Euclid in a Taxicab: Sparse Blind Deconvolution with Smoothed ℓ_1/ℓ_2 Regularization

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Optimization problem

Observation measurements $y = (y_n)_{1\le n\le N} \in \mathbb{R}^N$:

$$y = \overline{h} * \overline{x} + w$$

- $\overline{x} \in \mathbb{R}^N$ ⇝ original **unknown** sparse signal
- $\overline{h} \in \mathbb{R}^S$ ⇝ original **unknown** blur kernel
- $w \in \mathbb{R}^N$ ⇝ realization of an additive noise

Objective

Find an estimation $(\widehat{x}, \widehat{h}) \in \mathbb{R}^N \times \mathbb{R}^S$ of $(\overline{x}, \overline{h})$ from y.

Contributions

✓ New parametrized Smoothed One-Over-Two (SOOT) penalty term
⇝ Efficient for sparse blind deconvolution problems.

✓ Accelerated alternating minimization strategy
⇝ Convergence ensured for nonconvex problems.

Minimization problem

$$\text{Find } (\widehat{x}, \widehat{h}) \in \underset{(x,h)\in\mathbb{R}^N\times\mathbb{R}^S}{\text{Argmin}} \underbrace{\rho(x,h)}_{\text{Data fidelity}} + \underbrace{\varphi(x) + g(x,h)}_{\text{Regularization}} \qquad (1)$$

Least-squares objective function

$$(\forall (x,h) \in \mathbb{R}^N \times \mathbb{R}^S) \quad \rho(x,h) = \frac{1}{2}\|h * x - y\|^2.$$

⇝ Smooth function.

Smoothed One-Over-Two norm ratio penalty function

$$(\forall x \in \mathbb{R}^N) \quad \varphi(x) = \lambda \log\left(\frac{\ell_{1,\alpha}(x) + \beta}{\ell_{2,\eta}(x)}\right),$$

with $(\lambda, \alpha, \beta, \eta) \in]0, +\infty[^4$ and

$\ell_{1,\alpha}(x) = \sum_{n=1}^N \left(\sqrt{x_n^2 + \alpha^2} - \alpha\right)$ ⇝ smooth approximation of ℓ_1,

$\ell_{2,\eta}(x) = \sqrt{\sum_{n=1}^N x_n^2 + \eta^2}$ ⇝ smooth approximation of ℓ_2.

⇝ Smooth nonconvex function.

Additional a priori information

$g(x,h) = g_1(x) + g_2(h)$, where $g_1 \colon \mathbb{R}^N \to]-\infty, +\infty]$ and $g_2 \colon \mathbb{R}^S \to]-\infty, +\infty]$ are proper, lsc, semi-algebraic, convex, and continuous on their domain.

⇝ Non necessarily smooth convex function.

Optimization tools

Proximity operator

Let $A \in \mathbb{R}^{M\times M}$ be a Symmetric Positive Definite (SPD) matrix, $\gamma > 0$, and $\psi \colon \mathbb{R}^M \to]-\infty, +\infty]$ be a proper, lsc, convex function.

$$(\forall \widetilde{u} \in \mathbb{R}^M) \quad \text{prox}_{\gamma^{-1}A,\psi}(\widetilde{u}) = \underset{u\in\mathbb{R}^M}{\text{argmin}}\ \psi(u) + \frac{1}{2\gamma}(u - \widetilde{u})^\top A(u - \widetilde{u}).$$

Majorize-Minimize (MM) principle

Let $\psi \colon \mathbb{R}^M \to]-\infty, +\infty[$ be a differentiable function, and $\widetilde{u} \in \mathbb{R}^M$. An SPD matrix $A(\widetilde{u}) \in \mathbb{R}^{M\times M}$ satisfies the majoration condition w.r.t. ψ if

$$(\forall u \in \mathbb{R}^M) \quad \psi(\widetilde{u}) \le \psi(u) + (\widetilde{u} - u)^\top \nabla\psi(\widetilde{u}) + \frac{1}{2}(\widetilde{u} - u)^\top A(\widetilde{u})(\widetilde{u} - u).$$

