On general criteria for when the spectrum of a combination of random matrices depends only on the spectra of the components

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Main question

Given $\mathbf{A}$, $\mathbf{B}$ two $n \times n$ independent square Hermitian (or symmetric) random matrices

1. What can we say about the eigenvalue distribution of $\mathbf{A}$, once we know those of $\mathbf{A}+\mathbf{B}$ and $\mathbf{B}$?
2. What can we say about the eigenvalue distribution of $\mathbf{A}$, once we know those of $\mathbf{AB}$ and $\mathbf{B}$?

Such questions can also be asked starting with any functional of $\mathbf{A}$ and $\mathbf{B}$. When we can infer on the mentioned eigenvalue distributions, the corresponding operation is called *deconvolution*. Two main techniques used in the literature:

- The Stieltjes transform method,
- The method of moments.

We will focus on the latter.

Moments and mixed moments

Many probability distributions are uniquely determined by their moments $\int t^n d\mu(t)$ (Carlemans theorem), and can thus be used to characterize the spectrum of a random matrix.

- Let tr be the normalized trace, and $\mathbb{E}[\cdot]$ the expectation.
- The quantities $A_k = \mathbb{E}[\mathrm{tr}(\mathbf{A}^k)]$ are the moments (or individual moments) of $\mathbf{A}$.
- More generally, if $\mathbf{A}_i$ are random matrices,

  $$\mathbb{E}[\mathrm{tr}(\mathbf{A}_{i_1}\mathbf{A}_{i_2}\cdots\mathbf{A}_{i_k})]$$

  is called a mixed moment in the $\mathbf{A}_i$, when $i_1 \neq i_2$, $i_2 \neq i_3, \ldots$.
- More generally, we can define a mixed moment in terms of algebras: if $\mathcal{A}_i$ are algebras, $\mathbf{A}_i \in \mathcal{A}_{k_i}$ with $k_i \neq k_{i+1}$ for all i.

Freeness: a computational rule for mixed moments

Definition

A family of unital $*$-subalgebras $\{A_i\}_{i \in I}$ is called a free family if

$$\left\{ \begin{array}{c} a_j \in A_{i_j} \\ i_1 \neq i_2, i_2 \neq i_3, \cdots, i_{n-1} \neq i_n \\ \phi(a_1) = \phi(a_2) = \cdots = \phi(a_n) = 0 \end{array} \right\} \Rightarrow \phi(a_1 \cdots a_n) = 0. \quad (1)$$

- Defined at the algebraic level. Can be thought of as "spectral separation".
- A concrete rule for computing mixed moments in terms of individual moments ($\mathbb{E}[\mathrm{tr}(\cdot)]$ replaced with general ϕ).
- Defining σ as the partition where $k \sim l$ if and only if $i_k = i_l$, the same formula for the mixed moment applies for any $a_1 \cdots a_n$ giving rise to σ. Is in this way a particularly nice type of spectral separation.

Instead of free algebras, assume that we have subalgebras A_i of random matrices, where any random matrix from one algebra is independent from those in the other algebras.

- For which collection of algebras do mixed moments
$$\mathbb{E}[\mathrm{tr}(\mathbf{A}_{i_1}\mathbf{A}_{i_2}\cdots\mathbf{A}_{i_k})], \tag{2}$$
depend only on individual moments? In other words: when do we have spectral separation?
- The question is often more easily answered in the large n-limit:
$$\lim_{n\to\infty}\mathbb{E}[\mathrm{tr}(\mathbf{A}_{i_1}^{(n)}\mathbf{A}_{i_2}^{(n)}\cdots\mathbf{A}_{i_k}^{(n)})],$$
where we now assume that we have ensembles of random matrices, their dimensions growing so that $\lim_{N\to\infty}\frac{N}{L}=c$.
- In the large n-limit, the problem is coupled with finding what modes of convergence apply. Almost sure convergence?
- When is the computational rule for computing (2) the same for any choice of matrices from the algebras, as for freeeness?

If positive answers: good starting point for deconvolution.

Gaussian matrices

- If the $\mathbf{A}_i$ are Gaussian matrices, there exist results in the finite regime [1], on computational rules for mixed moments of Gaussian matrices and matrices independent from them.
- combinations of Gaussian matrices converge almost surely.
- Asymptotically free, so same convenient computational rule in the limit as for freeness.
- No need to expect that the same computational rule applies in the finite regime!

Vandermonde matrices

An $N \times L$ Vandermonde matrix with entries on the unit circle [2] is on the form

$$\mathbf{V} = \frac{1}{\sqrt{N}} \left(\begin{array}{ccc} 1 & \cdots & 1 \\ e^{-j\omega_1} & \cdots & e^{-j\omega_L} \\ \vdots & \ddots & \vdots \\ e^{-j(N-1)\omega_1} & \cdots & e^{-j(N-1)\omega_L} \end{array} \right) \tag{3}$$

ω_1,...,ω_L, also called phases, are assumed i.i.d., taking values in $[0, 2\pi)$. N and L go to infinity at the same rate, $c = \lim_{N\to\infty} \frac{L}{N}$ (the aspect ratio).

