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Multi-user communications: power allocation and impact of synchronism

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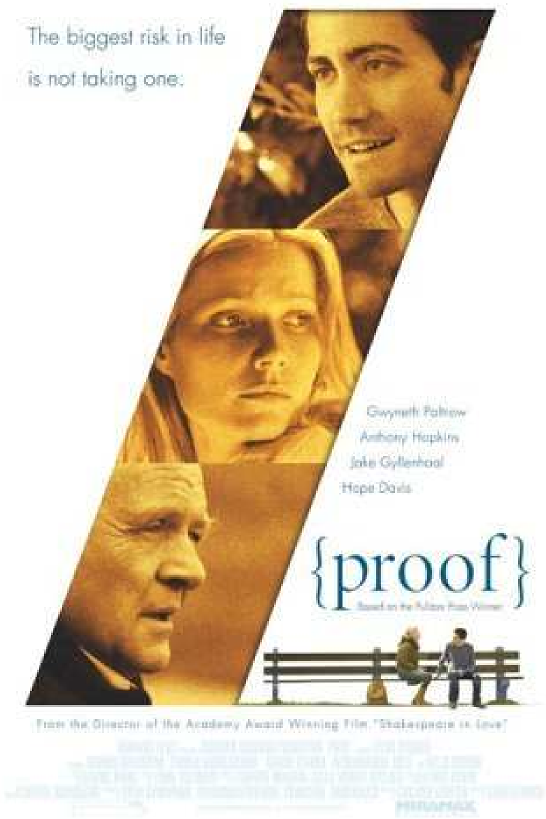
Random Matrix Theory



Cited as **one of the ”modern tools”** in mathematics and used in the proof of an important result in prime number theory

Presentation

Power allocation

Multiuser System Random Matrix Model

Multiuser system: K users, N dimensions

Received signal

$$\mathbf{y} = \mathbf{H}\mathbf{P}^{\frac{1}{2}}\mathbf{s} + \mathbf{n}$$

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & \ldots & h_{1K} \\ h_{21} & h_{22} & \ldots & h_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N1} & \ldots & \ldots & h_{NK} \end{bmatrix}, \quad \mathbf{P}^{\frac{1}{2}} = \begin{bmatrix} p_1^{\frac{1}{2}} & 0 & \ldots & 0 \\ 0 & p_2^{\frac{1}{2}} & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & p_K^{\frac{1}{2}} \end{bmatrix}$$

h_{ik} are independent zero mean Gaussian variables with variance $\mid g^k(i) \mid^2$. In particular:

$$\mathbf{H} = \mathbf{G} \odot \mathbf{W}$$

where $\mathbf{W}$ is an $N \times K$ i.i.d zero mean Gaussian matrix.

The G-model

The pattern mask $\mathbf{G}$ is a neat way of modelling all the systems (especially for cross-system resource allocation problems) in order to have a unified framework based on random matrices.

- **OFDM** systems: $g^k(i) = 0$ if $k \neq i$.
- **SIMO** systems: $g^1(i) = g^2(i) = \ldots g^K(i)$ where $g^l(i)$ represents the l^{th} eigenvalue of the correlation matrix $\mathbf{R}$ ($\mathbf{H} = \mathbf{RW}$).
- **MIMO** with Kronecker model: $g^l(i) = \lambda^T(l).\lambda^R(i)$.
- **CDMA** systems in frequency selective channels: $g^l(i)$ represents the frequency response of user l on carrier i

$$\mathbf{y} = \mathbf{H}_1\mathbf{w}_1\sqrt{P}_1 s_1 + \mathbf{H}_2\mathbf{w}_2\sqrt{P}_2 s_2 + \ldots + \mathbf{H}_K\mathbf{w}_K\sqrt{P}_k s_K + \mathbf{n}$$

Toeplitz structure

$$\mathbf{H}_i \sim \mathbf{F}^H\mathbf{D}_i\mathbf{F}$$

$$\begin{aligned}\tilde{\mathbf{y}} &= \mathbf{D}_1\tilde{\mathbf{w}}_1\sqrt{P}_1 s_1 + \mathbf{D}_2\tilde{\mathbf{w}}_2\sqrt{P}_2 s_2 + \ldots + \tilde{\mathbf{D}}_K\tilde{\mathbf{w}}_K\sqrt{P}_k s_K + \tilde{\mathbf{n}} \\ &= \left(\mathbf{G}\odot\tilde{\mathbf{W}}\right)\mathbf{P}^{\frac{1}{2}}\mathbf{s} + \tilde{\mathbf{n}}\end{aligned}$$

- **Ad-hoc networks**....

Multiuser ressource allocation: problem statement

- Based on a given set of target rates, what should be the adequate power allocations?
- Can the required rates be always satisfied?
- What is the minimum required knowledge such as each user determines solely the power to satistfy his rate (centralized versus non centralized system)?
- For the MMSE-SIC, what is the decoding order for a given set of rates?