SOOT algorithm

Let $x^0 \in \text{dom}\, g_1$ and $h^0 \in \text{dom}\, g_2$.
For $k = 0, 1, \ldots$
- $x^{k,0} = x^k,\ h^{k,0} = h^k,$
- For $j = 0, \ldots, J_k - 1$
  - Let $\gamma_x^{k,j} > 0,$
  - $\widetilde{x}^{k,j} = x^{k,j} - \gamma_x^{k,j} A_1(x^{k,j}, h^k)^{-1}\left(\nabla_x \rho(x^{k,j}, h^k) + \nabla\varphi(x^{k,j})\right),$
  - $x^{k,j+1} = \text{prox}_{(\gamma_x^{k,j})^{-1}A_1(x^{k,j},h^k),g_1}\left(\widetilde{x}^{k,j}\right),$
- $x^{k+1} = x^{k,J_k}.$
- For $i = 0, \ldots, I_k - 1$
  - Let $\gamma_h^{k,i} > 0,$
  - $\widetilde{h}^{k,i} = h^{k,i} - \gamma_h^{k,i} A_2(x^{k+1}, h^{k,i})^{-1}\nabla_h \rho(x^{k+1}, h^{k,i}),$
  - $h^{k,i+1} = \text{prox}_{(\gamma_h^{k,i})^{-1}A_2(x^{k+1},h^{k,i}),g_2}\left(\widetilde{h}^{k,i}\right),$
- $h^{k+1} = h^{k,I_k}.$

Convergence result

Assume that $(\forall k \in \mathbb{N})$

- $(\forall j \in \{1, \ldots, J_k\})$ $A_1(x^{k,j}, h^k)$ is an SPD matrix satisfying the majoration condition w.r.t. $\rho(\cdot, h^k) + \varphi$.
- $(\forall i \in \{1, \ldots, I_k\})$ $A_2(x^{k+1}, h^{k,i})$ is an SPD matrix satisfying the majoration condition w.r.t. $\rho(x^{k+1}, \cdot)$.

Then, the sequence $(x^k, h^k)_{k\in\mathbb{N}}$ converges to a critical point $(\widehat{x}, \widehat{h})$ of (1).

Application to seismic data deconvolution

Observation model

★ Sparse seismic signal $\overline{x}$ of length $N = 784$

★ Band-pass "Ricker" seismic wavelet $\overline{h}$ of length $S = 41$
⇝ spectrum concentrated between 10 and 40 Hz

★ Additive noise w ⇝ realization of W $\sim \mathcal{N}(0, \sigma^2 \text{I}_N)$

Implementation details

★ $g_1 = \iota_{[x_{\min}, x_{\max}]^N}$ with $(x_{\min}, x_{\max}) \in \mathbb{R}^2$

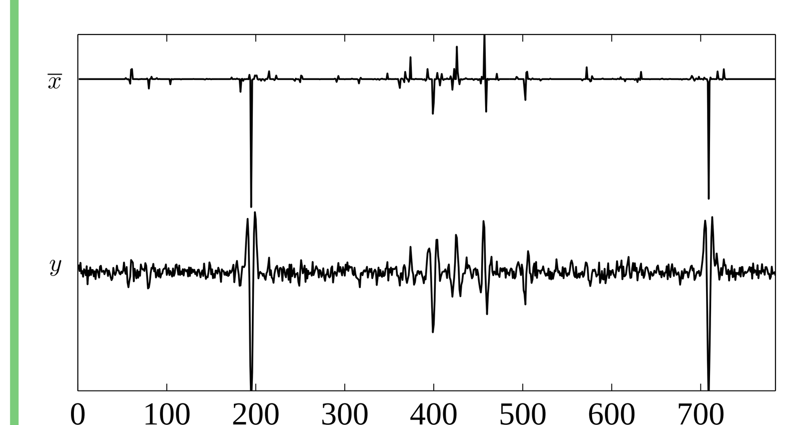
★ $g_2 = \iota_{\mathcal{C}}$ where $\mathcal{C} = \{h \in [h_{\min}, h_{\max}]^S \mid \|h\| \le \delta\}$, with $\delta > 0$ and $(h_{\min}, h_{\max}) \in \mathbb{R}^2$

