Heavy-tailed random matrices and the Poisson Weighted Infinite Tree

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PART I : RANDOM MATRICES

SPECTRAL MEASURE

Let $X = (X_{ij})_{1 \le i,j \le n}$ be a $n \times n$ complex matrix. Let $\lambda_1, \cdots, \lambda_n$ be its eigenvalues, the spectral measure of X is

$$\mu_X = \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k}.$$

- (random hermitian) : the array $(X_{ij})_{i \ge j \ge 1}$ is i.i.d., $X_{ij} = \bar{X}_{ji}$,
- (random non-hermitian) : the array $(X_{ij})_{i,j \ge 1}$ is i.i.d..

$\Longrightarrow$ As n goes to infinity, does the spectral measure converge ?

WIGNER'S SEMI-CIRCULAR LAW

Theorem 1. *If* $\mathbb{E}X_{11} = 0, \mathbb{E}|X_{11}|^2 = 1$ *and*

$$A_n = X/\sqrt{n},$$

then, almost surely,

$$\mu_{A_n} \Longrightarrow \mu_{sc},$$

where $\mu_{sc}(dx) = \frac{1}{2\pi}\sqrt{4 - x^2}dx$.

GIRKO'S CIRCULAR LAW

Theorem 2 (Tao & Vu (2008))**.** *If* $\mathbb{E}X_{11} = 0, \mathbb{E}|X_{11}|^2 = 1$ *and*

$$A_n = X/\sqrt{n},$$

then, almost surely,

$$\mu_{A_n} \Longrightarrow \textit{Unif}(D).$$

where $\textit{Unif}(D)$ *is the uniform distribution on the unit complex disc.*

$\Longrightarrow$ Found by Girko, earlier versions due to Edelman, Bai, Pan & Zhou, Götze & Tikhomirov ...

HEAVY-TAILED ENTRIES

We now assume that

$$\mathbb{P}(|X_{11}| > t) \sim t^{-\alpha}$$

for some

$$0 < \alpha < 2.$$

Define

$$A_n = X/n^{1/\alpha},$$

$\Longrightarrow$ In the hermitian and non-hermitian cases, does μ_{A_n} converge to a measure μ ?

HERMITIAN CASE

Theorem 3 (Ben Arous & Guionnet, 2008)**.** *There exists a probability measure* μ_{bc} *depending only on* α *such that, with the above assumptions, almost surely,*

$$\mu_{A_n} \Longrightarrow \mu_{bc}.$$

$\Longrightarrow$ Found non-rigourously by Bouchaud-Cizeau (1994).

PROPERTIES OF THE LIMIT MEASURE

Theorem 4. *For all* $0 < \alpha < 2$*, the probability measure* μ_{bc}

(i) is symmetric and has a bounded density f_{bc} *on* $\mathbb{R}$*,*

(ii) $f_{bc}(0) = \frac{1}{\pi}\Gamma\left(1 + \frac{2}{\alpha}\right)\left(\frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)}\right)^{\frac{1}{\alpha}}$,

(iii) $f_{bc}(t) \sim_{t\to\infty} \frac{\alpha}{2} t^{-\alpha-1}$.

$\Longrightarrow$ Summarizes properties obtained by Ben Arous & Guionnet, Belinschi, Dembo & Guionnet, Bordenave, Caputo & Chafaï.

NON-HERMITIAN CASE

Theorem 5. *Assume that* X_{11} *has a bounded density on* $\mathbb{R}$ *or* $\mathbb{C}$*, and is asymptotically radial :*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{X_{11}}{|X_{11}|} \in \cdot \;\middle|\; |X_{11}| \geq t \right) = \theta,$$

for some probability distribution θ *on* S^1*.*

Then, there exists a probability measure μ *depending only on* α *such that, with the above assumptions, almost surely,*

$$\mu_{A_n} \Longrightarrow \mu.$$

PROPERTIES OF THE LIMIT SPECTRAL MEASURE

Theorem 6. *The measure μ has radial bounded density $\mu(dz) = f(|z|)dz$, where*

$$f(0) = \frac{1}{\pi} \frac{\Gamma(1+\frac{2}{\alpha})^2 \Gamma(1+\frac{\alpha}{2})^{\frac{2}{\alpha}}}{\Gamma(1-\frac{\alpha}{2})^{\frac{2}{\alpha}}},$$

and as $r \to \infty$

$$f(r) \sim c\, r^{2(\alpha-1)} e^{-\frac{\alpha}{2} r^{\alpha}}.$$

PART II : OBJECTIVE METHOD AND LOCAL OPERATOR CONVERGENCE

HERMITIAN CASE : SPECTRAL MEASURE AT A VECTOR

There exists a probability measure on $\mathbb{R}$ such that, for all t integers,

$$(e_k, A_n^t e_k) = \int x^t d\mu_{A_n}^{(k)},$$

$$\mu_{A_n} = \frac{1}{n} \sum_{k=1}^{n} \mu_{A_n}^{(k)} \quad \text{and} \quad \mu_{A_n}^{(k)} = \sum_{i=1}^{n} |(e_k, u_i)|^2 \delta_{\lambda_i}.$$

The spectral measure at a vector is well defined for all self-adjoint operators.