MMSE Receiver

Model example :

$$\begin{aligned} \mathbf{y} &= \mathbf{W}\mathbf{s} + \mathbf{n} \\ &= \mathbf{u}s_1 + \mathbf{U}\mathbf{x} + \mathbf{n} \\ &= \mathbf{u}s_1 + \mathbf{n}' \end{aligned}$$

$$\mathbb{E}(\mathbf{n}'\mathbf{n}'^H) = (\mathbf{U}\mathbf{U}^H + \sigma^2\mathbf{I}) = \mathbf{Q}\Lambda\mathbf{Q}^H$$

Whitening filter:

$$\begin{aligned} \tilde{\mathbf{y}} = \mathbf{\Lambda}^{-\frac{1}{2}}\mathbf{Q}^H\mathbf{y} &= \mathbf{\Lambda}^{-\frac{1}{2}}\mathbf{Q}^H\mathbf{u}s_1 + \mathbf{\Lambda}^{-\frac{1}{2}}\mathbf{Q}^H\mathbf{n}' \\ &= \mathbf{g}s_1 + \mathbf{b} \end{aligned}$$

$\mathbf{b}$ is a white Gaussian noise.

MMSE Receiver

$$\tilde{\mathbf{y}} = \mathbf{\Lambda}^{-\frac{1}{2}}\mathbf{Q}^H\mathbf{u}s_1 + \mathbf{b}$$

Define $\mathbf{g} = \mathbf{\Lambda}^{-\frac{1}{2}}\mathbf{Q}^H\mathbf{u}$

The output SINR is maximized with:

$$\mathbf{g}^H\tilde{\mathbf{y}} = \mathbf{g}^H\mathbf{g}s_1 + \mathbf{g}^H\mathbf{b}$$

As a consequence, the receiver is:

$$\mathbf{g}^H\mathbf{\Lambda}^{-\frac{1}{2}}\mathbf{Q}^H = \mathbf{u}^H\left(\mathbf{Q}\mathbf{\Lambda}^{-1}\mathbf{Q}^H\right) = \mathbf{u}^H\left(\mathbf{U}\mathbf{U}^H + \sigma^2\mathbf{I}_N\right)^{-1}$$

Remark: The usual MMSE receiver is the unbiased one:

$$\mathbf{u}^H\left(\mathbf{W}\mathbf{W}^H + \sigma^2\mathbf{I}_N\right)^{-1} = \frac{1}{1 + \mathbf{u}^H\left(\mathbf{U}\mathbf{W}\mathbf{U}^H + \sigma^2\mathbf{I}_N\right)^{-1}\mathbf{u}}\mathbf{u}^H\left(\mathbf{U}\mathbf{U}^H + \sigma^2\mathbf{I}_N\right)^{-1}$$

MMSE Receiver

After MMSE filtering, we obtain:

$$\mathbf{g}^H\tilde{\mathbf{y}} = \mathbf{g}^H\mathbf{g}s_1 + \mathbf{g}^H\mathbf{b}$$

with $\mathbf{g} = \mathbf{\Lambda}^{-\frac{1}{2}}\mathbf{Q}^H\mathbf{u}$

Signal to Interference plus Noise Ratio (SINR):

$$\beta_N = \frac{(\mathbf{g}^H\mathbf{g})^2\mathbb{E}(\mid s_1 \mid^2)}{\mathbf{g}^H\mathbf{g}} = \mathbf{g}^H\mathbf{g} = \mathbf{u}^H\left(\mathbf{U}\mathbf{U}^H + \sigma^2\mathbf{I}_N\right)^{-1}\mathbf{u}$$

Multiuser receiver: MMSE and MMSE-SIC

- MMSE (Mininum Mean-Square Error receiver)
  - Output: $\hat{\text{s}} = \mathbf{H}^H (\mathbf{A})^{-1} \mathbf{y} \quad \text{with} \quad \mathbf{A} = \mathbf{H}\mathbf{H}^H + \sigma^2 \mathbf{I}_N$
  - Maximizes SINR over linear receivers

$$\text{SINR}_k = p_k \mathbf{h}_k^H \left(\mathbf{U}\mathbf{P}^k\mathbf{U}^H + \sigma^2 \mathbf{I}_N \right)^{-1} \mathbf{h}_k$$

  where $\mathbf{U}$ is the matrix resulting after extracting $\mathbf{h}_k$ from $\mathbf{H}$ and $\mathbf{P}^k$ is the $K-1 \times K-1$ diagonal matrix of powers.

- MMSE-SIC (Successive Interference Cancellation)
  - MMSE Sequential detection of the incoming block
  - Advantage

$$\text{SINR}_{(K-1)}^{\text{SIC}} \geq \text{SINR}_{(K-1)}^{\text{MMSE}}$$

Power allocation based on target rate constraints

Expression of the SINR:

- **MMSE:**

$$\mathrm{SINR}_i = \gamma_i = P_i \mathbf{h}_i^H \left(\sum_{l=1, l\neq i}^{K} P_l \mathbf{h}_l \mathbf{h}_l^H + \sigma^2 \mathbf{I} \right)^{-1} \mathbf{h}_i$$

- **MMSE-SIC:**
  - Last iteration, user K

$$\mathrm{SINR}_K = \gamma_K = \frac{\mathbf{h_K^H h_K} P_K}{\sigma^2}$$

  - Iteration k

$$\mathrm{SINR}_k = \gamma_k = P_k \mathbf{h}_k^H \left(\sum_{l=k+1}^{K} P_l \mathbf{h}_l \mathbf{h}_l^H + \sigma^2 \mathbf{I} \right)^{-1} \mathbf{h}_k$$

- In general, difficult to solve and requires knowledge of all the channel realizations.

