A Simple Approach to the Global Regime of Gaussian Ensembles of Random Matrices

L. Pastur
Mathematical Division
Institute for Low Temperature Physics, Kharkiv

Abstract

We present simple proofs of several basic facts of the global regime (the existence and the form of the non-random limiting Normalized Counting Measure of eigenvalues, and the Central Limit Theorem for the trace of the resolvent) for ensembles of random matrices, whose probability law involves the Gaussian distribution. The main difference with previous proofs (see e.g. [8, 12]) is the systematic use of the Poincare-Nash inequality, allowing us to obtain the $O(n^{-2})$ bounds for the variance of the normalized trace of the resolvent that are valid up to the real axis in the spectral parameter.

1 Introduction

Numerous problems of the Random Matrix Theory can be roughly divided in three groups or regimes, according to the order of magnitude of intervals of the spectral axis with respect to the matrix size n. The regime in which there exists a well defined limit of the Normalized Counting Measure (Density of States) of eigenvalues as $n \to \infty$ is known as global or macroscopic regime. The regime, dealing with intervals whose length is $O(n^{-1})$ (mean eigenvalue spacing) with respect to the scale, fixed by the global regime, is known as the local or microscopic regime. This is where the repulsion of levels, important in many applications, manifests itself. The regime in which intervals of the length $O(n^{-\alpha})$, $0 < \alpha < 1$ are relevant is known as the intermediate. Corresponding results were used in explanations of universal conductance fluctuations of small metallic particles [2].

In obtaining results on the global and intermediate regimes the bounds of the order $o(1)$, $n \to \infty$ on the variance of linear statistics

$$N_n[\varphi] := \sum_{l=1}^{n} \varphi(\lambda_l^{(n)}) \tag{1.1}$$

of eigenvalues $\{\lambda_l^{(n)}\}_{l=1}^n$ of random matrix in question play an important role. Most precise bounds (up to exact asymptotic form) have the order $O(n^{-2})$ and valid for sufficiently smooth test functions φ in (1.1). These bounds, showing that eigenvalues of random matrices are strongly dependent, appeared first in the physics literature (see e.g. reviews [3, 4]) and were then rigorously proved for a number of random matrix ensembles (see e.g. [13, 7] for ensembles with invariant probability law, and [6, 9] for the Wigner ensembles, whose entries are independent or weakly dependent random variables modulo symmetry conditions).

In this paper we will confine ourselves to the case $\varphi(\lambda) = (\lambda - z)^{-1}$, $\Im z \neq 0$ in (1.1), corresponding to the normalized trace of the resolvent of a random matrix H

$$g_n(z) := n^{-1}\mathrm{Tr}(H - z)^{-1} \tag{1.2}$$

as a linear statistic.

An important ingredient of proofs in the case of invariant ensembles is an explicit form of the joint probability law of eigenvalues (called often the Weyl formulas) and related variational and/or orthogonal polynomials techniques. As for the Wigner and other ensembles with independent or weakly dependent entries, here the $O(n^{-2})$ bounds result from an analysis of certain recurrence relations for the moments of g_n. This method is rather efficient and self-contained, but leads to $O(n^{-2})$ bounds and related asymptotic formulas only if $|\Im z| \geq Cw^2$, where w^2 is the variance of the matrix entries $\{H_{jk}\}_{j,k=1}^n$ and C is an absolute constant (see e.g. [9]).

The goal of this paper is to show that if the entries are Gaussian (even dependent) random variables, then $O(n^{-2})$ bounds for the variance of (3.8) can be obtained by a rather direct application of an inequality for a C^1 function of a family of Gaussian random variables. The inequality dates back to Poincare and Nash and is widely used in statistics and analysis (see [5, 10] and references therein).

The paper is organized as follows. In Section 2 we present our technical means, in particular, an identity for expectations of differentiable functions of Gaussian random variables and the Poincare–Nash inequality. In Section 3 we find the limit of the Normalized Counting Measure of eigenvalues for the deformed Gaussian Ensembles, corresponding to matrices that are sums of a non-random matrix and the matrix of the Gaussian Unitary Ensemble (GUE) or the Gaussian Orthogonal Ensemble (GOE). An important element of the proofs are $O(n^{-2})$ bounds (3.21) and (3.40), proved by using the Poincare–Nash inequality. In Section 4 we derive the asymptotic formula for the variance of (1.2) and prove the central limit theorem for this class of linear statistics of eigenvalues of the GUE and the GOE by applying similar techniques. Section 5 contains a collection of related results and outlined proofs for several other ensembles, involving Gaussian random variables: random matrices, whose entries are dependent Gaussian random variables, the deformed Wishart and Laguerre ensembles, ensembles, appearing in the telecommunications, and the Wigner ensembles.

2 Technical Means

Definition 2.1 (Stieltjes Transform) *Let m be a finite non-negative measure on $\mathbb{R}$. The function*

$$f(z) = \int_{\mathbb{R}} \frac{m(d\lambda)}{\lambda - z} \tag{2.1}$$

defined for all non-real z, $\Im z \neq 0$ is called the Stieltjes transform *of m.*

Proposition 2.2 *Let f be the Stieltjes transform of a finite non-negative measure m, $m(\mathbb{R}) < \infty$. Then:*

(i) f is analytic in $\mathbb{C} \setminus \mathbb{R}$, and $\overline{f(z)} = f(\overline{z})$;

(ii) $\Im f(z) \cdot \Im z > 0$ for $\Im z \neq 0$;

(iii) $\lim_{\eta\to\infty} \eta |f(i\eta)| < \infty$;

(iv) for any function f, possessing the above properties there exists a finite non-negative measure m on $\mathbb{R}$ such that f is its Stieltjes transform;

(v) if Δ is an interval of $\mathbb{R}$ whose edges are not atoms of the measure m, then we have the Stieltjes - Perron inversion formula

$$m(\Delta) = \lim_{\epsilon\to 0^+} \frac{1}{\pi} \int_{\Delta} \Im f(\lambda + i\epsilon) d\lambda; \tag{2.2}$$

(vi) *the above one-to-one correspondence between non-negative measures and their Stieltjes transforms is continuous in the weak topology of measures and in the topology of the uniform convergence on compact subsets of $\mathbb{C}\setminus\mathbb{R}$ of analytic functions;*

(vii) *we have* $\lim_{\eta\to\infty}\eta|f(i\eta)| = m(\mathbb{R})$.

The next proposition presents elementary facts of linear algebra that will be often used below.

Proposition 2.3 *Let $\mathcal{M}_n$ be the algebra of $n\times n$ matrices with complex entries, equipped with the Euclidean norm $||...||$. We have:*

(i) *if $M = \{M_{jk}\}_{j,k=1}^n \in \mathcal{M}_n$, then*

$$|M_{jk}| \le ||M||; \tag{2.3}$$

(ii) *if $M\in\mathcal{M}_n$ and* $\mathrm{Tr}M = \sum_{j=1}^n M_{jj}$ *is the* trace *of a matrix M, then for any $M_1, M_2 \in \mathcal{M}_n$*

$$|\mathrm{Tr}M_1M_2| \le (\mathrm{Tr}M_1M_1^*)^{1/2}(\mathrm{Tr}M_2M_2^*)^{1/2}, \tag{2.4}$$

where M^* is the Hermitian conjugate of M;

(iii) *if $M\in\mathcal{M}_n$, then*

$$|\mathrm{Tr}M| \le n||M||; \tag{2.5}$$

(iv) *for any Hermitian or real symmetric matrix M its* resolvent

$$G(z) = (M-z)^{-1},\ G(z) = \{G_{jk}(z)\}_{j,k=1}^n \tag{2.6}$$

is defined for all non-real z, $\Im z \neq 0$, and verifies the inequalities

$$||G(z)|| \le |\Im z|^{-1},\ |G_{jk}(z)| \le |\Im z|^{-1}; \tag{2.7}$$

(v) *if M_1 and M_2 are two Hermitian or real symmetric matrices and $G_r(z), r = 1,2$ are their resolvents, then*

$$G_2(z) = G_1(z) - G_1(z)(M_2 - M_1)G_2(z) \tag{2.8}$$

(the resolvent identity);

(vi) *if $G(z) = (M-z)^{-1}$ is viewed as a function of a Hermitian or a real symmetric matrix M, then its derivative $G'(z)$ with respect to M verifies the relation*

$$G'(z)\cdot X = -G(z)XG(z) \tag{2.9}$$

for any Hermitian or real symmetric X, and

$$||G'(z)|| \le ||G(z)||^2 \le |\Im z|^{-2}. \tag{2.10}$$

We present now several facts on expectations of functions of Gaussian random variables. Recall first the form of the Gaussian Orthogonal (GOE) and Gaussian Unitary (GUE) Ensembles. These are measures, defined on the sets $\mathcal{M}_\beta$ of $n\times n$ real symmetric ($\beta = 1$, GOE) and ($\beta = 2$, GUE) Hermitian matrices $M = \{M_{jk}\}_{j,k=1}^n$ respectively, and given by

$$\mathbf{P}_\beta(d_\beta M) = \frac{1}{Z_{n,\beta}}\exp\left(-\frac{n\beta}{4w^2}\mathrm{Tr}M^2\right)d_\beta M,\ \beta = 1,2, \tag{2.11}$$

where $Z_{n,\beta}$ is a normalizing constant and

$$d_1M = \prod_{1\le j\le k\le n} dM_{j,k},\quad d_2M = \prod_{j=1}^n dM_{j,j}\prod_{1\le j<k\le n} d\Re M_{j,k}d\Im M_{j,k}. \tag{2.12}$$

Proposition 2.4 *Consider the GOE (the GUE) and let $\Phi : \mathcal{M}_n \to \mathbb{C}$ be a $\mathcal{C}^1$ function, bounded together with its derivative. Then for any Hermitian (real symmetric) matrix X we have*

$$\mathbf{E}\{\Phi'(M) \cdot X\} = \frac{\beta n}{2w^2}\mathbf{E}\{\Phi(M)\mathrm{Tr}(MX)\}, \tag{2.13}$$

where the symbol $\mathbf{E}\{\dots\}$ denotes the expectation with respect to the GOE ($\beta = 1$) and the GUE ($\beta = 2$) measures (2.11).

Proof. Consider the integral

$$I = \int_{\mathcal{M}_\beta} \Phi(M) \exp\{-\beta n \mathrm{Tr} M^2/4w^2\} d_\beta M.$$

Since the measures $d_\beta M$, $\beta = 1, 2$ are invariant with respect to translations $M \to M + \varepsilon X$ for any $X \in \mathcal{M}_\beta$ and $\varepsilon \in \mathbb{R}$, we have

$$I = \int_{\mathcal{M}_\beta} \Phi(M + \varepsilon X) \exp\{-\beta n \mathrm{Tr}(M + \varepsilon X)^2/4w^2\} d_\beta M.$$

Differentiating this expression with respect to ε and then setting $\varepsilon = 0$, we obtain the assertion. ■

Remarks. 1. Taking the case $n = 1, \beta = 1$ in the proposition and denoting $2w^2 = \sigma^2$ we obtain

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} x\Phi(x)e^{-x^2/2\sigma^2}dx = \sigma^2 \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} \Phi'(x)e^{-x^2/2\sigma^2}dx \tag{2.14}$$

or

$$\mathbf{E}\{\xi\Phi(\xi)\} = \mathbf{E}\{\xi^2\}\mathbf{E}\{\Phi'(\xi)\}, \tag{2.15}$$

where ξ is the Gaussian random variable of zero mean and of variance σ^2. The first formula shows that the proposition is a matrix version of the integration by parts. The second formula makes explicit the "decoupling" nature of (2.13), whose analogs are widely used in various domains of mathematical physics.

2. It is easy to prove a multivariate version of (2.15). Namely, if $X = \{\xi_j\}_{j=1}^q \in \mathbb{R}^q$ is a random Gaussian vector such that

$$\mathbf{E}\{\xi_j\} = 0,\ \mathbf{E}\{\xi_j\xi_k\} = C_{jk},\ j,k = 1, ..., q, \tag{2.16}$$

and $\Phi : \mathbb{R}^q \to \mathbb{C}$ has bounded partial derivatives, then

$$\mathbf{E}\{\xi_j\Phi\} = \sum_{k=1}^q C_{jk}\mathbf{E}\{(\nabla\Phi)_k\},\ (\nabla\Phi)_k = \frac{\partial\Phi}{\partial x_k}. \tag{2.17}$$

Next result is known as the Poincare–Nash inequality (see e.g. [5, 10] and references therein) and will also play an important role below.

