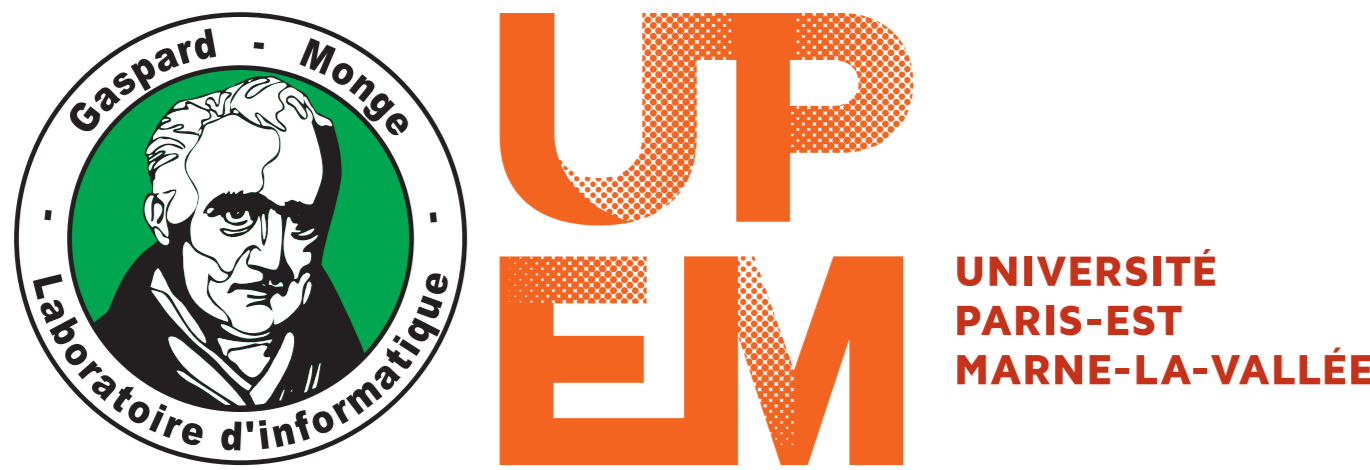


EUCLID IN A TAXICAB: SPARSE BLIND DECONVOLUTION WITH SMOOTHED ℓ_1/ℓ_2 REGULARIZATION



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OPTIMIZATION PROBLEM

Observation measurements $y = (y_n)_{1 \leq n \leq N} \in \mathbb{R}^N$:

$$y = \bar{h} * \bar{x} + w$$

- $\bar{x} \in \mathbb{R}^N$ \rightsquigarrow original **unknown** sparse signal
- $\bar{h} \in \mathbb{R}^S$ \rightsquigarrow original **unknown** blur kernel
- $w \in \mathbb{R}^N$ \rightsquigarrow realization of an additive noise

OBJECTIVE

Find an estimation $(\hat{x}, \hat{h}) \in \mathbb{R}^N \times \mathbb{R}^S$ of (\bar{x}, \bar{h}) from y .

CONTRIBUTIONS

- ✓ New parametrized Smoothed One-Over-Two (SOOT) penalty term
 \rightsquigarrow Efficient for sparse blind deconvolution problems.
- ✓ Accelerated alternating minimization strategy
 \rightsquigarrow Convergence ensured for nonconvex problems.

MINIMIZATION PROBLEM

$$\text{Find } (\hat{x}, \hat{h}) \in \underset{(x,h) \in \mathbb{R}^N \times \mathbb{R}^S}{\text{Argmin}} \underbrace{\rho(x,h)}_{\text{Data fidelity}} + \underbrace{\varphi(x) + g(x,h)}_{\text{Regularization}} \quad (1)$$

LEAST-SQUARES OBJECTIVE FUNCTION

$$(\forall (x,h) \in \mathbb{R}^N \times \mathbb{R}^S) \quad \rho(x,h) = \frac{1}{2} \|h * x - y\|^2.$$

\rightsquigarrow **Smooth** function.

