Deterministic equivalents for Haar matrices

Romain Couillet[1,2], Jakob Hoydis[1], Mérouane Debbah[1]
[1]Alcatel-Lucent Chair on Flexible Radio, Supélec, Gif sur Yvette, FRANCE
[2]ST-Ericsson, Sophia-Antipolis, FRANCE
romain.couillet@supelec.fr

Random Matrix Theory Symposium

Outline

Outline

Problem statement

We wish to characterize a deterministic equivalent for the following types of matrices

- *"sum of correlated Haar"*

$$\mathbf{B}_N = \sum_{k=1}^{K} \mathbf{H}_k \mathbf{W}_k \mathbf{P}_k \mathbf{W}_k^{\mathsf{H}} \mathbf{H}_k^{\mathsf{H}}$$

  with $\mathbf{H}_k \in \mathbb{C}^{N \times N_k}$ deterministic, $\mathbf{W}_k \in \mathbb{C}^{N_k \times n_k}$ unitary isometric, $\mathbf{P}_k \in \mathbb{C}^{n_k \times n_k}$ deterministic. Possible uses in wireless communications are
  - multi-cell frequency selective CDMA/SDMA with a single user per cell
  - single-cell downlink CDMA/SDMA with colored noise
  - capacity and MMSE SINR

- *"Haar matrices with a correlation profile"*

$$\mathbf{B}_N = \mathbf{X}\mathbf{X}^{\mathsf{H}}$$

  with $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n] \in \mathbb{C}^{N \times n}$ and

$$\mathbf{x}_k = \mathbf{R}_k^{\frac{1}{2}} \mathbf{w}_k$$

  with $\mathbf{R}_k$ deterministic and $\mathbf{W} = [\mathbf{w}_1, \ldots, \mathbf{w}_n] \in \mathbb{C}^{N \times n}$ isometric. Possible uses in wireless communications are
  - single/multi-cell frequency selective CDMA/SDMA uplink and downlink
  - beamforming with unitary precoding in frequency selective channels
  - capacity and MMSE SINR

Problem statement

We wish to characterize a deterministic equivalent for the following types of matrices

- *"sum of correlated Haar"*

$$\mathbf{B}_N = \sum_{k=1}^{K} \mathbf{H}_k \mathbf{W}_k \mathbf{P}_k \mathbf{W}_k^{\mathsf{H}} \mathbf{H}_k^{\mathsf{H}}$$

  with $\mathbf{H}_k \in \mathbb{C}^{N \times N_k}$ deterministic, $\mathbf{W}_k \in \mathbb{C}^{N_k \times n_k}$ unitary isometric, $\mathbf{P}_k \in \mathbb{C}^{n_k \times n_k}$ deterministic. Possible uses in wireless communications are

  - multi-cell frequency selective CDMA/SDMA with a single user per cell
  - single-cell downlink CDMA/SDMA with colored noise
  - capacity and MMSE SINR

- *"Haar matrices with a correlation profile"*

$$\mathbf{B}_N = \mathbf{X}\mathbf{X}^{\mathsf{H}}$$

  with $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n] \in \mathbb{C}^{N \times n}$ and

$$\mathbf{x}_k = \mathbf{R}_k^{\frac{1}{2}} \mathbf{w}_k$$

  with $\mathbf{R}_k$ deterministic and $\mathbf{W} = [\mathbf{w}_1, \ldots, \mathbf{w}_n] \in \mathbb{C}^{N \times n}$ isometric. Possible uses in wireless communications are

  - single/multi-cell frequency selective CDMA/SDMA uplink and downlink
  - beamforming with unitary precoding in frequency selective channels
  - capacity and MMSE SINR

Problem statement

We wish to characterize a deterministic equivalent for the following types of matrices

- *"sum of correlated Haar"*

$$\mathbf{B}_N = \sum_{k=1}^{K} \mathbf{H}_k \mathbf{W}_k \mathbf{P}_k \mathbf{W}_k^{\mathrm{H}} \mathbf{H}_k^{\mathrm{H}}$$

  with $\mathbf{H}_k \in \mathbb{C}^{N \times N_k}$ deterministic, $\mathbf{W}_k \in \mathbb{C}^{N_k \times n_k}$ unitary isometric, $\mathbf{P}_k \in \mathbb{C}^{n_k \times n_k}$ deterministic. Possible uses in wireless communications are

  - multi-cell frequency selective CDMA/SDMA with a single user per cell
  - single-cell downlink CDMA/SDMA with colored noise
  - capacity and MMSE SINR

- *"Haar matrices with a correlation profile"*

$$\mathbf{B}_N = \mathbf{X}\mathbf{X}^{\mathrm{H}}$$

  with $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n] \in \mathbb{C}^{N \times n}$ and

$$\mathbf{x}_k = \mathbf{R}_k^{\frac{1}{2}} \mathbf{w}_k$$

  with $\mathbf{R}_k$ deterministic and $\mathbf{W} = [\mathbf{w}_1, \ldots, \mathbf{w}_n] \in \mathbb{C}^{N \times n}$ isometric. Possible uses in wireless communications are

  - single/multi-cell frequency selective CDMA/SDMA uplink and downlink
  - beamforming with unitary precoding in frequency selective channels
  - capacity and MMSE SINR

Problem statement

We wish to characterize a deterministic equivalent for the following types of matrices

- *"sum of correlated Haar"*

$$\mathbf{B}_N = \sum_{k=1}^{K} \mathbf{H}_k \mathbf{W}_k \mathbf{P}_k \mathbf{W}_k^{\mathsf{H}} \mathbf{H}_k^{\mathsf{H}}$$

  with $\mathbf{H}_k \in \mathbb{C}^{N \times N_k}$ deterministic, $\mathbf{W}_k \in \mathbb{C}^{N_k \times n_k}$ unitary isometric, $\mathbf{P}_k \in \mathbb{C}^{n_k \times n_k}$ deterministic. Possible uses in wireless communications are
  - multi-cell frequency selective CDMA/SDMA with a single user per cell
  - single-cell downlink CDMA/SDMA with colored noise
  - capacity and MMSE SINR
- *"Haar matrices with a correlation profile"*

$$\mathbf{B}_N = \mathbf{X}\mathbf{X}^{\mathsf{H}}$$

  with $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n] \in \mathbb{C}^{N \times n}$ and

$$\mathbf{x}_k = \mathbf{R}_k^{\frac{1}{2}} \mathbf{w}_k$$

  with $\mathbf{R}_k$ deterministic and $\mathbf{W} = [\mathbf{w}_1, \ldots, \mathbf{w}_n] \in \mathbb{C}^{N \times n}$ isometric. Possible uses in wireless communications are
  - single/multi-cell frequency selective CDMA/SDMA uplink and downlink
  - beamforming with unitary precoding in frequency selective channels
  - capacity and MMSE SINR

Outline

Fundamental equations

Theorem (Theorem 1)

Let $\mathbf{P}_i \in \mathbb{C}^{n_i \times n_i}$ and $\mathbf{R}_i \in \mathbb{C}^{N \times N}$ be Hermitian nonnegative matrices and $\bar{c}_1, \ldots, \bar{c}_K$ be positive scalars. Then the following system of equations in $(\bar{e}_1, \ldots, \bar{e}_K)$

$$\begin{aligned} \bar{e}_i &= \frac{1}{N} \operatorname{tr} \mathbf{P}_i \left(e_i \mathbf{P}_i + [\bar{c}_i - e_i \bar{e}_i] \mathbf{I}_{n_i} \right)^{-1} \\ e_i &= \frac{1}{N} \operatorname{tr} \mathbf{R}_i \left(\sum_{j=1}^{K} \bar{e}_j \mathbf{R}_j - z \mathbf{I}_N \right)^{-1} . \end{aligned} \tag{1}$$

has a unique functional solution $(\bar{e}_1(z), \ldots, \bar{e}_K(z))$ with $z \mapsto e_i(z)$, $\mathbb{C} \setminus \mathbb{R}^+ \to \mathbb{C}$, the Stieltjes transform of a distribution function with support on $\mathbb{R}^+$.