Proposition 2.5 . *Consider a random Gaussian vector $X = \{\xi_j\}_{j=1}^p$, satisfying (2.16) with $p = q$, and $\Phi_{1,2} : \mathbb{R}^p \longrightarrow \mathbb{C}$, having bounded partial derivatives. Then*

$$\begin{aligned} \mathbf{Cov}\{\Phi_1, \Phi_2\} : &= \mathbf{E}\{\Phi_1\Phi_2)\} - \mathbf{E}\{\Phi_1\}\mathbf{E}\{\Phi_2\} \\ &\le \mathbf{E}\left\{(C\nabla\Phi_1, \nabla\overline{\Phi_1})\right\}^{1/2} \mathbf{E}(C\nabla\Phi_2, \nabla\overline{\Phi_2})^{1/2}, \end{aligned} \tag{2.18}$$

where

$$(C\nabla\Phi, \nabla\Phi) := \sum_{j,k=1}^{p} C_{jk}(\nabla\Phi)_j(\nabla\Phi)_k. \tag{2.19}$$

In particular, if $\Phi : \mathbb{R}^p \longrightarrow \mathbb{C}$ *has bounded partial derivatives, then*

$$\begin{aligned}\mathbf{Var}\{\Phi\} : &= \mathbf{E}\left\{|\Phi|^2)\right\} - |\mathbf{E}\{\Phi\}|^2 \\ &\leq \mathbf{E}\left\{(C\nabla\Phi, \nabla\Phi)\right\}.\end{aligned} \tag{2.20}$$

Proof. We will outline a proof, based on (2.17). Consider two q-component independent Gaussian vectors $X^{(1)}$ and $X^{(2)}$ with zero means and the covariance matrices $C^{(1)}$ and $C^{(2)}$. Define the "interpolating" Gaussian vector

$$X(t) = \sqrt{t}X^{(1)} + \sqrt{1-t}X^{(2)},\ t \in [0,1]. \tag{2.21}$$

Then for any $\Psi : \mathbb{R}^q \longrightarrow \mathbb{C}$ with bounded first and second partial derivatives we have

$$\mathbf{E}\left\{\Psi(X^{(1)})\right\} - \mathbf{E}\left\{\Psi(X^{(2)})\right\} = \frac{1}{2}\int_0^1 \mathbf{E}\left\{\big((C^{(1)}\nabla\Psi, \nabla\Psi) - (C^{(2)}\nabla\Psi, \nabla\Psi)\big)(X(t))\right\} dt. \tag{2.22}$$

Indeed, write the l.h.s. of (2.22) as

$$\int_0^1 \frac{d}{dt}\mathbf{E}\left\{\Psi(X(t)\right\} dt = \int_0^1 \mathbf{E}\left\{\Big(((2\sqrt{t})^{-1}X^{(1)} - (2\sqrt{1-t})^{-1}X^{(2)}), \nabla\Psi(X(t))\Big)\right\} dt.$$

Now, by using (2.17) in each term of the r.h.s., we obtain (2.22).

To prove (2.18) – (2.19), we choose

- $X^{(1)} = (X', Y')$, where (X', Y') is the $q = 2p$-component Gaussian vector, whose distribution is concentrated on the "diagonal" $X' = Y'$ and has there zero mean and the covariance matrix C,
- $X^{(2)} = (X'', Y'')$, where X'' and Y'' are independent p-component Gaussian vectors of zero mean and of covariance matrix C,
- $\Psi(X, Y) = \Phi_1(X)\Phi_2(Y)$.

In other words, $X^{(1)}$ and $X^{(2)}$ are $q = 2p$ - component Gaussian vectors with zero mean and with covariance matrices

$$C^{(1)} = \begin{pmatrix} C & C \\ C & C \end{pmatrix}, \quad C^{(2)} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}.$$

It is easy to see that for this choice of $C^{(1)}$ and $C^{(1)}$ the covariance $\mathbf{Cov}\{\Phi_1, \Phi_2\}$ has the form of the l.h.s. of (2.22), and we obtain

$$\mathbf{Cov}\{\Phi_1, \Phi_2\} = \int_0^1 \mathbf{E}\left\{(C\nabla\Phi_1(\widehat{X}(t)), \nabla\Phi_2(\widehat{Y}(t)))\right\} dt,$$

where $\widehat{X}(t) = \sqrt{t}X' + \sqrt{1-t}X''$, $\widehat{Y}(t) = \sqrt{t}Y' + \sqrt{1-t}Y''$, $t \in [0,1]$. Now, to obtain (2.18), we use Schwarz inequality $|(CX, Y)|^2 \leq (CX, \overline{X})(CY, \overline{Y})$, valid for a positive definite matrix C and any two vectors $X, Y \in \mathbb{C}^p$, Schwarz inequality for mathematical expectations, and the fact that $\widehat{X}(t)$ and $\widehat{Y}(t)$ are identically distributed Gaussian vectors, whose common law is determined by the matrix C, hence does not depend on t. ■

3 Deformed Semi-circle Law

Denote $\{\lambda_l^{(n)}\}_{l=1}^n$ eigenvalues of a $n \times n$ real symmetric or Hermitian matrix H and introduce the Normalized Counting Measure of eigenvalues (NCM)

$$N_n(\Delta) = \frac{1}{n} \sum_{l=1}^{n} \chi_\Delta(\lambda_l^{(n)}), \tag{3.1}$$

where χ_Δ is the indicator of an interval $\Delta \subset \mathbb{R}$. The NCM is a particular case of linear statistics (1.1), corresponding to $\varphi = \chi_\Delta$.

We will consider in this section the convergence of the Normalized Counting Measures of eigenvalues of the Gaussian Ensembles, a basic result of the global regime, pertinent for any subsequent study of the eigenvalue distribution of random matrix in question. In particular, we are going to prove that NCM converges with probability 1 to a non-random measure, known as the deformed semi-circle or the Wigner law.

We begin with Hermitian $n \times n$ matrices and consider the ensemble of the form

$$H = H^{(0)} + M, \tag{3.2}$$

where $H^{(0)}$ is a non-random Hermitian matrix and M is a random matrix, distributed according to the GUE law, defined by (2.11)–(2.12) with $\beta = 2$. Random matrices of this form can be viewed as "perturbations" or "deformations" of the GUE matrix by a non-random matrix $H^{(0)}$. We will call (3.2) the *deformed GUE*.

Writing

$$M = W/n^{1/2}, \tag{3.3}$$

we find from (2.11) that the entries $W_{jj},\ j = 1, ...n,\ \Re W_{j,k}$, and $\Im M_{j,k},\ 1 \le j < k \le n$ are independent Gaussian random variables, defined by the equalities:

$$\mathbf{E}\{W_{jk}\} = 0,\ \mathbf{E}\{W_{jk}^2\} = 0,\ \mathbf{E}\{|W_{jk}|^2\} = w^2(1 + \delta_{jk})/2. \tag{3.4}$$

This shows that the random variables $\{W_{jk}^n\}_{j,k=1}^n$ can be viewed as the upper left corner of the semi-infinite Hermitian matrix $\{W_{jk}^n\}_{j,k=1}^\infty$, whose entries are complex Gaussian random variables, defined by (3.4) for $1 \le j, k \le \infty$. This observation will allow us to use the convergence with probability 1 in the probability space, defined by $\{W_{jk}^n\}_{j,k=1}^\infty$.

Theorem 3.1 (deformed semicircle law) *Given $n \in \mathbb{N}$, consider the deformed Gaussian Unitary Ensemble (3.2) of $n \times n$ random Hermitian matrices, defined by (3.2) – (3.4). Assume that the Normalized Counting Measure of eigenvalues $N_n^{(0)}$ of $H^{(0)}$ converges weakly to a non-negative unit measure $N^{(0)}$ and denote*

$$f^{(0)}(z) = \int \frac{N^{(0)}(d\lambda)}{\lambda - z},\ \Im z \ne 0, \tag{3.5}$$

be the Stieltjes transform of $N^{(0)}$. Let N_n be the Normalized Counting Measure of the ensemble. Then there exists a non-negative unit measure N_{dsc} such that with probability 1 we have the weak convergence:

$$\lim_{n\to\infty} N_n = N_{dsc}, \tag{3.6}$$

and the Stieltjes transform f_{dsc} of N_{dsc} is a unique solution of the functional equation

$$f(z) = f^{(0)}(z + w^2 f(z)), \tag{3.7}$$

in the class of functions, analytic for $\Im z \ne 0$ and such that $\Im f(z) \cdot \Im z > 0$.

In view of the one-to-one correspondence between measures and their Stieltjes transforms (see Proposition 2.2) it suffices to study the Stieltjes transform

$$g_n(z) = \int \frac{N_n(d\lambda)}{\lambda - z}, \ \Im z \neq 0 \tag{3.8}$$

of the Normalized Counting Measure N_n. The spectral theorem for Hermitian matrices implies the formula

$$g_n(z) = n^{-1}\mathrm{Tr}G(z), \tag{3.9}$$

where

$$G(z) = (H - z)^{-1} \tag{3.10}$$

is the resolvent of H (see Proposition 2.3(iv)). This link between the NCM of a Hermitian (real symmetric) matrix and its resolvent will play an important role in what follows. In particular, it motivates the next lemma.

Lemma 3.2 *Let $G(z) = (H - z)^{-1}$ be the resolvent of the matrix (3.2), $G^{(0)}(z)$ be the resolvent of $H^{(0)}$, and g_n be defined by (3.8) – (3.9). Then we have for any non-real z*

$$\mathbf{E}\{G(z)\} = G^{(0)}(\widetilde{z}(z)) + w^2\mathbf{E}\left\{\overset{\circ}{g}_n(z)G(z)\right\}G^{(0)}(\widetilde{z}(z)), \tag{3.11}$$

where

$$\overset{\circ}{g}_n(z) = g_n(z) - f_n(z), \ f_n(z) = \mathbf{E}\{g_n(z)\}, \tag{3.12}$$

and

$$\widetilde{z} = z + w^2 f_n(z). \tag{3.13}$$

Proof. (i). Let j, k be two indexes varying between 1 and n. Applying Proposition 2.4 to the function $\Phi(M) = (H^{(0)} + M - z)^{-1}_{jk} := G_{jk}$, and using (2.9), we obtain for any Hermitian matrix X

$$\mathbf{E}\{(GXG)_{jk}\} + \frac{n}{w^2}\mathbf{E}\{G_{jk}\mathrm{Tr}MX\} = 0. \tag{3.14}$$

We choose here X as

$$X = \left\{X^{(p,q)}_{jk}\right\}^n_{p,q=1}, \ \ X^{(p,q)}_{jk} = a\delta_{jp}\delta_{kq} + \overline{a}\delta_{jq}\delta_{kp}, \quad a \in \mathbb{C}, \tag{3.15}$$

where p and q are two given indexes, varying between 1 and n. This yields

$$\mathbf{E}\{G_{jk}M_{qp}\} = -\frac{w^2}{n}\mathbf{E}\{G_{jp}G_{kq}\}, \tag{3.16}$$

where $\{M_{jk}\}^n_{j,k=1}$ are the entries of M of (3.2). By using the resolvent identity (2.8) for the pair $(H, H^{(0)})$

$$G = G^{(0)} - GMG^{(0)}, \tag{3.17}$$

we can write the equality

$$\mathbf{E}\{G_{jk}\} = G^{(0)}_{jk} - \sum_{p,q=1}^{n} \mathbf{E}\{G_{jq}M_{qp}\}G^{(0)}_{pk}. \tag{3.18}$$

Replacing the expectation in the sum by the r.h.s. of (3.16) with $k = q$, and using notation (3.12) – (3.13), we obtain the following matrix form of the previous relation:

$$\mathbf{E}\{G(z)\}\big(1 - w^2 f_n(z)G^{(0)}(z)\big) = G^{(0)}(z) + w^2\mathbf{E}\{\overset{\circ}{g}_n(z)G(z)\}G^{(0)}(z). \tag{3.19}$$

We have also:

$$1 - w^2 f_n(z) G^{(0)}(z) = (H^{(0)} - z - w^2 f_n(z)) G^{(0)}(z). \tag{3.20}$$

It follows from (3.12) that f_n is the Stieltjes transform of a probability measure, and by (2.7) we have for $\Im z \neq 0$: $|\Im(z + w^2 f(z))| > |\Im z| > 0$. Hence the matrix $H^{(0)} - z - w^2 f_n(z)$ is invertible uniformly in n if $\Im z \neq 0$. Its inverse is $G^{(0)}(\widetilde{z})$, and we can write the r.h.s. of (3.20) as $(G^{(0)}(\widetilde{z}))^{-1} G^{(0)}(z)$. This and (3.19) imply (3.11). ■

