AN IMPROVED MUSIC ALGORITHM BASED ON LOW RANK PERTURBATION OF LARGE RANDOM MATRICES

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ABSTRACT

This paper is devoted to subspace DoA estimation, when the number of available snapshots N is of the same order of magnitude as the number of sensors M. In this context, traditional subspace methods fail because the empirical covariance matrix of the observations is a poor estimate of the true covariance matrix. The goal of the paper is to propose a new consistent estimator of the DoAs in the case where $M, N \to +\infty$ at the same rate, using large random matrix theory. It is assumed that the number of sources is constant, and recent results on the so called spiked matrix models are used. First and second order results are provided.

Index Terms— DoA, MUSIC, Random matrices, Spiked model

1. INTRODUCTION

Subspace DoA (Direction Of Arrival) estimation methods using antenna arrays (such as MUSIC) have been extensively studied in the past because they offer a good trade-off between performance and complexity. Their statistical performance has mainly been characterized in the case where the number of snapshots N converges to $+\infty$ while the number of antennas M remains fixed. In practice, the corresponding conclusions are valid in finite sample size if N is much larger than M. However, this assumption is often not realistic if the number of antennas is large because in practice, the number of available snapshots is limited. In order to study the statistical performance of the subspace estimates in this context, Mestre et al. [1] proposed to consider the asymptotic regime in which M and N converge to $+\infty$ at the same rate, i.e. $M, N \to +\infty$, $\frac{M}{N}$ converges towards a positive constant. Using large random matrix theory (RMT) results, [1] proved that the traditional DoAs subspace estimators are asymptotically biased, and proposed, assuming the source signals are i.i.d, consistent estimators which outperform the standard ones, for realistic values of M and N. Later, [2] proposed a similar estimator assuming the source signals are unknown deterministic quantities. Neither one of these papers imposes restrictive conditions on the number of signals K, which may scale-up with M, N. The purpose of this paper is to propose consistent subspace estimators, in the context of deterministic signals, when the number of sources K is small compared to M, N. The present approach is again based on RMT results, but in contrast to [2], we use here recent results [3] concerning the singular values and the singular vectors of large random matrices, perturbed by low rank deterministic matrices. Under certain conditions, the low rank perturbation gives birth to "spiked" eigenvalues, located outside the bulk of eigenvalues of the non perturbed model. RMT techniques which describe the behavior of these eigenvalues and their related eigenvectors suggest many interesting applications in estimation theory and in signal processing. This paper, which describes one such application, is organized as follows. In section 3, we review some basic results in RMT and tools needed for our approach. In section 4, we propose the new estimator and analyze its asymtptotic behavior. Finally, in section 5, we present some numerical examples.

2. MODEL AND PROBLEM STATEMENT

We consider K narrow band deterministic source signals impinging on an antenna array of M elements with $K < M$. At time n, the received snapshot (of size M) writes

$$\mathbf{y}_n = \mathbf{A}\mathbf{s}_n + \mathbf{x}_n, \tag{1}$$

where $\mathbf{A} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_K)]$ is the $M \times K$ matrix containing the K linearly independent steering vectors, $\mathbf{x}_n$ is a Gaussian vector satisfying $\mathbb{E}[\mathbf{x}_n \mathbf{x}_n^H] = \sigma^2 \mathbf{I}_M$, and $\mathbf{s}_n$ is a vector containing the K source signals received at time n. Assuming we collect N independent observations of the previous model, (1) is equivalent to $\mathbf{\Sigma} = \mathbf{B} + \mathbf{W}$, with $\mathbf{\Sigma} = N^{-1/2}[\mathbf{y}_1, \ldots, \mathbf{y}_N]$, $\mathbf{B} = N^{-1/2}[\mathbf{A}\mathbf{s}_1, \ldots, \mathbf{A}\mathbf{s}_N]$, and $\mathbf{W}$ built as $\mathbf{\Sigma}$ and $\mathbf{B}$. In this paper, we will assume that $M < N$. In order to construct subspace methods, it is always assumed that matrix $\mathbf{S} = [\mathbf{s}_1, \ldots, \mathbf{s}_N]$ has full rank K which implies that $\mathbf{B}$ has K non null singular values. Write the SVD of $\mathbf{B}$ as $\mathbf{B} = \mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{V}^H$ with $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_K]$