Point-wise uniqueness

Theorem (Theorem 2)

For each z real negative, the system of equations (1) *has a unique scalar-valued solution* $(\bar{e}_1, \ldots, \bar{e}_K)$ *with* $\bar{e}_i = \lim_{t\to\infty} \bar{e}_i^{(t)}$, *where* $\bar{e}_i^{(t)}$ *is the unique solution of*

$$\bar{e}_i^{(t)} = \frac{1}{N} \operatorname{tr} \mathbf{P}_i \left(e_i^{(t)} \mathbf{P}_i + [\bar{c}_i - e_i^{(t)} \bar{e}_i^{(t)}] \mathbf{I}_{n_i} \right)^{-1} \tag{2}$$

within the interval $[0, c_i \bar{c}_i / e_i^{(t)})$, $e_i^{(0)}$ *can take any positive value and* $e_i^{(t)}$ *is recursively defined by:*

$$e_i^{(t+1)} = \frac{1}{N} \operatorname{tr} \mathbf{R}_i \left(\sum_{j=1}^{K} \bar{e}_j^{(t)} \mathbf{R}_j - z \mathbf{I}_N \right)^{-1}.$$

The solution $\bar{e}_i^{(t)}$ *of* (2) *is explicitly given by*

$$\bar{e}_i^{(t)} = \lim_{k\to\infty} \bar{e}_i^{(t,k)},$$

with $\bar{e}_i^{(t,0)} \in [0, c_i \bar{c}_i / e_i^{(t)})$ *and, for* $k \geq 1$,

$$\bar{e}_i^{(t,k)} = \frac{1}{N} \operatorname{tr} \mathbf{P}_i \left(e_i^{(t)} \mathbf{P}_i + [\bar{c}_i - e_i^{(t)} \bar{e}_i^{(t,k-1)}] \mathbf{I}_{n_i} \right)^{-1}.$$

Convergence in distribution

Theorem (Theorem 3)

Let $\mathbf{P}_i \in \mathbb{C}^{n_i \times n_i}$ be a Hermitian nonnegative matrix with spectral norm bounded uniformly along n_i and $\mathbf{W}_i \in \mathbb{C}^{N_i \times n_i}$ be the $n_i \leq N_i$ columns of a unitary Haar distributed random matrix. We also consider $\mathbf{H}_i \in \mathbb{C}^{N \times N_i}$ a random matrix such that $\mathbf{R}_i \triangleq \mathbf{H}_i \mathbf{H}_i^{\mathsf{H}} \in \mathbb{C}^{N \times N}$ has uniformly bounded spectral norm along N, almost surely. Denote

$$\mathbf{B}_N = \sum_{i=1}^{K} \mathbf{H}_i \mathbf{W}_i \mathbf{P}_i \mathbf{W}_i^{\mathsf{H}} \mathbf{H}_i^{\mathsf{H}}.$$

Then, as $N, N_1, \ldots, N_K, n_1, \ldots, n_K$ grow to infinity with $\bar{c}_i \triangleq \frac{N_i}{N}$ satisfying $0 < \liminf \bar{c}_i \leq \limsup \bar{c}_i < \infty$ and $0 \leq \frac{n_i}{N_i} \triangleq c_i \leq 1$ for all i, we have

$$F^{\mathbf{B}_N} - F_N \Rightarrow 0$$

almost surely, where F_N is the distribution function with Stieltjes transform $m_N(z)$ defined by

$$m_N(z) = \frac{1}{N} \operatorname{tr} \left(\sum_{i=1}^{K} \bar{e}_i(z) \mathbf{R}_i - z \mathbf{I}_N \right)^{-1}, \tag{3}$$

with $z \mapsto \bar{e}_i(z)$, $\mathbb{C} \setminus \mathbb{R}^+ \to \mathbb{C}$, defined in Theorem 1.

Deterministic equivalent of the Shannon transform

Theorem (Theorem 4)

Let $\mathbf{B}_N \in \mathbb{C}^{N\times N}$ *be defined as in Theorem 3 with* $z = -1/x$ *for some* $x > 0$*. Denoting* $\mathcal{V}_{\mathbf{B}_N}(x) = \frac{1}{N}\log\det(x\mathbf{B}_N + \mathbf{I}_N)$ *the Shannon-transform of* $F^{\mathbf{B}_N}$*, we have*

$$\mathcal{V}_{\mathbf{B}_N}(x) - \mathcal{V}_N(x) \xrightarrow{\text{a.s.}} 0, \tag{4}$$

as $N, N_1, \ldots, N_K, n_1, \ldots, n_K$ *grow to infinity with* $0 < \liminf \bar{c}_i \leq \limsup \bar{c}_i < \infty$*, where*

$$\mathcal{V}_N(x) = \frac{1}{N}\log\det\left(\mathbf{I}_N + x\sum_{i=1}^{K}\bar{e}_i\mathbf{R}_i\right) + \sum_{i=1}^{K}\left[\frac{1}{N}\log\det\left([\bar{c}_i - e_i\bar{e}_i]\mathbf{I}_{n_i} + e_i\mathbf{P}_i\right) + (1-c_i)\bar{c}_i\log(\bar{c}_i - e_i\bar{e}_i) - \bar{c}_i\log(\bar{c}_i)\right]. \tag{5}$$

Deterministic equivalent of the MMSE SINR

Theorem (Theorem 5)

Under the conditions of Theorem 3, we have

$$\mathbf{w}_{ij}^{\mathrm{H}}\mathbf{H}_i^{\mathrm{H}}\left(\mathbf{B}_N - p_{ij}\mathbf{H}_i\mathbf{w}_{ij}\mathbf{w}_{ij}^{\mathrm{H}}\mathbf{H}_i^{\mathrm{H}} - z\mathbf{I}_N\right)^{-1}\mathbf{H}_i\mathbf{w}_{ij} - \frac{e_i}{\bar{c}_i - e_i\bar{e}_i} \xrightarrow{\mathrm{a.s.}} 0,$$

with $\mathbf{w}_{ij} \in \mathbb{C}^{N_i}$ *the* j^{th} *column of* $\mathbf{W}_i$*.*

Outline

1. Main Results
   - Deterministic Equivalent for a sum of independent Haar
   - Comparison with the i.i.d. case
2. Sketch of Proof
   - First deterministic equivalent
   - Second deterministic equivalent
   - Uniqueness and convergence of the det. eq.
3. Simulation plots
4. Haar matrix with correlated columns

Comparison with the i.i.d. case

Assume $\bar{c}_i = c_i = 1$ for every i. Then,

- for $\mathbf{W}_i$ Haar,

$$m_N(z) = \frac{1}{N} \operatorname{tr} \left(\sum_{j=1}^{K} \bar{e}_j \mathbf{R}_j - z\mathbf{I}_N \right)^{-1}, \text{ with}$$

$$\bar{e}_i = \frac{1}{N} \operatorname{tr} \mathbf{P}_i \left(e_i \mathbf{P}_i + [1 - e_i \bar{e}_i] \mathbf{I}_N \right)^{-1} \tag{6}$$

$$e_i = \frac{1}{N} \operatorname{tr} \mathbf{R}_i \left(\sum_{j=1}^{K} \bar{e}_j \mathbf{R}_j - z\mathbf{I}_N \right)^{-1}.$$

- for $\mathbf{W}_i$ i.i.d.,[1]

$$m_N(z) = \frac{1}{N} \operatorname{tr} \left(\sum_{j=1}^{K} \bar{e}_j \mathbf{R}_j - z\mathbf{I}_N \right)^{-1}, \text{ with}$$

$$\bar{e}_i = \frac{1}{N} \operatorname{tr} \mathbf{P}_i \left(e_i \mathbf{P}_i + \mathbf{I}_N \right)^{-1} \tag{7}$$

$$e_i = \frac{1}{N} \operatorname{tr} \mathbf{R}_i \left(\sum_{j=1}^{K} \bar{e}_j \mathbf{R}_j - z\mathbf{I}_N \right)^{-1}$$

[1] R. Couillet, J. W. Silverstein, M. Debbah, “A deterministic equivalent for the analysis of MIMO multiple access channels”, http://arxiv.org/abs/0906.3667v3, IEEE Trans. on Inf. Theory, *to appear*.

Comparison with the i.i.d. case

Assume $\bar{c}_i = c_i = 1$ for every i. Then,

- for $\mathbf{W}_i$ Haar,

$$m_N(z) = \frac{1}{N} \operatorname{tr} \left(\sum_{j=1}^{K} \bar{e}_j \mathbf{R}_j - z \mathbf{I}_N \right)^{-1}, \text{ with}$$

$$\bar{e}_i = \frac{1}{N} \operatorname{tr} \mathbf{P}_i \left(e_i \mathbf{P}_i + [1 - {\color{red} e_i \bar{e}_i}] \mathbf{I}_N \right)^{-1} \tag{6}$$

$$e_i = \frac{1}{N} \operatorname{tr} \mathbf{R}_i \left(\sum_{j=1}^{K} \bar{e}_j \mathbf{R}_j - z \mathbf{I}_N \right)^{-1}.$$

- for $\mathbf{W}_i$ i.i.d.,[1]

$$m_N(z) = \frac{1}{N} \operatorname{tr} \left(\sum_{j=1}^{K} \bar{e}_j \mathbf{R}_j - z \mathbf{I}_N \right)^{-1}, \text{ with}$$

$$\bar{e}_i = \frac{1}{N} \operatorname{tr} \mathbf{P}_i \left(e_i \mathbf{P}_i + \mathbf{I}_N \right)^{-1} \tag{7}$$

$$e_i = \frac{1}{N} \operatorname{tr} \mathbf{R}_i \left(\sum_{j=1}^{K} \bar{e}_j \mathbf{R}_j - z \mathbf{I}_N \right)^{-1}$$

[1] R. Couillet, J. W. Silverstein, M. Debbah, "A deterministic equivalent for the analysis of MIMO multiple access channels", http://arxiv.org/abs/0906.3667v3, IEEE Trans. on Inf. Theory, *to appear*.

Outline

Outline (First det. eq.)