SMOOTHED ONE-OVER-TWO NORM RATIO PENALTY FUNCTION

$$(\forall x \in \mathbb{R}^N) \quad \varphi(x) = \lambda \log \left(\frac{\ell_{1,\alpha}(x) + \beta}{\ell_{2,\eta}(x)} \right),$$

with $(\lambda, \alpha, \beta, \eta) \in]0, +\infty[^4$ and

$$\ell_{1,\alpha}(x) = \sum_{n=1}^N \left(\sqrt{x_n^2 + \alpha^2} - \alpha \right) \rightsquigarrow \text{smooth approximation of } \ell_1,$$

$$\ell_{2,\eta}(x) = \sqrt{\sum_{n=1}^N x_n^2 + \eta^2} \rightsquigarrow \text{smooth approximation of } \ell_2.$$

\rightsquigarrow **Smooth nonconvex** function.

ADDITIONAL A PRIORI INFORMATION

$g(x,h) = g_1(x) + g_2(h)$, where $g_1: \mathbb{R}^N \rightarrow]-\infty, +\infty]$ and $g_2: \mathbb{R}^S \rightarrow]-\infty, +\infty]$ are proper, lsc, semi-algebraic, convex, and continuous on their domain.

\rightsquigarrow **Non necessarily smooth convex** function.

OPTIMIZATION TOOLS

PROXIMITY OPERATOR

Let $A \in \mathbb{R}^{M \times M}$ be a Symmetric Positive Definite (SPD) matrix, $\gamma > 0$, and $\psi: \mathbb{R}^M \rightarrow]-\infty, +\infty]$ be a proper, lsc, convex function.
($\forall \tilde{u} \in \mathbb{R}^M$) $\text{prox}_{\gamma^{-1}A, \psi}(\tilde{u}) = \underset{u \in \mathbb{R}^M}{\text{argmin}} \psi(u) + \frac{1}{2\gamma} (u - \tilde{u})^\top A (u - \tilde{u})$.

MAJORIZE-MINIMIZE (MM) PRINCIPLE

Let $\psi: \mathbb{R}^M \rightarrow]-\infty, +\infty]$ be a differentiable function, and $\tilde{u} \in \mathbb{R}^M$. An SPD matrix $A(\tilde{u}) \in \mathbb{R}^{M \times M}$ satisfies the **majoration condition** w.r.t. ψ if

$$(\forall u \in \mathbb{R}^M) \quad \psi(\tilde{u}) \leq \psi(u) + (\tilde{u} - u)^\top \nabla \psi(\tilde{u}) + \frac{1}{2} (\tilde{u} - u)^\top A(\tilde{u}) (\tilde{u} - u).$$

SOOT ALGORITHM

Let $x^0 \in \text{dom } g_1$ and $h^0 \in \text{dom } g_2$.

For $k = 0, 1, \dots$

$$x^{k,0} = x^k, \quad h^{k,0} = h^k,$$

For $j = 0, \dots, J_k - 1$

Let $\gamma_x^{k,j} > 0$,

$$\tilde{x}^{k,j} = x^{k,j} - \gamma_x^{k,j} A_1(x^{k,j}, h^k)^{-1} (\nabla_x \rho(x^{k,j}, h^k) + \nabla \varphi(x^{k,j})),$$

$$x^{k,j+1} = \text{prox}_{(\gamma_x^{k,j})^{-1} A_1(x^{k,j}, h^k), g_1}(\tilde{x}^{k,j}),$$

$$x^{k+1} = x^{k, J_k}.$$

For $i = 0, \dots, I_k - 1$

Let $\gamma_h^{k,i} > 0$,

$$\tilde{h}^{k,i} = h^{k,i} - \gamma_h^{k,i} A_2(x^{k+1}, h^{k,i})^{-1} \nabla_h \rho(x^{k+1}, h^{k,i}),$$

$$h^{k,i+1} = \text{prox}_{(\gamma_h^{k,i})^{-1} A_2(x^{k+1}, h^{k,i}), g_2}(\tilde{h}^{k,i}),$$

$$h^{k+1} = h^{k, I_k}.$$

CONVERGENCE RESULT

Assume that $(\forall k \in \mathbb{N})$

- $(\forall j \in \{1, \dots, J_k\})$ $A_1(x^{k,j}, h^k)$ is an SPD matrix satisfying the majoration condition w.r.t. $\rho(\cdot, h^k) + \varphi$.
- $(\forall i \in \{1, \dots, I_k\})$ $A_2(x^{k+1}, h^{k,i})$ is an SPD matrix satisfying the majoration condition w.r.t. $\rho(x^{k+1}, \cdot)$.