Algebraic result for Vandermonde matrices [3]

Theorem

Let $\{\mathbf{V}_i\}_{i\in I}, \{\mathbf{V}_j\}_{j\in J}$ be independent Vandermonde matrices, with arbitrary phase distributions $\{\omega_i\}_{i\in I}$ and $\{\omega_j\}_{j\in J}$, respectively, with continuous density.

- *Let $\mathcal{A}_I$ be the algebra generated by $\{(\mathbf{V}_{i_1})^H\mathbf{V}_{i_2}\}_{i_1,i_2\in I}$.*
- *Let $\mathcal{A}_J$ be the algebra generated by $\{(\mathbf{V}_{j_1})^H\mathbf{V}_{j_2}\}_{j_1,j_2\in J}$.*

We have that any mixed moment

$$\lim_{N\to\infty} \mathbb{E}\left[\mathrm{tr}\left(a_{i_1}a_{j_1}a_{i_2}a_{j_2}\cdots a_{i_n}a_{j_n}\right)\right] \text{ with } a_{i_k}\in\mathcal{A}_I, a_{j_k}\in\mathcal{A}_J, \qquad (4)$$

depends only on individual moments of the form

$$\lim_{N\to\infty} \mathbb{E}\left[\mathrm{tr}(a)\right] \text{ with } a\in\mathcal{A}_I,$$

$$\lim_{N\to\infty} \mathbb{E}\left[\mathrm{tr}(a)\right] \text{ with } a\in\mathcal{A}_J. \qquad (5)$$

Sketch of proof

We need to compute

$$\lim_{N\to\infty} \mathbb{E}\left[\mathrm{tr}\left(\mathbf{V}_{k_1}^H \mathbf{V}_{k_2} \cdots \mathbf{V}_{k_{2n-1}}^H \mathbf{V}_{k_{2n}}\right)\right].$$

- Define $\sigma \in \mathcal{P}(2n)$ defined by $r \sim_\sigma s$ if and only if $\omega_{k_r} = \omega_{k_s}$,
- let σ_j be the block of σ where $\omega_{k_i} = \omega_j$ for $i \in \sigma_j$.
- For $\pi \in \mathcal{P}(n)$, define $\rho(\pi) \in \mathcal{P}(2n)$ as the partition in $\mathcal{P}(2n)$ generated by the relations:

$$k \sim_{\rho(\pi)} l \text{ if } \begin{cases} \lfloor k/2 \rfloor + 1 \sim_\pi \lfloor l/2 \rfloor + 1 \text{ and} \\ k \sim_{\sigma_1} l \end{cases}$$

  where σ_1 defined by $r \sim_{\sigma_1} s$ if and only if $\mathbf{V}_{k_r} = \mathbf{V}_{k_s}$.
- $\mathcal{B}(n) \subset \mathcal{P}(n)$ be defined as in [3],
- write $\rho(\pi) \vee [0,1]_n = \{\rho_1, ..., \rho_{r(\pi)}\}$, with each $\rho_i \geq [0,1]_{\|\rho_i\|/2}$ ($r(\pi)$ the number of blocks). Can be written so in a unique way.

By carefully collecting terms we obtain in the limit

$$\sum_{\pi\in\mathcal{B}(n)} K_{\rho,u}(2\pi)^{|\rho|-1}\prod_{i=1}^{r(\pi)}\int\prod_j p_{\omega_j}(x)^{|\rho_i\cap\sigma_j|}dx, \qquad (6)$$

- Here p_ω is the density of the phase distribution ω.
- The $K_{\rho,u}$ are called *Vandermonde mixed moment expansion coefficients*
- When each $\mathbf{V}_{k_{2j-1}}^H\mathbf{V}_{k_{2j}}$ is in either $\mathcal{A}_I$ or $\mathcal{A}_J$, in each integral $\int\prod_j p_{\omega_j}(x)^{|\rho_i\cap\sigma_j|}dx$, all ω_j are either contained in $\{\omega_i\}_{i\in I}$, or in $\{\omega_j\}_{j\in J}$,
- Each such integral can be written in terms of moments from either $\mathcal{A}_I$ or $\mathcal{A}_J$, showing that we have spectral separation.

Due to (6), the moments of Vandermonde matrices are in the large n-limit essentially determined from

$$I_{k,\omega} = (2\pi)^{k-1} \int_0^{2\pi} p_\omega(x)^k dx. \tag{7}$$

- Reduces the dimensionality of the problem.
- In the finite regime, the moments are probably not uniquely determined from such simple quantities.

