Aalto University
School of Science
and Technology

On Demmel Condition Number Distributions with Applications in Telecommunications

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Outline

Demmel Condition Number

- Definition
- Existing results

Derivations for DCN Distributions

- General framework
- Exact distribution
- Asymptotic distribution

Applications in Wireless Communications

- Adaptive transmission
- Adaptive detection

Definition

- Define a $K \times N$ dimension matrix $\mathbf{X}$ with independent and identically distributed (i.i.d) complex Gaussian entries, each with zero mean and unit variance.
- The $K \times K$ Hermitian matrix $\mathbf{R} = \mathbf{X}\mathbf{X}^\dagger$ follows a complex Wishart distribution with N degree of freedom (d.o.f).
- We denote the ordered eigenvalues of $\mathbf{R}$ as $\lambda_1 > \lambda_2 > ... > \lambda_K > 0$, and the trace of $\mathbf{R}$ as $T = \text{tr}\{\mathbf{R}\} = ||\mathbf{X}||_F^2 = \sum_{i=1}^{K} \lambda_i$, where $|| \cdot ||_F$ is the Frobenius norm.

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- The Demmel Condition Number (DCN) of $\mathbf{R}$ is defined as the ratio of its trace to its smallest eigenvalue λ_K,

$$X := \frac{\sum_{i=1}^{K} \lambda_i}{\lambda_K} = \frac{T}{\lambda_K}, \tag{1}$$

where $x \in [K, \infty]$.

Existing results

- Limited results on the DCN distribution exist in the literature.
- A. Edelman, "On the distribution of a scaled condition number," *Math. Comp., vol. 58, pp. 185-190, 1992.*
  - *Exact DCN distributions for the special case $K = N$ (both real and complex cases).*
  - *Mainly based on the fact that λ_K has tractable expressions when $K = N$ (e.g. exponentially distributed in complex case).*
  - *Using an equality (A. W. Davis,* 1972*) between Laplace transforms of PDFs of X and λ_K.*

Existing results

- M. Matthaiou, M. R. McKay, P. J. Smith, and J. A. Nossek, "On the condition number distribution of complex Wishart matrices," *IEEE Tran. Commun., vol. 58, no. 6, pp. 1705-1711, Jun. 2010.*
  - *Exact DCN distributions for $K = 2$ with arbitrary N.*
  - *Established through standard condition number distribution ($\frac{\lambda_1+\lambda_2}{\lambda_2} = 1 + \frac{\lambda_1}{\lambda_2}$).*
- *Above two results are exact. No asymptotic results w.r.t. matrix dimension are available.*

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- *Above two results are exact. No asymptotic results w.r.t. matrix dimension are available.*
- *In this work, both exact and asymptotic DCN distributions for arbitrary* K *and* N *are derived.*

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- Thus, λ_K equals the product of the independent r.v Y and T. Define $f(x)$, $g(x)$ and $h(x)$ as the PDFs of λ_K, T and Y respectively.
- By this independence, it holds that

$$M_z[f(x)] = M_z[g(x)] M_z[h(x)], \tag{2}$$

where $M_z[\cdot]$ denotes Mellin transform.

General framework

- By Mellin inversion integral, the distribution of $h(x)$ can be uniquely determined by

$$h(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-z} \frac{M_z[f(x)]}{M_z[g(x)]} \mathrm{d}z. \tag{3}$$

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- This framework provides the possibility to obtain both exact and asymptotic DCN distributions.

Exact distribution

- C. S. Park and K. B. Lee "Statistical multimode transmit antenna selection for limited feedback MIMO systems," *IEEE Tran. Wireless Commun., vol. 7, no. 11, pp. 4432-4438, Nov. 2008.*
  - *PDF of λ_K represented as a weighted sum of polynomials as*

$$f(x) = e^{-Kx} \sum_{n=N-K}^{(N-K)K} c_n^{(N,K)} x^n. \tag{4}$$

  - *Coefficients $c_n^{(N,K)}$ is determined by the symmetry of the integral representation of λ_K (A. Edelman,* 1989*).*