Asymptotic analysis based on random matrix theory

Result. (based on Girko's result) Based on the set of target rates $(\gamma_1, \dots, \gamma_K) = (2^{R_1} - 1, \dots, 2^{R_k} - 1)$, the power allocation statisfies the following equations:

- SINR at the output of the MMSE receiver:

$$\gamma^k = \frac{P_k}{N} \sum_{i=1}^{N} \frac{\mid g^k(i) \mid^2}{\sigma^2 + \frac{1}{N} \sum_{l=1, l \neq k}^{K} \frac{P_l |g^l(i)|^2}{1+\gamma^l}}$$

- SINR at the output of the MMSE-SIC receiver

$$\gamma^k = \frac{P_k}{N} \sum_{i=1}^{N} \frac{\mid g^k(i) \mid^2}{\sigma^2 + \frac{1}{N} \sum_{l=k+1}^{K} \frac{P_l |g^l(i)|^2}{1+\gamma^l}}$$

- The result is independent of the channel realization and based only on target rates.

Simple case: i.i.d. channel

- $g^k(i) = 1 \qquad \forall\ i, k$
- SINR of MMSE receiver:

$$\gamma_k = \frac{P_k}{N} \sum_{i=1}^{N} \left(\sigma^2 + \frac{1}{N} \sum_{l=1, l \neq k}^{K} \frac{P_l}{1 + \gamma_l} \right)^{-1}$$

- For general rate requirements:

$$P_k = \gamma_k \sigma^2 (1 - \frac{1}{N} \sum_{l=1, l \neq k}^{K} \frac{\gamma_l}{1 + \gamma_l})^{-1},$$

  - Rates satisfied if $K - \sum_{l=1}^{K} \frac{1}{1+\gamma^l} < N$
- Decentralized ressource allocation possible if only the statics of the rates are known.
  - Discrete set of available rates $\{r_1, \dots, r_m\}$ and $\{k_1, \dots, k_m\}$ users with each rate.
  - Large system approximation $K_i \approx K_i^* = pr(r = r_i)K$

MMSE-SIC Power allocation: i.i.d. channel

What is the decoding order of the users in order to minimize the total transmitted power of the users?

A. Suarez, L. Cottatellucci and M. Debbah, ”Optimal decoding order under target rate constraints”, submitted to SPAWC 2007

Result: Let us assume, without loss of generality that we have ordered the users according to increasing requested rates $\gamma^1 \leq \gamma^2 \leq ... \leq \gamma^K$. Then, for the MMSE-SIC receiver, the users should be decoded in precisely that order and the assigned power to each of them is given by

$$p_k = \gamma^k \sigma^2 \prod_{i=k+1}^{K} [1 + \frac{1}{N} \frac{\gamma^i}{1+\gamma^i}].$$

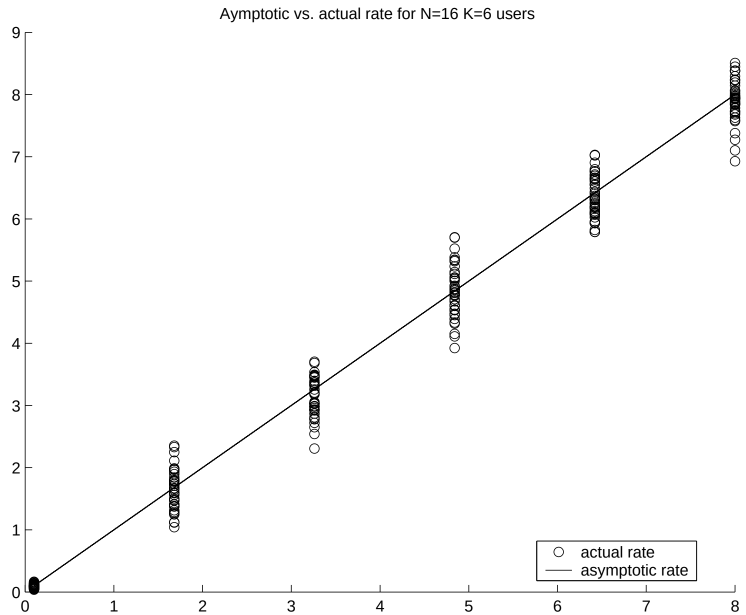
Moreover, by using Groupwise MMSE-SIC, decentralized power allocation can be performed.

General case

A. Suarez, L. Cottatellucci and M. Debbah, ”Optimal decoding order under target rate constraints”, submitted to SPAWC 2007

- Different user gains: (e.g. flat-fading CDMA) $|g^k(i)|^2 = |g^k|^2$
  - Same results with $\frac{\gamma_k}{|g^k|^2}$
  - Decoding order: $\frac{\gamma_1}{|g^1|^2} \leq \cdots \leq \frac{\gamma_K}{|g^K|^2}$
  - Power $P_k = \frac{\gamma_k}{|g^k|^2}\sigma^2 \prod_{l=k+1}^{K}[1 + \frac{1}{N}\frac{\gamma_l/|g^l|^2}{1+\gamma_l}]$
- General variance profile: (e.g. Frequency selective fading CDMA, MIMO) the expression is more complicated. However, at high SNR, only the ordering of $\frac{\gamma_k}{\sum_{i=1}^{N}|g^k(i)|^2}$ matters).