and $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_K]$ the left and right singular vector matrices, and $\mathbf{\Lambda}^{1/2}$ the $M \times M$ diagonal matrix of singular values $\sqrt{\lambda_1} > \ldots > \sqrt{\lambda_K}$ (in decreasing order), assumed to have multiplicity one. The MUSIC algorithm is based on the fact that the K angles $\theta_1, \ldots, \theta_K$ are solutions to the equation $\mathbf{a}(\theta)^H(\mathbf{I} - \mathbf{U}\mathbf{U}^H)\mathbf{a}(\theta) = 0$. To estimate these angles, it is necessary to estimate the quantity $\eta = \mathbf{a}^H\mathbf{U}\mathbf{U}^H\mathbf{a}$ for any vector $\mathbf{a}$ of size M. For this, the traditional approach consists in replacing the signal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_K$ of $\mathbf{B}\mathbf{B}^H$ by those of the empirical covariance matrix $\mathbf{\Sigma}\mathbf{\Sigma}^H$. We denote by $\hat{\lambda}_1 \geq \ldots \geq \hat{\lambda}_M$ the eigenvalues of $\mathbf{\Sigma}\mathbf{\Sigma}^H$ and by $\hat{\mathbf{U}} = [\hat{\mathbf{u}}_1, \ldots, \hat{\mathbf{u}}_M]$ and $\hat{\mathbf{V}} = [\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_M]$ the left and right singular vector matrices of $\mathbf{\Sigma}$. The traditional estimator thus writes $\eta_{\text{trad}} = \mathbf{a}^H \left[\hat{\mathbf{u}}_1 \cdots \hat{\mathbf{u}}_K\right] \left[\hat{\mathbf{u}}_1 \cdots \hat{\mathbf{u}}_K\right]^H \mathbf{a}$. Thanks to the law of large number, the previous estimator is consistent in the case where $N \to \infty$ while M is kept constant. However, when $M, N \to +\infty$ in such a way that $M/N \to c > 0$, the previous estimator does not converge anymore to η.

In this paper, we will assume that K is independent of N, and that $M/N \to c \in]0, 1[$, as $M, N \to \infty$. All the convergences in this paper will be considered under this regime and referred directly by the statement "$N \to +\infty$". For two scalar random variables sequences $(X_N), (Y_N)$, we will write "$X_N \asymp Y_N$" instead of $X_N - Y_N \to 0$ a.s. as $N \to \infty$.

3. LARGE RANDOM MATRICES AND SPIKED MODELS

We begin with some well-known results, concerning the convergence of the resolvent of $\mathbf{W}\mathbf{W}^H$, namely the matrix $\mathbf{Q}(x) = (\mathbf{W}\mathbf{W}^H - x\mathbf{I})^{-1}$.

Theorem 1 ([4]). *As $N \to \infty$, the smallest and largest eigenvalue of $\mathbf{W}\mathbf{W}^H$ converge respectively towards $\sigma^2(1-\sqrt{c})^2$ and $\sigma^2(1+\sqrt{c})^2$. Moreover, if $[a, b] \subset]\sigma^2(1+\sqrt{c})^2, +\infty[$, then, for deterministic vectors $\mathbf{c}, \mathbf{d}$ such that $\sup_N\{\|\mathbf{c}\|, \|\mathbf{d}\|\} < +\infty$, we have with probability one,*

$$\sup_{x\in[a,b]} \left|\mathbf{c}^H\mathbf{Q}(x)\mathbf{d} - m(x)\mathbf{c}^H\mathbf{d}\right| \to 0, \tag{2}$$

$$\sup_{x\in[a,b]} \left|\mathbf{c}^H\mathbf{Q}(x)\mathbf{W}\mathbf{d}\right| \to 0 \tag{3}$$

where

$$m(x) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(\lambda)}{\lambda - x} \tag{4}$$

is the Stieltjes transform of the Marcenko-Pastur distribution μ carried by $\mathrm{supp}(\mu) = [\sigma^2(1-\sqrt{c})^2, \sigma^2(1+\sqrt{c})^2]$. $m(x)$ is given by

$$m(x) = -\frac{x - \sigma^2(1-c) - \sqrt{(x - \sigma^2(1-c))^2 - 4\sigma^2 cx}}{2\sigma^2 cx}$$

for $x > \sigma^2(1+\sqrt{c})^2$.

