Random Matrices in Wireless Communications

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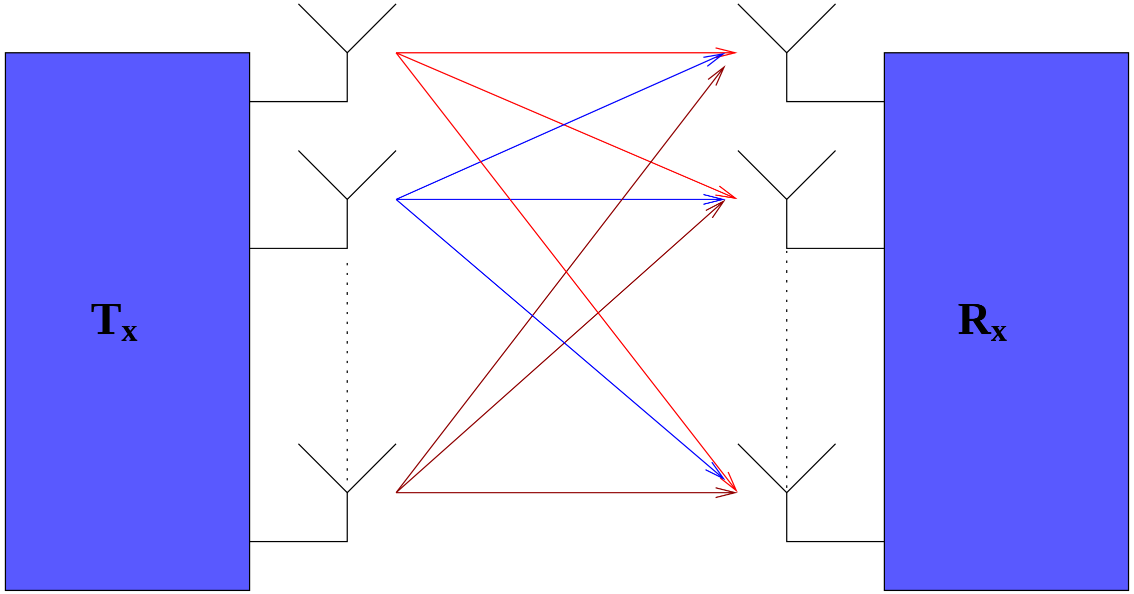
MIMO Representation



$$\mathbf{y}(t) = \sqrt{\frac{\rho}{n_{t_x}}} \int \mathbf{H}_{n_{r_x} \times n_{t_x}}(\tau)\mathbf{x}(t-\tau)d\tau + \mathbf{n}(t)$$

and

$$\mathbf{y}(f) = \sqrt{\frac{\rho}{n_{t_x}}} \mathbf{H}_{n_{r_x} \times n_{t_x}}(f)\mathbf{x}(f) + \mathbf{n}(f)$$

A useful metric: Mutual Information

- Mesasurements have shown that:

$$\lim_{n_{t_x} \to \infty, \frac{n_{r_x}}{n_{t_x}} = \beta} \log_2 \det \left(\mathbf{I}_{n_{t_x}} + \frac{\rho}{n_{t_x}} \mathbf{H}^H \mathbf{H} \right) - n_{t_x} \mu \to N(0, \sigma^2)$$