HERMITIAN CASE : REDUCTION TO LOCAL CONVERGENCE

$\Longrightarrow$ By exchangeability, we get

$$\mathbb{E}\mu_{A_n} = \mathbb{E}\mu_{A_n}^{(1)}$$

$\Longrightarrow$ From basic concentration inequality, $\mu_{A_n} - \mathbb{E}\mu_{A_n}$ converges a.s. to 0.

$\Longrightarrow$ It is enough to get the convergence of $\mu_{A_n}^{(1)}$.

HERMITIAN CASE : LOCAL OPERATOR CONVERGENCE

We look for a random operator A defined in $L^2(V)$ for some countable set V such that there exists a sequence of bijections $\sigma_n : V \to \mathbb{N}$, $\varnothing \in V$, $\sigma_n(\varnothing) = 1$ and, for all $\phi \in L^2(V)$ with compact support, weakly,

$$\sigma_n^{-1} A_n \sigma_n \phi \to A\phi.$$

If A is self-adjoint, then it would imply that

$$\mu_{A_n}^{(1)} \to \mu_A^{(\varnothing)}.$$

NON-HERMITIAN CASE

The above strategy does not work. The spectral measure at a vector does not exist for non-normal matrices.

The local operator convergence is not sufficient.

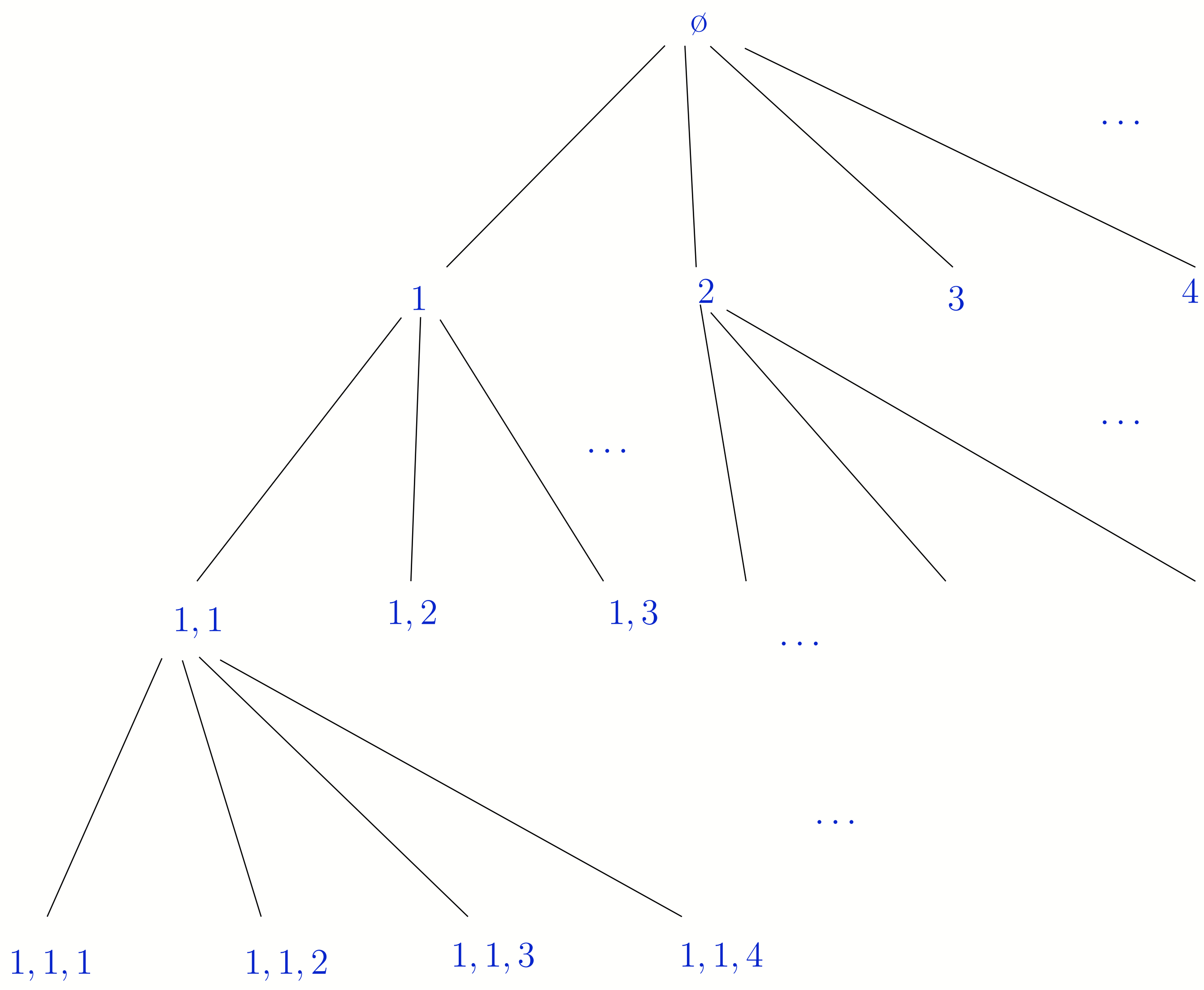
$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ & & \cdots & & \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \quad \text{vs} \quad \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ & & \cdots & & \\ 1 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

An additional ingredient is needed. (For the moment, we skip this very important part).