Theorem 3.3 *Let $g_n(z)$ be as in (3.9) – (3.10), where H is given by (3.2). Then*

$$\mathbf{Var}\{g_n(z)\} := \mathbf{E}\left\{|g_n(z) - \mathbf{E}\{g_n(z)\}|^2\right\} \leq \frac{w^2}{n^2 |\Im z|^4}. \tag{3.21}$$

Proof. We will use inequality (2.20), choosing the GUE matrix M as X and $g_n(z)$ as Φ. We have by (2.9):

$$\begin{aligned} \frac{\partial g_n(z)}{\partial M_{jj}} &= -\frac{1}{n}(G^2)_{jj}, \ \frac{\partial g_n(z)}{\partial \Re M_{jk}} = -\frac{1}{n}\left[(G^2)_{jk} + (G^2)_{kj}\right], \\ \frac{\partial g_n(z)}{\partial \Im M_{jk}} &= -\frac{i}{n}\left[(G^2)_{jk} - (G^2)_{kj}\right]. \end{aligned}$$

According to (2.11) $M_{jj},\ j = 1,...,n,\ \Re M_{jk},\ 1 \leq j < k \leq n,\ \Im M_{jk},\ 1 \leq j < k \leq n$ are independent Gaussian random variables of zero mean and of variance

$$\mathbf{E}\{{M_{jj}}^2\} = \frac{w^2}{n},\ \mathbf{E}\{(\Re M_{jk})^2\} = \mathbf{E}\{(\Im M_{jk})^2\} = \frac{w^2}{2n}, \tag{3.22}$$

Hence, the r.h.s. of (2.20) will be in this case:

$$\begin{aligned} &\mathbf{E}\left\{\frac{w^2}{n^3}\sum_{j=1}^{n}\left|(G^2)_{jj}\right|^2 + \frac{w^2}{2n^3}\sum_{1\leq j<k\leq n}\left|(G^2)_{jk} + (G^2)_{kj}\right|^2 + \left|(G^2)_{jk} - (G^2)_{kj}\right|^2\right\} \\ = \ &\frac{w^2}{n^3}\mathbf{E}\left\{\sum_{j,k=1}^{n}\left|(G^2)_{jk}\right|^2\right\} = \frac{w^2}{n^3}\mathbf{E}\left\{\mathrm{Tr} G^2(z) G^2(z^*)\right\}. \end{aligned}$$

In view of (2.7) the r.h.s. admits the bound $w^2/n^2|\Im z|^4$, coinciding with the r.h.s. of (3.21). ■

Proof of Theorem 3.1. According to (3.8) $g_n(z) = n^{-1}\mathrm{Tr}G(z)$ is the Stieltjes transform of the Normalized Counting Measure N_n. By applying the operation $n^{-1}\mathrm{Tr}$ to formula (3.11), we obtain for $f_n(z) = \mathbf{E}\{g_n(z)\}$:

$$f_n(z) = f_n^{(0)}(z + w^2 f_n(z)) + w^2 \mathbf{E}\left\{\overset{\circ}{g}_n(z) n^{-1}\mathrm{Tr} G(z) G^{(0)}(\widetilde{z})\right\}, \tag{3.23}$$

where

$$f_n^{(0)}(z) = \int \frac{N_n^{(0)}(d\lambda)}{\lambda - z},\ \Im z \neq 0. \tag{3.24}$$

By using (3.21), (2.7), and Schwarz inequality, we estimate the second term in the r.h.s. of (3.23) by the expression $w^2|\Im z|^{-2}\mathbf{E}\left\{|\overset{\circ}{g}_n(z)|^2\right\}^{1/2}$, bounded from above by $w^3/n|\Im z|^4$ in view of Theorem 3.3. We obtain the inequality

$$\left|f_n(z) - f_n^{(0)}(z + w^2 f_n(z))\right| \leq \frac{w^3}{n|\Im z|^4}. \tag{3.25}$$

In view of (2.7) the sequence $\{f_n\}$ consists of functions, analytic and uniformly bounded in n and in z by $\eta_0^{-1} < \infty$ if $|\Im z| \geq \eta_0 > 0$. Hence, there exists a function f and an infinite subsequence $\{f_{n_j}\}_{j\geq 1}$ that converges to f uniformly on any compact set of $\mathbb{C} \setminus \mathbb{R}$. According to Proposition 2.2(iii) we have

$$\Im f_n(z) \cdot \Im z > 0, \quad \Im z \neq 0, \tag{3.26}$$

thus $\Im f(z) \cdot \Im z \geq 0, \quad \Im z \neq 0$. In addition, according to Proposition 2.2 and the hypothesis of the theorem on the convergence of the sequence $\{N_n^{(0)}\}$ to $N^{(0)}$, the sequence $\{f_n^{(0)}\}$ of (3.24) is analytic in $\mathbb{C} \setminus \mathbb{R}$ and converges uniformly on compact sets of $\mathbb{C} \setminus \mathbb{R}$ to the Stieltjes transform $f^{(0)}$ of the limiting counting measure $N^{(0)}$ of "unperturbed" matrices $H^{(0)}$. This allows us to pass to the limit $n_j \to \infty$ in (3.25) and to obtain that the limit of any converging subsequence of the sequence $\{f_n\}$ satisfies the functional equation (3.7). According to Lemma 3.4 below, the equation is uniquely soluble in the class of functions, analytic for $\Im z \neq 0$ and such that $\Im f(z) \cdot \Im z \geq 0, \quad \Im z \neq 0$, and the solution possesses the property $\Im f(z) \cdot \Im z > 0, \quad \Im z \neq 0$. Hence the whole sequence $\{f_n\}$ converges to the unique solution f_{dsc} of the equation.

In addition, the Tchebyshev inequality and Theorem 3.3 imply that for any $\varepsilon > 0$,

$$\mathbf{P}\{|f_n(z) - g_n(z)| > \varepsilon\} \leq \frac{1}{\varepsilon^2}\mathbf{Var}\{g_n(z)\} \leq \frac{w^2}{\varepsilon^2 |\Im z|^4 n^2}.$$

Hence the series $\sum_{n=1}^{\infty} \mathbf{P}\{|f_n(z) - g_n(z)| > \varepsilon\}$ converges for any $\varepsilon > 0$, and $|\Im z| \geq \eta_0 > 0$, and by the Borel-Cantelli lemma we have for any fixed z, $|\Im z| \geq \eta_0 > 0$ with probability 1 $\lim_{n\to\infty} g_n(z) = f(z)$. Let us show that g_n converges to f uniformly on any compact of $\mathbb{C} \setminus \mathbb{R}$ with probability 1. Because of the uniqueness of analytic continuation it suffices to prove that with the same probability the limiting relation $\lim_{n\to\infty} g_n(z_j) = f(z_j)$ is valid for all points of an infinite sequence $\{z_j\}_{j\geq 1}, \ z_j, \ |\Im z_j| \geq \eta_0 > 0$, possessing an accumulation point. Indeed, according to the above $\mathbf{P}\{\Omega(z_j\} = 1, \ \ \forall j$. Hence

$$\mathbf{P}\left\{\bigcap_{j\geq 1} \Omega(z_j)\right\} = 1,$$

and the last assertion is proved.

Denote by N_{dsc} the non-negative measure, whose Stieltjes transform is f_{dsc}. Then, by Proposition 2.2(vi), we have with probability 1 the weak convergence (3.6), and, in view of Lemma 3.4, N_{dsc} is a unit measure. The measure can be found by using the inversion formula of Proposition 2.2(v). ■

Remark. If in the conditions of the above theorem we assume additionally that the sequence $\{N_n^{(0)}\}$ is tight, then the sequence $\{N_n\}$ is also tight with probability 1. Indeed, consider first the case of the GUE itself, corresponding to $H^{(0)} = 0$ in (3.2). In this case we have by definition of the NCM and by (3.3)

$$\int \lambda^2 N_n(d\lambda) = \frac{1}{n^2} \sum_{j,k=1}^{n} |W_{jk}|^2. \tag{3.27}$$

It is easy to prove that the sum on the r.h.s. of (3.27) converges with probability 1 to $\mathbf{E}\{|W_{12}|^2\} = w^2$ (this is the strong law of large numbers for the Gaussian random variables $\{W_{jk}\}_{j,k=1}^{\infty}$). Hence, the second moment of N_n is bounded uniformly in n with probability 1 and the sequence $\{N_n\}$ is tight with probability 1.

In a general case of the deformed GUE (3.2)–(3.3) we can argue as follows. We first use the resolvent identity (3.17) and inequalities (2.4) – (2.7), according to which

$$|g_n(z) - g_n^{(0)}(z)| \le \left|n^{-1}\mathrm{Tr}G(z)MG^{(0)}(z)\right| \le \frac{1}{|\Im z|^2}\left(n^{-2}\sum_{j,k=1}^{n}|W_{jk}|^2\right)^{1/2}.$$

Next, we note that if m is a unit non-negative measure and f is its Stieltjes transform then

$$-(1+\eta)^{-1} + m(\{\lambda : |\lambda| \le \eta^{1/2}\}) \le \eta\Im f(i\eta) \le \eta|f(i\eta)| \le m(\{\lambda : |\lambda| \le \eta^{3/2}\}) + (1+\eta)^{-1/2}.$$

By using these inequalities we obtain the bounds

$$N_n(\{\lambda : |\lambda| \le \eta^{3/2}\}) \ge N_n^{(0)}(\{\lambda : |\lambda| \le \eta^{1/2}\}) - (1+\eta)^{-1} - (1+\eta)^{-1/2} - W_n^{1/2}\eta^{-1},$$

where W_n is the r.h.s. of (3.27). Since, according to the above, W_n is bounded with probability 1, the bound and the tightness of the sequence $\{N_n^{(0)}\}$ implies the tightness of $\{N_n\}$.

Lemma 3.4 *Let $f^{(0)}$ be the Stieltjes transform of a unit non-negative measure, and w be a positive number. Then the functional equation*

$$f(z) = f^{(0)}(z + w^2 f(z)) \tag{3.28}$$

has at most one solution, analytic for $\Im z \ne 0$ and such that

$$\Im f \cdot \Im z \ge 0. \tag{3.29}$$

The solution is the Stieltjes transform of a unit non-negative measure N, in particular, inequality (3.29) is strict: $\Im f \cdot \Im z > 0$.

Proof. Let us prove first that for any solution of (3.28) – (3.29) the inequality (3.29) is strict. Assume that $\Im f(z_0) = 0$ for some z_0, $\Im z_0 \ne 0$. Then (3.28) implies that $\Im f^{(0)}(z_0) = 0$. This is impossible because, according to (3.5),

$$\Im f^{(0)}(z) = \Im z \int \frac{N^{(0)}(d\mu)}{|\mu - z|^2}$$

is strictly positive for any non-real z, and any non-negative unit measure $N^{(0)}$.

Let us prove now that (3.28) is uniquely soluble. Assume that (3.28) – (3.29) possesses two solutions f_1 and f_2. Since in any bounded part of the upper half-plane they coincide at most at a finite number of points, there exists a subsequence $\{z_p\}$, such that $z_p \to \infty$ as $p \to \infty$ and $f_1(z_p) \ne f_2(z_p)$, $\forall p$. By using (3.28) and (3.5), we obtain the relation

$$1 = w^2 \int \frac{N^{(0)}(d\mu)}{(\mu - z_p - w^2 f_1(z_p))\,(\mu - z_p - w^2 f_2(z_p))}.$$

that has to be valid for all z_p. This is impossible, because the limit of the r.h.s. of the relation is zero as $p \to \infty$ (recall that $|f_{1,2}(z)| \le |\Im z|^{-1}$).