Asymptotic vs. Actual rates



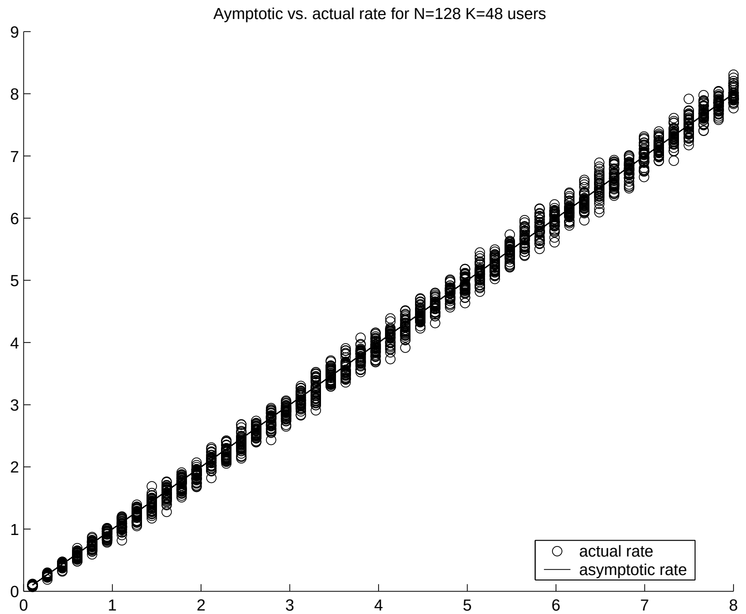
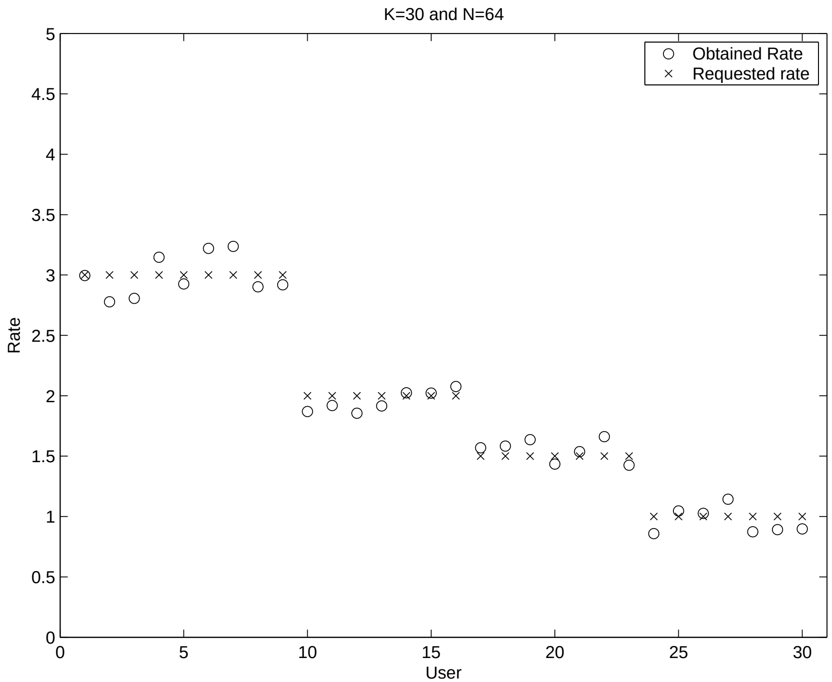


Figure 1: Comparison between the asymptotic and real rate for a system with N=16 and K=6

Figure 2: Comparison between the asymptotic and real rate for a system with N=128 and K=48