- Denoting $\delta_i \triangleq \frac{1}{N_i - n_i} \operatorname{tr} (\mathbf{I}_{N_i} - \mathbf{W}_i \mathbf{W}_i^{\mathrm{H}}) \mathbf{H}_i^{\mathrm{H}} (\mathbf{B}_N - z\mathbf{I}_N)^{-1} \mathbf{H}_i$ and $f_i \triangleq \frac{1}{N} \operatorname{tr} \mathbf{R}_i (\mathbf{B}_N - z\mathbf{I}_N)^{-1}$, we prove

$$f_i - \frac{1}{N} \operatorname{tr} \mathbf{R}_i (\mathbf{G} - z\mathbf{I}_N)^{-1} \xrightarrow{\text{a.s.}} 0,$$

with $\mathbf{G} = \sum_{j=1}^{K} \bar{g}_j \mathbf{R}_j$ and

$$\bar{g}_i = \frac{1}{(1-c_i)\bar{c}_i + \frac{1}{N} \sum_{l=1}^{n_i} \frac{1}{1+p_{il}\delta_i}} \frac{1}{N} \sum_{l=1}^{n_i} \frac{p_{il}}{1+p_{il}\delta_i},$$

where p_{il} denotes the l^{th} eigenvalue of $\mathbf{P}_i$, and δ_i is linked to f_i through

$$f_i - \left((1-c_i)\bar{c}_i \delta_i + \frac{1}{N} \sum_{l=1}^{n_i} \frac{\delta_i}{1+p_{il}\delta_i} \right) \xrightarrow{\text{a.s.}} 0.$$

Outline (Second det. eq.)

- $\bar{g}_i$ is then shown to satisfy

$$\bar{g}_i - \frac{1}{N}\sum_{l=1}^{n_i}\frac{p_{il}}{\bar{c}_i + p_{il}f_i - f_i\bar{g}_i} = \bar{g}_i - \frac{1}{N}\operatorname{tr}\mathbf{P}_i\left(f_i\mathbf{P}_i + [\bar{c}_i - f_i\bar{g}_i]\mathbf{I}_{n_i}\right)^{-1} \xrightarrow{\text{a.s.}} 0,$$

which induces the $2K$-equation system

$$f_i - \frac{1}{N}\operatorname{tr}\mathbf{R}_i\left(\sum_{j=1}^{K}\bar{g}_j\mathbf{R}_j - z\mathbf{I}_N\right)^{-1} \xrightarrow{\text{a.s.}} 0$$

$$\bar{g}_i - \frac{1}{N}\operatorname{tr}\mathbf{P}_i\left(\bar{g}_i\mathbf{P}_i + [\bar{c}_i - f_i\bar{g}_i]\mathbf{I}_{n_i}\right)^{-1} \xrightarrow{\text{a.s.}} 0.$$

- Introducing $\mathbf{F} = \sum_{i=1}^{K}\bar{f}_i\mathbf{R}_i$, we finally prove

$$f_i - \frac{1}{N}\operatorname{tr}\mathbf{R}_i\left(\sum_{j=1}^{K}\bar{f}_j\mathbf{R}_j - z\mathbf{I}_N\right)^{-1} \xrightarrow{\text{a.s.}} 0$$

$$\bar{f}_i - \frac{1}{N}\operatorname{tr}\mathbf{P}_i\left(\bar{f}_i\mathbf{P}_i + [\bar{c}_i - f_i\bar{f}_i]\mathbf{I}_{n_i}\right)^{-1} = 0,$$

where, for $z < 0$, $\bar{f}_i$ lies in $[0, c_i\bar{c}_i/f_i)$ and is now uniquely determined by f_i.

Outline (Second det. eq.)

- $\bar{g}_i$ is then shown to satisfy

$$\bar{g}_i - \frac{1}{N}\sum_{l=1}^{n_i} \frac{p_{il}}{\bar{c}_i + p_{il} f_i - f_i \bar{g}_i} = \bar{g}_i - \frac{1}{N} \operatorname{tr} \mathbf{P}_i \left(f_i \mathbf{P}_i + [\bar{c}_i - f_i \bar{g}_i] \mathbf{I}_{n_i} \right)^{-1} \xrightarrow{\text{a.s.}} 0,$$

which induces the $2K$-equation system

$$f_i - \frac{1}{N} \operatorname{tr} \mathbf{R}_i \left(\sum_{j=1}^{K} \bar{g}_j \mathbf{R}_j - z \mathbf{I}_N \right)^{-1} \xrightarrow{\text{a.s.}} 0$$

$$\bar{g}_i - \frac{1}{N} \operatorname{tr} \mathbf{P}_i \left(\bar{g}_i \mathbf{P}_i + [\bar{c}_i - f_i \bar{g}_i] \mathbf{I}_{n_i} \right)^{-1} \xrightarrow{\text{a.s.}} 0.$$

- Introducing $\mathbf{F} = \sum_{i=1}^{K} \bar{f}_i \mathbf{R}_i$, we finally prove

$$f_i - \frac{1}{N} \operatorname{tr} \mathbf{R}_i \left(\sum_{j=1}^{K} \bar{f}_j \mathbf{R}_j - z \mathbf{I}_N \right)^{-1} \xrightarrow{\text{a.s.}} 0$$

$$\bar{f}_i - \frac{1}{N} \operatorname{tr} \mathbf{P}_i \left(\bar{f}_i \mathbf{P}_i + [\bar{c}_i - f_i \bar{f}_i] \mathbf{I}_{n_i} \right)^{-1} = 0,$$

where, for $z < 0$, $\bar{f}_i$ lies in $[0, c_i \bar{c}_i / f_i)$ and is now uniquely determined by f_i.

Outline

1. Main Results
   - Deterministic Equivalent for a sum of independent Haar
   - Comparison with the i.i.d. case
2. Sketch of Proof
   - First deterministic equivalent
   - Second deterministic equivalent
   - Uniqueness and convergence of the det. eq.
3. Simulation plots
4. Haar matrix with correlated columns

Fundamental lemma

To perform classical det. eq. techniques, we need a **trace lemma**.

- In the i.i.d. case, this is the classical

Theorem

Let $\mathbf{B}_N \in \mathbb{C}^{N \times N}$ have uniformly bounded spectral norm. Let $\mathbf{x}_N \in \mathbb{C}^N$ be random vectors of i.i.d. entries with zero mean, variance $1/N$ and finite eighth order moment, independent of $\mathbf{B}_N$. Then

$$\mathrm{E}\left[\left|\mathbf{x}_N^{\mathrm{H}}\mathbf{B}_N\mathbf{x}_N - \frac{1}{N}\operatorname{tr}\mathbf{B}_N\right|^4\right] \leq \frac{C}{N^2}, \tag{8}$$

as $N \to \infty$.

- In the Haar case, this is

Theorem

Let $\mathbf{W}$ be $n < N$ columns of an $N \times N$ Haar matrix and suppose $\mathbf{w}$ is a column of $\mathbf{W}$. Let $\mathbf{B}_N$ be an $N \times N$ random matrix, which is a function of all columns of $\mathbf{W}$ except $\mathbf{w}$ and $B = \sup_N \|\mathbf{B}_N\| < \infty$, then

$$\mathrm{E}\left[\left|\mathbf{w}^{\mathrm{H}}\mathbf{B}_N\mathbf{w} - \frac{1}{N-n}\operatorname{tr}(\mathbf{\Pi}\mathbf{B}_N)\right|^4\right] \leq \frac{C}{N^2},$$

where $\mathbf{\Pi} = \mathbf{I}_N - \mathbf{W}\mathbf{W}^{\mathrm{H}} + \mathbf{w}\mathbf{w}^{\mathrm{H}}$ and C is a constant which depends only on B and $\frac{n}{N}$.

Fundamental lemma

To perform classical det. eq. techniques, we need a **trace lemma**.

- In the i.i.d. case, this is the classical

Theorem

Let $\mathbf{B}_N \in \mathbb{C}^{N\times N}$ have uniformly bounded spectral norm. Let $\mathbf{x}_N \in \mathbb{C}^N$ be random vectors of i.i.d. entries with zero mean, variance $1/N$ and finite eighth order moment, independent of $\mathbf{B}_N$. Then

$$\mathrm{E}\left[\left|\mathbf{x}_N^{\mathsf{H}}\mathbf{B}_N\mathbf{x}_N - \frac{1}{N}\operatorname{tr}\mathbf{B}_N\right|^4\right] \leq \frac{C}{N^2}, \tag{8}$$

as $N \to \infty$.

- In the Haar case, this is

Theorem

Let $\mathbf{W}$ be $n < N$ columns of an $N \times N$ Haar matrix and suppose $\mathbf{w}$ is a column of $\mathbf{W}$. Let $\mathbf{B}_N$ be an $N \times N$ random matrix, which is a function of all columns of $\mathbf{W}$ except $\mathbf{w}$ and $B = \sup_N \|\mathbf{B}_N\| < \infty$, then

$$\mathrm{E}\left[\left|\mathbf{w}^{\mathsf{H}}\mathbf{B}_N\mathbf{w} - \frac{1}{N-n}\operatorname{tr}(\mathbf{\Pi}\mathbf{B}_N)\right|^4\right] \leq \frac{C}{N^2},$$

where $\mathbf{\Pi} = \mathbf{I}_N - \mathbf{W}\mathbf{W}^{\mathsf{H}} + \mathbf{w}\mathbf{w}^{\mathsf{H}}$ and C is a constant which depends only on B and $\frac{n}{N}$.

Fundamental lemma

To perform classical det. eq. techniques, we need a **trace lemma**.