To prove that the solution of (3.28) – (3.29) is the Stieltjes transform of a unit non-negative measure, we have to prove that $\lim_{\eta\to\infty}\eta|f(i\eta)| = 1$ (see Proposition 2.2 (vii)). Since $f^{(0)}$ possesses the same property, it suffices to prove the equality $\lim_{\eta\to\infty}\eta|f(i\eta)| = \lim_{\eta\to\infty}\eta|f^{(0)}(i\eta)|$. It follows readily from (3.28) and the inequality $\Im\widetilde{z} \ge \Im z$. ■

Corollary 3.5 *Consider the GUE, and let N_n be its Normalized Counting Measure of eigenvalues. Then there exists a unit measure N_{sc}, called the semi-circle law and such that the sequence $\{N_n\}$ converges tightly to N_{sc} with probability 1:*

$$\lim_{n\to\infty} N_n = N_{sc},$$

and

$$N_{sc}(\Delta) = \int_\Delta \rho_{sc}(\lambda)d\lambda,\ \rho_{sc}(\lambda) = \frac{1}{2\pi w^2}(4w^2 - \lambda^2)_+^{1/2}, \tag{3.30}$$

where we denote here and below

$$x_+ = \max(x, 0),\ x \in \mathbb{R}. \tag{3.31}$$

Proof. The case of the GUE corresponds to $H^{(0)} = 0$ in (3.2). The normalizing counting measure of this matrix is the unit measure, concentrated at 0, and its Stieltjes transform is $f_n^{(0)}(z) = -1/z$. Its limit is the same, hence equation (3.28) in this case is

$$f(z) = -\frac{1}{z + w^2 f(z)}, \tag{3.32}$$

or

$$w^2 f^2(z) + z f(z) + 1 = 0. \tag{3.33}$$

A solution of this quadratic equation that satisfies the condition $\Im f(z) \cdot \Im z \geq 0,\ \Im z \neq 0$ is unique and is given by

$$f(z) = \frac{1}{2w^2}\left(\sqrt{z^2 - 4w^2} - z\right), \tag{3.34}$$

where $\sqrt{z^2 - 4w^2}$ denotes the branch that has the asymptotic behavior $\sqrt{z^2 - 4w^2} = z + O(|z|^{-1}),\quad z \to \infty$. In particular, this branch assumes purely imaginary values with positive imaginary part on the upper edge of the cut $(-2w, 2w)$. Applying to (3.34) the inversion formula (2.2), we obtain (3.30). ■

Remarks. 1. The case of the GUE itself requires fewer technicalities, than the general case of matrices (3.2). Indeed, since in this case $G^{(0)} = -z^{-1}$, the operation $n^{-1}\mathrm{Tr}$, applied to (3.11) with this $G^{(0)}$, yields

$$w^2 f_n^2(z) + z f_n(z) + 1 = -w^2 \mathbf{E}\{\overset{\circ}{g}{}_n^2(z)\}. \tag{3.35}$$

Hence, Theorem (3.3) leads directly to the quadratic equation (3.33). The unique solubility of this equation in the class of analytic functions verifying (3.29) is immediate.

2. For the deformed Gaussian Orthogonal Ensemble the limiting NCM is the same, i.e., it is given by the deformed semi-circle law, although the proof is more involved. We outline the proof in the case of the GOE itself, indicating only moments that are different from those of the proof of Theorem 3.1.

Recall first that according to (2.11) the GOE corresponds to $n \times n$ real symmetric matrices of the form (cf (3.3))

$$M = W/n^{1/2}, \tag{3.36}$$

where $W = \{W_{jk}\}$ are Gaussian random variables, independent for $1 \leq j < k \leq n$ and such that (cf (3.4))

$$W_{jk} = W_{kj},\ \ \mathbf{E}\{W_{jk}\} = 0,\ \ \mathbf{E}\{W_{jk}^2\} = w^2(1 + \delta_{jk}). \tag{3.37}$$

Hence we can again view $\{W_{jk}\}_{j,k=1}^n$ as the $n \times n$ upper left corner of the semi-infinite real symmetric matrix $\{W_{jk}\}_{j,k=1}^\infty$ with Gaussian entries, defined by (3.37), and we can consider the convergence with probability 1 in the corresponding probability space.

By using Proposition 2.4 for the case $\beta = 1$ of real symmetric matrices and choosing as X the matrices $X^{(p,q)} = \{X^{(p,q)}_{jk}\}^n_{j,k=1}$, $p, q = 1, ..., n$ with $X^{(p,q)}_{jk} = \delta_{jp}\delta_{kq} + \delta_{jq}\delta_{kp}$ (cf (3.15)), we obtain instead of (3.16) the relation

$$\mathbf{E}\{G_{jk}M_{qp}\} = -\frac{w^2}{n}\mathbf{E}\{G_{jp}G_{qk}\} - \frac{w^2}{n}\mathbf{E}\{G_{jq}G_{pk}\}, \tag{3.38}$$

valid for $j, k, p, q = 1, ..., n$, and containing the additional cross term in the r.h.s.. This leads to the following analog of (3.35):

$$w^2 f_n^2(z) + z f_n(z) + 1 = -w^2\mathbf{E}\{(\overset{\circ}{g}_n(z))^2\} + n^{-2}w^2\mathbf{E}\{\mathrm{Tr}G^2(z)\}, \tag{3.39}$$

containing the term $w^2n^{-2}\mathbf{E}\{\mathrm{Tr}G^2(z)\}$, absent in the GUE case (see (3.35)). In view of (2.5) the term admits the bound $|w^2n^{-2}\mathbf{E}\{\mathrm{Tr}G^2(z)\}| \le w^2/n|\Im z|^2$, hence does not contribute to the limiting form of (3.39). The form coincides with (3.33) provided that the variance $\mathbf{E}\{|\overset{\circ}{g}_n(z)|^2\}$ vanishes as $n \to \infty$. This fact, namely an analog of bound (3.21) for the GOE, can be proved by the same argument as in the case of the GUE above. Indeed, applying again inequality (2.20) to $g_n(z)$, we obtain

$$\mathbf{Var}\{g_n(z)\} \le \frac{2w^2}{n^2|\Im z|^4}. \tag{3.40}$$

4 Variance and Central Limit Theorem for the Trace of the Resolvent

4.1 Variance

Next theorem is a more detailed version of Theorem 3.3. To avoid technicalities, we confine themselves the case of the GUE itself.

Theorem 4.1 *Consider the GUE. Let $g_n(z)$ be defined by (3.8) – (3.10) with $H^{(0)} = 0$. Then we have for $n \to \infty$*

$$\mathbf{Cov}\{g_n(z_1), g_n(z_2)\} = d_2(z_1, z_2)n^{-2} + r_n^{(2)}(z_1, z_2), \tag{4.1}$$

where

$$d_2(z_1, z_2) := -\frac{1}{2(z_1 - z_2)^2}\left(1 - \frac{z_1z_2 - 4w^2}{\sqrt{z_1^2 - 4w^2}\sqrt{z_2^2 - 4w^2}}\right), \tag{4.2}$$

and $r_n^{(2)}$ admits the bound

$$|r_n^{(2)}(z_1, z_2)| \le C/n^3, \tag{4.3}$$

where C is independent on n and finite if $\min\{|\Im z_1|, |\Im z_2|\} > 0$.

Proof. We can write by definition

$$\mathbf{Cov}\{g_n(z_1, z_2)\} = \mathbf{E}\{g_n(z_1)\overset{\circ}{g}_n(z_2)\}. \tag{4.4}$$

Applying Proposition 2.4 to the r.h.s., we obtain the identity

$$\mathbf{E}\{g_n(z_1)\overset{\circ}{g}_n(z_2)\} = -\frac{w^2}{z_1}\mathbf{E}\{g_n^2(z_1)\overset{\circ}{g}_n(z_2)\} - \frac{w^2}{z_1n^3}\mathbf{E}\{\mathrm{Tr}G(z_1)G^2(z_2)\}. \tag{4.5}$$

This, (3.12), and the relations

$$G(z_1)G(z_2)\} = \frac{G(z_1)-G(z_2)\}}{z_1-z_2}, \quad \frac{d}{dz}G(z) = G^2(z), \tag{4.6}$$

allow us to rewrite the r.h.s. of the identity as

$$-\frac{2w^2}{z_1}f_n(z_1)\mathbf{E}\{g_n(z_1)\overset{\circ}{g}_n(z_2)\} - \frac{w^2}{z_1 n^2}\frac{\partial}{\partial z_2}\frac{f_n(z_1)-f_n(z_2)}{z_1-z_2} - \frac{w^2}{z_1}\mathbf{E}\{\overset{\circ}{g}{}_n^2(z_1)\overset{\circ}{g}_n(z_2)\}.$$

Hence, we obtain from (4.4)

$$\mathbf{Cov}\{g_n(z_1), g_n(z_2)\} = -\frac{w^2}{z_1+2w^2 f_n(z_1)}\left\{\frac{1}{n^2}\frac{\partial}{\partial z_2}\frac{f_n(z_1)-f_n(z_2)}{z_1-z_2} + \mathbf{E}\{\overset{\circ}{g}{}_n^2(z_1)\overset{\circ}{g}_n(z_2)\}\right\}, \tag{4.7}$$

where $z_1 + 2w^2 f_n(z_1) \neq 0$ if $|\Im z_1| \neq 0$ because $\Im f_n(z)\cdot \Im z > 0$ for $\Im z \neq 0$. Moreover, we have the bound

$$|z+2w^2 f_n(z)| \geq |\Im(z+2w^2 f_n(z))| \geq |\Im z|. \tag{4.8}$$

Consider the contribution of the first term of the r.h.s of (4.7). By (3.35), (3.33) and Theorem 3.3 we have

$$|f(z)-f_n(z)| \leq \frac{w^2}{z+w^2(f(z)+f_n(z))}\mathbf{Var}\{g_n(z)\} \leq \frac{w^4}{n^2|\Im z|^5},$$

where we took into account (4.8) and the inequality $\Im f(z)\cdot \Im z > 0$ for $\Im z \neq 0$, implying that $|z+w^2(f(z)+f_n(z))| \geq |\Im z|$.

Besides, since f_n is analytic for $\Im z \neq 0$, we have for $|\Im z_{1,2}| \geq \eta_0 > 0$:

$$\frac{\partial}{\partial z_2}\frac{f_n(z_1)-f_n(z_2)}{z_1-z_2} = \int_0^1 f_n''(z_1+t(z_2-z_1))t\,dt,$$

and

$$f_n''(z) = \frac{1}{\pi i}\int_{|\zeta-z|=\eta_0/2}\frac{f_n(\zeta)d\zeta}{(\zeta-z)^3}. \tag{4.9}$$

The above three relations imply that the replacement f_n by f in (4.7) yields an error term bounded by $C(\eta_0)/n^4$, where $C(\eta_0)$ is finite if $\eta_0 > 0$.

We have then

$$\begin{aligned}\mathbf{Cov}\{g_n(z_1), g_n(z_2)\} &= -\frac{w^2}{n^2(z_1+2w^2 f(z_1))}\frac{\partial}{\partial z_2}\frac{f(z_1)-f(z_2)}{z_1-z_2} \\ &\quad -\frac{w^2}{z_1+2w^2 f_n(z_1)}\mathbf{E}\{\overset{\circ}{g}{}_n^2(z_1)\overset{\circ}{g}_n(z_2)\} + O(n^{-4}),\end{aligned} \tag{4.10}$$

if $\min\{|\Im z_1|, |\Im z_2|\} > 0$. Now it is easy to show by using (3.34) that the first term in the r.h.s. coincides with the first term of the r.h.s. of (4.1).

To finish the proof we have to show that $\mathbf{E}\{\overset{\circ}{g}{}_n^2(z_1)\overset{\circ}{g}_n(z_2)\}$ is of the order $O(n^{-3})$. Indeed, by Schwarz inequality

$$\left|\mathbf{E}\{\overset{\circ}{g}{}_n^2(z_1)\overset{\circ}{g}_n(z_2)\}\right| \leq \mathbf{Var}^{1/2}\{\overset{\circ}{g}{}_n^2(z_1)\}\mathbf{Var}^{1/2}\{g_n(z_2)\}.$$

The second factor of the r.h.s. is estimated in Theorem 3.3. To estimate the first term we use again the Poincare-Nash bound (2.18) – (2.19). This and Theorem 3.3 yield

$$\mathbf{Var}\{\overset{\circ}{g}{}_n^2(z_1)\} \leq \frac{4w^2}{n^3}\mathbf{E}\{|\overset{\circ}{g}_n(z_1)|^2 \mathrm{Tr} G^2(z_1)G^2(z_1^*)\} \leq \frac{4w^4}{n^4|\Im z_1|^8}, \tag{4.11}$$

and we obtain the inequality $\left|\mathbf{E}\{\overset{\circ}{g}{}_n^2(z_1)\overset{\circ}{g}_n(z_2)\}\right| \leq 2w^3/n^3|\Im z_1|^4|\Im z_2|^2$. This proves bound (4.3), hence the theorem. ■

Remarks. 1. Similar argument shows that in the case of the GOE we have

$$\mathbf{Cov}\{g_n(z_1), g_n(z_2)\} = d_1(z_1, z_2)n^{-2} + r_n^{(1)}(z_1, z_2), \tag{4.12}$$

where

$$d_1(z_1, z_2) := -\frac{1}{(z_1 - z_2)^2}\left(1 - \frac{z_1 z_2 - 4w^2}{\sqrt{z_1^2 - 4w^2}\sqrt{z_2^2 - 4w^2}}\right), \tag{4.13}$$

and $r_n^{(1)}$ admits the same bound as (4.3).