PART III : CONVERGENCE TO POISSON WEIGHTED INFINITE TREE

ALDOUS' PWIT

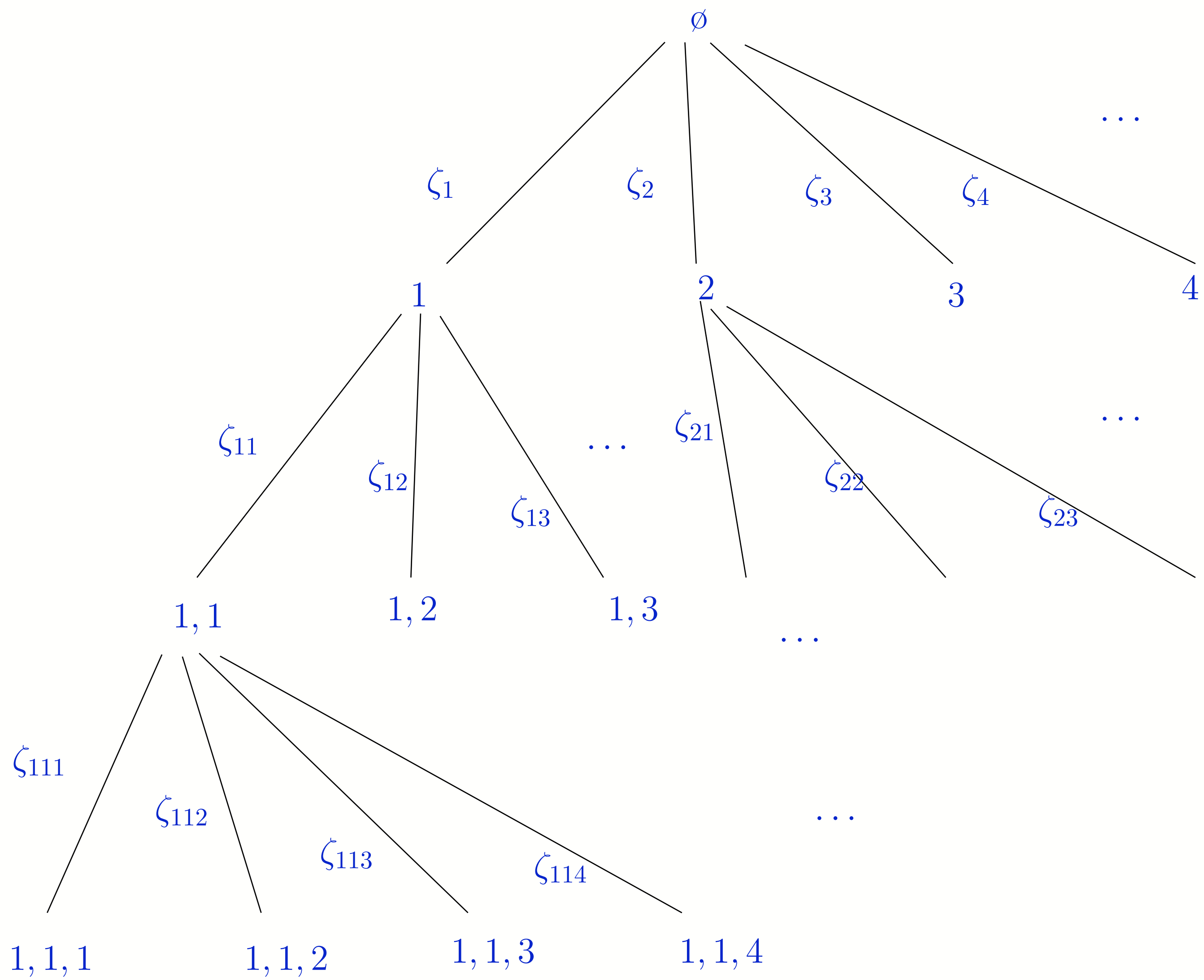
Let $V = \cup_{k\in\mathbb{N}}\mathbb{N}^k$ with $\mathbb{N}^0 = \emptyset$. Consider the infinite tree on V:



ALDOUS' PWIT

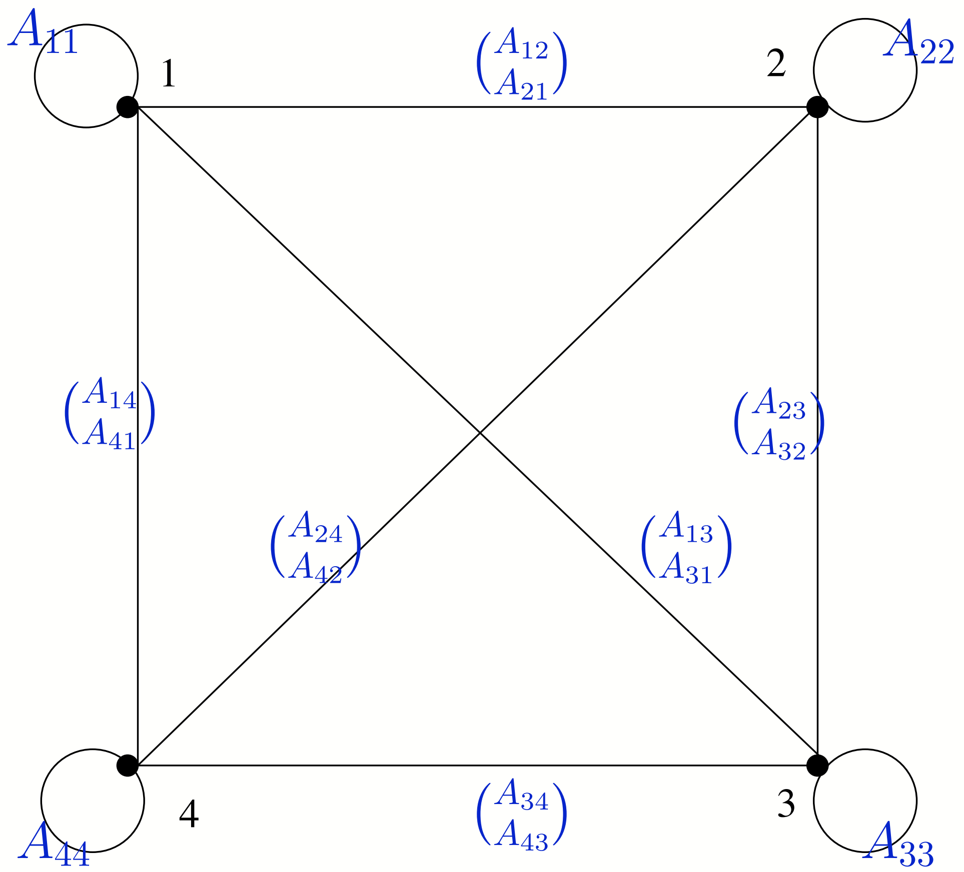
Let $(Z_v)_{v\in V}$ be iid Poisson processes of intensity λ on $\mathbb{R}_+$,

$Z_v = \{0 \le \zeta_{v1} \le \zeta_{v2} \le \cdots\}$



GRAPHIC REPRESENTATION OF A MATRIX

We think of the matrix A_n as an oriented weighted graph on n vertices.



ORDERED STATISTICS

The vector $\left(\binom{A_{11}}{A_{11}}, \binom{A_{12}}{A_{21}}, \cdots, \binom{A_{1n}}{A_{n1}}\right)$ is reordered non-increasingly in

$$\left(\begin{pmatrix} A_{1\sigma(1)} \\ A_{\sigma(1)1} \end{pmatrix}, \begin{pmatrix} A_{1\sigma_1(2)} \\ A_{\sigma(2)1} \end{pmatrix}, \cdots, \begin{pmatrix} A_{1\sigma(n)} \\ A_{\sigma(n)1} \end{pmatrix}\right)$$

with

$$\|\begin{pmatrix} A_{1\sigma(1)} \\ A_{\sigma(1)1} \end{pmatrix}\|_1 \geq \|\begin{pmatrix} A_{1\sigma(2)} \\ A_{\sigma(2)1} \end{pmatrix}\|_1 \geq \cdots .$$

$\Longrightarrow$ We restrict ourselves to non-hermetian case and non-negative random variables.

CONVERGENCE OF ORDERED STATISTICS

$$\left(\begin{pmatrix} A_{1\sigma(1)} \\ A_{\sigma(1)1} \end{pmatrix}, \begin{pmatrix} A_{1\sigma(2)} \\ A_{\sigma(2)1} \end{pmatrix}, \dots , \begin{pmatrix} A_{1\sigma(n)} \\ A_{\sigma(n)1} \end{pmatrix} \right)$$

converges to

$$\left(\begin{pmatrix} \varepsilon_1 \\ 1-\varepsilon_1 \end{pmatrix} \zeta_1^{-\frac{1}{\alpha}}, \begin{pmatrix} \varepsilon_2 \\ 1-\varepsilon_2 \end{pmatrix} \zeta_2^{-\frac{1}{\alpha}}, \dots \right),$$

where $(\zeta_k)_{k\geq 1}, \zeta_1 \leq \zeta_2 \leq \cdots$, is a Poisson point process of intensity

$$\Lambda(dx) = 2\mathbb{1}_{x>0} dx$$

and (ε_k) iid $\mathrm{Ber}(1/2)$ random variables.

CONVERGENCE OF ORDERED STATISTICS

For fixed i, the vector

$$\left(\begin{pmatrix} A_{j\sigma(i)} \\ A_{\sigma(i)j} \end{pmatrix}\right)_{j\neq 1}$$