Then, the sequence $(x^k, h^k)_{k \in \mathbb{N}}$ converges to a critical point (\hat{x}, \hat{h}) of (1).

APPLICATION TO SEISMIC DATA DECONVOLUTION

OBSERVATION MODEL

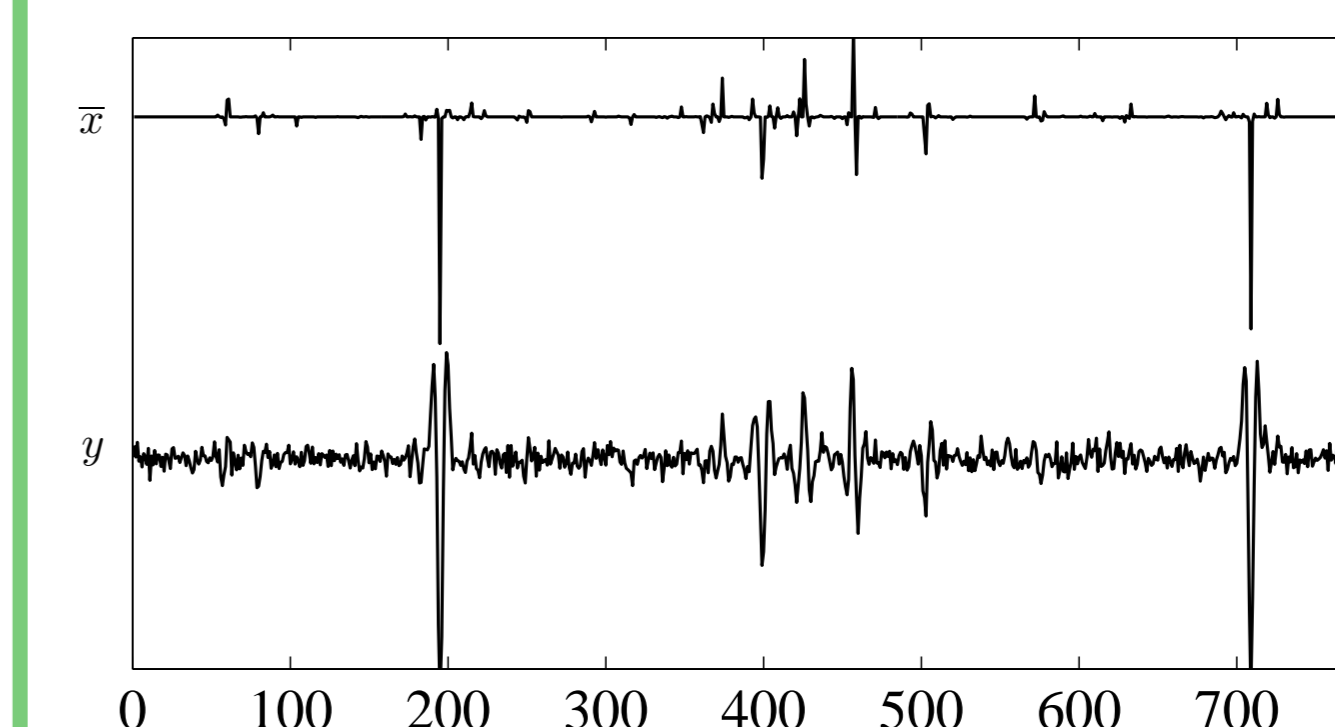
- ★ Sparse seismic signal \bar{x} of length $N = 784$
- ★ Band-pass ‘‘Ricker’’ seismic wavelet \bar{h} of length $S = 41$
 \rightsquigarrow spectrum concentrated between 10 and 40 Hz
- ★ Additive noise w \rightsquigarrow realization of $W \sim \mathcal{N}(0, \sigma^2 I_N)$

