt{d_n}}\right)^{\ell}\mathbf{1}_n$$

has no influence on the positivity of $\mathbf{x}_n$, a direct mathematical proof is currently beyond our reach for $d_n \ll n$.

4. DISCUSSION

The results presented here lie between Random Matrix Theory (RMT) and perturbation theory, slightly outside the range of RMT. In fact, consider

$$\mathbf{x}_n = \left(I_n - \frac{M_n}{\alpha_n\sqrt{\beta_n}}\right)^{-1}\mathbf{1}_n$$

In RMT, the random matrix part is supposed to have a limiting macroscopic effect, and this is indeed the case if $\alpha_n = \alpha$ is a constant and

$$\left\|\frac{M_n}{\sqrt{\beta_n}}\right\| \sim \mathcal{O}(1) \quad \text{as} \quad n \to \infty\,.$$

From a perturbation theory point of view, the random matrix part vanishes asymptotically as it is the case if $\alpha_n \to \infty$:

$$\frac{1}{\alpha_n}\left\|\frac{M_n}{\sqrt{\beta_n}}\right\| \xrightarrow[n\to\infty]{} 0\,.$$

As demonstrated in Table 1, the vanishing effect of the random part $\alpha_n^{-1}\beta_n^{-1/2}M_n$ is extremely slow.

n	10^2	10^3	10^4	10^5	10^6
$\frac{1}{\alpha_n^*}$	0.33	0.27	0.23	0.21	0.19

Table 1. The quantity $\frac{1}{\alpha_n^*} = \frac{1}{\sqrt{2\log n}}$ vanishes extremely slowly as n increases.

5. SIMULATIONS

In this section, we illustrate the phase transition phenomenon toward a positive solution $\mathbf{x}_N$ depending on the scaling α_N, β_N being either fixed at N (FMM) or d_N (SMM).

In Figure 1, we consider the transition toward feasibility for the full matrix model. We consider different values of N, respectively 400 (dashed), 1000 (solid). For each N and each κ on the x-axis, we simulate 10000 $N \times N$ matrices M_N and compute the solution $\mathbf{x}_N$ of (6) at the scalings $\alpha_N(\kappa) = \kappa\sqrt{\log(N)}$ and $\beta_N = N$. Each curve represents the proportion of feasible solutions $\mathbf{x}_N$ obtained for 10000 simulations. The red dotted vertical line corresponds to the

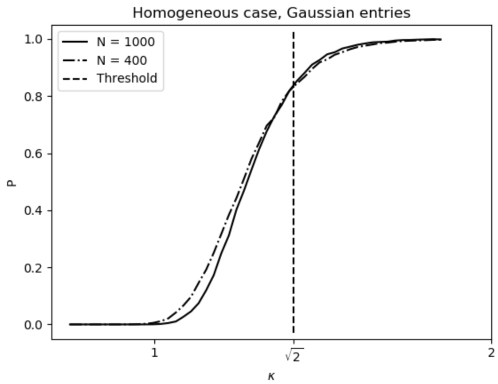

Fig. 1. Transition toward feasibility for the FMM

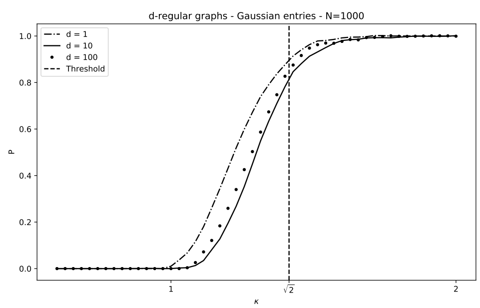

Fig. 2. Transition toward feasibility for the SMM

critical scaling $\alpha_N^* = \sqrt{2\log(N)}$ for $\kappa = \sqrt{2}$. The proportion of feasible solutions ranges from 0 for $\kappa \leq 1$ to 1 for $\kappa \geq 2$.

In Figure 2, we consider the transition toward feasibility for the SMM. In this case, N is fixed $N = 1000$ while d_n varies from 1 to 500. The phase transition is similar to the FMM. Notice in particular that in this case simulations tend to validate the phase transition phenomenon even for $d < \log(1000) = 6,90$.