- In the i.i.d. case, this is the classical

Theorem

Let $\mathbf{B}_N \in \mathbb{C}^{N\times N}$ have uniformly bounded spectral norm. Let $\mathbf{x}_N \in \mathbb{C}^N$ be random vectors of i.i.d. entries with zero mean, variance $1/N$ and finite eighth order moment, independent of $\mathbf{B}_N$. Then

$$\mathrm{E}\left[\left|\mathbf{x}_N^{\mathrm{H}}\mathbf{B}_N\mathbf{x}_N - \frac{1}{N}\operatorname{tr}\mathbf{B}_N\right|^4\right] \leq \frac{C}{N^2}, \tag{8}$$

as $N \to \infty$.

- In the Haar case, this is

Theorem

Let $\mathbf{W}$ be $n < N$ columns of an $N \times N$ Haar matrix and suppose $\mathbf{w}$ is a column of $\mathbf{W}$. Let $\mathbf{B}_N$ be an $N \times N$ random matrix, which is a function of all columns of $\mathbf{W}$ except $\mathbf{w}$ and $B = \sup_N \|\mathbf{B}_N\| < \infty$, then

$$\mathrm{E}\left[\left|\mathbf{w}^{\mathrm{H}}\mathbf{B}_N\mathbf{w} - \frac{1}{N-n}\operatorname{tr}(\boldsymbol{\Pi}\mathbf{B}_N)\right|^4\right] \leq \frac{C}{N^2},$$

where $\boldsymbol{\Pi} = \mathbf{I}_N - \mathbf{W}\mathbf{W}^{\mathrm{H}} + \mathbf{w}\mathbf{w}^{\mathrm{H}}$ and C is a constant which depends only on B and $\frac{n}{N}$.

Inference step

- we first suppose $\limsup_N c_i < 1$ in order to apply the trace lemma.
- as usual, denoting $\mathbf{G} = \sum_{i=1}^K \bar{g}_i \mathbf{R}_i$, we take the difference

$$\frac{1}{N}\operatorname{tr} \mathbf{A}(\mathbf{B}_N - z\mathbf{I}_N)^{-1} - \frac{1}{N}\operatorname{tr} \mathbf{A}(\mathbf{G} - z\mathbf{I}_N)^{-1}$$

$$= \frac{1}{N}\operatorname{tr}\left[\mathbf{A}(\mathbf{B}_N - z\mathbf{I}_N)^{-1}\sum_{i=1}^K \mathbf{H}_i\left(-\mathbf{W}_i\mathbf{P}_i\mathbf{W}_i^{\mathsf{H}} + \bar{g}_i\mathbf{I}_{N_i}\right)\mathbf{H}_i^{\mathsf{H}}(\mathbf{G} - z\mathbf{I}_N)^{-1}\right]$$

$$\overset{(a)}{=} \sum_{i=1}^K \bar{g}_i \frac{1}{N}\operatorname{tr} \mathbf{A}(\mathbf{B}_N - z\mathbf{I}_N)^{-1}\mathbf{R}_i(\mathbf{G} - z\mathbf{I}_N)^{-1} - \frac{1}{N}\sum_{i=1}^K\sum_{l=1}^{n_i} p_{il}\mathbf{w}_{il}^{\mathsf{H}}\mathbf{H}_i^{\mathsf{H}}(\mathbf{G} - z\mathbf{I}_N)^{-1}\mathbf{A}(\mathbf{B}_N - z\mathbf{I}_N)^{-1}\mathbf{H}_i\mathbf{w}_{il}$$

$$\overset{(b)}{=} \sum_{i=1}^K \bar{g}_i\bar{c}_i \frac{1}{N_i}\operatorname{tr} \mathbf{A}(\mathbf{B}_N - z\mathbf{I}_N)^{-1}\mathbf{R}_i(\mathbf{G} - z\mathbf{I}_N)^{-1} - \frac{1}{N}\sum_{i=1}^K\sum_{l=1}^{n_i} \frac{p_{il}\mathbf{w}_{il}^{\mathsf{H}}\mathbf{H}_i^{\mathsf{H}}(\mathbf{G} - z\mathbf{I}_N)^{-1}\mathbf{A}(\mathbf{B}_{(i,l)} - z\mathbf{I}_N)^{-1}\mathbf{H}_i\mathbf{w}_{il}}{1 + p_{il}\mathbf{w}_{il}^{\mathsf{H}}\mathbf{H}_i^{\mathsf{H}}(\mathbf{B}_{(i,l)} - z\mathbf{I}_N)^{-1}\mathbf{H}_i\mathbf{w}_{il}},$$

with $\mathbf{w}_{il} \in \mathbb{C}^{N_i}$ the l^{th} column of $\mathbf{W}_i$, $p_{i1}, \ldots, p_{in_i}$ the eigenvalues of $\mathbf{P}_i$ and $\mathbf{B}_{(i,l)} = \mathbf{B}_N - p_{il}\mathbf{H}_i\mathbf{w}_{il}\mathbf{w}_{il}^{\mathsf{H}}\mathbf{H}_i^{\mathsf{H}}$.

- In the i.i.d. case, due to the trace lemma, we identify easily $\bar{g}_i$.
- But things are not as simple in the Haar case!, since

$$\mathbf{w}_{il}^{\mathsf{H}}\mathbf{H}_i^{\mathsf{H}}(\mathbf{G} - z\mathbf{I}_N)^{-1}\mathbf{A}(\mathbf{B}_{(i,l)} - z\mathbf{I}_N)^{-1}\mathbf{H}_i\mathbf{w}_{il} \sim \frac{1}{N_i - n_i}\operatorname{tr}(\mathbf{I}_N - \mathbf{W}_i\mathbf{W}_i^{\mathsf{H}})\mathbf{H}_i^{\mathsf{H}}(\mathbf{G} - z\mathbf{I}_N)^{-1}\mathbf{A}(\mathbf{B}_N - z\mathbf{I}_N)^{-1}\mathbf{H}_i$$

Inference step

- we first suppose $\limsup_N c_i < 1$ in order to apply the trace lemma.
- as usual, denoting $\mathbf{G} = \sum_{i=1}^K \bar{g}_i \mathbf{R}_i$, we take the difference

$$\frac{1}{N}\operatorname{tr}\mathbf{A}(\mathbf{B}_N - z\mathbf{I}_N)^{-1} - \frac{1}{N}\operatorname{tr}\mathbf{A}(\mathbf{G} - z\mathbf{I}_N)^{-1}$$

$$= \frac{1}{N}\operatorname{tr}\left[\mathbf{A}(\mathbf{B}_N - z\mathbf{I}_N)^{-1}\sum_{i=1}^K \mathbf{H}_i\left(-\mathbf{W}_i\mathbf{P}_i\mathbf{W}_i^{\mathsf{H}} + \bar{g}_i\mathbf{I}_{N_i}\right)\mathbf{H}_i^{\mathsf{H}}(\mathbf{G} - z\mathbf{I}_N)^{-1}\right]$$

$$\overset{(a)}{=} \sum_{i=1}^K \bar{g}_i \frac{1}{N}\operatorname{tr}\mathbf{A}(\mathbf{B}_N - z\mathbf{I}_N)^{-1}\mathbf{R}_i(\mathbf{G} - z\mathbf{I}_N)^{-1} - \frac{1}{N}\sum_{i=1}^K\sum_{l=1}^{n_i} p_{il}\mathbf{w}_{il}^{\mathsf{H}}\mathbf{H}_i^{\mathsf{H}}(\mathbf{G} - z\mathbf{I}_N)^{-1}\mathbf{A}(\mathbf{B}_N - z\mathbf{I}_N)^{-1}\mathbf{H}_i\mathbf{w}_{il}$$

$$\overset{(b)}{=} \sum_{i=1}^K \bar{g}_i\bar{c}_i \frac{1}{N_i}\operatorname{tr}\mathbf{A}(\mathbf{B}_N - z\mathbf{I}_N)^{-1}\mathbf{R}_i(\mathbf{G} - z\mathbf{I}_N)^{-1} - \frac{1}{N}\sum_{i=1}^K\sum_{l=1}^{n_i} \frac{p_{il}\mathbf{w}_{il}^{\mathsf{H}}\mathbf{H}_i^{\mathsf{H}}(\mathbf{G} - z\mathbf{I}_N)^{-1}\mathbf{A}(\mathbf{B}_{(i,l)} - z\mathbf{I}_N)^{-1}\mathbf{H}_i\mathbf{w}_{il}}{1 + p_{il}\mathbf{w}_{il}^{\mathsf{H}}\mathbf{H}_i^{\mathsf{H}}(\mathbf{B}_{(i,l)} - z\mathbf{I}_N)^{-1}\mathbf{H}_i\mathbf{w}_{il}},$$

  with $\mathbf{w}_{il} \in \mathbb{C}^{N_i}$ the l^{th} column of $\mathbf{W}_i$, $p_{i1}, \ldots, p_{in_i}$ the eigenvalues of $\mathbf{P}_i$ and $\mathbf{B}_{(i,l)} = \mathbf{B}_N - p_{il}\mathbf{H}_i\mathbf{w}_{il}\mathbf{w}_{il}^{\mathsf{H}}\mathbf{H}_i^{\mathsf{H}}$.
- In the i.i.d. case, due to the trace lemma, we identify easily $\bar{g}_i$.
- But things are not as simple in the Haar case!, since

$$\mathbf{w}_{il}^{\mathsf{H}}\mathbf{H}_i^{\mathsf{H}}(\mathbf{G} - z\mathbf{I}_N)^{-1}\mathbf{A}(\mathbf{B}_{(i,l)} - z\mathbf{I}_N)^{-1}\mathbf{H}_i\mathbf{w}_{il} \sim \frac{1}{N_i - n_i}\operatorname{tr}(\mathbf{I}_N - \mathbf{W}_i\mathbf{W}_i^{\mathsf{H}})\mathbf{H}_i^{\mathsf{H}}(\mathbf{G} - z\mathbf{I}_N)^{-1}\mathbf{A}(\mathbf{B}_N - z\mathbf{I}_N)^{-1}\mathbf{H}_i$$