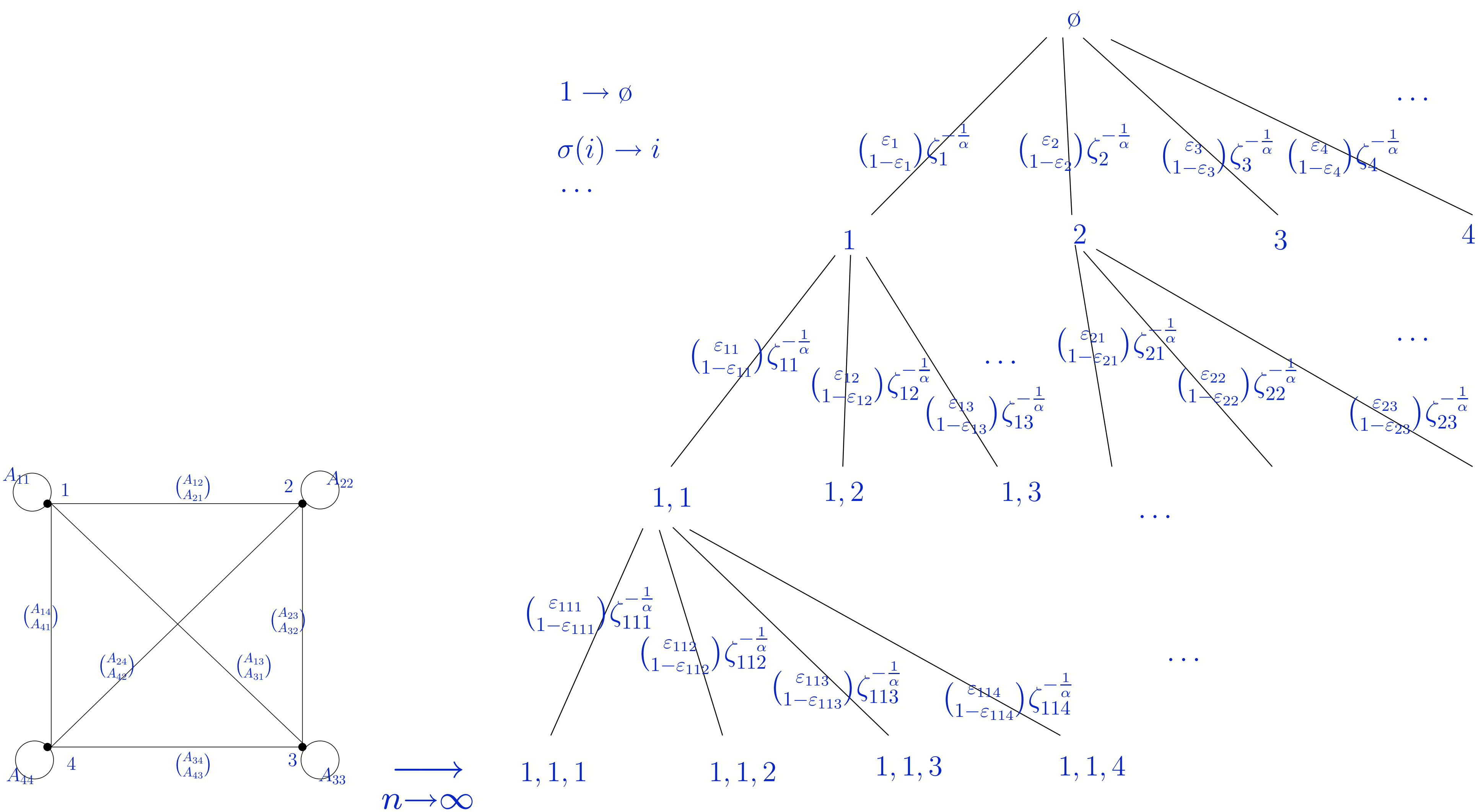
is reordered non-increasingly.

It converges again to

$$\left(\begin{pmatrix} \varepsilon_{i1} \\ 1-\varepsilon_{i1} \end{pmatrix}\zeta_{i1}^{-\frac{1}{\alpha}}, \begin{pmatrix} \varepsilon_{i2} \\ 1-\varepsilon_{i2} \end{pmatrix}\zeta_{i2}^{-\frac{1}{\alpha}}, \dots\right),$$

where $(\zeta_{ik})_{k\geq 1}$ are independent Poisson processes of intensity Λ and $(\varepsilon_{ik})_{i,k}$ iid $\mathrm{Ber}(1/2)$ r.v.

LOCAL CONVERGENCE TO ALDOUS' PWIT



OPERATOR ON THE PWIT

Define the operator on compactly supported function of $L^2(V)$,

$$A\delta_v = \sum_{k\geq 1}(1-\varepsilon_{vk})\zeta_{vk}^{-\frac{1}{\alpha}}\delta_{vk} + \varepsilon_v\zeta_v^{-\frac{1}{\alpha}}\delta_{a(v)},$$

where $a(v)$ is the ancestor of $v \neq \emptyset$.

$\Longrightarrow$ There exists a sequence of bijections $\sigma_n : V \to \mathbb{N}$, $\emptyset \in V$, $\sigma_n(\emptyset) = 1$ and, for all $\phi \in L^2(V)$ with compact support, weakly,

$$\sigma_n^{-1} A_n \sigma_n \phi \to A\phi.$$

HERMITIAN CASE : OPERATOR ON THE PWIT

In the hermitian case, the operator is defined similarly, we simply forget about, the ε_v's :

$$A\delta_v = \sum_{k\geq 1} \zeta_{vk}^{-\frac{1}{\alpha}} \delta_{vk} + \zeta_v^{-\frac{1}{\alpha}} \delta_{a(v)}.$$

$\longrightarrow$ Again, for all $\phi \in L^2(V)$ with compact support, weakly, $\sigma_n^{-1} A_n \sigma_n \phi \to A\phi.$

Theorem 7. *With probability one, the operator A is (essentially) self-adjoint.*

$\Longrightarrow$ As a corollary, we obtain the convergence of μ_{A_n} to $\mu_{bc} := \mathbb{E}\mu_A^{(\emptyset)}$.