IMPLEMENTATION DETAILS

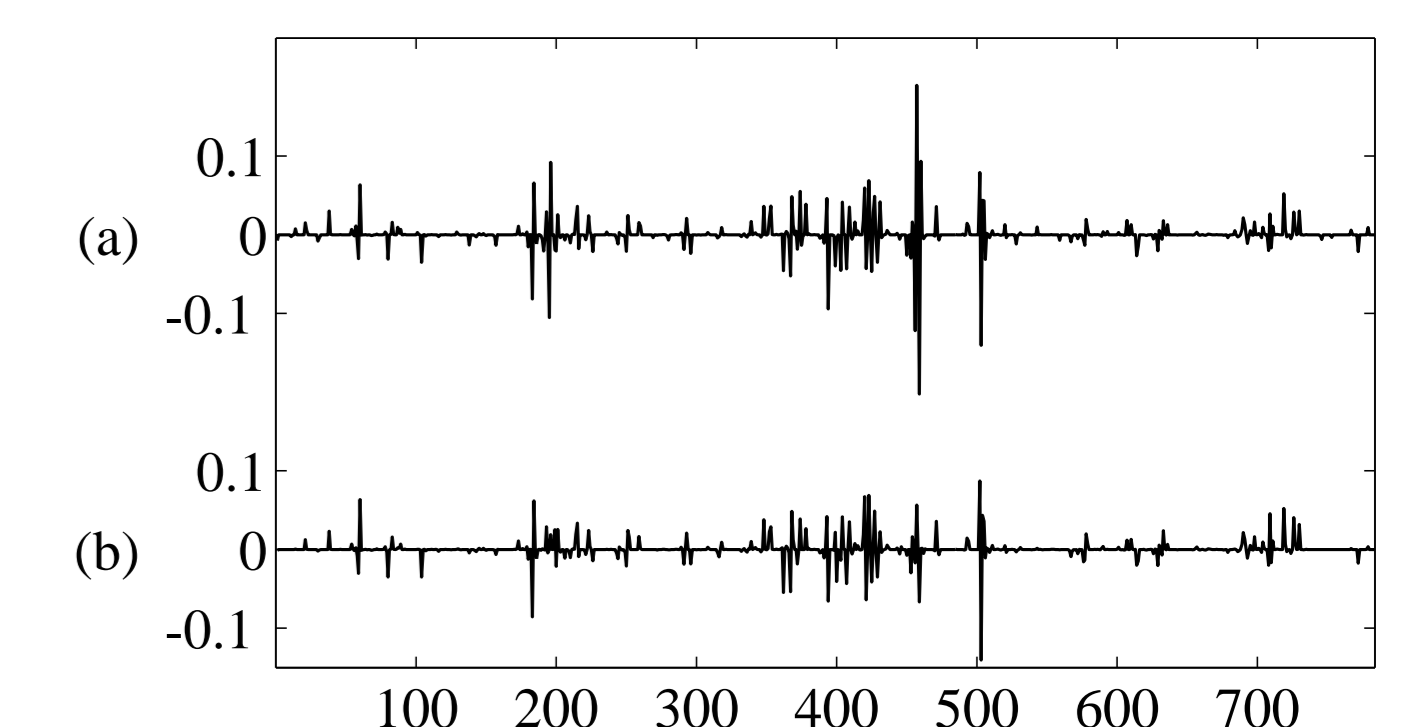
- ★ $g_1 = \iota_{[x_{\min}, x_{\max}]^N}$ with $(x_{\min}, x_{\max}) \in \mathbb{R}^2$
- ★ $g_2 = \iota_{\mathcal{C}}$ where $\mathcal{C} = \{h \in [h_{\min}, h_{\max}]^S \mid \|h\| \leq \delta\}$,
with $\delta > 0$ and $(h_{\min}, h_{\max}) \in \mathbb{R}^2$
- ★ $I_k \equiv 1$ and $J_k \equiv J$
- ★ Stopping criterion: $\|x^k - x^{k-1}\| \leq \sqrt{N} \times 10^{-6}$