Decentralized power allocation



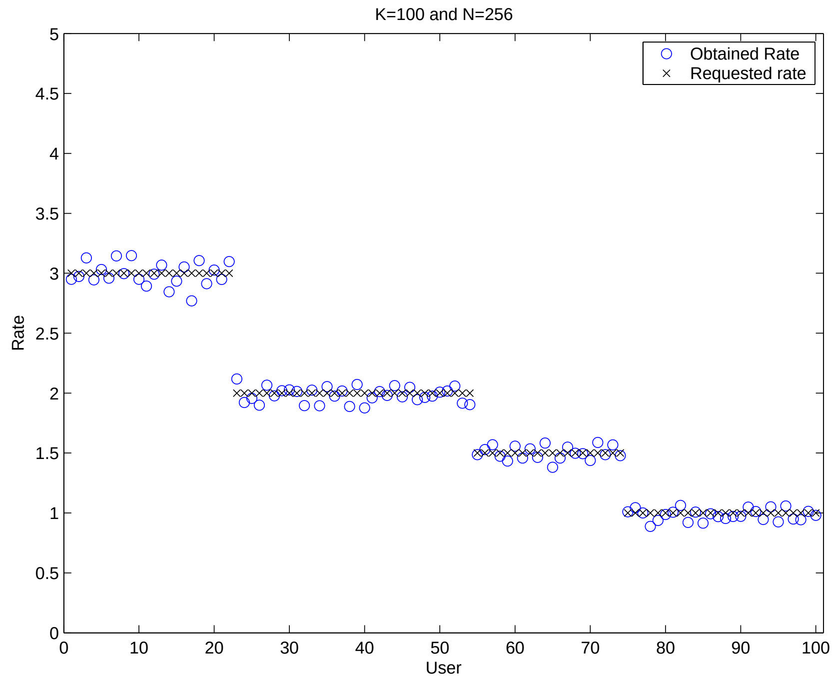


Figure 3: Comparison between the asymptotic and real rate for a system with N=16 and K=6

Figure 4: Comparison between the asymptotic and real rate for a system with N=128 and K=48

MMSE vs. MMSE-SIC required power

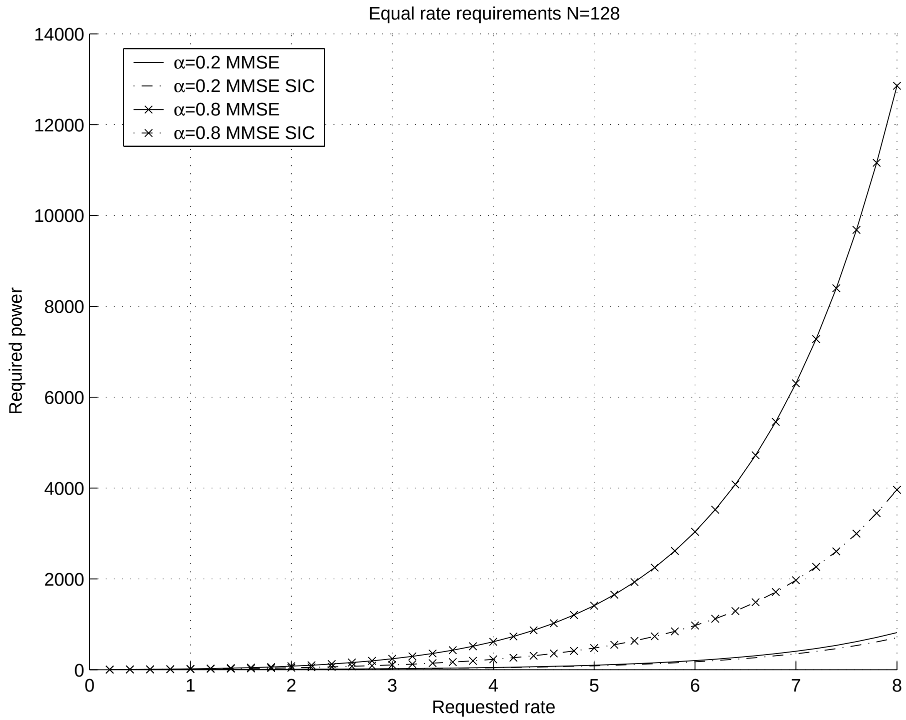


Figure 5: Needed powers for MMSE and MMSE-SIC for different values of $\alpha = N/K$ and fixed number of dimensions N=128

Power allocation with random matrix theory

Many papers are actually dealing with random matrix theory in the setting of game theory (one against many)

- F. Meshkati, H. Poor, S. Schwarz and N. Mandayam, ”A non cooperative power control game in delay-contrained multiple access networks,” Proceedings of the IEEE International Symposium on Information Theory (ISIT), Adelaide, Australia, September, 2005.
- N. Bonneau, M. Debbah and E. Altman, ”Wardrop equilibrium in CDMA networks”, submitted to the 3rd workshop on Resource Allocation in Wireless Networks 2007

Impact of synchronism

Uplink frequency selective channel

Received signal

$$\mathbf{y} = \mathbf{H}\mathbf{P}^{\frac{1}{2}}\mathbf{s} + \mathbf{n}$$

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1K} \\ h_{21} & h_{22} & \dots & h_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N1} & \dots & \dots & h_{NK} \end{bmatrix}, \quad \mathbf{P}^{\frac{1}{2}} = \begin{bmatrix} p_1^{\frac{1}{2}} & 0 & \dots & 0 \\ 0 & p_2^{\frac{1}{2}} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & p_K^{\frac{1}{2}} \end{bmatrix}$$

In particular:

$$\mathbf{H} = \mathbf{G} \odot \mathbf{W}$$

where $\mathbf{W}$ is an $N \times K$ matrix extracted from a random unitary matrix haar distributed.

Uplink CDMA: should we synchronize users?

N. Bonneau, M. Debbah, E. Altman and G. Caire, ”When to Synchronize in Uplink CDMA”, 2005 IEEE International Symposium on Information Theory, 4-9 September 2005, Adelaide, Australia.

Question: When is it useful to use orthogonal uplink signaling?

Answer: Intuitively,

- In the case of flat fading, orthogonality is preserved.
- In the case of frequency-selective fading, orthogonality is destroyed.