HERMITIAN CASE : RECURSIVE DISTRIBUTIONAL EQUATION

The resolvent formula and the recursive structure of the PWIT implies a RDE for, $z \in \mathbb{C}_+ = \{z \in \mathbb{C} : \Im(z) > 0\}$,

$$g_\varnothing(z) := \langle \delta_\varnothing, (A - z)^{-1} \delta_\varnothing \rangle$$

$$g_\varnothing \stackrel{d}{=} -\left(z + \sum_{k \in \mathbb{N}} \xi_k g_k \right)^{-1},$$

where $g_\varnothing$, $(g_k)_{k \in \mathbb{N}}$ are i.i.d. independent of $\{\xi_k\}_{k \in \mathbb{N}}$, a independent Poisson point process of $\mathbb{R}_+$ with intensity $\frac{\alpha}{2} x^{-\frac{\alpha}{2}-1} dx$.

RECURSIVE DISTRIBUTIONAL EQUATION

If S is a positive $\alpha/2$-stable random variable,

$$\sum_{k\in\mathbb{N}} \xi_k g_k \stackrel{d}{=} \mathbb{E}[g_\varnothing^{\frac{\alpha}{2}}]^{\frac{2}{\alpha}} S.$$

$\Longrightarrow$ The RDE can be solved in terms of a scalar fixed point equation for $\mathbb{E}[g_\varnothing^{\frac{\alpha}{2}}]^{\frac{2}{\alpha}}$.

$\Longrightarrow$ Since g is the Cauchy-Stieltjes transform of $\mu_\varnothing$, we deduce the properties of $\mu_{bc} = \mathbb{E}\mu_A^{(\varnothing)}$.

PART IV : CONVERGENCE IN THE NON-HERMITIAN CASE

SINGULAR VALUES

$0 \leq \sigma_n \leq \cdots \leq \sigma_1$: square roots of the eigenvalues of A^*A.

Since $|\det A| = \sqrt{\det(A^*A)}$,

$$\prod_{k=1}^{n} |\lambda_k| = \prod_{k=1}^{n} \sigma_k$$

For $z \in \mathbb{C}$, let $\sigma_n(z) \leq \cdots \leq \sigma(z)$ be the singular values of $A - z$ and

$$\nu_A(z) = \frac{1}{n} \sum_{k=1}^{n} \delta_{\sigma_k(z)}.$$

$\Longrightarrow$ for all $z \in \mathbb{C} \backslash \mathrm{supp}(\mu_A)$,

$$U_{\mu_A}(z) = \int_{\mathbb{C}} \ln|\lambda - z| \mu_A(d\lambda) \quad = \quad \int_{\mathbb{R}_+} \ln(x) \nu_A(z, dx)$$

LOGARITHMIC POTENTIAL

$$U_\mu(z) = \int_{\mathbb{C}} \ln|\lambda - z| \mu(d\lambda).$$

Define

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Laplacian differential operator

$$\Delta = \bar{\partial}\partial = \frac{1}{4}\left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right).$$

In $\mathcal{D}'(\mathbb{C})$,

$$\Delta U_\mu = \pi\mu.$$

$\Longrightarrow$ The logarithmic potential characterizes the measure.

GIRKO'S METHOD

Theorem 8 (Criterion of convergence)**.** *Let* (A_n) *be a sequence of matrices. Assume that for almost all* $z \in \mathbb{C}$*,*

(i) $\nu_{A_n}(z)$ *converges weakly to* $\nu(z)$*, a probability measure on* $\mathbb{R}_+$*.*

(ii) $\int \ln(x) \nu_{A_n}(z, dx)$ *is uniformly integrable.*

Then there exists a probability measure μ *on* $\mathbb{C}$*, such that for almost all* $z \in \mathbb{C}$*,* $U_\mu(z) = \int \ln(x) \nu(z, dx)$ *and* μ_{A_n} *converges weakly to* μ*.*

CONVERGENCE OF THE SINGULAR VALUES

Theorem 9. *For all $z \in \mathbb{C}$, there exists a measure $\nu(z, \cdot)$, depending only on α and z, such that almost surely*

$$\lim_n \nu_{A_n}(z) = \nu(z).$$

(For $z = 0$, Belinschi, Dembo & Guionnet (2009). For an explanation of this result, wait a few slides)

UNIFORM INTEGRABILITY

Bai (1999) has developped the first method to prove the uniform integrability of

$$\int \ln(x)\nu_{A_n}(z,dx) = \frac{1}{n}\sum_{i=1}^{n} \ln \sigma_i(z).$$

Here, we adapt the argument of Tao & Vu (2008) for the circular law.

For the large singular values, we use the inequality, for any $0 < p \leq 2$,

$$\sum_{i=1}^{n} \sigma_i^p \leq \sum_{i=1}^{n} \|R_i\|_2^p = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |A_{ij}|^2\right)^{\frac{p}{2}},$$

where $R_1, \cdots, R_n$ are rows of A.

UNIFORM INTEGRABILITY

For the smallest singular value, the bounded density assumption implies easily, for some $p \geq 1$, almost surely,

$$\sigma_n(z) = \Omega\left(n^{-p}\right).$$

$\Longrightarrow$ we may lower bound $\sigma_{n-i}(z)$ by n^{-p} for $0 \leq i \leq n^{1-\gamma}$.