Remark 1. *The same results hold for $\mathbf{Q}(x)^2$, and they are obtained by replacing $\mathbf{Q}(x)$ with $\mathbf{Q}(x)^2$ and $m(x)$ with its derivative $m'(x)$ in* (2), (3)*, e.g.* $\sup_{x\in[a,b]} |\mathbf{c}^H\mathbf{Q}(x)^2\mathbf{d} - m'(x)\mathbf{c}^H\mathbf{d}| \to 0$ *and* $\sup_{x\in[a,b]} |\mathbf{c}^H\mathbf{Q}(x)^2\mathbf{W}\mathbf{d}| \to 0$ *a.s. Similarly, for the co-resolvent $\tilde{\mathbf{Q}}(x) = (\mathbf{W}^H\mathbf{W} - x\mathbf{I})^{-1}$, we have, e.g., $\mathbf{c}^H\tilde{\mathbf{Q}}(x)\mathbf{d} \asymp \tilde{m}(x)\mathbf{c}^H\mathbf{d}$, with $\tilde{m}(x) = cm(x) - \frac{1-c}{x}$, the Stieltjes transform of the probability measure $\tilde{\mu} = c\mu + (1-c)\delta_0$. Note that from the identity $\tilde{\mathbf{Q}}\mathbf{W}^H = \mathbf{W}^H\mathbf{Q}$, we directly obtain $\sup_{x\in[a,b]} |\mathbf{c}^H\mathbf{W}^H\tilde{\mathbf{Q}}(x)\mathbf{d}| \to 0$ using eq.* (3).

Theorem 2 ([3],[5]). *Assume $\lambda_K > \sigma^2\sqrt{c}$. Then, as $N \to \infty$, for $k = 1, \ldots, K$, we have with probability one*

$$\hat{\lambda}_k \to \psi(\lambda_k) = \frac{(\lambda_k + \sigma^2 c)(\lambda_k + \sigma^2)}{\lambda_k}, \tag{5}$$

and for $k > K$ and each $\epsilon > 0$, $\hat{\lambda}_k \in [\sigma^2(1-\sqrt{c})^2 - \epsilon, \sigma^2(1+\sqrt{c})^2 + \epsilon]$ almost surely for all large N.

Remark 2. *Let $\Gamma(x) = xm(x)\tilde{m}(x)$. Then we have the equality $\lambda_k\Gamma(\psi(\lambda_k)) = 1$ for $k = 1, \ldots, K$.*

From the previous theorem, if the eigenvalues $\lambda_1, \ldots, \lambda_K$ are large enough, then for all $\epsilon > 0$, the sample eigenvalues $\hat{\lambda}_{M-K+1}, \ldots, \hat{\lambda}_M$ belong to $[\sigma^2(1-\sqrt{c})^2 - \epsilon, \sigma^2(1+\sqrt{c})^2 + \epsilon]$ while $\hat{\lambda}_1, \ldots, \hat{\lambda}_K$ will be bounded away of $\sigma^2(1+\sqrt{c})^2$ a.s. for all large N. This ensures that we can estimate the number of sources K (detection) by counting the sample eigenvalues above $\sigma^2(1+\sqrt{c})^2 + T_H$, with T_H a certain threshold value. Moreover, this theorem provides a way to consistently estimate the signal eigenvalues $\lambda_1, \ldots, \lambda_K$.