Using auxiliary variables (1)

The idea now is to express terms of the form $\frac{1}{N_i - n_i} \operatorname{tr}(\mathbf{I}_N - \mathbf{W}_i\mathbf{W}_i^{\mathrm{H}})\mathbf{D}$ as a function of $\frac{1}{N} \operatorname{tr} \mathbf{D}$.
In particular,

- Denoting

$$\delta_i \triangleq \frac{1}{N_i - n_i} \operatorname{tr}\left(\mathbf{I}_{N_i} - \mathbf{W}_i\mathbf{W}_i^{\mathrm{H}}\right) \mathbf{H}_i^{\mathrm{H}} \left(\mathbf{B}_N - z\mathbf{I}_N\right)^{-1} \mathbf{H}_i$$

$$f_i \triangleq \frac{1}{N} \operatorname{tr} \mathbf{R}_i \left(\mathbf{B}_N - z\mathbf{I}_N\right)^{-1},$$

we have

$$\begin{aligned}
(1 - c_i)\bar{c}_i\delta_i &= f_i - \frac{1}{N}\sum_{l=1}^{n_i} \mathbf{w}_{il}^{\mathrm{H}}\mathbf{H}_i^{\mathrm{H}} \left(\mathbf{B}_N - z\mathbf{I}_N\right)^{-1} \mathbf{H}_i\mathbf{w}_{il} \\
&= f_i - \frac{1}{N}\sum_{l=1}^{n_i} \frac{\mathbf{w}_{il}^{\mathrm{H}}\mathbf{H}_i^{\mathrm{H}} \left(\mathbf{B}_{(i,l)} - z\mathbf{I}_N\right)^{-1} \mathbf{H}_i\mathbf{w}_{il}}{1 + p_{il}\mathbf{w}_{il}^{\mathrm{H}}\mathbf{H}_i^{\mathrm{H}} \left(\mathbf{B}_{(i,l)} - z\mathbf{I}_N\right)^{-1} \mathbf{H}_i\mathbf{w}_{il}} \\
&\simeq f_i - \frac{1}{N}\sum_{l=1}^{n_i} \frac{\delta_i}{1 + p_{il}\delta_i}.
\end{aligned}$$

Using auxiliary variables (2)

- Similarly, denoting

$$\beta_i \triangleq \frac{1}{N_i - n_i} \operatorname{tr} \left(\mathbf{I}_{N_i} - \mathbf{W}_i \mathbf{W}_i^{\mathsf{H}} \right) \mathbf{H}_i \left(\mathbf{G} - z\mathbf{I}_N \right)^{-1} \mathbf{A} \left(\mathbf{B}_N - z\mathbf{I}_N \right)^{-1} \mathbf{H}_i,$$

we have

$$\beta_i \left((1 - c_i)\bar{c}_i + \frac{1}{N} \sum_{l=1}^{n_i} \frac{1}{1 + p_{il}\delta_i} \right) \simeq \frac{1}{N} \operatorname{tr} \mathbf{H}_i^{\mathsf{H}} \left(\mathbf{G} - z\mathbf{I}_N \right)^{-1} \mathbf{A} \left(\mathbf{B}_N - z\mathbf{I}_N \right)^{-1} \mathbf{H}_i$$

or equivalently

$$\frac{\beta_i}{1 + p_{il}\delta_i} \simeq \frac{\frac{1}{N} \operatorname{tr} \mathbf{H}_i^{\mathsf{H}} \left(\mathbf{G} - z\mathbf{I}_N \right)^{-1} \mathbf{A} \left(\mathbf{B}_N - z\mathbf{I}_N \right)^{-1} \mathbf{H}_i}{\left((1 - c_i)\bar{c}_i + \frac{1}{N} \sum_{l=1}^{n_i} \frac{1}{1+p_{il}\delta_i} \right) (1 + p_{il}\delta_i)}.$$

Plugging pieces together

Choosing

$$\bar{g}_i = \frac{1}{(1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i}}\frac{1}{N}\sum_{l=1}^{n_i}\frac{p_{il}}{1+p_{il}\delta_i},$$

we then have

$$\frac{1}{N}\operatorname{tr}\mathbf{A}(\mathbf{B}_N - z\mathbf{I}_N)^{-1} - \frac{1}{N}\operatorname{tr}\mathbf{A}(\mathbf{G}-z\mathbf{I}_N)^{-1}$$

$$= \sum_{i=1}^{K}\frac{1}{(1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i}}\frac{1}{N}\sum_{l=1}^{n_i}\frac{p_{il}}{1+p_{il}\delta_i}\frac{1}{N}\operatorname{tr}\mathbf{H}_i^{\mathsf{H}}(\mathbf{G}-z\mathbf{I}_N)^{-1}\mathbf{A}(\mathbf{B}_N-z\mathbf{I}_N)^{-1}\mathbf{H}_i$$

$$-\frac{1}{N}\sum_{i=1}^{K}\sum_{l=1}^{n_i}\frac{p_{il}\mathbf{w}_{il}^{\mathsf{H}}\mathbf{H}_i^{\mathsf{H}}(\mathbf{G}-z\mathbf{I}_N)^{-1}\mathbf{A}(\mathbf{B}_{(i,l)}-z\mathbf{I}_N)^{-1}\mathbf{H}_i\mathbf{w}_{il}}{1+p_{il}\mathbf{w}_{il}^{\mathsf{H}}\mathbf{H}_i^{\mathsf{H}}(\mathbf{B}_{(i,l)}-z\mathbf{I}_N)^{-1}\mathbf{H}_i\mathbf{w}_{il}}$$

$$= \sum_{i=1}^{K}\sum_{l=1}^{n_i}\frac{p_{il}}{N}\left[\frac{\frac{1}{N}\operatorname{tr}\mathbf{H}_i^{\mathsf{H}}(\mathbf{G}-z\mathbf{I}_N)^{-1}\mathbf{A}(\mathbf{B}_N-z\mathbf{I}_N)^{-1}\mathbf{H}_i}{((1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l'=1}^{n_i}\frac{1}{1+p_{i,l'}\delta_i})(1+p_{il}\delta_i)} - \frac{\mathbf{w}_{il}^{\mathsf{H}}\mathbf{H}_i^{\mathsf{H}}(\mathbf{G}-z\mathbf{I}_N)^{-1}\mathbf{A}(\mathbf{B}_{(i,l)}-z\mathbf{I}_N)^{-1}\mathbf{H}_i\mathbf{w}_{il}}{1+p_{il}\mathbf{w}_{il}^{\mathsf{H}}\mathbf{H}_i^{\mathsf{H}}(\mathbf{B}_{(i,l)}-z\mathbf{I}_N)^{-1}\mathbf{H}_i\mathbf{w}_{il}}\right] \to 0.$$

so that

$$f_i - \frac{1}{N}\operatorname{tr}\mathbf{R}_i\left(\sum_{k=1}^{K}\frac{1}{(1-c_k)\bar{c}_k + \frac{1}{N}\sum_{l=1}^{n_k}\frac{1}{1+p_{kl}\delta_k}}\frac{1}{N}\sum_{l=1}^{n_k}\frac{p_{kl}}{1+p_{kl}\delta_k}\mathbf{R}_k - z\mathbf{I}_N\right)^{-1} \to 0$$

$$f_i - \frac{1}{N}\sum_{l=1}^{n_i}\frac{\delta_i}{1+p_{il}\delta_i} - (1-c_i)\bar{c}_i\delta_i \to 0.$$

Outline

A detour to free probability theory

- The case $K = 1$, $c_1 = \bar{c}_1 = 1$ can be treated using free probability theory and in particular the R- and S-transform.
- The result is not the same as above. Instead we have[2]

$$\bar{e} = \frac{1}{N} \operatorname{tr} \mathbf{P} \left(e\mathbf{P} + [1 - e\bar{e}]\mathbf{I}_n \right)^{-1}$$
$$e = \frac{1}{N} \operatorname{tr} \mathbf{R} \left(\bar{e}\mathbf{R} - z\mathbf{I}_N \right)^{-1}.$$

- the next step is to show that both expressions are consistent.

[2] see the note from W. Hachem, “An expression for $\int \log(t/\sigma^2 + 1)\mu \boxplus \tilde{\mu}(dt)$”

A detour to free probability theory

- The case $K = 1$, $c_1 = \bar{c}_1 = 1$ can be treated using free probability theory and in particular the R- and S-transform.
- The result is not the same as above. Instead we have[2]

$$\bar{e} = \frac{1}{N} \operatorname{tr} \mathbf{P} \left(e\mathbf{P} + [1 - e\bar{e}]\mathbf{I}_n \right)^{-1}$$
$$e = \frac{1}{N} \operatorname{tr} \mathbf{R} \left(\bar{e}\mathbf{R} - z\mathbf{I}_N \right)^{-1}.$$

- the next step is to show that both expressions are consistent.