Noise level (σ)		0.01	0.02	0.03	
Observation error ($\times 10^{-2}$)	ℓ_2	7.14	7.35	7.68	
	ℓ_1	2.85	3.44	4.09	
Signal error ($\times 10^{-2}$)	[Krishnan et al., 2011]	ℓ_2	1.23	1.66	1.84
		ℓ_1	0.38	0.47	0.53
	SOOT	ℓ_2	1.09	1.63	1.83
		ℓ_1	0.34	0.43	0.48
Kernel error ($\times 10^{-2}$)	[Krishnan et al., 2011]	ℓ_2	1.88	2.51	3.21
		ℓ_1	1.44	1.96	2.53
	SOOT	ℓ_2	1.62	2.26	2.93
		ℓ_1	1.22	1.77	2.31
Time (s.)	[Krishnan et al., 2011]	106	61	56	
	SOOT	56	22	18	

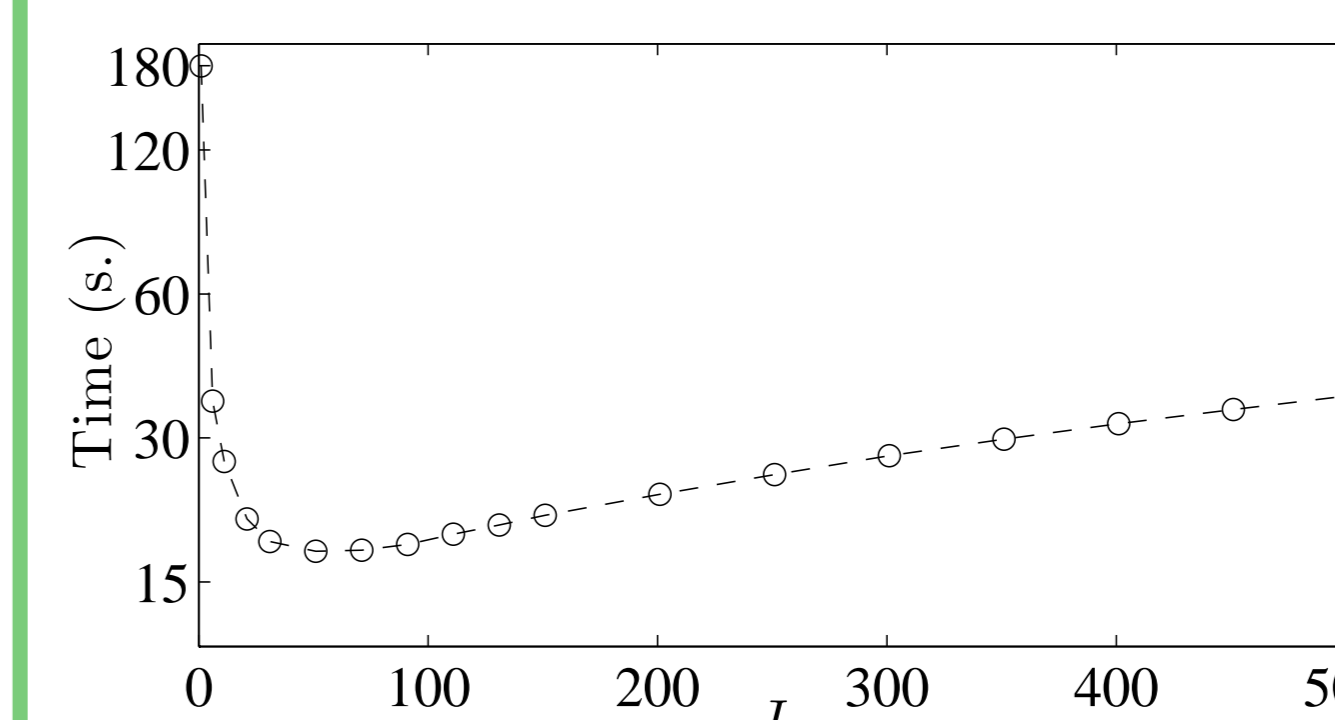
Comparison between
[Krishnan et al., 2011] and
SOOT for \bar{x} and \bar{h} estimates
($J = 71$, averaged over 200
noise realizations)