Theorem 3 ([3]). *Assume $\lambda_K > \sigma^2\sqrt{c}$. Then, for $k, l \leq K$, as $N \to \infty$, we have with probability one*

$$|\mathbf{u}_k^H\hat{\mathbf{u}}_l|^2 \to \begin{cases} \frac{m(\psi(\lambda_k))}{\lambda_k\Gamma'(\psi(\lambda_k))} & \text{if } k = l \\ 0 & \text{else} \end{cases}, \tag{6}$$

with $\Gamma'(x)$ the derivative of $\Gamma(x)$, satisfying $\Gamma'(x) < 0$ for $x > \sigma^2(1+\sqrt{c})^2$.

Proof. The proof can be found in [3], but we give here the main steps because our estimation approach strongly relies on it. First, note that from [6, Th. 7.3.7], we have for $k = 1, \ldots, K$

$$\begin{bmatrix} \mathbf{0} & \mathbf{\Sigma} \\ \mathbf{\Sigma}^H & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{u}}_k \\ \hat{\mathbf{v}}_k \end{bmatrix} = \hat{\lambda}_k \begin{bmatrix} \hat{\mathbf{u}}_k \\ \hat{\mathbf{v}}_k \end{bmatrix}.$$

Thanks to theorem 2, $\hat{\lambda}_k \to \psi(\lambda_k) > \sigma^2(1+\sqrt{c})^2$, and thus a.s., $\hat{\lambda}_k$ is not an eigenvalue of $\mathbf{W}\mathbf{W}^H$ for N large enough, because $\|\mathbf{W}\mathbf{W}^H\| \to \sigma^2(1+\sqrt{c})^2$ a.s. from theorem 1.

Therefore, using the block matrix inversion formula,

$$\begin{bmatrix}\hat{\mathbf{u}}_k\\ \hat{\mathbf{v}}_k\end{bmatrix} = -\begin{bmatrix}\mathbf{W}\tilde{\mathbf{Q}}(\hat{\lambda}_k)\mathbf{V}\mathbf{\Lambda}^{1/2} & \sqrt{\hat{\lambda}_k}\mathbf{Q}(\hat{\lambda}_k)\mathbf{U}\mathbf{\Lambda}^{1/2}\\ \sqrt{\hat{\lambda}_k}\tilde{\mathbf{Q}}(\hat{\lambda}_k)\mathbf{V}\mathbf{\Lambda}^{1/2} & \tilde{\mathbf{Q}}(\hat{\lambda}_k)\mathbf{W}^H\mathbf{U}\mathbf{\Lambda}^{1/2}\end{bmatrix}\begin{bmatrix}\mathbf{U}^H\hat{\mathbf{u}}_k\\ \mathbf{V}^H\hat{\mathbf{v}}_k\end{bmatrix}. \tag{7}$$

Let $\mathbf{c}\in\mathbb{C}^M,\mathbf{d}\in\mathbb{C}^N$ such that $\sup_N\{\|\mathbf{c}\|,\|\mathbf{d}\|\}<+\infty$. Using theorem 1 eq. (3) with the co-resolvent $\tilde{\mathbf{Q}}(\hat{\lambda}_k)$ (see remark 1), it is easy to show that $\mathbf{c}^H\mathbf{W}\tilde{\mathbf{Q}}(\hat{\lambda}_k)\mathbf{V}\mathbf{\Lambda}^{1/2}\mathbf{U}^H\hat{\mathbf{u}}_k \asymp 0$ and $\mathbf{d}^H\tilde{\mathbf{Q}}(\hat{\lambda}_k)\mathbf{W}^H\mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{V}^H\hat{\mathbf{v}}_k\asymp 0$. Consequently, the convergence $\hat{\lambda}_k\asymp\psi(\lambda_k)$ (theorem 2) and the uniform convergences in theorem 1 imply

$$\mathbf{c}^H\hat{\mathbf{u}}_k \asymp -\sqrt{\psi(\lambda_k)}m(\psi(\lambda_k))\mathbf{c}^H\mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{V}^H\hat{\mathbf{v}}_k, \tag{8}$$
$$\mathbf{d}^H\hat{\mathbf{v}}_k \asymp -\sqrt{\psi(\lambda_k)}\tilde{m}(\psi(\lambda_k))\mathbf{d}^H\mathbf{V}\mathbf{\Lambda}^{1/2}\mathbf{U}^H\hat{\mathbf{u}}_k, \tag{9}$$

which, by taking $\mathbf{c}=\mathbf{u}_l$ and $\mathbf{d}=\mathbf{V}\mathbf{\Lambda}^{1/2}\mathbf{U}^H\mathbf{u}_l$, lead to

$$\mathbf{u}_l^H\hat{\mathbf{u}}_k \asymp \psi(\lambda_k)m(\psi(\lambda_k))\tilde{m}(\psi(\lambda_k))\lambda_l\mathbf{u}_l^H\hat{\mathbf{u}}_k.$$