[2] see the note from W. Hachem, "An expression for $\int \log(t/\sigma^2 + 1)\mu \boxplus \tilde{\mu}(dt)$"

A detour to free probability theory

- The case $K = 1$, $c_1 = \bar{c}_1 = 1$ can be treated using free probability theory and in particular the R- and S-transform.
- The result is not the same as above. Instead we have[2]

$$\bar{e} = \frac{1}{N} \operatorname{tr} \mathbf{P} \left(e\mathbf{P} + [1 - e\bar{e}]\mathbf{I}_n \right)^{-1}$$
$$e = \frac{1}{N} \operatorname{tr} \mathbf{R} \left(\bar{e}\mathbf{R} - z\mathbf{I}_N \right)^{-1}.$$

- the next step is to show that both expressions are consistent.

[2] see the note from W. Hachem, "An expression for $\int \log(t/\sigma^2 + 1)\mu \boxplus \tilde{\mu}(dt)$"

Simplifying $\bar{g}_k$

$$\bar{g}_i = \frac{1}{N}\sum_{l=1}^{n_i} \frac{p_{il}}{\left((1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i}\right) + p_{il}\delta_i\left((1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i}\right)},$$

- We first remind that

$$f_i \simeq \delta_i\left((1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i}\right).$$

- From

$$\bar{c}_i - \bar{g}_i\delta_i\left((1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i}\right) = (1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i},$$

we also find that

$$\bar{c}_i - \bar{g}_i f_i \simeq (1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i}$$

- Together, this gives

$$\bar{g}_i \simeq \frac{1}{N}\sum_{l=1}^{n_i}\frac{p_{il}}{\bar{c}_i - f_i\bar{g}_i + p_{il}f_i} = \frac{1}{N}\operatorname{tr}\mathbf{P}_i\left(f_i\mathbf{P}_i + (\bar{c}_i - f_i\bar{g}_i)\mathbf{I}_{n_i}\right)$$

where δ_i no longer appears and $\bar{g}_i$ is now related to itself.

Simplifying $\bar{g}_k$

$$\bar{g}_i = \frac{1}{N}\sum_{l=1}^{n_i} \frac{p_{il}}{\left((1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i}\right) + p_{il}\delta_i\left((1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i}\right)},$$

- We first remind that

$$f_i \simeq \delta_i\left((1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i}\right).$$

- From

$$\bar{c}_i - \bar{g}_i\delta_i\left((1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i}\right) = (1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i},$$

  we also find that

$$\bar{c}_i - \bar{g}_i f_i \simeq (1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i}$$

- Together, this gives

$$\bar{g}_i \simeq \frac{1}{N}\sum_{l=1}^{n_i}\frac{p_{il}}{\bar{c}_i - f_i\bar{g}_i + p_{il}f_i} = \frac{1}{N}\operatorname{tr}\mathbf{P}_i\left(f_i\mathbf{P}_i + (\bar{c}_i - f_i\bar{g}_i)\mathbf{I}_{n_i}\right)$$

  where δ_i no longer appears and $\bar{g}_i$ is now related to itself.

Simplifying $\bar{g}_k$

$$\bar{g}_i = \frac{1}{N}\sum_{l=1}^{n_i} \frac{p_{il}}{\left((1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i}\right) + p_{il}\delta_i\left((1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i}\right)},$$

- We first remind that

$$f_i \simeq \delta_i\left((1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i}\right).$$

- From

$$\bar{c}_i - \bar{g}_i\delta_i\left((1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i}\right) = (1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i},$$

  we also find that

$$\bar{c}_i - \bar{g}_i f_i \simeq (1-c_i)\bar{c}_i + \frac{1}{N}\sum_{l=1}^{n_i}\frac{1}{1+p_{il}\delta_i}$$

- Together, this gives

$$\bar{g}_i \simeq \frac{1}{N}\sum_{l=1}^{n_i}\frac{p_{il}}{\bar{c}_i - f_i\bar{g}_i + p_{il}f_i} = \frac{1}{N}\operatorname{tr}\mathbf{P}_i\left(f_i\mathbf{P}_i + (\bar{c}_i - f_i\bar{g}_i)\mathbf{I}_{n_i}\right)$$

  where δ_i no longer appears and $\bar{g}_i$ is now related to itself.

Final convergence step

- From the above, we finally have

$$f_i - \frac{1}{N}\operatorname{tr}\mathbf{R}_i\left(\sum_{j=1}^{K}\bar{g}_j\mathbf{R}_j - z\mathbf{I}_N\right)^{-1} \xrightarrow{\text{a.s.}} 0$$

$$\bar{g}_i - \frac{1}{N}\operatorname{tr}\mathbf{P}_i\left(\bar{g}_i\mathbf{P}_i + [\bar{c}_i - f_i\bar{g}_i]\mathbf{I}_{n_i}\right)^{-1} \xrightarrow{\text{a.s.}} 0.$$

- We then take $\bar{f}_i$ to be the unique solution within $[0, \bar{c}_i c_i/f_i)$ of the equation in x

$$x = \frac{1}{N}\sum_{l=1}^{n_i}\frac{p_{il}}{\bar{c}_i + p_{il}f_i - xf_i}$$

(uniqueness is easy to check) and show that $\bar{f}_i - \bar{g}_i \to 0$.

- For this, notice that

$$|\bar{g}_i - \bar{f}_i| \leq \left|\bar{g}_i - \frac{1}{N}\sum_{l=1}^{n_i}\frac{p_{il}}{\bar{c}_i - f_i\bar{g}_i + p_{il}f_i}\right| + |\bar{g}_i - \bar{f}_i| \cdot \left|\frac{1}{N}\sum_{l=1}^{n_i}\frac{p_{il}f_i}{(\bar{c}_i - f_i\bar{f}_i + p_{il}f_i)(\bar{c}_i - f_i\bar{g}_i + p_{il}f_i)}\right|.$$

Since $\bar{f}_i \in [0, \bar{c}_i c_i/f_i)$, $\bar{c}_i - f_i\bar{f}_i + p_{il}f_i \geq (1 - c_i)\bar{c}_i$. For $|z|$ large, $f_i \to 0$ and then the second RHS term is small. Since the first RHS term tends to zero, $\bar{f}_i - \bar{g}_i \to 0$.

Final convergence step

- From the above, we finally have

$$f_i - \frac{1}{N}\operatorname{tr}\mathbf{R}_i\left(\sum_{j=1}^{K}\bar{g}_j\mathbf{R}_j - z\mathbf{I}_N\right)^{-1} \xrightarrow{\text{a.s.}} 0$$

$$\bar{g}_i - \frac{1}{N}\operatorname{tr}\mathbf{P}_i\left(\bar{g}_i\mathbf{P}_i + [\bar{c}_i - f_i\bar{g}_i]\mathbf{I}_{n_i}\right)^{-1} \xrightarrow{\text{a.s.}} 0.$$

- We then take $\bar{f}_i$ to be the unique solution within $[0, \bar{c}_i c_i/f_i)$ of the equation in x

$$x = \frac{1}{N}\sum_{l=1}^{n_i}\frac{p_{il}}{\bar{c}_i + p_{il}f_i - xf_i}$$

(uniqueness is easy to check) and show that $\bar{f}_i - \bar{g}_i \to 0$.

- For this, notice that

$$|\bar{g}_i - \bar{f}_i| \leq \left|\bar{g}_i - \frac{1}{N}\sum_{l=1}^{n_i}\frac{p_{il}}{\bar{c}_i - f_i\bar{g}_i + p_{il}f_i}\right| + |\bar{g}_i - \bar{f}_i|\cdot\left|\frac{1}{N}\sum_{l=1}^{n_i}\frac{p_{il}f_i}{(\bar{c}_i - f_i\bar{f}_i + p_{il}f_i)(\bar{c}_i - f_i\bar{g}_i + p_{il}f_i)}\right|.$$

Since $\bar{f}_i \in [0, \bar{c}_i c_i/f_i)$, $\bar{c}_i - f_i\bar{f}_i + p_{il}f_i \geq (1 - c_i)\bar{c}_i$. For $|z|$ large, $f_i \to 0$ and then the second RHS term is small. Since the first RHS term tends to zero, $\bar{f}_i - \bar{g}_i \to 0$.

Final formula

- We finally have

$$f_i - \frac{1}{N}\operatorname{tr}\mathbf{R}_i\left(\sum_{j=1}^{K}\bar{f}_j\mathbf{R}_j - z\mathbf{I}_N\right)^{-1} \xrightarrow{\text{a.s.}} 0$$

$$\bar{f}_i - \frac{1}{N}\operatorname{tr}\mathbf{P}_i\left(\bar{f}_i\mathbf{P}_i + [\bar{c}_i - f_i\bar{f}_i]\mathbf{I}_{n_i}\right)^{-1} = 0$$

with $\bar{f}_i \in [0, \bar{c}_i c_i / f_i)$, unique.