Determining the coefficients $c_n^{(N,K)}$

- Define

$$I_n(m) := \sum_{k=0}^{n} \binom{n}{k} (m+n-k)!x^k. \tag{5}$$

- $K = 2$, PDF of λ_K is

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- $K = 4$, PDF of λ_K is

$$c_4 e^{-4x} x^{N-4}[I_{N-4}(6) I_{N-4}(4) I_{N-4}(2) - I_{N-4}(6)(I_{N-4}(3))^2 \quad (8)$$
$$+2 I_{N-4}(5) I_{N-4}(4) I_{N-4}(3) - (I_{N-4}(5))^2 I_{N-4}(2) - (I_{N-4}(4))^2].$$

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- Note:
  - After some basic manipulations, the expressions for coefficients of x can be obtained.
  - Although tedious, coefficients for arbitrary K can be similarly calculated.

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\begin{aligned}
M_z[f(x)] &= \sum_{n=N-K}^{(N-K)K} c_n^{(N,K)} \frac{\Gamma(z+n)}{K^{z+n}}, \\
M_z[g(x)] &= \frac{1}{\Gamma(m/2)} \Gamma(z + \frac{m}{2} - 1), \quad (m = 2KN).
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- Using the residue theorem, $h(x)$ is uniquely determined to be

$$h(x) = \frac{\Gamma(m/2)}{(1-Kx)^{2-m/2}} \sum_{n=N-K}^{(N-K)K} \frac{c_n^{(N,K)}}{\Gamma(m/2-n-1)} \left(\frac{x}{1-Kx}\right)^n. \tag{9}$$

Exact distribution

- By a simple transform, PDF of DCN is obtained as,

$$d(x) = \frac{\Gamma(m/2)x^{-m/2}}{(x-K)^{2-m/2}} \sum_{n=N-K}^{(N-K)K} \frac{c_n^{(N,K)}}{\Gamma(m/2-n-1)}(x-K)^{-n}. \tag{10}$$

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- Then CDF of DCN is calculated to be,

$$D(y) = \Gamma(\frac{m}{2}) \sum_{n=N-K}^{(N-K)K} \frac{K^{-n-1}c_n^{(N,K)}}{\Gamma(m/2-n-1)} \left(B(a,b) - B_{\frac{K}{y}}(a,b)\right) \tag{11}$$

$B_x(a,b)$ and $B(a,b)$ are incomplete and complete Beta function respectively and $a = n+1$, $b = \frac{m}{2} - n - 1$.

Special cases

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- $K = N$ (A. Edelman, 1992)
  - The only coefficient left is $c_0^{(K,K)} = K$.
  - Inserting this coefficient into the derived PDF, $d(x)$ simplifies to

$$d(x) = K(K^2 - 1)x^{-K^2}(x - K)^{K^2-2}. \tag{12}$$

  - Agrees with the known result.

Special cases

- $K = 2$, with arbitrary N (M. Matthaiou, 2010)
  - The coefficient in this case is

$$c_n^{(N,2)} = \frac{\Gamma(2N - n - 1)}{\Gamma(N)\Gamma(n - N + 3)\Gamma(2N - n - 3)}. \tag{13}$$

  - Inserting $c_n^{(N,2)}$ into the derived PDF, $d(x)$ simplifies to

$$d(x) = \frac{\Gamma(2N)}{\Gamma(N)\Gamma(N - 1)}(x - 2)^2 x^{-2N}(x - 1)^{N-2}. \tag{14}$$

  - In agreement with the known result.

One numerical example

- $K = 4$, $N = 5$.
- $d(x)$ is calculated to be

$$3420(x-4)^{14}(x^3+5x^2-20x+4)x^{-20}. \tag{15}$$

- $D(y)$ is calculated to be

$$\begin{aligned} &1-213.75B_{\frac{4}{y}}(2,18)-908.438B_{\frac{4}{y}}(3,17) \\ &-908.438B_{\frac{4}{y}}(4,16)-227.109B_{\frac{4}{y}}(5,15). \end{aligned} \tag{16}$$