Since $m(\psi(\lambda_i))\tilde{m}(\psi(\lambda_i))\psi(\lambda_i))\lambda_j=1$ iff $i=j$ (see remark 2), we deduce $\mathbf{u}_l^H\hat{\mathbf{u}}_k\asymp 0$ for $k\neq l$. It can be shown similarly that $\mathbf{v}_l^H\hat{\mathbf{v}}_k\asymp 0$ if $k\neq l$. Now, getting back to (7), we have $\|\hat{\mathbf{u}}_k\|^2=\chi_1+\chi_2+\chi_3+\chi_3^*$ with

$$\chi_1=\hat{\mathbf{u}}_k^H\mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{V}^H\widetilde{\mathbf{Q}}(\hat{\lambda}_k)\mathbf{W}^H\mathbf{W}\widetilde{\mathbf{Q}}(\hat{\lambda}_k)\mathbf{V}\mathbf{\Lambda}^{1/2}\mathbf{U}^H\hat{\mathbf{u}}_k,$$
$$\chi_2=\hat{\lambda}_k\hat{\mathbf{v}}_k^H\mathbf{V}\mathbf{\Lambda}^{1/2}\mathbf{U}^H\mathbf{Q}(\hat{\lambda}_k)^2\mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{V}^H\hat{\mathbf{v}}_k,$$
$$\chi_3=\hat{\mathbf{u}}_k^H\mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{V}^H\widetilde{\mathbf{Q}}(\hat{\lambda}_k)\mathbf{W}^H\sqrt{\hat{\lambda}_k}\mathbf{Q}(\hat{\lambda}_k)\mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{V}^H\hat{\mathbf{v}}_k.$$

Since $\widetilde{\mathbf{Q}}(\hat{\lambda}_k)\mathbf{W}^H\mathbf{Q}(\hat{\lambda}_k)=\mathbf{W}^H\mathbf{Q}(\hat{\lambda}_k)^2$, theorem 1 eq. (2) applied to $\mathbf{Q}(\hat{\lambda}_k)^2$ (see remark 1) imply $\chi_3\asymp 0$. Using the identity $\tilde{\mathbf{Q}}(\hat{\lambda}_k)\mathbf{W}^H\mathbf{W}=\mathbf{I}+\hat{\lambda}_k\tilde{\mathbf{Q}}(\hat{\lambda}_k)$ and the convergences $\mathbf{u}_l^H\hat{\mathbf{u}}_k\asymp 0$ and $\mathbf{v}_l^H\hat{\mathbf{v}}_k\asymp 0$ for $k\neq l$, we obtain

$$\chi_1\asymp\lambda_k|\hat{\mathbf{u}}_k^H\mathbf{u}_k|^2\left(\mathbf{v}_k^H\tilde{\mathbf{Q}}(\hat{\lambda}_k)\mathbf{v}_k+\hat{\lambda}_k\mathbf{v}_k^H\tilde{\mathbf{Q}}(\hat{\lambda}_k)^2\mathbf{v}_k\right),$$
$$\chi_2\asymp\hat{\lambda}_k\lambda_k|\hat{\mathbf{v}}_k^H\mathbf{v}_k|^2\mathbf{u}_k^H\mathbf{Q}(\hat{\lambda}_k)^2\mathbf{u}_k.$$