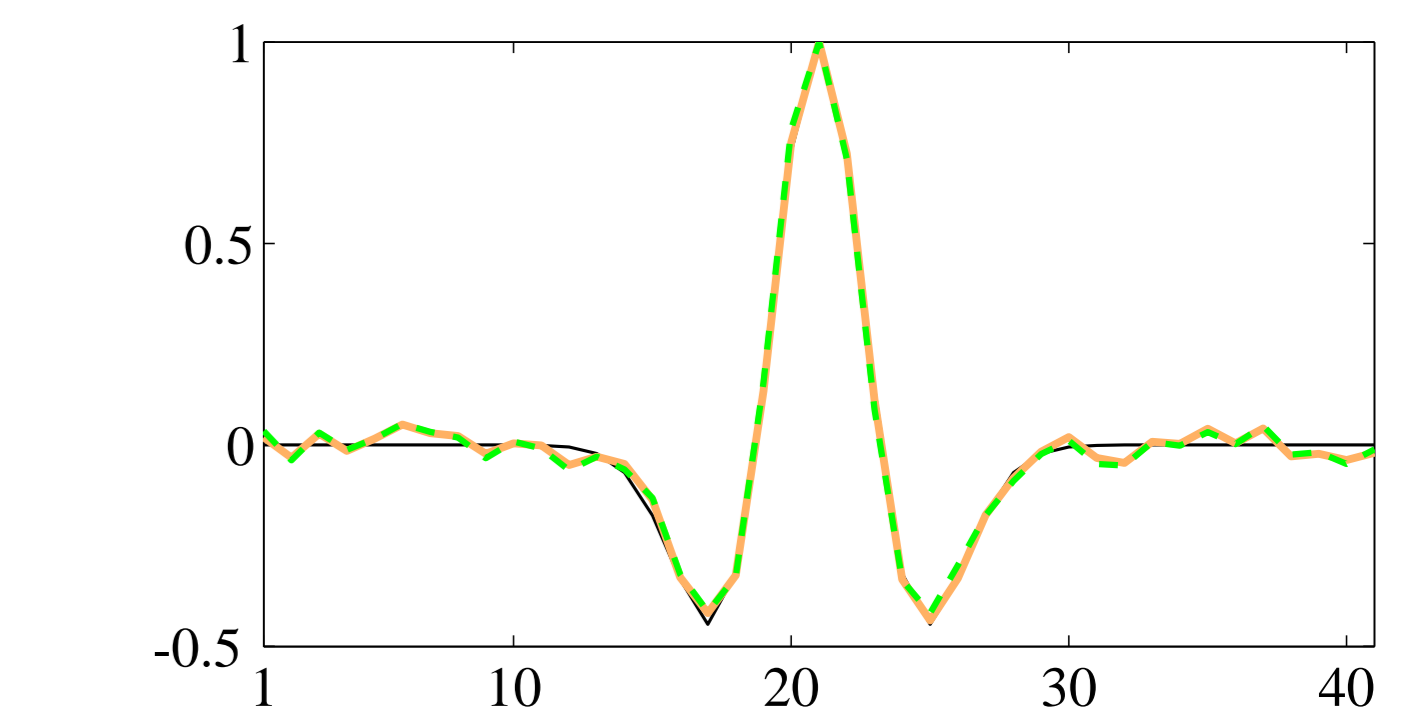
Unknown seismic signal \bar{x} (top)
and blurred/noisy observation y (bottom)



Signal estimation error $\bar{x} - \hat{x}$ with estimates \hat{x}
given by (a) [Krishnan et al., 2011] and (b) SOOT



Reconstruction time for different numbers of
inner-loops J (averaged over 30 noise realizations)



Original blur \bar{h} (continuous thin), estimated blur
 \hat{h} with [Krishnan et al., 2011] (dashed thick) and
SOOT (continuous thick)

TOOLBOX: <http://lc.cx/soot>