Outline

Uniqueness of the fixed-point equation

Define

$$h_i : (x_1, \ldots, x_K) \mapsto \frac{1}{N} \operatorname{tr} \mathbf{R}_i \left(\sum_{j=1}^{K} \bar{x}_j \mathbf{R}_j - z \mathbf{I}_N \right)^{-1}$$

with $\bar{x}_j$ the unique solution of the equation in y

$$y = \frac{1}{N} \sum_{l=1}^{n_j} \frac{p_{jl}}{\bar{c}_j + x_j p_{jl} - x_j y}$$

such that $0 \leq y < c_j \bar{c}_j / x_j$.

For uniqueness and convergence of the fixed-point algorithm, it is sufficient to prove that the vector $\mathbf{h} \triangleq (h_1, \ldots, h_K)$ is a standard function,[3] i.e. it satisfies the conditions

- *Positivity:* if $x_1, \ldots, x_K > 0$, then $h(x_1, \ldots, x_K) > 0$,
- *Monotonicity:* if $x_1 > x_1', \ldots, x_K > x_K'$, then for all j, $h_j(x_1, \ldots, x_K) > h_j(x_1', \ldots, x_K')$,
- *Scalability:* for all $\alpha > 0$ and j, $\alpha h_j(x_1, \ldots, x_K) > h_j(\alpha x_1, \ldots, \alpha x_K)$.

The only non-trivial step is to show monotonicity.

[3] Theorem 2 of R. D. Yates, "A framework for uplink power control in cellular radio systems," IEEE J. Sel. Areas Commun., vol. 13, no. 7, pp. 1341-1347, 1995.

Uniqueness of the fixed-point equation

Define

$$h_i : (x_1, \ldots, x_K) \mapsto \frac{1}{N} \operatorname{tr} \mathbf{R}_i \left(\sum_{j=1}^{K} \bar{x}_j \mathbf{R}_j - z\mathbf{I}_N \right)^{-1}$$

with $\bar{x}_j$ the unique solution of the equation in y

$$y = \frac{1}{N} \sum_{l=1}^{n_j} \frac{p_{jl}}{\bar{c}_j + x_j p_{jl} - x_j y}$$

such that $0 \leq y < c_j \bar{c}_j / x_j$.
For uniqueness and convergence of the fixed-point algorithm, it is sufficient to prove that the vector $\mathbf{h} \triangleq (h_1, \ldots, h_K)$ is a standard function,[3] i.e. it satisfies the conditions

- *Positivity:* if $x_1, \ldots, x_K > 0$, then $h(x_1, \ldots, x_K) > 0$,
- *Monotonicity:* if $x_1 > x_1', \ldots, x_K > x_K'$, then for all j, $h_j(x_1, \ldots, x_K) > h_j(x_1', \ldots, x_K')$,
- *Scalability:* for all $\alpha > 0$ and j, $\alpha h_j(x_1, \ldots, x_K) > h_j(\alpha x_1, \ldots, \alpha x_K)$.

The only non-trivial step is to show monotonicity.

[3] Theorem 2 of R. D. Yates, "A framework for uplink power control in cellular radio systems," IEEE J. Sel. Areas Commun., vol. 13, no. 7, pp. 1341-1347, 1995.

Monotonicity

- we introduce the auxiliary variables $\Delta_1, \ldots, \Delta_K$, with the properties

$$x_i = \Delta_i \left((1 - c_i)\bar{c}_i + \frac{1}{N} \sum_{l=1}^{n_i} \frac{1}{1 + p_{il}\Delta_i} \right)$$
$$= \Delta_i \left(\bar{c}_i - \frac{1}{N} \sum_{l=1}^{n_i} \frac{p_{il}\Delta_i}{1 + p_{il}\Delta_i} \right).$$

and

$$\bar{c}_i - x_i\bar{x}_i = (1 - c_i)\bar{c}_i + \frac{1}{N} \sum_{l=1}^{n_i} \frac{1}{1 + p_{il}\Delta_i}$$
$$= \bar{c}_i - \frac{1}{N} \sum_{l=1}^{n_i} \frac{p_{il}\Delta_i}{1 + p_{il}\Delta_i}.$$

- It is not difficult to prove these Δ_i are uniquely defined.

Monotonicity (2)

- We show first that $\frac{d}{dx_i}\bar{x}_i < 0$
- This unfolds from

$$\frac{d}{d\Delta_i}\bar{x}_i = \frac{1}{{\Delta_i}^2\left(\bar{c}_i - \frac{1}{N}\sum_{l=1}^{n_i}\frac{p_{il}\Delta_i}{1+p_{il}\Delta_i}\right)^2}\left[\left(\frac{1}{N}\sum_{l=1}^{n_i}\frac{p_{il}\Delta_i}{1+p_{il}\Delta_i}\right)^2 - \frac{\bar{c}_i}{N}\sum_{l=1}^{n_i}\frac{(p_{il}\Delta_i)^2}{(1+p_{il}\Delta_i)^2}\right]$$

which is negative from Cauchy-Schwarz.

- From this, we have for two sets $x_1, \ldots, x_K$ and $x'_1, \ldots, x'_K$ of positive values such that $x_j > x'_j$

$$\begin{aligned} & h_j(x_1, \ldots, x_K) - h_j(x'_1, \ldots, x'_K) \\ &= \sum_{i=1}^{K}(\bar{x}'_i - \bar{x}_i)\frac{1}{N}\operatorname{tr}\mathbf{R}_j\left(\sum_{k=1}^{K}\bar{x}_k\mathbf{R}_k - z\mathbf{I}_N\right)^{-1}\mathbf{R}_i\left(\sum_{k=1}^{K}\bar{x}'_k\mathbf{R}_k - z\mathbf{I}_N\right)^{-1} \\ &> 0. \end{aligned}$$

Monotonicity (2)

- We show first that $\frac{d}{dx_i}\bar{x}_i < 0$
- This unfolds from

$$\frac{d}{d\Delta_i}\bar{x}_i = \frac{1}{{\Delta_i}^2\left(\bar{c}_i - \frac{1}{N}\sum_{l=1}^{n_i}\frac{p_{il}\Delta_i}{1+p_{il}\Delta_i}\right)^2}\left[\left(\frac{1}{N}\sum_{l=1}^{n_i}\frac{p_{il}\Delta_i}{1+p_{il}\Delta_i}\right)^2 - \frac{\bar{c}_i}{N}\sum_{l=1}^{n_i}\frac{(p_{il}\Delta_i)^2}{(1+p_{il}\Delta_i)^2}\right]$$

  which is negative from Cauchy-Schwarz.
- From this, we have for two sets $x_1,\ldots,x_K$ and $x_1',\ldots,x_K'$ of positive values such that $x_j > x_j'$

$$\begin{aligned} & h_j(x_1,\ldots,x_K) - h_j(x_1',\ldots,x_K') \\ &= \sum_{i=1}^{K}(\bar{x}_i' - \bar{x}_i)\frac{1}{N}\operatorname{tr}\mathbf{R}_j\left(\sum_{k=1}^{K}\bar{x}_k\mathbf{R}_k - z\mathbf{I}_N\right)^{-1}\mathbf{R}_i\left(\sum_{k=1}^{K}\bar{x}_k'\mathbf{R}_k - z\mathbf{I}_N\right)^{-1} \\ &> 0. \end{aligned}$$

Convergence of the det. eq.

- call e_i the solution of the fixed-point equation in x_i. The last step is to show that

$$f_i - e_i \xrightarrow{\text{a.s.}} 0.$$

- this unfolds from classical arguments by showing

$$|f_i - e_i| \leq \alpha |f_i - e_i| + \varepsilon$$

  with $\varepsilon \xrightarrow{\text{a.s.}} 0$ when the dimension grows large and $0 < \alpha < 1$ for some $|z|$ large enough. Vitali theorem completes the proof for all z.

- For the case $c = 1$, we write

$$\left|\frac{1}{N}\operatorname{tr}\mathbf{R}_i\left(\mathbf{B}_N - z\mathbf{I}_N\right)^{-1} - e_i(z)\right| \leq \left|\frac{1}{N}\operatorname{tr}\mathbf{R}_i\left(\mathbf{B}_N - z\mathbf{I}_N\right)^{-1} - \frac{1}{N}\operatorname{tr}\mathbf{R}_i\left(\mathbf{B}_N^{(n)} - z\mathbf{I}_N\right)^{-1}\right| + \left|\frac{1}{N}\operatorname{tr}\mathbf{R}_i\left(\mathbf{B}_N^{(n)} - z\mathbf{I}_N\right)^{-1} - e_i^{(n)}(z)\right| + \left|e_i^{(n)}(z) - e_i(z)\right|,$$

  with $e_i^{(n)}$, $\mathbf{B}_N^{(n)}$ the values of e_i, $\mathbf{B}_N$ if the $\mathbf{P}_k$ are truncated into $n \times n$ matrices.

- We then show that the limsup of all terms are less than any $\varepsilon > 0$ as $n, N \to \infty$ for some $c < 1$.

Convergence of the det. eq.

- call e_i the solution of the fixed-point equation in x_i. The last step is to show that

$$f_i - e_i \xrightarrow{\text{a.s.}} 0.$$

- this unfolds from classical arguments by showing

$$|f_i - e_i| \leq \alpha |f_i - e_i| + \varepsilon$$

with $\varepsilon \xrightarrow{\text{a.s.}} 0$ when the dimension grows large and $0 < \alpha < 1$ for some $|z|$ large enough. Vitali theorem completes the proof for all z.