Consequently, theorem 1 eq. (2) and remark 1 imply as above,

$$\chi_1\asymp\lambda_k\left(\tilde{m}(\psi(\lambda_k))+\psi(\lambda_k)\tilde{m}'(\psi(\lambda_k))\right)|\hat{\mathbf{u}}_k^H\mathbf{u}_k|^2,$$
$$\chi_2\asymp\lambda_k\psi(\lambda_k)m'(\psi(\lambda_k))|\hat{\mathbf{v}}_k^H\mathbf{v}_k|^2.$$

By taking $\mathbf{d}=\mathbf{v}_k$ in (9), we obviously have $|\mathbf{v}_k^H\hat{\mathbf{v}}_k|^2\asymp\psi(\lambda_k)\lambda_k\tilde{m}(\psi(\lambda_k))^2|\hat{\mathbf{u}}_k^H\mathbf{u}_k|^2$. Since $\hat{\mathbf{u}}_k$ is a unit norm vector, $\chi_1+\chi_2\asymp 1$ and we finally get

$$1\asymp|\hat{\mathbf{u}}_k^H\mathbf{u}_k|^2\Big[\lambda_k\tilde{m}(\psi(\lambda_k))+\lambda_k\psi(\lambda_k)\tilde{m}'(\psi(\lambda_k)) + \psi(\lambda_k)^2\lambda_k^2m'(\psi(\lambda_k))\tilde{m}(\psi(\lambda_k))^2\Big].$$

Using the equality $\lambda_k\psi(\lambda_k)m(\psi(\lambda_k))\tilde{m}(\psi(\lambda_k))=1$, we obtain the result of the theorem. From the integral representation of Stieltjes transform (see theorem 1 eq. (4)) applied to $m(x)$ and $\tilde{m}(x)$, it is easily seen that the function Γ is decreasing on the interval $]\sigma^2(1+\sqrt{c})^2,+\infty[$, and thus $\Gamma'(\psi(\lambda_k))<0$. □

4. MAIN RESULTS

Theorem 4. *Assume $\lambda_K>\sigma^2\sqrt{c}$. Then, as $N\to\infty$, for all $\mathbf{b}\in\mathbb{C}^M$ such that $\sup_N\|\mathbf{b}\|<+\infty$, we have with probability one*

$$\mathbf{b}^H\mathbf{U}\mathbf{U}^H\mathbf{b}\asymp\sum_{k=1}^K|\mathbf{b}^H\hat{\mathbf{u}}_k|^2\frac{\Gamma'(\hat{\lambda}_k)}{\Gamma(\hat{\lambda}_k)m(\hat{\lambda}_k)}.$$

Proof. Using equation (8) and the convergence $\mathbf{v}_k^H\hat{\mathbf{v}}_l\asymp 0$ iff $k\neq l$, given in the proof of theorem 3, we get

$$\mathbf{b}^H\hat{\mathbf{u}}_k\asymp-\sqrt{\psi(\lambda_k)}\sqrt{\lambda_k}m(\psi(\lambda_k))\mathbf{b}^H\mathbf{u}_k\mathbf{v}_k^H\hat{\mathbf{v}}_k.$$

Taking $\mathbf{d}=\mathbf{v}_k$ in equation (9), we obtain

$$\mathbf{v}_k^H\hat{\mathbf{v}}_k\asymp-\sqrt{\psi(\lambda_k)}\sqrt{\lambda_k}\tilde{m}(\psi(\lambda_k))\mathbf{u}_k\hat{\mathbf{u}}_k.$$

Therefore,

$$|\mathbf{b}^H\hat{\mathbf{u}}_k|^2\asymp \psi(\lambda_k)^2\lambda_k^2|\mathbf{b}^H\mathbf{u}_k|^2m(\psi(\lambda_k))^2\tilde{m}(\psi(\lambda_k))^2|\mathbf{u}_k^H\hat{\mathbf{u}}_k|^2.$$

Using the equality $1=\lambda_k\psi(\lambda_k)m(\hat{\lambda}_k)\tilde{m}(\hat{\lambda}_k)$ (see remark 2) and convergence (6) of theorem 3, we finally get

$$|\mathbf{b}^H\hat{\mathbf{u}}_k|^2\asymp|\mathbf{b}^H\mathbf{u}_k|^2\frac{\psi(\lambda_k)m(\psi(\lambda_k))^2\tilde{m}(\psi(\lambda_k))}{\Gamma'(\psi(\lambda_k))}.$$