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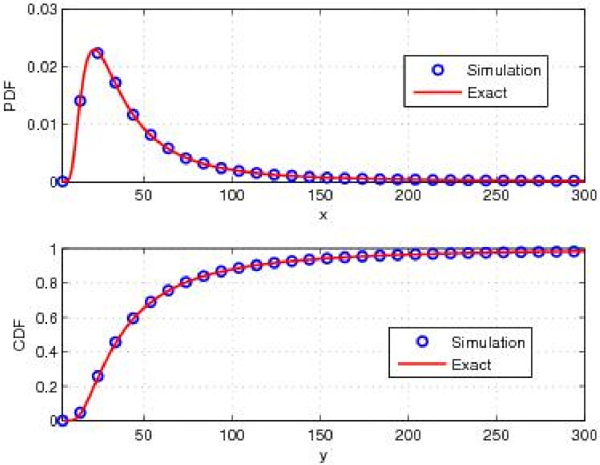

Asymptotic distribution

- Motivation:
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- Motivation:
  - Determining the coefficients may appear a problem for large dimensional matrices.
  - We would like to gain insight into the behavior of DCN distribution when the dimension K, N are large.
- We derive a closed-form asymptotic DCN distribution, which circumvents the need to calculate the coefficients.
- The asymptotic result falls in the developed Mellin transform framework as well.

An asymptotic result on λ_k distribution

- For λ_k, there exists sequences $a(K, N)$ and $b(K, N)$ such that the distribution of the random variable

$$\Lambda_K = \frac{\lambda_K - a(K, N)}{b(K, N)} \tag{17}$$

  converges to the Tracy-Widom distribution of order two (O. N. Feldheim, 2010), denoted as F_{TW2}.
- This result provides an approximation to λ_k for large K and N,

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- Numerical burden to calculate $F_{\text{TW2}}(\cdot)$, simpler closed-form approximation is more desirable.

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- It was stated in (A. Edelman, 2005) that λ_k can be well approximated by a Gamma distribution.
- Motivated by this, we propose a Gamma approximation by calculating the first two asymptotic moments via Tracy-Widom distribution.
  - $E[\lambda_K] = a(K,N) + b(K,N)E[\Lambda_K]$.
  - $V[\lambda_K] = \big(b(K,N)\big)^2 V[\Lambda_K]$.
- Convergence in distribution implies

$$E[\Lambda_K] \to E[x_{\text{TW2}}] = -1.7711, \tag{19}$$

$$V[\Lambda_K] \to V[x_{\text{TW2}}] = 0.8132. \tag{20}$$

Gamma approximation to λ_k distribution

- For a Gamma distribution with parameters θ and k, by matching the two moments of λ_k, θ and k is obtained as

$$k = \frac{\big(a(K,N) + b(K,N)E[x_{TW2}]\big)^2}{\big(b(K,N)\big)^2 V[x_{TW2}]}, \quad (21)$$

$$\theta = \frac{\big(b(K,N)\big)^2 V[x_{TW2}]}{a(K,N) + b(K,N)E[x_{TW2}]}. \quad (22)$$

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- By the residue theorem and a variable transform, the PDF of asymptotic DCN is calculated as

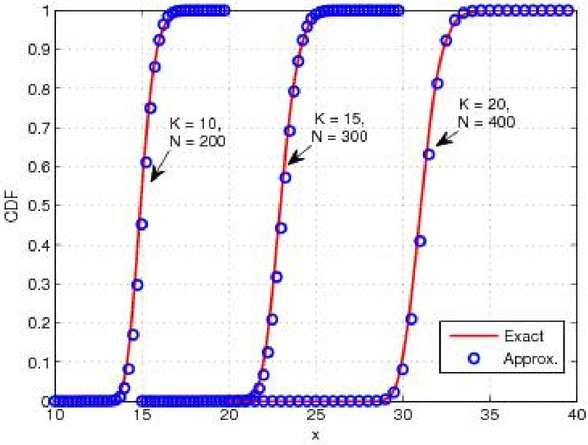
$$d(x) = c_1 x^{-m/2}(\theta x - 1)^{m/2-k-1}. \tag{24}$$

- Then, CDF of asymptotic DCN is calculated as

$$D(y) = c_2\big(C(K) - C(y)\big), \tag{25}$$

$C(x) = {}_2F_1(k, 1+k-\frac{m}{2}; k+1; \frac{1}{\theta x})x^{-k}$.

Numerical results



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- Performance analysis and design of MIMO techniques relies on the statistical properties of the random MIMO channels.
- DCN reflects the eigenvalue spread of the random MIMO channel – indicates multipath richness for a given channel realization.
- Using this fact, several MIMO transmit and receive schemes can be proposed.

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