2. It is convenient to write a unique formula for (4.2) and (4.13):

$$d_\beta(z_1, z_2) := -\frac{1}{\beta(z_1 - z_2)^2}\left(1 - \frac{z_1 z_2 - 4w^2}{\sqrt{z_1^2 - 4w^2}\sqrt{z_2^2 - 4w^2}}\right),\ \beta = 1, 2. \tag{4.14}$$

3. We mention also another expression for $d_\beta(z_1, z_2)$:

$$d_\beta(z_1, z_2) = \frac{2w^2}{\beta(1 - w^2 f^2(z_1))(1 - w^2 f^2(z_2))}\left(\frac{f(z_1) - f(z_2)}{z_1 - z_2}\right)^2, \tag{4.15}$$

where f is the Stieltjes transform (3.34) of the semicircle law, the limiting Normalized Counting Measure of eigenvalues for the GUE and the GOE.

4. According to physics literature (see e.g. [4]) the expectation of any unitary invariant and smooth function of the GUE matrix admits an expansion in n^{-2}. Since $\overset{\circ}{g}_n(z_1)\overset{\circ}{g}_n(z_2)$ is smooth and unitary invariant, we have to expect in this case that the error term in (4.2) is of the order $O(n^{-4})$. This requires a bound $O(n^{-4})$ for the second term of the r.h.s. of (4.10) that can be proved as follows. By repeating the argument that led from (4.5) to (4.10), we obtain

$$\mathbf{E}\{\overset{\circ}{g}{}_n^2(z_1)\overset{\circ}{g}_n(z_2)\} = -\frac{w^2}{z_1 + 2w^2 f_n(z_1)}\mathbf{E}\left\{\left(\overset{\circ}{g}{}_n^2(z_1) - \mathbf{E}\{\overset{\circ}{g}{}_n^2(z_1)\}\right)\overset{\circ}{g}_n(z_1)\overset{\circ}{g}_n(z_2)\right\} + O(n^{-4.})$$

The expectation in the r.h.s. is estimated by $(\mathbf{Var}\{\overset{\circ}{g}{}_n^2(z_1)\}\mathbf{Var}\{\overset{\circ}{g}_n(z_1)\overset{\circ}{g}_n(z_2)\})^{1/2}$. Now, by applying (2.20), we find that the both variances are of the order $O(n^{-4})$ (cf (4.11)), hence this expression is $O(n^{-4})$ as well. This implies the same order of magnitude of the error term in (4.2).

4.2 Central Limit Theorem

The results of Section 3 can be viewed as an analog of the strong law of large numbers for the linear statistics (1.2). In this section we consider the central limit theorem for these linear statistics.

According to (3.21) and (3.40) the variance of $g_n := n^{-1}\mathrm{Tr}(M - z)^{-1}$, the normalized trace of the resolvent of M of (3.3) – (3.4), is of the order $O(n^{-2})$ for the Gaussian Ensembles. Hence the central limit theorem should be valid for the trace itself

$$\gamma_n(z) := \sum_{l=0}^{n} \frac{1}{\lambda_l - z} = n g_n(z). \tag{4.16}$$

This has to be compared with the case of the i.i.d. random variables with the finite second moment, where the variance of linear statistics is always of the order $O(n^{-1})$, and the central limit theorem is valid for linear statistics multiplied by $\sqrt{n}$.

We will begin from the technically simplest case of the random variable

$$\gamma_{R,n}(z) := \Re\gamma_n(z), \tag{4.17}$$

and the Gaussian Unitary Ensemble.

Theorem 4.2 *Consider the GUE. Then for any fixed z such that $|\Im z| \neq 0$ the random variable*

$$\overset{\circ}{\gamma}_{R,n}(z) := \gamma_{R,n}(z) - \mathbf{E}\{\gamma_{R,n}(z)\} \tag{4.18}$$

converges in distribution to the Gaussian random variable $\overset{\circ}{\gamma}_R(z)$ whose expectation is zero and whose variance is

$$v_2(z,\overline{z}) := \mathbf{Var}\{\gamma_R(z)\} = \frac{1}{4}\left(d_2(z,z) + d_2(\overline{z},\overline{z}) + 2d_2(z,\overline{z})\right), \tag{4.19}$$

where $d_2(z_1,z_2)$ is given by (4.2).

Proof. Consider the characteristic function of $\overset{\circ}{\gamma}_{R,n}(z)$:

$$F_n(t) := \mathbf{E}\{\exp\{it\overset{\circ}{\gamma}_{R,n}(z)\}\},\ t \in \mathbb{R}.$$

It suffices to prove that for any fixed $t \in \mathbb{R}$ $F_n(t)$ converges to $e^{-v_2t^2/2}$ as $n \to \infty$, where v_2 is given by (4.19).

We have evidently

$$\frac{d}{dt}F_n(t) = \frac{i}{2}[A_n(t) + B_n(t)], \tag{4.20}$$

where

$$A_n(t) := \mathbf{E}\{\overset{\circ}{\gamma}_n(z)\exp\{it\overset{\circ}{\gamma}_{R,n}(z)\}\},\ B_n(t) := \mathbf{E}\{\overline{\overset{\circ}{\gamma}_n(z)}\exp\{it\overset{\circ}{\gamma}_{R,n}(z)\}\}, \tag{4.21}$$

and

$$\overset{\circ}{\gamma}_n(z) =: \gamma_n(z) - \mathbf{E}\{\gamma_n(z)\} \tag{4.22}$$

is the centralized trace of the resolvent (cf (4.18)). Applying Proposition 2.4 to $\Phi = G_{jk}\exp\{it\overset{\circ}{\gamma}_{R,n}\}$ and performing simple transformations, we obtain

$$\begin{aligned}&n[1 + zf_n(z)]F_n(t) + zA_n(t) + w^2\mathbf{E}\{n^{-1}\gamma_n^2(z)\exp\{it\overset{\circ}{\gamma}_{R,n}(z)\}\}\\ &+\frac{it}{2}w^2\mathbf{E}\{[n^{-1}\mathrm{Tr}G^3(z) + n^{-1}\mathrm{Tr}G^2(z)G(\overline{z})]\exp\{it\overset{\circ}{\gamma}_{R,n}(z)\}\} = 0,\end{aligned}$$

where $f_n(z) = \mathbf{E}\{n^{-1}\mathrm{Tr}G(z)\}$ (see (3.12)) and we took into account that $\overline{\gamma_n(z)} = \gamma_n(\overline{z})$. Set here $t = 0$, multiply the result by $F_n(t)$, taking into account that $F_n(0) = 1,\ A_n(0) = 0$, and subtract the obtained equality from the above. This and the identity

$$\gamma_n^2 - \mathbf{E}\{\gamma_n^2(z)\} = \overset{\circ}{\gamma}{}_n^2 - \mathbf{E}\{\overset{\circ}{\gamma}{}_R^2(z)\} + 2\overset{\circ}{\gamma}_n nf_n$$

yield

$$\begin{aligned}(z + 2w^2f_n(z))A_n(t) = -w^2\mathbf{E}\left\{n^{-1}\overset{\circ}{\gamma}{}_n^2(z)\exp\{it\overset{\circ}{\gamma}_{R,n}(z)\}\right\} \qquad (4.23)\\ +w^2\mathbf{E}\left\{n^{-1}\overset{\circ}{\gamma}{}_n^2(z)\right\}F_n(t) - \frac{it}{2}w^2\mathbf{E}\left\{[n^{-1}\mathrm{Tr}G^3(z) + n^{-1}\mathrm{Tr}G^2(z)G(\overline{z})]\exp\{it\overset{\circ}{\gamma}_{R,n}(z)\right\}.\end{aligned}$$

According to Theorem 3.21, we have

$$|\mathbf{E}\{n^{-1}(\overset{\circ}{\gamma}_n(z))^2\}| \le n\mathbf{Var}\{g_n(z)\} \le \frac{w^2}{n|\Im z|^2},$$

(recall that according to (3.8) $g_n(z) := n^{-1}\mathrm{Tr}G(z) = n^{-1}\gamma_n(z)$). Hence, the first and the second terms of the r.h.s. of (4.23) vanish as $n \to \infty$. This, (4.6), and the obvious formula $G^3(z) = G''(z)/2$ allow us to write (4.23) in the following asymptotic form (cf the r.h.s. of (4.7))

$$(z + 2w^2 f_n(z))A_n(t) = -\frac{it}{2}w^2\mathbf{E}\left\{\left[\frac{1}{2}\frac{d^2}{dz^2}g_n(z) + \frac{\partial}{\partial \overline{z}}\frac{g_n(z) - g_n(\overline{z})}{z - \overline{z}}\right]\exp\{it\overset{\circ}{\gamma}_{R,n}(z)\}\right\} + o(1). \tag{4.24}$$

Since $g_n(z)$ is analytic for $|\Im z| \neq 0$, we have by the Cauchy formula for any $|\Im z| \ge \eta_0 > 0$

$$\frac{d^2}{dz^2}\overset{\circ}{g}_n(z) = \frac{1}{\pi i}\int_{|\zeta - z| = \eta_0/2}\frac{\overset{\circ}{g}_n(\zeta)d\zeta}{(\zeta - z)^3}.$$

This, (3.21), and Schwarz inequality imply

$$\mathbf{Var}\left\{\frac{d^2}{dz^2}g_n(z)\right\} \le \frac{Aw^2}{n^2\eta_0^8}, \tag{4.25}$$

where A is an absolute constant. This estimate, (4.8), and Corollary 3.5, according to which the limit $\lim_{n\to\infty} f_n(z) = f(z)$ is uniform on a compact set of $\mathbb{C} \setminus \mathbb{R}$, allow us to replace (4.24) by

$$A_n(t) = itL(z, \overline{z})F_n(t) + o(1),\ n \to \infty,$$

where

$$L(z, \overline{z}) = -\frac{w^2}{2(z + 2w^2 f(z))}\left(\frac{1}{2}\frac{d^2 f(z)}{dz^2} + \frac{\partial}{\partial \overline{z}}\frac{f(z) - f(\overline{z})}{z - \overline{z}}\right), \tag{4.26}$$

and f is given by (3.34).

Similar argument leads to the asymptotic relation for the second term $B_n(t)$ of (4.20), defined in (4.21):

$$B_n(t) = it\overline{L(z, \overline{z})}F_n(t) + o(1),\ |\Im z| \ge \eta_0 > 0,\ n \to \infty.$$

We obtain now from (4.20):

$$\frac{d}{dt}F_n(t) = -tv_2(z, \overline{z})F_n(t) + r_n(t, z, \overline{z}), \tag{4.27}$$

where

$$v_2(z, \overline{z}) = \Re L(z, \overline{z}), \tag{4.28}$$

and

$$\lim_{n\to\infty} r_n(t, z, \overline{z}) = 0 \tag{4.29}$$

uniformly in $|\Im z| \ge \eta_0 > 0$ and in t, varying in any finite interval.

Since $F_n(0) = 1$, we can write (4.27) as

$$F_n(t) = e^{-\frac{v_2 t^2}{2}} + \int_0^t e^{-\frac{v_2}{2}(t^2 - s^2)}r_n(s, z, \overline{z})ds,$$

implying, together with (4.29), that uniformly in $|\Im z| \geq \eta_0 > 0$ and in t, varying in any finite interval,

$$\lim_{n\to\infty} F_n(t) = \exp\left\{-\frac{v_2(z,\overline{z})t^2}{2}\right\}.$$

This and the standard continuity theorem for characteristic functions proves that the random variable $\overset{\circ}{\gamma}_{R,n}(z)$ converges in distribution to a Gaussian random variable $\overset{\circ}{\gamma}_R(z)$ of zero mean and of variance $v_2(z,\overline{z})$ given by (4.28). In view of (4.26) and (3.34) we obtain after a simple algebra the expression (4.19) – (4.2) for the variance. ■

Remarks. **1.** According to Theorem 4.1 $d_2(z_1,z_2)$ is the leading term of the covariance of $\gamma_n(z_1)$ and $\gamma_n(z_2)$ as $n\to\infty$. This is in complete correspondence with the relation

$$\begin{aligned} v_2(z,\overline{z}) &= \lim_{n\to\infty} \mathbf{E}\{(\Re\overset{\circ}{\gamma}_n(z))^2\} \\ &= \lim_{n\to\infty} \frac{1}{4}\mathbf{E}\{(\overset{\circ}{\gamma}_n(z))^2 + (\overset{\circ}{\gamma}_n(\overline{z}))^2 + 2\overset{\circ}{\gamma}_n(z)\overset{\circ}{\gamma}_n(\overline{z})\} \\ &= \frac{1}{2}\Re\left(d_2(z,z) + d_2(z,\overline{z})\right), \end{aligned}$$

where $\overset{\circ}{\gamma}_n(z)$ is defined in (4.16) and in (4.22), and d_2 is given by (4.2).