The result of the theorem follows from $\hat{\lambda}_k\asymp\psi(\lambda_k)$ (th. 2). □

Given $D\in\mathbb{R}$, we assume henceforth that the steering vectors write $\mathbf{a}(\theta)=M^{-1/2}\left[\exp(-\mathbf{i}D\ell\sin(\theta))\right]_{\ell=0}^{M-1}$. Writing $\eta_{\text{spike}}(\theta)=\sum_{k=1}^K|\mathbf{a}(\theta)^H\hat{\mathbf{u}}_k|^2\frac{\Gamma'(\hat{\lambda}_k)}{\Gamma(\hat{\lambda}_k)m(\hat{\lambda}_k)}$, this theorem can be used to show that for any $k=1,\ldots,r$, there exists a local maximum $\hat{\theta}_k$ of $\eta_{\text{spike}}(\theta)$ such that $\hat{\theta}_k\to\theta_k$ almost surely. The next result that we provide without proof characterizes the fluctuations of this estimator:

Theorem 5 (CLT)**.** *Assume the setting of Theorem 4. Then the vector $\mathbf{e}_N=[\hat{\theta}_k-\theta_k]_{k=1}^K$ satisfies*

$$N^{3/2}\mathbf{e}_N\xrightarrow[N\to\infty]{\mathcal{L}}\mathcal{N}\left(0,\begin{bmatrix}\omega_1^2 & & \\ & \ddots & \\ & & \omega_K^2\end{bmatrix}\right)$$

where

$$\omega_k^2=\frac{6}{c^2D^2\cos(\theta_k)^2}\left(\frac{m'(\psi(\lambda_k))-m(\psi(\lambda_k))^2}{cm(\psi(\lambda_k))^2} + \lambda_k\left(m(\psi(\lambda_k))+\psi_k m'(\psi(\lambda_k))\right)\right)$$

5. DISCUSSIONS AND NUMERICAL RESULTS

In this section, we compare the performance of the traditional estimator η_{trad}, the spike estimator η_{spike}, and the recent estimator provided in [2] (referred as "G-MUSIC" estimator), also based on large random matrix results.

We consider steering vectors $\mathbf{a}(\theta)$ with $D = \pi$. The signals are realizations of mutually independent Gaussian AR(1) processes with correlation coefficient 0.9 and the SNR is defined here as $10\log(\sigma^{-2})$.

In experiment 1, we consider two sources located at $\theta_1 = 16°$ and $\theta_1 = 18°$. The number of antennas is $M = 20$ and the number of snapshots is $N = 40$. The "separation condition" ($\lambda_K > \sigma^2\sqrt{c}$) holds for all values of SNR between 6 dB and 30 dB. In figure 1, we evaluate by Monte-Carlo simulations the quantity $0.5(\mathbb{E}|\hat{\theta}_1 - \theta_1|^2 + \mathbb{E}|\hat{\theta}_2 - \theta_2|^2)$, which is the mean of the MSE of the two estimated angles, versus the SNR. The performance of the spike and G-MUSIC estimators are very close. In figure 2, we compute by Monte-Carlo simulations $\frac{1}{K}\sum_{k=1}^{K}\mathbb{E}|\hat{\eta}(\theta_k) - \eta(\theta_k)|^2$, i.e the mean over the MSE of the localization function, evaluated at the true angles. For an SNR greater that 10 dB, the performance of the spike estimator is close once again to G-MUSIC.

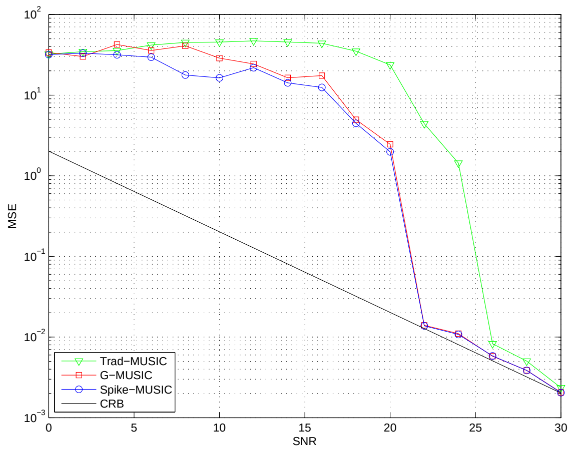

Fig. 1. Mean of the MSE of the two estimated angles.