6. ADDITIONAL RESULTS

We now illustrate two aspects of the phase transition not covered by the results presented so far.

In Figure 3, the phase transition is shown to hold for the FMM with Bernoulli ± 1 entries. Although the Gaussiannity of the entries is mathematically important for the proofs, these simulations tend to show that this assumption is merely technical but not necessary.

In Figure 4, we illustrate the phase transition phenomenon for the FMM for a non-homogeneous linear system:

$$\mathbf{x}_n = \mathbf{r}_n + \frac{1}{\alpha_n\sqrt{n}} M_n \mathbf{x}_n \ , \tag{8}$$

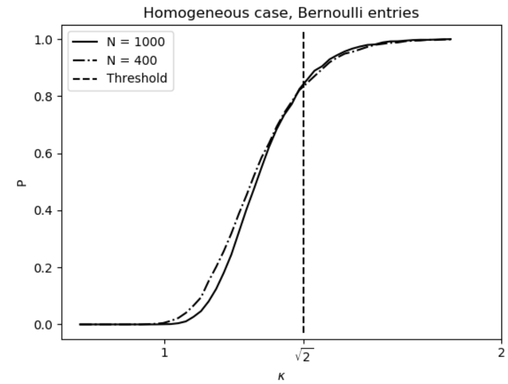

Fig. 3. Non-Gaussian entries, FMM

where $\mathbf{r}_n = (r_k)$ a $n \times 1$ deterministic vector with positive components. In this non-homogeneous case, the phase transition is not as clean-cut as in the homogeneous case but there is a buffer zone where the transition occurs. We formalize this with the help of the following notations:

$$\begin{cases} r_{\min} = \min_{1\leq k\leq n} r_k \ , \\ r_{\max} = \max_{1\leq k\leq n} r_k \end{cases} \quad \text{and} \quad \sigma_{\mathbf{r}}(n) = \sqrt{n^{-1}\sum_{k\in[n]} r_k^2} \ .$$

Assume that $\rho_{\min}, \rho_{\max}$ are independent from n and

$$0 \ < \ \rho_{\min} \ \leq \ r_{\min} \ \leq \ \sigma_{\mathbf{r}} \ \leq \ r_{\max} \ \leq \ \rho_{\max} < \infty \, .$$

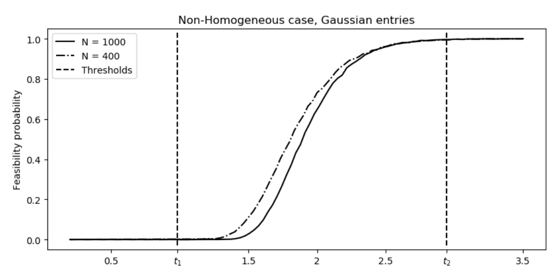

Fig. 4. Non-Homogeneous system, full matrix model with the buffer zone $[t_1, t_2]$ where $t_1 = \frac{\alpha_n^*\sigma_{\mathbf{r}}(n)}{r_{\max}(n)}$ and $t_2 = \frac{\alpha_n^*\sigma_{\mathbf{r}}(n)}{r_{\min}(n)}$.

Theorem 2 (Bizeul et al. [1]). *Let* $\alpha_n \xrightarrow[n\to\infty]{} \infty$ *and denote by* $\alpha_n^* = \sqrt{2\log n}$. *Let* $\mathbf{x}_n = (x_k)_{k\in[n]}$ *be the solution of* (8).

- *If there exists* $\varepsilon > 0$ *such that eventually* $\alpha_n \leq (1-\varepsilon)\frac{\alpha_n^*\sigma_{\mathbf{r}}(n)}{r_{\max}(n)}$ *then* $\mathbb{P}\left\{\min_{k\in[n]} x_k > 0\right\} \xrightarrow[n\to\infty]{} 0\,.$
- *If there exists* $\varepsilon > 0$ *such that eventually* $\alpha_n \geq (1+\varepsilon)\frac{\alpha_n^*\sigma_{\mathbf{r}}(n)}{r_{\min}(n)}$ *then* $\mathbb{P}\left\{\min_{k\in[n]} x_k > 0\right\} \xrightarrow[n\to\infty]{} 1\,.$

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