Eigenvectors of some large sample covariance matrix ensembles

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- Supp(H) bounded away from 0 and $+\infty$

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Marčenko and Pastur (1967), Silverstein (1995):

$$\exists F \quad \text{s.t.} \quad F_N(\lambda) \xrightarrow{\text{a.s.}} F(\lambda)$$

at all points of continuity of F

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Inversion formula: if F is continuous at a and b:

$$F(b) - F(a) = \lim_{\eta \to 0^+} \frac{1}{\pi} \int_a^b \mathrm{Im}\left[m_F(\xi + i\eta) \right] d\xi$$

MP67/Silverstein (1995) Equation

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$\forall z \in \mathbb{C}^+$, $m = m_F(z)$ is the unique solution in $\{m \in \mathbb{C} : \frac{c-1}{z} + cm \in \mathbb{C}^+\}$ to

$$m = \int_{-\infty}^{+\infty} \frac{1}{\tau\left(1 - c - czm\right) - z} dH(\tau)$$

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- $\forall \lambda \in \mathbb{R} - \{0\}$, $\lim_{z \in \mathbb{C}^+ \to \lambda} m_F(z) \equiv \breve{m}_F(\lambda)$ exists
- F has continuous derivative $F' = \frac{1}{\pi}\mathrm{Im}\left[\breve{m}_F\right]$ on $\mathbb{R} - \{0\}$

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- The higher the c, the more spreading there is
- (Not obvious) Large eigenvalues get more spread out than small ones

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$$\Theta_N^g(z) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\lambda_i - z}\sum_{j=1}^{N}|u_i^* v_j|^2 \times g(\tau_j)$$

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$$g(\tau) \equiv 1 \quad \Longleftrightarrow \quad \Theta_N^g = m_{F_N}$$

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\begin{aligned}
m_{F_N}(z) &= \frac{1}{N}\mathrm{Tr}\left[(S_N - zI)^{-1}\right] \\
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Same integration kernel!

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$$\Omega_N^g(\lambda) \overset{\text{a.s.}}{\to} \Omega^g(\lambda) = \lim_{\eta\to 0^+}\frac{1}{\pi}\int_{-\infty}^{\lambda}\mathrm{Im}\left[\Theta^g\left(l + i\eta\right)\right]dl$$

wherever Ω^g is continuous

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$$N|u_i^* v_j|^2 \approx \frac{c\lambda_i\tau_j}{|\tau_j\left[1-c-c\lambda_i\breve{m}_F(\lambda_i)\right]-\lambda_i|^2}$$

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$$|P_k u_i|^2 \approx \frac{n_k c \lambda_i t_k}{N \left| t_k \left[1 - c - c\lambda_i \breve{m}_F(\lambda_i) \right] - \lambda_i \right|^2}$$

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Solution:

$$\widetilde{D}_N = \mathrm{Diag}(\widetilde{d}_1, \ldots, \widetilde{d}_N) \quad \text{where} \quad \widetilde{d}_i = u_i^* \, \Sigma_N \, u_i$$

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$$u_i^* \Sigma_N u_i \approx \frac{\lambda_i}{|1 - c - c\lambda_i \breve{m}_F(\lambda_i)|^2}$$

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Percentage Relative Improvement in Average Loss:

$$PRIAL = 100 \times \left[1 - \frac{\mathbb{E} \left\| \widetilde{S}_N - U_N \widetilde{D}_N U_N^* \right\|^2}{\mathbb{E} \left\| S_N - U_N \widetilde{D}_N U_N^* \right\|^2} \right]$$

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Compare with Ledoit-Wolf (2004) *linear* shrinkage estimator