In experiment 2, we consider $K = 5$ sources located at $-20°$, $-10°$, $0°$, $10°$ and $20°$ and M and N are still equal to 20 and 40. The separation condition is verified for all values of SNR between 10 dB and 30 dB. In figure 3, we plot the same graph as in figure 2. We notice that the spike estimator is not close anymore to the G-MUSIC because the ratio K/N is not small enough. However, it still outperforms the traditional MUSIC estimator.

Concerning the complexity issue, the spike MUSIC and the G-MUSIC are comparable, the latter requiring one additional SVD of a $M \times M$ matrix. Yet, the spike approach is interesting thanks to the simplicity of its analysis which will make it applicable to many estimation problems.

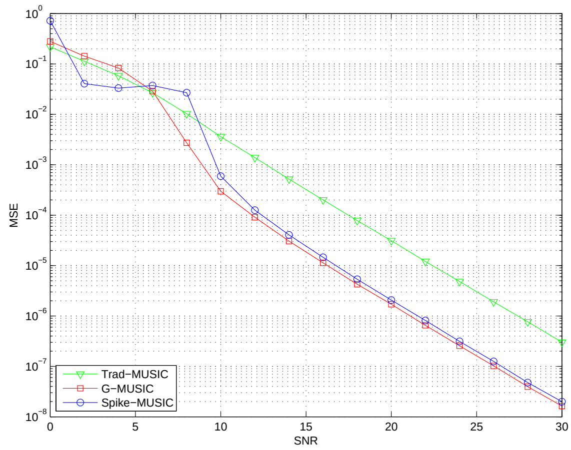

Fig. 2. Mean of the MSE (localization function), $K = 2$.

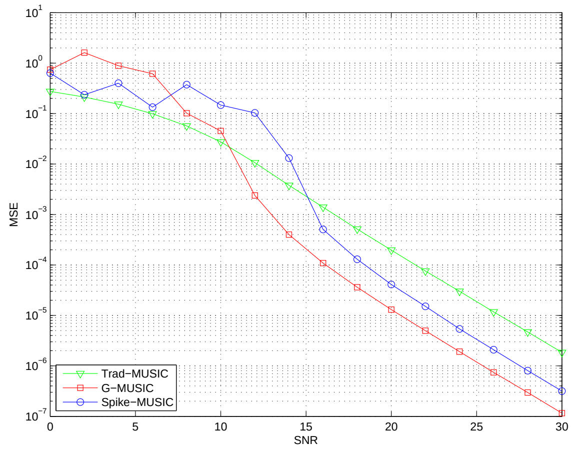

Fig. 3. Mean of the MSE (localization function), $K = 5$.

6. REFERENCES

[1] X. Mestre and M. Lagunas, “Modified subspace algorithms for DoA estimation with large arrays,” *IEEE Transactions on Signal Processing*, vol. 56, no. 2, p. 598, 2008.

[2] P. Vallet, P. Loubaton, and X. Mestre, “Improved Subspace Estimation for Multivariate Observations of High Dimension: The Deterministic Signal Case,” *submitted*, 2010, arXiv: 1002.3234.

[3] F. Benaych-Georges and R. Rao, “The eigenvalues and eigenvectors of finite, low rank perturbations of large random matrices,” *submitted*, 2009, arXiv: 0910.2120.

[4] Z. Bai and J. Silverstein, *Spectral analysis of large dimensional random matrices*. Springer Verlag, 2010.

[5] P. Loubaton and P. Vallet, “Almost sure localization of the eigenvalues in a gaussian information plus noise model. Application to the spiked models.” *submitted*, 2010, arXiv: 1009.5807.

[6] R. Horn and C. Johnson, *Matrix analysis*. Cambridge University Press, 1990.