2. We will formulate now a general statement of similar nature. Its proof follows the same strategy but is more tedious.

Theorem 4.3 *Consider the GUE defined by (2.11) with $\beta=2$, and denote $\gamma_n(z) = \mathrm{Tr}(M-z)^{-1}$, $\Im z \neq 0$. Given integers $p\geq 0$ and $q\geq 0$ and $\eta_0>0$, take a set of points $(z_1,...,z_p,z_{p+1},...,z_{p+q})$, such that $|\Im z_j| \geq \eta_0 > 0$, $j=1,...,p+q$ and denote*

$$\begin{aligned} \overset{\circ}{\gamma}_{R,n}(z_j) &= \Re[\gamma_n(z_j) - \mathbf{E}\{\gamma_n(z_j)\}],\ j=1,...,p, \\ \overset{\circ}{\gamma}_{I,n}(z_k) &= \Im[\gamma_n(z_k) - \mathbf{E}\{\gamma_n(z_k)\}],\ k=p+1,...,p+q. \end{aligned}$$

Then the collection

$$\Gamma_{pq}^{(n)} = \{\overset{\circ}{\gamma}_{R,n}(z_j),\ j=1,...,p;\ \overset{\circ}{\gamma}_{I,n}(z_k),\ k=p+1,...,p+q.\}$$

of $p+q$ random variables converges in distribution as $n\to\infty$ to the set of Gaussian random variables

$$\Gamma_{pq} = \{\overset{\circ}{\gamma}_R(z_j),\ j=1,...,p;\ \overset{\circ}{\gamma}_I(z_k),\ k=p+1,...,p+q\},$$

whose expectations are equal to zero and whose covariances are

$$\begin{aligned} \mathbf{Cov}\{\overset{\circ}{\gamma}_R(z_{j_1}),\overset{\circ}{\gamma}_R(z_{j_2})\} &= \frac{1}{2}\Re\left(d_2(z_{j_1},z_{j_2}) + d_2(z_{j_1},\overline{z_{j_2}})\right),\ j_1,j_2=1,...,p, \\ \mathbf{Cov}\{\overset{\circ}{\gamma}_I(z_{k_1}),\overset{\circ}{\gamma}_I(z_{k_2})\} &= -\frac{1}{2}\Re\left(d_2(z_{k_1},z_{k_2}) - d_2(z_{k_1},\overline{z_{k_2}})\right),\ k_1,k_2=p+1,...,p+q, \\ \mathbf{Cov}\{\overset{\circ}{\gamma}_R(z_j),\overset{\circ}{\gamma}_I(z_k)\} &= \frac{1}{2}\Im\left(d_2(z_j,z_k) - d_2(z_j,\overline{z_k})\right),\ j=1,...,p,\ k=p+1,...,p+q. \end{aligned}$$

3. The above proof establishes the central limit theorem for the GUE and we will give below a similar fact for the GOE. For a more general case see [7].

Consider now analogous results for the GOE, corresponding to $\beta=1$ in (2.11). Our arguments will again be based on the differential formula (2.13), this time for the case of real symmetric matrices, i.e. for $\beta=1$.

It was explained above (see Remarks 1 – 2 after Corollary 3.5) that the limiting form (3.30) of the Normalized Counting Measure of eigenvalues as well as the order of magnitude of its variance are the same for the GUE and the GOE (cf (3.21) and (3.40)). The next theorem is the central limit theorem for the GOE.

Theorem 4.4 *Consider the GOE, defined in (2.11) with $\beta = 1$. Then for any z such that $\Im|z| \geq \eta_0 > 0$ the random variable*

$$\overset{\circ}{\gamma}_{R,n}(z) := \gamma_{R,n}(z) - \mathbf{E}\{\gamma_{R,n}\},$$

where $\gamma_{R,n}(z) = \Re\mathrm{Tr}(M - z)^{-1}$, converges in distribution to the Gaussian random variable whose expectation is zero and whose variance is

$$\mathbf{Var}\{\gamma_R(z)\} = \frac{1}{4}\left(d_1(z,z) + d_1(\overline{z},\overline{z}) + 2d_1(z,\overline{z})\right),$$

where $d_1(z_1, z_2)$ is given by (4.13).

Proof. We will follow the scheme of the proof of Theorem 4.2, but using the differentiation formula (2.13) for the case $\beta = 1$ of real symmetric matrices. We obtain the following analog of the relation (4.23):

$$\begin{aligned}(z + 2w^2 f_n(z))A_n(t) = -w^2\mathbf{E}\{n^{-1}\overset{\circ}{\gamma}{}^2_n(z)\exp\{it\overset{\circ}{\gamma}_{R,n}(z)\}\} \\ +w^2\mathbf{E}\{n^{-1}\overset{\circ}{\gamma}{}^2_n(z)\}F_n(t) - itw^2\mathbf{E}\{[n^{-1}\mathrm{Tr}G^3(z) + n^{-1}\mathrm{Tr}G^2(z)G(\overline{z})]\exp\{it\overset{\circ}{\gamma}_{R,n}(z)\} \\ -w^2\mathbf{E}\{[n^{-1}\mathrm{Tr}G^2(z) - \mathbf{E}\{n^{-1}\mathrm{Tr}G^2(z)\}]\exp\{it\overset{\circ}{\gamma}_{R,n}(z)\}\}.\end{aligned}$$

The relation differs from (4.23) by the factor 1/2 in front of the second term of the r.h.s. and by the last line (cf the analogous term in (3.39)). In view of (4.6) the line can be rewritten as

$$-w^2\mathbf{E}\{\frac{d}{dz}\overset{\circ}{g}_n(z)\exp\{it\overset{\circ}{\gamma}_{R,n}(z)\}\}.$$

Now by using an analogue of (4.25) for the derivative $g'_n(z)$, we can prove that this expression vanishes as $n \to \infty$. The rest of the proof repeats literally that of Theorem 4.2 ■

We note that an analog of Theorem 4.3 is also valid for the GOE.

5 Other Ensembles

We outline here analogs of Theorems 3.1, and 3.3 for certain other ensembles, involving Gaussian random variables.

1). *Ensembles with correlated Gaussian entries.* We write again $M = n^{-1/2}W$, where now M and W are $n \times n$, $n = 2m + 1$ real symmetric matrices and $W = \{W_{jk}\}_{|j|,|k|\leq m}$ is the $n \times n$ central square of the double infinite real symmetric matrix $\{W_{jk}\}_{j,k\in\mathbb{Z}}$, whose entries are Gaussian random variables such that

$$\mathbf{E}\{W_{jk}\} = 0,\ \mathbf{E}\{W_{j_1k_1}W_{j_2k_2}\} = B_{j_1-j_2,k_1-k_2} + B_{j_1-k_2,k_1-j_2}, \tag{5.1}$$

where $B_{j,k}$ satisfies the conditions

$$B_{j,k} = B_{j,-k},\ B_{j,k} = B_{k,j},\ \sum_{j,k\in\mathbb{Z}} |B_{j,k}| := b < \infty. \tag{5.2}$$

The case $B_{jk} = w^2\delta_{j0}\delta_{k0}$ is the GOE.

Under these conditions the NCM of the corresponding ensemble converges weakly with probability 1 to the limiting unit measure whose Stieltjes transform f is uniquely defined by the relations:

$$f(z) = \int_0^1 e^{2\pi ip}\widehat{f}(z,p)dp, \tag{5.3}$$

where $\widehat{f}$ is analytic in z, $\Im\widehat{f}(z,p)\cdot\Im z > 0$, $\Im z \neq 0$, $|\widehat{f}(z,p)| \leq |\Im z|^{-1}$, $\forall p \in [0,1)$ and is uniquely determined by the equation, generalizing (3.32):

$$\widehat{f}(z,p) = -\left(z + \int_0^1 \widehat{B}(p,q)\widehat{f}(z,q)dq\right)^{-1}, \tag{5.4}$$

where

$$\widehat{B}(p,q) = \sum_{j,k\in\mathbb{Z}} e^{2\pi ip(j-k)}B_{j,k}. \tag{5.5}$$

To prove these facts we write as above, by using (2.17):

$$\mathbf{E}\{G_{jk}(z)\} = -\frac{\delta_{jk}}{z} - \frac{1}{z}\sum_{|q|\leq m}\mathbf{E}\{\Delta_{j-q}^{(n)}G_{qk}(z)\} - \mathbf{E}\{T\}, \tag{5.6}$$

where

$$\Delta_{j-q}^{(n)} = \frac{1}{n}\sum_{l,p\in[-m,m]} B_{j-q,l-p}G_{lp}(z), \tag{5.7}$$

and

$$T = \frac{1}{zn}\sum_{l,p,q\in[-m,m]} B_{j-p,l-q}G_{lp}(z)G_{qk}(z).$$

We have by Schwarz inequality, (5.2), and by the inequality $\sum_{|l|\in m}|G_{lk}(z)|^2 \leq |\Im z|^{-2}$:

$$\begin{aligned} |T| &\leq \frac{1}{|\Im z|n}\sum_{|p|\leq m}\left(\sum_{l,q\in[-m,m]}|B_{j-p,l-q}|\cdot|G_{lp}(z)|^2\sum_{l,q\in[-m,m]}|B_{j-p,l-q}|\cdot|G_{qk}(z)|^2\right)^{1/2} \\ &\leq \frac{1}{|\Im z|n}\sum_{|p|\leq m}\left(\sum_{q\in\mathbb{Z}}|B_{j-p,q}|\sum_{|l|\leq m}|G_{lp}(z)|^2\sum_{l\in\mathbb{Z}}|B_{j-p,l}|\sum_{|q|\leq m}|G_{qk}(z)|^2\right)^{1/2} \\ &\leq \frac{1}{n|\Im z|^3}B_{j-q}, \end{aligned} \tag{5.8}$$

where

$$B_j = \sum_{l\in\mathbb{Z}}|B_{j,l}|.$$

Hence, the third term in the r.h.s. of (5.6) vanishes as $n \to \infty$.

Let us show that the variance of (5.7) is of the order $O(n^{-1})$. We use again the Poincare–Nash inequality (2.20). This require the derivatives of $\Delta_{j-q}^{(n)}$ with respect to M_{ab}. We have by (2.9)

$$\frac{\partial\Delta_{j-q}^{(n)}}{\partial M_{ab}} = -\frac{1}{n}\sum_{l,p\in[-m,m]} B_{j-q,l-p}G_{la}(z)G_{bp}(z),$$

and by an argument similar to that in the proof of (5.8) we obtain

$$\left| \frac{\partial \Delta^{(n)}_{j-q}}{\partial M_{ab}} \right| \le \frac{B_{j-q}}{n|\Im z|^2}.$$

Thus, we have by (2.20) and (5.2):

$$\mathbf{Var}\{\Delta^{(n)}_{j-q}\} \le \frac{B^2_{j-q}}{n^3|\Im z|^4} \sum_{a_1,b_1,a_2,b_2\in[-m,m]} (|B_{a_1-a_2,b_1-b_2}| + |B_{a_1-b_2,b_1-a_2}|) \le \frac{2bB^2_{j-q}}{n|\Im z|^4}.$$

This and (5.8) allow us to write (5.6) as

$$\mathbf{E}\{G_{jk}(z)\} = -\frac{\delta_{jk}}{z} - \frac{1}{z}\sum_{|q|\le m} \mathbf{E}\{\Delta^{(n)}_{j-q}\}\mathbf{E}\{G_{qk}(z)\} + O(n^{-1/2}),\ |\Im z| \neq 0.$$

By using the above relation, it is possible to obtain the following formulas for the limit f of the expectation $f_n = \mathbf{E}\{n^{-1}\mathrm{Tr}G(z)\}$:

$$\begin{aligned} f(z) &= f_0(z), \\ f_j(z) &= -\frac{1}{z} - \frac{1}{z}\sum_{l\in\mathbb{Z}} \Delta_{j-l}(z) f_l(z),\ \ \Delta_j(z) = \sum_{l\in\mathbb{Z}} B_{j,l}(z) f_l(z),\ j\in\mathbb{Z}. \end{aligned}$$

Now, passing to the Fourier transforms in these formulas, we obtain (5.3) – (5.5). To prove the convergence with probability 1 of $g_n(z) := n^{-1}\mathrm{Tr}G(z)$ to the same limit, we use again the Poincare–Nash inequality, leading to the $O(n^{-2})$ bound of the variance of $g_n(z)$ for $|\Im z| \neq 0$. We have by (2.20), (5.1) – (5.2), and (2.9):

$$\mathbf{Var}\{g_n(z)\} \le n^{-3} \sum_{a_1,b_1,a_2,b_2\in[-m,m]} \big(|B_{a_1-a_2,b_1-b_2}| + |B_{a_1-b_2,b_1-a_2}|\big)\mathbf{E}\{|(G^2)_{a_1b_1}|\cdot|(G^2)_{a_1b_1}|(z)\}.$$

Consider the contribution of the first term in the parentheses of the r.h.s.. By using Schwarz inequality for sums and for expectations, we obtain the bound

$$n^{-3}\left(\sum_{a_1,b_1,a_2,b_2\in[-m,m]} |B_{a_1-a_2,b_1-b_2}|\mathbf{E}\{|(G^2)_{a_1b_1}|^2\} \sum_{a_1,b_1,a_2,b_2\in[-m,m]} |B_{a_1-a_2,b_1-b_2}|\mathbf{E}\{|(G^2)_{a_1b_1}|^2\} \right)^{1/2}$$

$$\le \frac{b}{n^3}\left(\sum_{a_1,b_1\in[-m,m]} \mathbf{E}\{|(G^2)_{a_1b_1}|^2\} \sum_{a_2,b_2\in[-m,m]} \mathbf{E}\{|(G^2)_{a_1b_1}|^2\} \right)^{1/2} \le \frac{b}{n^2|\Im z|^4}.$$

The contribution of the second term of the r.h.s. of the above inequality for $\mathbf{Var}\{g_n(z)\}$ admits the same bound, implying an $O(n^{-2})$ bound for $\mathbf{Var}\{g_n(z)\}$. This finishes an outline of proof of the announced result for ensembles of random matrices with correlated Gaussian entries. For earlier proofs see e.g. [8].