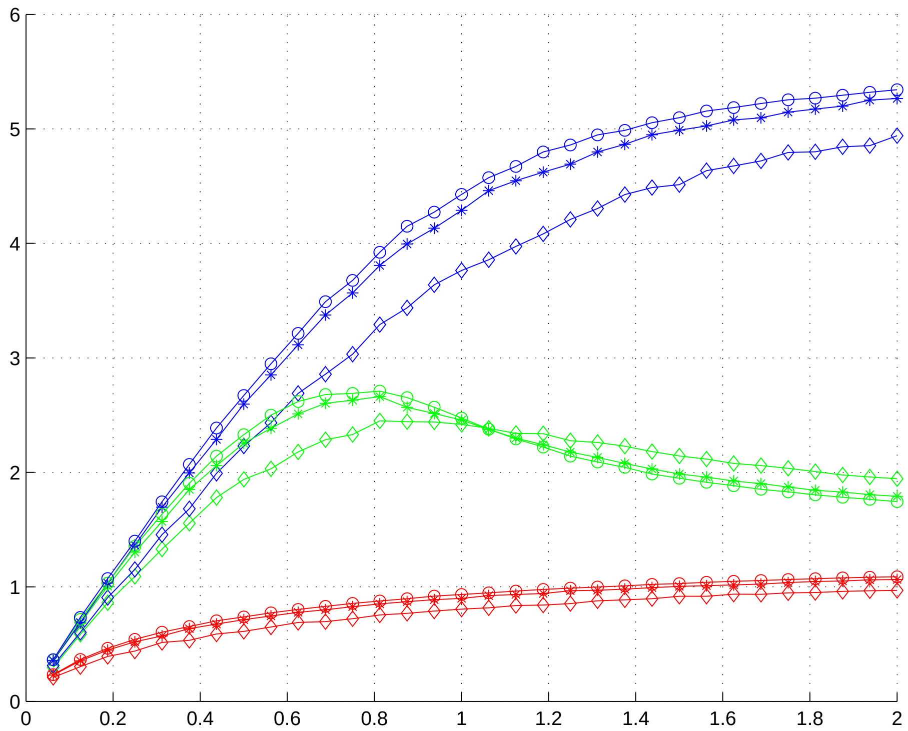
Uplink i.i.d CDMA: Frequency Selective fading model

MMSE receiver: For the high number of users finite case, let $h^k(i)$ be the frequency response of user k on frequency i, then the SINR β^k at the output of the MMSE receiver is given by:

$$\beta^k = \frac{P_k}{N}\sum_{i=1}^{N}\frac{\mid h^k(i) \mid^2}{\sigma^2 + \frac{1}{N}\sum_{l=1,l\neq k}^{K}\frac{P_l|h^l(i)|^2}{1+\beta^l}}$$

By the way, even for i.i.d codes, is it better to have frequency selective or flat fading channels?

Uplink CDMA: Flat fading versus frequency selective



Flat fading represented by diamond curve
Frequency selective with 5 paths represented by star curve
Frequency selective with 128 paths represented by circle curve.

Orthogonal Uplink CDMA: multipath channel

For user k, the model of the channel is given by

$$c_k(\tau) = \sum_{p=0}^{L-1} c_{pk}\phi(\tau - \tau_{pk})$$

where ϕ is the transmit pulse filter.

The Fourier transform of $c(\tau)$ after pulse matched filtering at the receiver is
$h_k(f) = \sum_{p=0}^{L-1} c_{pk}e^{-j2\pi f\tau_{kp}} \mid \Phi(f) \mid^2$ where $\Phi(f) = \begin{matrix} 1 & \text{if } -\frac{W}{2} \leq f \leq \frac{W}{2} \\ 0 & \text{otherwise} \end{matrix}$

Sampling at frequencies $f_1 = -\frac{W}{2}$, $f_2 = -\frac{W}{2} + \frac{1}{N}W, \ldots, f_N = -\frac{W}{2} + \frac{N-1}{N}W$, we obtain the coefficients $h_k(i)$:

$$h_k(i) = h_k(f_i) = \sum_{p=0}^{L-1} c_{pk}e^{-j2\pi\frac{i}{N}W\tau_{pk}}e^{j\pi W\tau_{pk}}$$

Orthogonal Uplink CDMA: SINR with Matched Filter

Proposition: When $N \to \infty$ and $\frac{K}{N} \to \alpha$, the SINR with Matched filter is:

$$\text{Orthogonal codes :} \mathrm{SINR}^{\text{orth}} = \frac{\frac{1}{W}\int_{-\frac{W}{2}}^{\frac{W}{2}} |h_1(f)|^2 df}{\sigma^2 + \alpha(\varrho - \xi_1)}$$

$$\text{i.i.d. codes :} \mathrm{SINR}^{\text{iid}} = \frac{\frac{1}{W}\int_{-\frac{W}{2}}^{\frac{W}{2}} |h_1(f)|^2 df}{\sigma^2 + \alpha\varrho}$$

where

$$\varrho = \mathbb{E}_{h_k}\left[\mid h_k(f) \mid^2\right] \text{ and } \xi_1 = \frac{\mathbb{E}_{h_k}\left[\mid \frac{1}{W}\int_{-\frac{W}{2}}^{\frac{W}{2}} h_1(f) h_k^*(f) df \mid^2\right]}{\frac{1}{W}\int_{-\frac{W}{2}}^{\frac{W}{2}} \mid h_1(f) \mid^2 df}$$

The asymptotic SINR depends only on a few meaningful parameters: α, σ^2, and the distribution of the elements of $\mathbf{H}$!