Vandermonde mixed moment expansion coefficients

- Write $\rho(\pi) = \{W_1, ..., W_{|\rho(\pi)|}\}$,
- write $W_j = W_j^{\cdot} \cup W_j^H$, with $W_j^{\cdot}$ the even elements of W_j (the $\mathbf{V}$-terms), W_j^H the odd elements of W_j (the $\mathbf{V}^H$-terms).
- Form the $|\rho(\pi)|$ equations

$$\sum_{k \in W_r^H} x_{(k+1)/2+1} = \sum_{k \in W_r^{\cdot}} x_{k/2+1} \tag{8}$$

  in n variables $x_1, \ldots, x_n$.
- $K_{\rho,u}$ is the volume of the solution set to (8), when all x_i are constrained to $[0, 1]$.
- $K_{\rho,u}$ can be found with Fourier-Motzkin elmimination, and always computes to a rational number in $[0, 1]$.
- Matrices such as Hankel and Toeplitz matrices also have asymptotic eigenvalue distributions which can be determined from such quantities.

Generalized Vandermonde matrices

Similar result exists

$$\mathbf{V} = \frac{1}{\sqrt{N}} \begin{pmatrix} e^{-j\lfloor Nf(0)\rfloor\omega_1} & \cdots & e^{-j\lfloor Nf(0)\rfloor\omega_L} \\ e^{-j\lfloor Nf(\frac{1}{N})\rfloor\omega_1} & \cdots & e^{-j\lfloor Nf(\frac{1}{N})\rfloor\omega_L} \\ \vdots & \ddots & \vdots \\ e^{-j\lfloor Nf(\frac{N-1}{N})\rfloor\omega_1} & \cdots & e^{-j\lfloor Nf(\frac{N-1}{N})\rfloor\omega_L} \end{pmatrix}, \quad (9)$$

where f is called the *power distribution* (a function from $[0,1)$ to $[0,1)$). Theorem 2 will hold for such matrices also, as long as

- The power distribution is "sufficiently uniform",
- all $\{\mathbf{V}_i\}_{i\in I}$ have the same power distribution,
- all $\{\mathbf{V}_j\}_{j\in J}$ have the same power distribution.

Note that the power distribution governing each algebra may be different!

Related matrices: Euclidean matrices [4]

- Entry (k, l) has the form $\frac{1}{n}F(\omega_k - \omega_l)$, where $\omega_1, \ldots, \omega_n$ are i.i.d. with uniform distribution.
- A generalized Vandermonde matrix where the power distribution is a sum of Dirac measures corresponds to a Euclidean matrix.
- Empirical eigenvalue distribution converges to a counting measure with an accumulation point at 0, expressible in terms of the Fourier transform of F [4].
- Unknown whether we have spectral separation in the same nice way as for Vandermonde matrices, although an Euclidean matrix corresponds to a generalized Vandermonde matrix (the problem is that the power distribution is not uniform enough!).

Permutational invariance

Definition

$\mathbf{A}$ is called permutationally invariant if the distribution of $\mathbf{A}$ is the same as that of $P\mathbf{A}P^{-1}$ for any permutation matrix P.

Assume that $\mathbf{D}_i$ are diagonal matrices, and consider

$$\mathrm{tr}\left(\mathbf{D}_1\mathbf{A}_1 \cdots \mathbf{D}_n\mathbf{A}_n\right). \tag{10}$$

- If $\mathbf{A}_i$ are permutationally invariant, the moments of the $\mathbf{D}_i$-matrices can be factored out in (10). Used for Vandermonde matrices. Permutational invariance thus ensures spectral separation partially.
- Permutational invariance also ensures that (10) can be split into a sum, the sum indexed over $\mathcal{P}(n)$.
- Spectral separation in $\mathbf{A}_i$ depends on the joint distribution of the entries of the $\mathbf{A}_i$, which perhaps only in the limit factors into nice expressions, some which may involve only moments.

Random Matrix Library [5]

1. Implementation with support for many types of matrices: Vandermonde matrices, Gaussian matrices, Toeplitz matrices, Hankel matrices e.t.c.
2. Can generate symbolic formulas for several convolution operations,
3. Can compute the convolution with a given set of moments numerically, as would be needed in real-time applications.
4. Can perform deconvolution, where this is possible, to infer on the parameters in an underlying model.

Further work

- Find as general criteria as possible for when spectral separation is possible.
- Support for more matrices in the Random Matrix Library [5]
- Optimization of the methods in the Random Matrix Library.

- This talk is available at
  `http://folk.uio.no/oyvindry/talks.shtml`
- My publications are listed at
  `http://folk.uio.no/oyvindry/publications.shtml`

THANK YOU!

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