For the moderately small singular values, Tao & Vu prove that almost surely, for all $n^{1-\gamma} \leq i \leq n$,

$$\sigma_{n-i}(z) = \Omega\left(\frac{i}{n}\right).$$

We only prove that, in a weaker sense, that

$$\sigma_{n-i}(z) = \Omega\left(\frac{i}{n}\right)^{\frac{1}{\alpha}+\frac{1}{2}}.$$

A NEW LOOK AT THE SPECTRAL MEASURE

We want to study the limit spectral measure μ.

The measure $\nu(z)$ is not explicit and we only know that

$$U_\mu(z) = \int \ln(x)\nu(z, dx)$$

and

$$\mu = \frac{1}{\pi}\Delta U_\mu.$$

$\Longrightarrow$ We need another characterization of the spectral measure : some have appeared in the physics literature, Feinberg & Zee, Jarosz & Nowak, Rogers & Castillo...

BIPARTIZATION

Define

$$B_{ij} = \begin{pmatrix} 0 & A_{ij} \\ \bar{A}_{ji} & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}).$$

$\Longrightarrow B = (B_{ij})_{1\le i,j\le n}$ is an hermitian matrix in $\mathcal{M}_n(\mathcal{M}_2(\mathbb{C})) \simeq \mathcal{M}_{2n}(\mathbb{C})$.

BIPARTIZATION

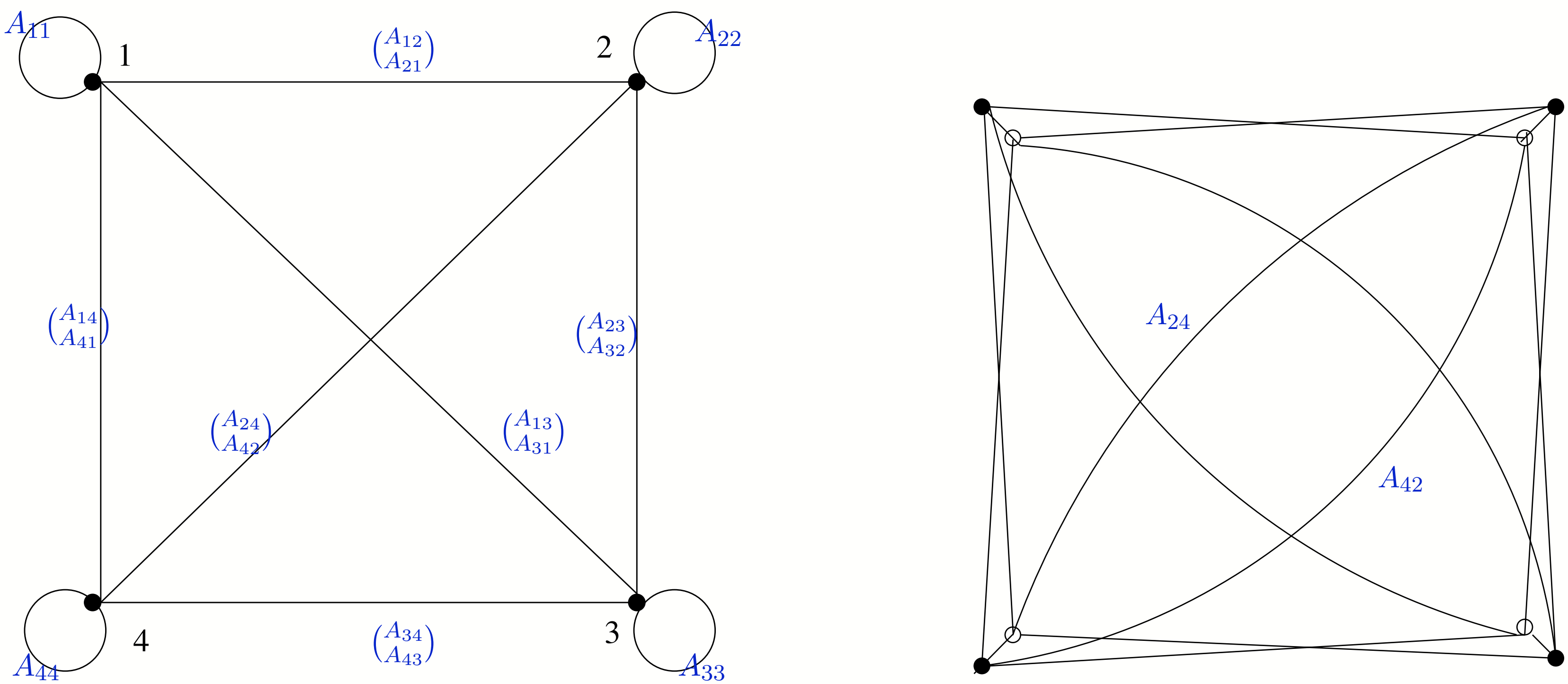


Figure 1: Graphical interpretation of bipartization.

RESOLVENT

Define the quaternionic set $\mathbb{H}_+ = \left\{ U = \begin{pmatrix} \eta & z \\ \bar{z} & \eta \end{pmatrix}, \eta \in \mathbb{C}_+, z \in \mathbb{C} \right\}$.