$\mathrm{SINR}^{\text{iid}}$ is always inferior to $\mathrm{SINR}^{\text{orth}}$.

Haar Matrices

Useful Properties:

$$\mathbb{E}[\mid \theta_{ij} \mid^2] = \frac{1}{N}$$

$$\mathbb{E}[\mid \theta_{ij} \mid^4] = \frac{2}{N(N+1)}$$

$$\mathbb{E}[\mid \theta_{ij} \mid^2 \mid \theta_{kj} \mid^2] = \mathbb{E}[\mid \theta_{ij} \mid^2 \mid \theta_{il} \mid^2] = \frac{1}{N(N+1)}$$

$$\mathbb{E}[\mid \theta_{ij} \mid^2 \mid \theta_{kl} \mid^2] = \frac{2}{N^2-1}$$

$$\mathbb{E}[\theta_{ij}\theta_{kl}\theta_{il}^*\theta_{kj}^*] = -\frac{1}{N(N^2-1)}$$

Orthogonal Uplink CDMA: Simplifying assumptions

Channel model: We will suppose that for the channel model

1. The fading coefficients are i.i.d. Gaussian with

$$\mathbb{E}\left[c_{pk}\right] = 0 \text{ and } \mathbb{E}\left[\mid c_{pk} \mid^2\right] = \frac{\varrho}{L}$$

2. The delays are uniformly distributed according to the bandwidth

$$\tau_{pk} = \frac{p}{W}$$

Orthogonal Uplink CDMA: Simplification of the asymptotic expressions of the SINR

The SINR for the Matched filter in this case becomes

$$\begin{aligned} \mathrm{SINR}^{\mathrm{orth}} &= \frac{\sum_{p=0}^{L-1} |c_{p1}|^2}{\sigma^2 + \alpha\varrho\left(1 - \frac{1}{L}\right)} \\ \mathrm{SINR}^{\mathrm{iid}} &= \frac{\sum_{p=0}^{L-1} |c_{p1}|^2}{\sigma^2 + \alpha\varrho} \end{aligned}$$

As a consequence:

$$\frac{\mathrm{SINR}^{\mathrm{orth}}}{\mathrm{SINR}^{\mathrm{iid}}} = \frac{\sigma^2 + \alpha\varrho}{\sigma^2 + \alpha\varrho\left(1 - \frac{1}{L}\right)}$$

When $\sigma^2 \longrightarrow 0$,

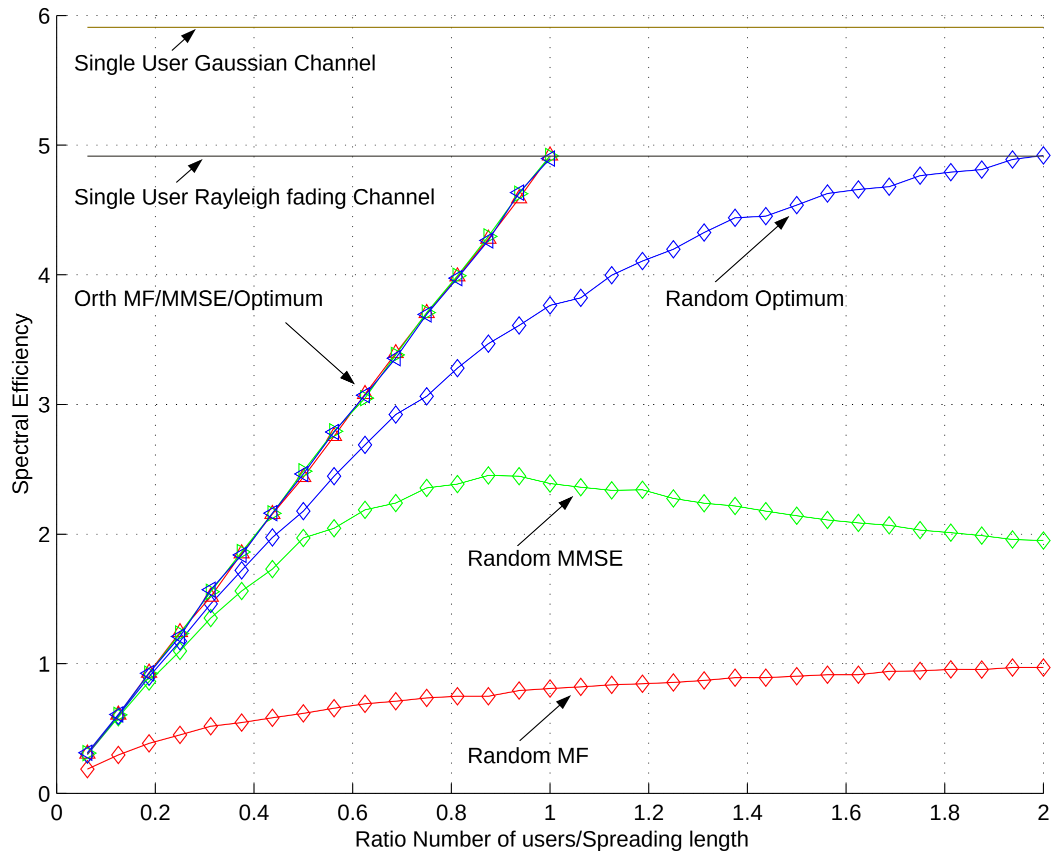
$$\frac{\mathrm{SINR}^{\mathrm{orth}}}{\mathrm{SINR}^{\mathrm{iid}}} \longrightarrow \frac{L}{L-1}$$