2). *Deformed Wishart Ensemble.* The ensemble is defined by (3.2) in which now

$$M = \frac{1}{n}XX', \tag{5.9}$$

where $X = \{X_{j\mu}\}_{j=1,\mu=1}^{n,p}$ is $n \times p$ matrix, X' is its transposed, and the entries of X are i.i.d. Gaussian random variables of zero mean and of the variance x^2:

$$\mathbf{E}\{X_{j\mu}\} = 0, \quad \mathbf{E}\{X_{j\mu}^2\} = x^2. \tag{5.10}$$

Denoting again $G(z)$ the resolvent of H, and $g_n(z) = n^{-1}\mathrm{Tr}G(z)$, we find easily that in this case

$$\frac{\partial g_n}{\partial X_{j\mu}} = -\frac{2}{n^2}\sum_{k=1}^{n}(G^2(z))_{jl}X_{l\mu}, \tag{5.11}$$

hence we have in view of Proposition 2.5 and (5.10):

$$\mathbf{Var}\{g_n(z)\} \le \frac{4x^2}{n^4}\sum_{\mu=1}^{p}\mathbf{E}\Big\{\sum_{l,m=1}^{n}(G^2(z))_{jl}X_{l\mu}\overline{(G^2(z))_{jm}X_{m\mu}}\Big\}. \tag{5.12}$$

Setting $X_\mu = \{X_{j\mu}\}_{j=1}^{n} \in \mathbb{R}^n$, we can write the r.h.s. of the inequality as

$$\frac{4x^2}{n^4}\sum_{\mu=1}^{p}\mathbf{E}\{||G^2(z)X_\mu||^2\}.$$

By Schwarz inequality and (2.7) we have

$$||G^2(z)X_\mu||^2 \le ||G(z)||^4||X_\mu||^2 \le \frac{1}{|\Im z|^4}||X_\mu||^2 = \frac{1}{|\Im z|^4}\sum_{j=1}^{n}|X_{j\mu}|^2.$$

This, (5.10) and (5.12) imply

$$\mathbf{Var}\{g_n(z)\} \le \frac{4x^4p}{n^3|\Im z|^4} \le \frac{8x^4c}{n^2|\Im z|^4}, \tag{5.13}$$

if n is large enough, since $p/n \to c < \infty$.

Note that the above bound is valid for any $\Im z \neq 0$ and any "unperturbed" matrix $H^{(0)}$ in (3.2). However, if $H^{(0)}$ is positively definite, then H is also positive definite and we have the bound $O(1/n^2)$ also for negative values of the spectral parameter: $z = -\sigma^2$. In this case we have to replace $|\Im z|$ by σ^2 in (5.13), because for any real symmetric (Hermitian) matrix H we have $||(H-z)^{-1}|| \le (\mathrm{dist}(\text{spectrum } H, z))^{-1}$, in particular, if H is positive definite, $||(H+\sigma^2)^{-1}|| \le \sigma^{-2}$. We will use this observation below.

It can be shown, by using (5.13) and an argument similar to that of the previous section, that the Normalized Counting Measure of eigenvalues of ensemble (3.2) – (5.9) converge with probability 1 to the limiting distribution, whose Stieltjes transform solves the equation (cf (3.7)):

$$f(z) = f^{(0)}\left(z - \frac{x^2}{1+x^2f(z)}\right), \tag{5.14}$$

where $f^{(0)}$ is defined in (3.5). Analogous result is valid for complex valued Gaussian matrices X and X^* instead of X' in (5.9). This ensemble is called sometimes the Laguerre Ensemble because of use of Laguerre polynomials in the orthogonal polynomials approach.

A bit more involved argument allows one to study a more general random matrix than (5.10):

$$\frac{1}{n}XTX',$$

where T is a $p \times p$ real symmetric matrix, whose NCM τ_p converges weakly to a non-negative measure τ. In this case we have instead of (5.14)

$$f(z) = f^{(0)} \left(z - \int \frac{x^2 t \tau(dt)}{1 + x^2 t f(z)} \right).$$

Notice that the matrix T can also be random but independent of X. In this case we have to assume that τ is non-random and that τ_p converges to τ with probability 1. A particular case of this problem, corresponding to a diagonal T with i.i.d. diagonal entries, was studied in [11] by another method.

3) *Random matrices of the telecommunication theory.* We consider the real symmetric version, where the corresponding matrices have the form [14]:

$$B = X'T'(TXX'T' + \sigma^2)^{-1}TX, \tag{5.15}$$

in which X is as in (5.9), X' is its transposed, T is a $n \times n$ matrix such that the Normalized Eigenvalue Measure of eigenvalues τ_n of TT' converges to a unit measure τ. We outline an argument, showing that $n^{-1}\mathrm{Tr}B$ converges with probability one uniformly on compact sets of $]0, \infty[$ in σ to a non-random limit, given by formulas (5.25) – (5.26) below.

Assume first that the norms of TT' are uniformly bounded in n and note that

$$n^{-1}\mathrm{Tr}B = 1 - \sigma^2 n^{-1}\mathrm{Tr}(A + \sigma^2)^{-1}, \tag{5.16a}$$

where

$$A = TXX'T'. \tag{5.17}$$

Hence $\mathbf{Var}\{n^{-1}\mathrm{Tr}B\} = \mathbf{Var}\{n^{-1}\mathrm{Tr}(A{+}\sigma^2)^{-1}\}$. Denoting as before $g_n(z) = n^{-1}\mathrm{Tr}G(z),\ G(z) = (A - z)^{-1}$, we have for $g_n(-\sigma^2)$ (cf (5.11))

$$\frac{\partial g_n(-\sigma^2)}{\partial X_{j\mu}} = -\frac{2}{n^2}(T'G^2(-\sigma^2)TX_\mu)_j.$$

Thus, by Proposition 2.5,

$$\mathbf{Var}\{g_n(-\sigma^2)\} \leq \frac{8x^4 c}{n^2\sigma^8}||T||^4. \tag{5.18}$$

To find the limit of $\mathbf{E}\{g_n(-\sigma^2)\}$ we can use the scheme similar to that of Section 3. We will use however another scheme, outlined below and dated to [11] (see also [6, 12]).

For any two vectors X, Y of $\mathbb{C}^n$ denote by $L_{X,Y}$ the rank one matrix, acting on $\Psi \in \mathbb{C}^n$ by the formula

$$L_{X,Y}\Psi = (\Psi, X)Y. \tag{5.19}$$

Then we can write (5.9) as $M = n^{-1}\sum_{\mu=1}^p L_{X_\mu,X_\mu}$ and A of (5.17) as

$$A = \frac{1}{n}\sum_{\mu=1}^p L_{Y_\mu,Y_\mu},\ Y_\mu = TX_\mu. \tag{5.20}$$

Hence, we have by the resolvent formula

$$\mathbf{E}\{G(-\sigma^2)\} = \frac{1}{\sigma^2} - \frac{1}{n\sigma^2}\sum_{\mu=1}^p \mathbf{E}\{L_{Y_\mu,GY_\mu}\}. \tag{5.21}$$

It is easy to show that for any Hermitian (real symmetric) matrix H, any $Y \in \mathbb{C}^n$, and any z that do not belong to the spectrum of $H + L_{Y,Y}$, we have

$$G_Y := (H + L_{Y,Y} - z)^{-1} = G - \frac{GL_{Y,Y}G}{1 + (GY, Y)}, \quad G := (H - z)^{-1},$$

in particular

$$G_Y Y = \frac{1}{1 + (GY, Y)} GY. \tag{5.22}$$

This formula and (5.21) yield

$$\mathbf{E}\{G(-\sigma^2)\} = \frac{1}{\sigma^2} - \frac{1}{n\sigma^2} \sum_{\mu=1}^{p} \mathbf{E}\left\{\frac{1}{1 + n^{-1}(G_\mu Y_\mu, Y_\mu)} L_{Y_\mu, G_\mu Y_\mu}\right\}, \ G_\mu = G(-\sigma^2)\big|_{Y_\mu=0}. \tag{5.23}$$

Since the vectors $\{X_\nu\}_{\nu=1}^{p}$ are i.i.d., $\{Y_\nu\}_{\nu=1}^{p}$ have the same property and since G_μ does not contain Y_μ, we have by (5.10) and (5.20)

$$\mathbf{E}_\mu\{(G_\mu Y_\mu, Y_\mu)\} = x^2 \mathrm{Tr} T' G_\mu T, \ \mathbf{E}_\mu\{L_{Y_\mu, G_\mu Y_\mu}\} = G_\mu T T',$$

where $\mathbf{E}_\mu\{...\}$ denotes the expectation with respect to X_μ, and if $\mathbf{Var}_\mu\{...\}$ is the corresponding variance, then

$$\mathbf{Var}_\mu\{n^{-1}(G_\mu Y_\mu, Y_\mu)\} \le \frac{Cx^4}{n\sigma^4} ||T||^4,$$

where C is an absolute constant. This and the inequality $(G_\mu Y_\mu, Y_\mu) \ge 0$ allow us to replace $n^{-1}(G_\mu Y_\mu, Y_\mu)$ by $n^{-1}\mathrm{Tr} G_\mu$ in (5.23), and then, after applying once more (5.22), by $n^{-1}\mathrm{Tr} T' G T$. We arrive to the relation

$$\mathbf{E}\{G(-\sigma^2)\} = \frac{1}{\sigma^2} - \frac{x^2}{n\sigma^2} \sum_{\mu=1}^{p} \mathbf{E}\left\{\frac{1}{1 + x^2 n^{-1} \mathrm{Tr} T' G T} G T T'\right\} + O(1/n).$$

Now, by using again Proposition 2.5, we prove an $O(1/n^2)$ bound for the variance of $n^{-1}\mathrm{Tr} T' G T$ (cf (5.18)), which leads to the asymptotic formula

$$\mathbf{E}\{G(-\sigma^2)\} = \frac{1 + x^2 h_n}{x^2 c} \left(T T' + \frac{\sigma^2(1 + x^2 h_n)}{x^2 c}\right)^{-1} + O(1/n), \tag{5.24}$$

where

$$h_n = \mathbf{E}\{n^{-1} \mathrm{Tr} T' G(-\sigma^2) T\}.$$

Applying the operation $n^{-1}\mathrm{Tr}$ to (5.24) and to this relation, multiplied by T' from the left and by T from the right, and by using an argument, similar to that in the proof of Theorem 3.1, we prove that g_n converges with probability 1 to a non - random limit f, h_n converges to h and f and h satisfy the following system of functional equations:

$$\begin{aligned} f &= \frac{(1 + x^2 h)}{x^2 c} f_0\left(-\frac{\sigma^2(1 + x^2 h)}{x^2 c}\right), \\ h &= \frac{(1 + x^2 h)}{x^2 c} f_1\left(-\frac{\sigma^2(1 + x^2 h)}{x^2 c}\right), \end{aligned} \tag{5.25}$$

where

$$f_0(z) = \int_0^\infty \frac{\tau(d\lambda)}{\lambda - z}, \ f_1(\sigma^2) = \int_0^\infty \frac{\lambda \tau(d\lambda)}{\lambda - z}, \tag{5.26}$$

and τ is the limiting Normalized Counting Measure of eigenvalues of TT'.