- For the case $c = 1$, we write

$$\left| \frac{1}{N} \operatorname{tr} \mathbf{R}_i \left(\mathbf{B}_N - z\mathbf{I}_N \right)^{-1} - e_i(z) \right| \leq \left| \frac{1}{N} \operatorname{tr} \mathbf{R}_i \left(\mathbf{B}_N - z\mathbf{I}_N \right)^{-1} - \frac{1}{N} \operatorname{tr} \mathbf{R}_i \left(\mathbf{B}_N^{(n)} - z\mathbf{I}_N \right)^{-1} \right| + \left| \frac{1}{N} \operatorname{tr} \mathbf{R}_i \left(\mathbf{B}_N^{(n)} - z\mathbf{I}_N \right)^{-1} - e_i^{(n)}(z) \right| + \left| e_i^{(n)}(z) - e_i(z) \right|,$$

with $e_i^{(n)}$, $\mathbf{B}_N^{(n)}$ the values of e_i, $\mathbf{B}_N$ if the $\mathbf{P}_k$ are truncated into $n \times n$ matrices.

- We then show that the limsup of all terms are less than any $\varepsilon > 0$ as $n, N \to \infty$ for some $c < 1$.

Outline

Scenario

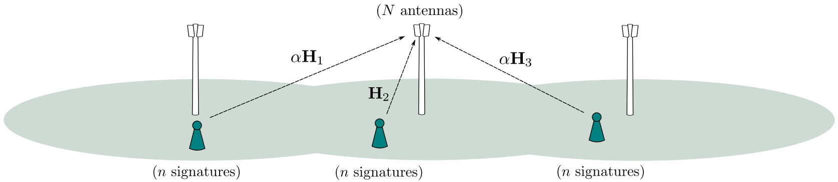

Figure: Three-cell example: BS_2 decodes the *n* streams from the UT in its own cell while treating the other signals as interference.

Deterministic equivalent of the Shannon transform

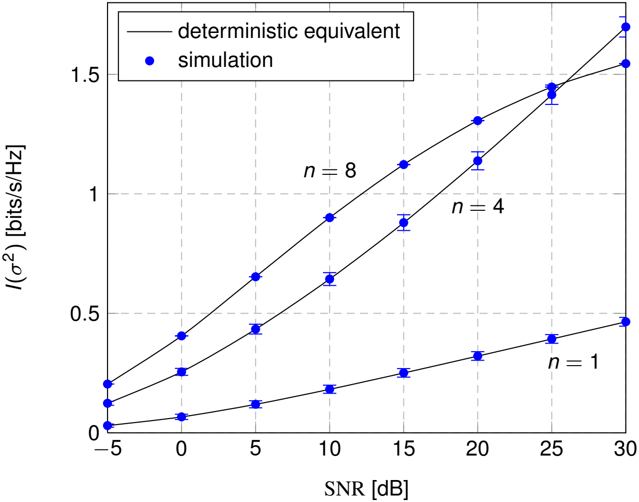

Figure: Mutual information $I(\sigma^2)$ versus SNR for different numbers of transmit signatures n, $N = 16$, $N_i = 8$, $\mathbf{P}_i = \mathbf{I}_n$, $\alpha = 0.5$. Error bars represent one standard deviation on each side.

Deterministic equivalent of the MMSE SINR

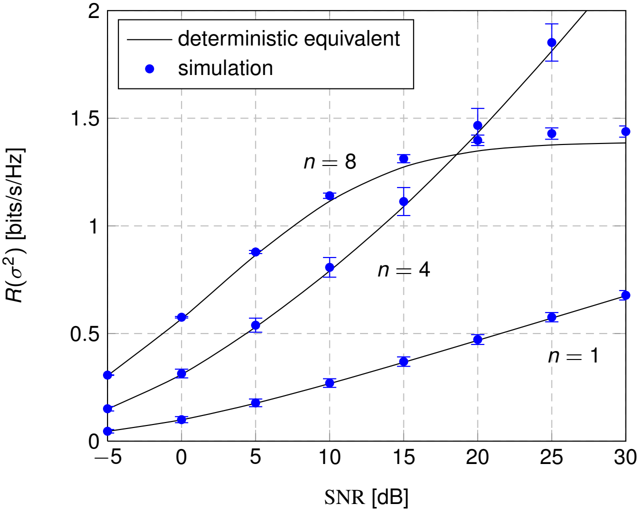

Figure: SUm rate $R(\sigma^2)$ at the output of the MMSE decoder for user 2 versus SNR for different numbers of transmit signatures n, $N = 16$, $N_i = 8$, $\mathbf{P}_i = \mathbf{I}_n$, $\alpha = 0.5$. Error bars represent one standard deviation on each side.

Outline

Main result... so far!

- We consider the model
$$\mathbf{B}_N = \sum_{k=1}^{K} \mathbf{R}_k \mathbf{w}_k \mathbf{w}_k^{\mathsf{H}} \mathbf{R}_k^{\mathsf{H}}$$
with $\mathbf{R}_k \in \mathbb{C}^{N\times N}$ deterministic with bounded spectral norm and $[\mathbf{w}_1, \ldots, \mathbf{w}_K]$ the $K \leq N$ columns of a unitary Haar matrix.
- We have the following det. eq. for $\mathbf{A}$ with bounded spectral norm

Theorem

$$\frac{1}{N}\operatorname{tr}\mathbf{A}(\mathbf{B}_N - z\mathbf{I}_N)^{-1} - \frac{1}{N}\operatorname{tr}\mathbf{A}(\mathbf{Q} - z\mathbf{I}_N)^{-1} \xrightarrow{\text{a.s.}} 0$$

with

$$\mathbf{Q} = \frac{1}{N}\sum_{k=1}^{K} \frac{\mathbf{R}_k \mathbf{R}_k^{\mathsf{H}}}{(1+e_{kk})\left(1 - \frac{1}{N}\sum_{i=1}^{K} \frac{\bar{e}_{ki} e_{ik}}{(1+e_{ii})\bar{e}_{kk}}\right)}$$

where, for $1 \leq k, l \leq K$

$$e_{kl} = \frac{\frac{1}{N}\operatorname{tr}\mathbf{R}_l\mathbf{R}_k^{\mathsf{H}}(\mathbf{Q} - z\mathbf{I}_N)^{-1}}{1 - \frac{1}{N}\sum_{i=1}^{K}\frac{e_{ki}e_{il}}{(1+e_{ii})e_{kl}}} \quad \text{and} \quad \bar{e}_{kl} = \frac{\frac{1}{N}\operatorname{tr}\mathbf{R}_l\mathbf{R}_k^{\mathsf{H}}(\mathbf{Q} - z\mathbf{I}_N)^{-2}}{1 - \frac{1}{N}\sum_{i=1}^{K}\frac{\bar{e}_{ki}e_{il}}{(1+e_{ii})\bar{e}_{kl}}}.$$

Comments and Conclusions

- Thanks to the trace lemma for Haar matrices, it is possible to extend techniques for matrices with independent entries to Haar matrices.
- The technique is more involved than the free probability approach but is fully consistent
- We introduced results that are non convenient to treat within the free probability framework alone
- The trace lemma technique leads to a first impractical expression, which may be refined by some sort of “guess-work”.
- Some open questions:
  - Can we apply this framework for more involved models based on Haar matrices?
  - Can we extend the technique to other matrix models (e.g., Euclidean, Vandermonde random matrices)?
  - Can we extend this study into moment formulas?

Comments and Conclusions

- Thanks to the trace lemma for Haar matrices, it is possible to extend techniques for matrices with independent entries to Haar matrices.
- The technique is more involved than the free probability approach but is fully consistent
- We introduced results that are non convenient to treat within the free probability framework alone
- The trace lemma technique leads to a first impractical expression, which may be refined by some sort of "guess-work".
- Some open questions:
  - Can we apply this framework for more involved models based on Haar matrices?
  - Can we extend the technique to other matrix models (e.g., Euclidean, Vandermonde random matrices)?
  - Can we extend this study into moment formulas?

Comments and Conclusions

- Thanks to the trace lemma for Haar matrices, it is possible to extend techniques for matrices with independent entries to Haar matrices.
- The technique is more involved than the free probability approach but is fully consistent
- We introduced results that are non convenient to treat within the free probability framework alone
- The trace lemma technique leads to a first impractical expression, which may be refined by some sort of “guess-work”.
- Some open questions:
  - Can we apply this framework for more involved models based on Haar matrices?
  - Can we extend the technique to other matrix models (e.g., Euclidean, Vandermonde random matrices)?
  - Can we extend this study into moment formulas?

Comments and Conclusions

- Thanks to the trace lemma for Haar matrices, it is possible to extend techniques for matrices with independent entries to Haar matrices.
- The technique is more involved than the free probability approach but is fully consistent
- We introduced results that are non convenient to treat within the free probability framework alone
- The trace lemma technique leads to a first impractical expression, which may be refined by some sort of "guess-work".
- Some open questions:
  - Can we apply this framework for more involved models based on Haar matrices?
  - Can we extend the technique to other matrix models (e.g., Euclidean, Vandermonde random matrices)?
  - Can we extend this study into moment formulas?