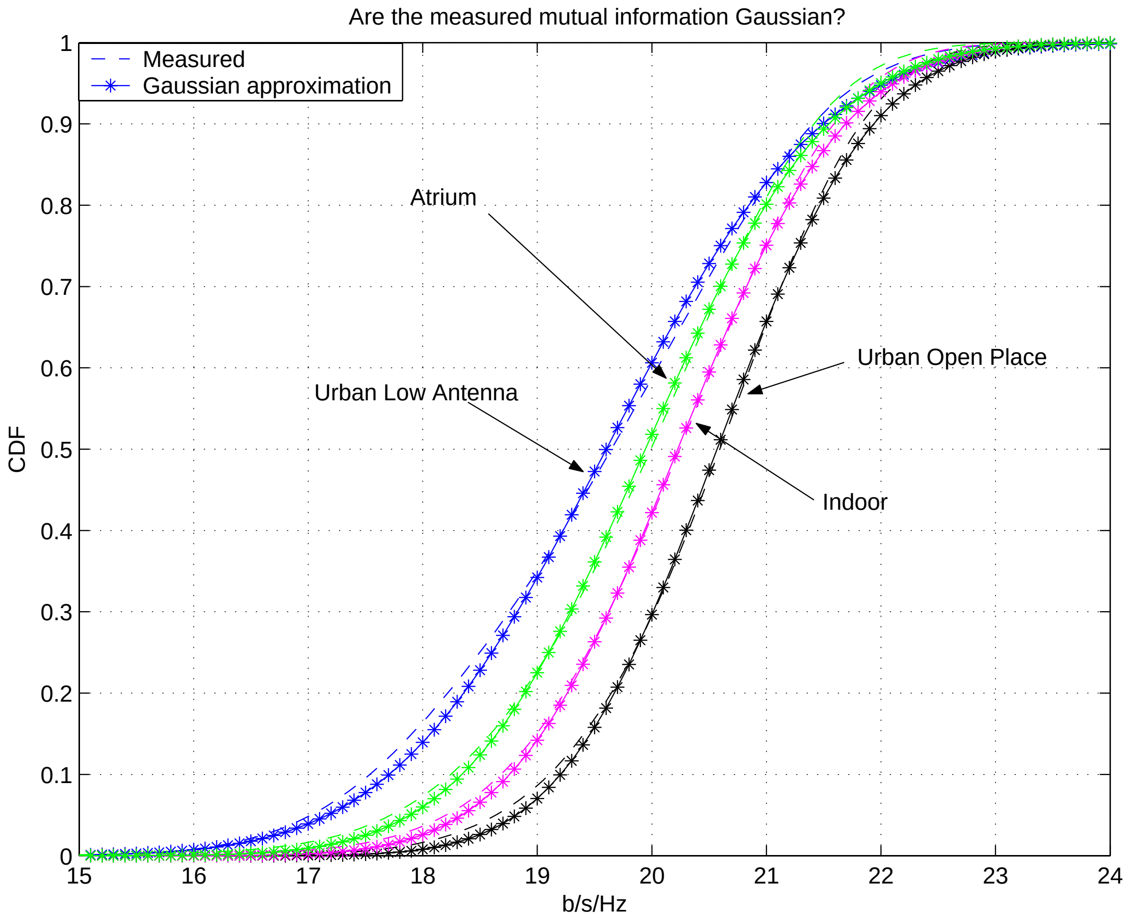
- The distribution of the mutual information ($M = \log_2 \det \left(\mathbf{I}_{n_{t_x}} + \frac{\rho}{n_{t_x}} \mathbf{H}^H \mathbf{H} \right)$ in b/s/Hz) is very useful for quality of service optimization.
- For example, if we impose the outage probability $q = 0.01$, then one can easily find the corresponding rate R:

$$q = \mathrm{CDF}(R) = P(M \leq R) = \int_{-\infty}^{R} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(u - n_{t_x}\mu)^2}{2\sigma^2}} du.$$

- The Cumulative Density function (CDF) is also used as a channel modelling metric.

- Explicit expressions of the mutual information ease the optimization of the "water-filling"' formula (To be explained in next meeting) .

Do Real Channels have a Gaussian behavior?



- The Gaussian behavior of the mutual information appears already for 8×8 MIMO systems.

How far is asymptotic?

- Results have engineering applications (some results of the mutual information distribution at $\rho = 10$):
  - With 6 antennas, we are at 0.02% of the asymptotic mean value while the variance is only at 1% of the asymptotic variance value!
  - With 3 antennas, we are at 0.6% of the asymptotic mean value while the variance is only at 4% of the asymptotic variance value!
- **Remark:** This speed of convergence does not hold for other metrics such as Signal to Interference plus Noise Ratio (SINR).
  - D.N.C Tse and O. Zeitouni, "Linear Multiuser Receivers in Random Environments", IEEE Trans. on Information Theory, pp.171-188, Jan. 200.

State of the art

- $\mathbf{H}$ zero mean i.i.d Gaussian:
  - Represents a very rich scaterring environment. Overestimates measured mutual information.
  - Problem solved for the mutual information distribution:
    - Z.D. Bai and J. W. Silverstein, "CLT of Linear Spectral Statistics of Large Dimensional Sample Covariance Matrices", Annals of Probability 32(1A) (2004), pp. 553-605.

$$\mu = \beta \ln(1 + \rho - \rho\alpha) + \ln(1 + \rho\beta - \rho\alpha) - \alpha$$

$$\sigma^2 = -\ln[1 - \frac{\alpha^2}{\beta}]$$

$$\alpha = \frac{1}{2}[1 + \beta + \frac{1}{\rho} - \sqrt{(1 + \beta + \frac{1}{\rho})^2 - 4\beta}]$$

State of the art

- $\mathbf{H}$ zero mean Gaussian Uncorrelated non-identically distributed entries:
  - Represents a very rich scaterring environment with different receiving powers on each antenna.
  - Overestimates measured mutual information.
  - Problem solved for the $\mathbf{mean}$ only:
    - V. L. Girko, "Theory of Random Determinants"', Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.
    - Applied by Tulino and Verdu (See monograph).
  - Distribution: open issue.