In a two-path channel, gain of 3 dB; in a 5-path channel, gain of less than 1 dB.

Orthogonal Uplink CDMA: SINR with MMSE receiver

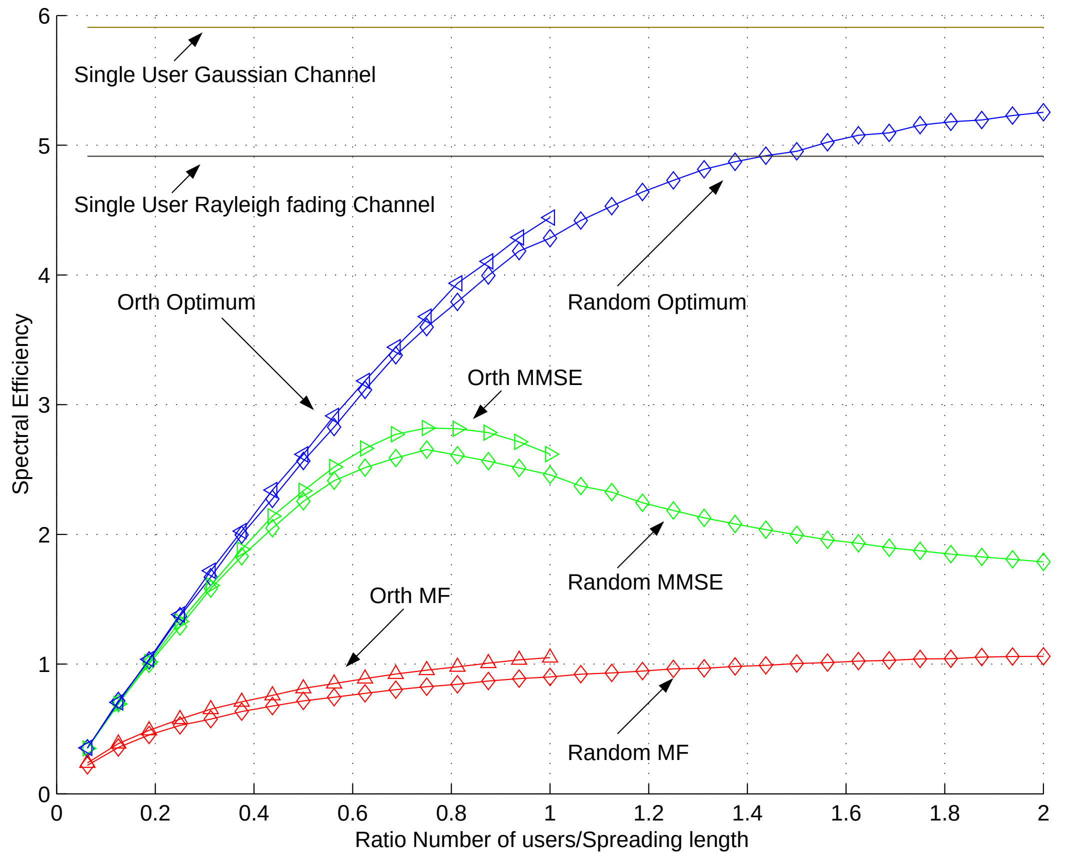
Result known but the proof is still under study.....

Orthogonal Uplink CDMA: Simulations, $\rho = 1$, $\mathrm{SNR} = 10\mathrm{dB}$, $L = 1$



In a one-path channel, orthogonality gain is maximal.

Orthogonal Uplink CDMA: Simulations, $\rho = 1$, $\mathrm{SNR} = 10\mathrm{dB}$, $L = 5$



As L increases, orthogonality gain decreases for any receiver.

Is it useful to synchronize in wireless systems?

Spectral efficiency always increases with the use of orthogonal codes.

The orthogonality gain depends mainly on the number of paths and the load of the system.

As a consequence, adaptive synchronization protocols for future multiple access CDMA schemes could be used to increase the rate.

To fully assess the gain, studies need to be conducted to determine the amount of overhead signaling for a given number of users and bandwidth.

Other type of models of asychronism (chip asynchronicity) are studied in L. Cottatellucci, M. Debbah and R. Müller, ”On the capacity of asynchronous CDMA systems”, submitted to IEEE International Symposium on Information Theory 2007.

Last Slide

THANK YOU!