Simulation Results

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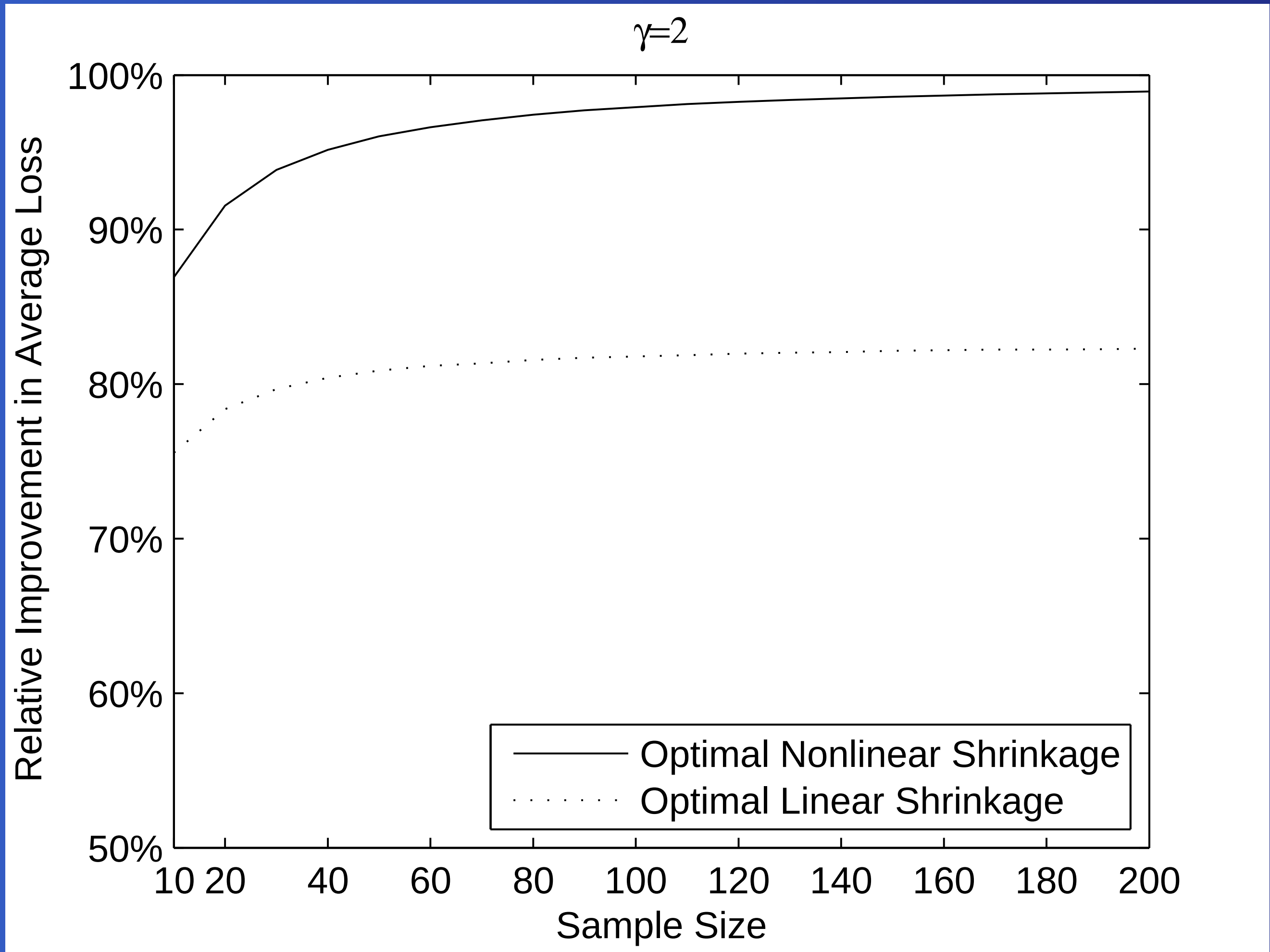

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$$u_i^* \, \Sigma_N^{-1} \, u_i \geq (u_i^* \, \Sigma_N \, u_i)^{-1}$$

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$$u_i^* \Sigma_N^{-1} u_i \approx \frac{1 - c - 2c\lambda_i \mathrm{Re}[\breve{m}_F(\lambda_i)]}{\lambda_i}$$

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- We do for sample eigenvectors what MP67/S95 did for sample eigenvalues

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2. Construct *bona fide* nonlinear shrinkage estimator of the inverse of the covariance matrix
3. Show that $N|u_i^* v_j|^2$ is even closer to

$$\frac{c\lambda_i \tau_j}{\left|\tau_j \left[1 - c - c\lambda_i \breve{m}_F(\lambda_i)\right] - \lambda_i\right|^2}$$

than we have shown in this paper