State of the art

- $\mathbf{H}$ zero mean Gaussian with correlation on one side ($\mathbf{\Theta}$ zero mean i.i.d Gaussian):
  - Correlation at the transmitter: $\mathbf{H} = \mathbf{R^{\frac{1}{2}}}_{t_x}\mathbf{\Theta}$.
  - Correlation at the receiver: $\mathbf{H} = \mathbf{\Theta}\mathbf{R^{\frac{1}{2}}}_{r_x}$.
  - In both cases ($\mathbf{R}_{t_x}$ and $\mathbf{R}_{r_x}$ are hermitian matrices), the model underestimates measured mutual information.
- Problem solved for the mutual information distribution with explicit expressions of μ and σ
  - Z.D. Bai and J. W. Silverstein, "CLT of Linear Spectral Statistics of Large Dimensional Sample Covariance Matrices", Annals of Probability 32(1A) (2004), pp. 553-605.

State of the art

- $\mathbf{H}=\mathbf{R}^{\frac{1}{2}}{}_{t_x}\mathbf{\Theta}\mathbf{R}^{\frac{1}{2}}{}_{r_x}$, with $\mathbf{\Theta}$ zero mean i.i.d Gaussian:
  - Represents correlation at both end.
  - Seperable correlation is not always fulfilled in reality.
  - mean mutual information: Tulino, Verdu (see monograph): in fact, an application of Girko.
  - variance (Sengupta and Mitra using replica method):
    - A. Sengupta and P. Mitra, "Capacity of Mutlivariante Channels with Multiplicative Noise: Random Matrix Techniques and Large-N Expansion for Full Transfer Matrices", LANL Archive Physics, oct. 2000
    - A. Moustakas, S. Simon and A. Sengupta, "MIMO Capacity through Correlated Channels in the presence of Correlated Interferers: A (Not so) Large-N Analysis, IEEE Transactions on Information Theory, oct. 2003

State of the art

- **Results:** Denote ξ and η the eigevenvalues of matrices $\mathbf{R}_{t_x}$ and $\mathbf{R}_{r_x}$ respectively:

$$\mu = \sum_{i=1}^{n_{t_x}} \log(1 + \rho\xi_i r) + \sum_{i=1}^{n_{r_x}} \log(1 + \rho\eta_i q) - n_{r_x} qr$$

$$\sigma^2 = -2\log(1 - g(r, q))$$

$$g(r, q) = \left[\frac{1}{n_{t_x}} \sum_{i=1}^{n_{t_x}} (\frac{\rho\eta_i}{1 + \eta_i\rho q})^2\right] \left[\frac{1}{n_{t_x}} \sum_{i=1}^{n_{r_x}} (\frac{\rho\xi_i}{1 + \xi_i\rho r})^2\right]$$

$$r = \frac{1}{n_{t_x}} \sum_{i=1}^{n_{t_x}} \frac{\rho\eta_i}{1 + \eta_i\rho q}$$

$$q = \frac{1}{n_{t_x}} \sum_{i=1}^{n_{r_x}} \frac{\rho\xi_i}{1 + \xi_i\rho r}$$

- The replica method has been introduced in Telecommunications for the first time by Tanaka: "A Statistical Mechanics Approach to Large System Analysis of CDMA Multiuser detectors, IEEE IT, vol.48, no11,p.2888-2910. nov.2002

State of the art

- $\mathbf{H}$ zero mean Gaussian with any type of correlation $C = \mathbb{E}(\mathrm{vec}(\mathbf{H})\mathrm{vec}(\mathbf{H})^H)$ (The operator $\mathrm{vec}(\mathbf{H})$ stacks all the columns of matrix $\mathbf{H}$ into a single column):
    - mean mutual information: open issue
    - distribution: open issue.

State of the art

- $\mathbf{H}$ Rice Channel:

$$\mathbf{H} = \sqrt{\frac{K}{K+1}}\mathbf{A} + \sqrt{\frac{1}{K+1}}\mathbf{B}$$

  - $\mathbf{A}$ represents the line of sight component (mean) of the channel.
  - $\mathbf{B}$ is the random component of the channel with zero mean Gaussian distributed entries.
  - K is the Ricean factor:
    - When $K \rightarrow 0$, $\mathbf{H}$ zero mean channel.
    - When $K \rightarrow 0$, $\mathbf{H}$ is a purely deterministic channel.

State of the art

- $\mathbf{H}$ Rice Channel:

$$\mathbf{H} = \sqrt{\frac{K}{K+1}}\mathbf{A} + \sqrt{\frac{1}{K+1}}\mathbf{B}$$

  - mean mutual information: problem solved
    - for B i.i.d Gaussian: B. Dozier and J. Silverstein “On the Empirical Distribution of Eigenvalues of Large Dimensional Information-Plus-Noise Type Matrices”, submitted.
    - for B Gaussian independent with different variances: Girko. ”Theory of Stochastic Canonical Equations”’, vol 1, Kluwer Academic publishers, Dordrecht
    - For B any correlation, open issue.
  - distribution: open issue.