Resolvent matrix :

$$R = (B - U \otimes I_n)^{-1} \qquad \in \ \mathcal{M}_n(\mathcal{M}_2(\mathbb{C})),$$

$$B - U \otimes I_n = \begin{pmatrix} B_{11} - U & B_{12} & \cdots \\ B_{12}^* & B_{22} - U & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}.$$

FROM THE RESOLVENT TO THE SINGULAR VALUES

$$R_{kk}(U) = \begin{pmatrix} a_k & b_k \\ b'_k & c_k \end{pmatrix}, \qquad \text{with} \quad U = \begin{pmatrix} \eta & z \\ \bar{z} & \eta \end{pmatrix}.$$

For $\nu_A(z)$: the trace of R is the Cauchy-Stieltjes transform of $\check{\nu}_A(z)$,

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2}(a_k + c_k) = \int \frac{1}{x-\eta}\check{\nu}_A(z,dx),$$

$$\check{\nu}_A(z) = \frac{1}{2n}\sum_{i=1}^{n}\delta_{\sigma_i(z)} + \delta_{-\sigma_i(z)}$$

FROM THE RESOLVENT TO THE SPECTRAL MEASURE

For μ_A : in $\mathcal{D}'(\mathbb{C})$,

$$\mu_A = -\frac{1}{\pi n}\sum_{k=1}^{n} \partial b_k(\cdot, 0) = \lim_{\eta \downarrow 0} -\frac{1}{\pi n}\sum_{k=1}^{n} \partial b_k(\cdot, \eta).$$

(Similar computation in Rogers & Costillo (2009))

LOCAL OPERATOR CONVERGENCE

$\Longrightarrow$ By exchangeability, we get

$$\mathbb{E}\mu_{A_n} = \lim_{\eta \downarrow 0} -\frac{1}{\pi} \partial \mathbb{E} b_1(\cdot, \eta).$$

$\Longrightarrow$ It is enough to get the convergence of $R_{11}(U)$.

The local convergence of A_n to an operator A

$$\sigma_n^{-1} A_n \sigma_n \phi \to A\phi$$

implies the local convergence of B_n to B, the bipartized operator of A.

LOCAL OPERATOR CONVERGENCE

We show that B is (essentially) self-adjoint $\Longrightarrow$ convergence of $\nu_{A_n}(z)$.

+ Uniform integrability, we have $\mu = \lim_{t \downarrow 0} -\frac{1}{\pi}\partial \mathbb{E} b(\cdot, it)$, where

$$R_{\emptyset\emptyset}(U) = \begin{pmatrix} a & b \\ b' & c \end{pmatrix},$$

and

$$R = (B - U \otimes I)^{-1}.$$

RECURSIVE DISTRIBUTIONAL EQUATION

The resolvent formula and the recursive structure of the PWIT implies a RDE for

$$R_{øø}(U) = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}.$$

$$\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \stackrel{d}{=} -\left(U + \sum_{k \in \mathbb{N}} \begin{pmatrix} \xi'_k c_k & 0 \\ 0 & \xi_k a_k \end{pmatrix} \right)^{-1},$$

where a, c, $(a_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$ are i.i.d. independent of $\{\xi_k\}_{k\in\mathbb{N}}$, $\{\xi'_k\}_{k\in\mathbb{N}}$ two independent Poisson point processes of $\mathbb{R}_+$ with intensity $\frac{\alpha}{2} x^{-\frac{\alpha}{2}-1} dx$.

RECURSIVE DISTRIBUTIONAL EQUATION

For $\eta = it$, $a = ih$ is pure imaginary and

$$h \stackrel{d}{=} \frac{t + \sum_{k\in\mathbb{N}} \xi_k h_k}{|z|^2 + \left(t + \sum_{k\in\mathbb{N}} \xi_k h_k\right)\left(t + \sum_{k\in\mathbb{N}} \xi'_k h'_k\right)}.$$

If S is a positive $\alpha/2$-stable random variable,

$$\sum_{k\in\mathbb{N}} \xi_k h_k \stackrel{d}{=} \mathbb{E}[h_1^{\frac{\alpha}{2}}]^{\frac{2}{\alpha}} S.$$

$\Longrightarrow$ The RDE can be solved in terms of a scalar fixed point equation for $\mathbb{E}[h_1^{\frac{\alpha}{2}}]^{\frac{2}{\alpha}}$.

$\Longrightarrow$ From $\mu = \lim_{t\downarrow 0} -\frac{1}{\pi}\partial \mathbb{E} b_1(\cdot, it)$, we get the properties of μ.

IN SUMMARY

- The objective method is an efficient framework to deal with sparse random matrices.
- Dependencies in the entries are allowed : all computations are done in the limit operator.
- In other sparse cases : how to prove the uniform integrability ?
- What about eigenvectors ? analogs of local Wigner's theorem ?

OPEN PROBLEM

Let G_n be a k-oriented regular graph on $\{1, \cdots, n\}$, drawn uniformly.

Consider its adjacency matrix

$$(A_n)_{ij} = \mathbb{1}(i \to j).$$

The limit operator is the adjacency operator of the k-oriented regular infinite tree. The computation on the 2×2 resolvent shows that

$$\mu(dz) = \frac{1}{\pi} \frac{k^2(k-1)}{(k^2 - |z|^2)^2} \mathbb{1}_{|z|<\sqrt{k}} dz.$$

(= Brown's measure of the free sum of k Haar unitary, Haagerup and Larsen (2000))

$\Longrightarrow$ How to prove the uniform integrability of the spectral measure in this case ?