The above proof of (5.25) was given under the assumption that the norms of TT' are uniformly bounded in n. The general case can be obtained by a standard truncation procedure, which is easy to carry out because the norm does not present in (5.25) and (5.26).

4). *Wigner Ensembles.* A natural question is to which extent the above results, obtained for ensembles with Gaussian variables, can be generalized. We will discuss shortly this question for the Wigner Ensembles, defined as follows (for technical convenience we will consider in this subsection real symmetric matrices). Write the matrix M in the form (3.3)

$$M = n^{-1/2}W, \tag{5.27}$$

where $W = \{W_{jk}^{(n)}\}_{j,k=1}^{n}$ with $W_{jk}^{(n)} = W_{kj}^{(n)} \in \mathbb{R},\ 1 \le j \le k \le n$. Suppose that the random variables $W_{jk}^{(n)},\ 1 \le j \le k \le n$ are independent and that

$$\mathbf{E}\{W_{jk}^{(n)}\} = 0,\ \ \mathbf{E}\{(W_{jk}^{(n)})^2\} = (1+\delta_{jk})w^2, \tag{5.28}$$

i.e. the two first moments of the entries are as in the GOE case (see (3.36) – (3.37)). In other words, the probability law of the matrix W is

$$\mathbf{P}(d_1W) = \prod_{1\le j\le k\le n} F_{j,k}^{(n)}(dW_{jk}), \tag{5.29}$$

where for any $1 \le j \le k \le n$ $F_{j,k}^{(n)}$ is a probability measure on the real line, satisfying condition (5.28).

A sufficiently detailed study of the global regime of these ensembles is rather involved (see e.g. [1, 6, 9, 12]). It is worth to note however that many of these results can be obtained by applying a generalization of the method, used in previous sections and based on resolvent identity and on differentiation formulas (2.13), and (2.17). The role of these formulas in general case of the Wigner Ensemble plays the following one [9].

Let ξ be a real valued and centered random variable, having $p+2$ finite moments for some positive integer p, and let $\Phi : \mathbb{R} \to \mathbb{C}$ be a function, whose first $(p+1)$ derivatives are bounded. Denote by $\kappa_l,\ l = 1, 2, ...$ the cumulants (semi-invariants) of ξ, i.e. the MacLaurin coefficients of logarithm of the characteristic function of ξ. Then

$$\mathbf{E}\{\xi\Phi(\xi)\} = \sum_{l=1}^{p} \frac{\kappa_{l+1}}{l!}\mathbf{E}\{\Phi^{(l)}(\xi)\} + \varepsilon_p, \tag{5.30}$$

where

$$|\varepsilon_p| \le C_p \sup_{x\in\mathbb{R}} |\Phi^{(p+1)}(x) \cdot |\mathbf{E}\{|\xi|^{p+2}\}, \tag{5.31}$$

and C_p depends on p only. The cumulants can be expressed via the moments of ξ. Namely, if $\mu_l = \mathbf{E}\{\xi^l\}$, and $\mu_1 = 0$, then

$$\kappa_1 = 0,\ \kappa_2 = \mu_2 = \mathbf{Var}\{\xi\},\ \kappa_3 = \mu_3,\ \kappa_4 = \mu_4 - 3\mu_2^2, \tag{5.32}$$

etc. For the Gaussian random variable all cumulants but κ_2 vanish, and the above formula reduces to (2.15). Note that κ_4 is called in statistics the excess of the random variable ξ. It is an important ingredient of a simple statistical test to find that a given random variable is not Gaussian.

We present now a result, whose proof is based on (5.30).

Theorem 5.1 *Let $n^{-1/2}\widehat{W}$ be the GOE matrix (3.36) – (3.37) and $n^{-1/2}W$ be the Wigner matrix (5.27) – (5.29), satisfying the condition*

$$\sup_n \max_{1\le j,k\le n} \mathbf{E}\{|W_{jk}^{(n)}|^3\} := w_3 < \infty. \tag{5.33}$$

Denote by $G_1(z)$ and $G_2(z)$ the resolvents of $n^{-1/2}\widehat{W}$ and $n^{-1/2}W$. Then

$$\left|\mathbf{E}\{n^{-1}\mathrm{Tr}G_1(z)\} - \mathbf{E}\{n^{-1}\mathrm{Tr}G_2(z)\}\right| \le Cw_3/n^{1/2}|\Im z|^4, \tag{5.34}$$

where C is an absolute constant.

Proof. Consider the "interpolating" random matrix (cf (2.21))

$$M(t) = \sqrt{t/n}\; W + \sqrt{(1-t)/n}\; \widehat{W}, \;\; 0 \le t \le 1, \tag{5.35}$$

viewed as defined on the product of probability spaces of matrices W and $\widehat{W}$. In other words, we assume that matrices W and $\widehat{W}$ in (5.35) are independent. Denote again by $\mathbf{E}\{...\}$ the corresponding expectation in the product space. It is evident that

$$M(1) = n^{-1/2}W, \;\; M(0) = n^{-1/2}\widehat{W}, \tag{5.36}$$

and if $G(z,t)$ is the resolvent of $M(t)$, then we have by (5.36):

$$n^{-1}\mathrm{Tr}G_1(z) - n^{-1}\mathrm{Tr}G_2(z) = \int_0^1 \frac{d}{dt} n^{-1}\mathrm{Tr}G(z,t)dt$$

and by (2.9):

$$\frac{d}{dt} n^{-1}\mathrm{Tr}G(z,t) = n^{-1}\mathrm{Tr}G^2(z,t)\left(\frac{1}{2\sqrt{nt}}W - \frac{1}{2\sqrt{n(1-t)}}\widehat{W}\right). \tag{5.37}$$

Now we will apply the differentiation formula (5.30) with $p = 1$ to transform the first term in parentheses. To this end we take into account that W and $G^2(z,t)$ are symmetric and write the term as

$$\frac{1}{2\sqrt{nt}}\left(\sum_{j=1}^n \mathbf{E}\{W_{jj}(G^2(z,t))_{jj}\} + 2\sum_{1\le j<k\le n} \mathbf{E}\{W_{jk}(G^2(z,t))_{kj}\}\right).$$

Since $\{W_{jk}\}_{1\le j<k\le n}$ are independent, we can apply (5.30) to every term of the sums. In view of (5.28) we obtain that the contribution of the first term in the parentheses of (5.37) is

$$\frac{w^2}{2\sqrt{n^3 t}}\sum_{j,k=1}^n (1+\delta_{jk})\mathbf{E}\left\{\frac{\partial}{\partial W_{jk}^{(n)}}(G^2(z,t))_{kj}\right\} + \mathcal{R}_n, \tag{5.38}$$

where

$$|\mathcal{R}|_n \le \frac{w_3}{2\sqrt{n^5 t}}\sum_{j,k=1}^n \sup_{M_{jk}\in\mathbb{R}}\left|\frac{\partial^2}{\partial\left(M_{jk}\right)^2}(G^2(z,t))_{kj}\right| \tag{5.39}$$

It follows from the Gaussian differential formula (2.13) that the contribution of the second term in the parentheses of (5.37) can be written in the same form as (5.38), but without the remainder term $\mathcal{R}_n$. By using formula (2.9), it is easy to show that the expressions are

$$-\frac{w^2}{n^2}\mathbf{E}\left\{\mathrm{Tr}G^2\mathrm{Tr}G + \mathrm{Tr}G^3\right\}. \tag{5.40}$$

Hence, the r.h.s of (5.37) is equal to $\mathcal{R}_n$. By using formula (2.9) twice, we find that the second derivative in (5.39) is the sum of terms of the form $n^{-1}(G^2)_{ab}G_{cd}G_{ef}$, where $a, b, .., f$ assume values j, k. Each of these terms is bounded by $n^{-1}|\Im z|^{-4}$ in view of (2.7). We obtain that the remainder (5.39), hence the derivative (5.37), admits the bound:

$$|\mathcal{R}|_n \le \frac{Cw_3}{n^{1/2}\,|\Im z|^4}, \tag{5.41}$$

where C is an absolute constant. This fact and the interpolating property (5.37) yield (5.34). ■

Remarks. 1. A similar argument and (3.40) imply an $O(n^{-1/2})$ bound for the variance of $n^{-1}\mathrm{Tr}(M-z)^{-1}$. This and a standard truncation procedure lead to the weak convergence in probability of the NCM of a Wigner Ensemble satisfying (5.28) and the condition

$$\lim_{n\to\infty} n^{-2} \sum_{1\le j\le k\le n} \int_{|W|\ge\tau\sqrt{n}} W^2 F_{j,k}^{(n)}(dW) = 0,\ \forall\tau > 0. \tag{5.42}$$

The condition is a matrix analog of the well known Lindeberg condition for the validity of the central limit theorem. Hence the semicircle law is a common form of the limiting eigenvalue counting measures for all Wigner ensembles, satisfying (5.28) and (5.42). For these and other numerous results for the Wigner ensembles see e.g. [1, 6, 9, 12] and references therein.

2.. As another example of application of (5.30) we mention the asymptotic form of the covariance of $\gamma_n = \mathrm{Tr}(M-z)^{-1}$ for the Wigner matrix (5.27) – (5.29), such that its moment $\mathbf{E}\{|W_{jk}^{(n)}|^5\}$ is bounded uniformly in j, k and n [9]:

$$n^2\mathbf{Cov}\{\gamma_n(z_1)\gamma_n(z_2)\} = d_1(z_1, z_2) + 2\kappa_4 h(z_1)h(z_2) + O(n^{-1/2}), \tag{5.43}$$

where $d_1(z_1, z_2)$ is the covariance for the GOE, given by (4.13), $h(z) = f^2(z)(z^2 - 4w^2)^{-1/2}$, and f is defined in (3.34).

Acknowledgements. I am grateful to Prof. P. Loubaton, whose questions motivated the work. It was supported in part by the Fonds National de la Science (France) via the ACI Program "Nouvelles Interfaces des Mathématiques", project N° NIM205.

References

[1] Z. D. Bai, *Methodologies in spectral analysis of large-dimensional random matrices: a review*, Statist. Sinica **9** (1999) 611–677.

[2] C. W. J. Beenakker, *Random-matrix theory of quantum transport*, Rev. Mod. Phys. **69** (1997) 731-847.

[3] D. Bessis, C. Itzykson, J.-B. Zuber, *Quantum field theory techniques in graphical enumeration*, Adv. in Appl. Math. **1** (1980) 109–157.

[4] T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, S.S.M. Wong, *Random-matrix physics: spectrum and strength fluctuations,* Rev. Mod. Phys. **53** (1981) 385–479.

[5] L. H. Y. Chen, *An inequality involving normal distribution*, J. Multivariate Anal. 50 (1982) 213–223, 585–604.

[6] V.L. Girko, *Theory of Stochastic Canonical Equations*, vols.I, II Kluwer, Dordrecht, 2001.

[7] K. Johansson, *On fluctuations of eigenvalues of random Hermitian matrices*, Duke Math. J. **91** (1998) 151–204.

[8] A. Khorunzhy, *Eigenvalue distribution of large random matrices with correlated entries*, Mat. Fiz. Anal. Geom. **3** (1996), 80–101

[9] A. Khorunzhy, B. Khoruzhenko, L. Pastur, *Random matrices with independent entries: asymptotic properties of the Green function*, J. Math. Phys. **37** (1996) 5033–5060.

[10] M. Ledoux, *Concentration of Measure Phenomenon*, AMS, Providence, R.I., 2001.

[11] V. Marchenko, L. Pastur, *Eigenvalue distribution in some ensembles of random matrices*, Math. USSR Sbornik, **1** (1967) 457–483.

[12] L. Pastur, *A simple approach to the global regime of the random matrix theory*, Mathematical Results in Statistical Mechanics, S. Miracle-Sole, J. Ruiz, V. Zagrebnov (Eds.), World Scientific, Singapore 1999, pp. 429-454.

[13] L. Pastur, M. Shcherbina, *Universality of the local eigenvalue statistics for a class of unitary invariant matrix ensembles*, J. Stat. Phys. **86** (1997) 109-147.

[14] A. M. Tulino, S. Verdu, *Random Matrix Theory and Wireless Communications*, Now Publishers Inc., Hanover, MA, 2004.