★ $I_k \equiv 1$ and $J_k \equiv J$

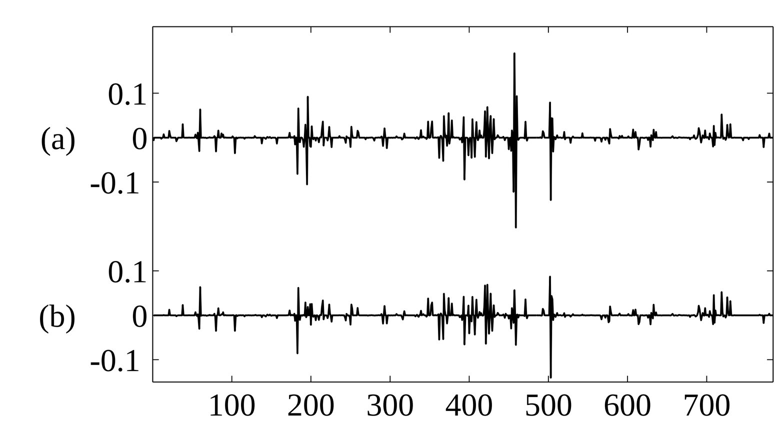
★ Stopping criterion: $\|x^k - x^{k-1}\| \le \sqrt{N} \times 10^{-6}$

Noise level (σ)			0.01	0.02	0.03
Observation error ($\times 10^{-2}$)		ℓ_2	7.14	7.35	7.68
		ℓ_1	2.85	3.44	4.09
Signal error ($\times 10^{-2}$)	[Krishnan *et al.*, 2011]	ℓ_2	1.23	1.66	1.84
		ℓ_1	0.38	0.47	0.53
	SOOT	ℓ_2	1.09	1.63	1.83
		ℓ_1	0.34	0.43	0.48
Kernel error ($\times 10^{-2}$)	[Krishnan *et al.*, 2011]	ℓ_2	1.88	2.51	3.21
		ℓ_1	1.44	1.96	2.53
	SOOT	ℓ_2	1.62	2.26	2.93
		ℓ_1	1.22	1.77	2.31
Time (s.)	[Krishnan *et al.*, 2011]		106	61	56
	SOOT		56	22	18

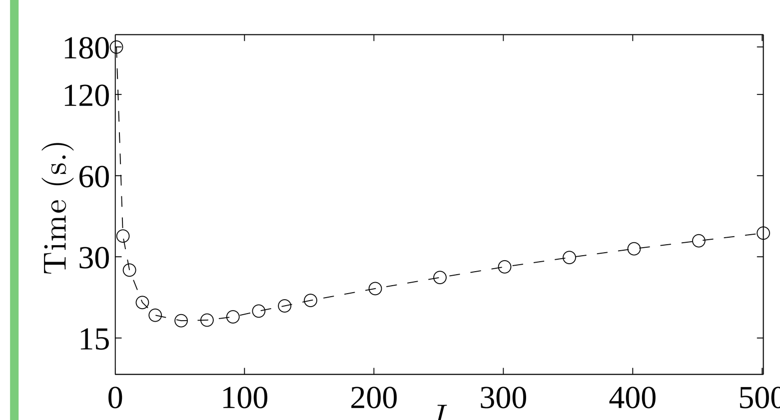
Comparison between [Krishnan *et al.*, 2011] and SOOT for $\overline{x}$ and $\overline{h}$ estimates ($J = 71$, averaged over 200 noise realizations)



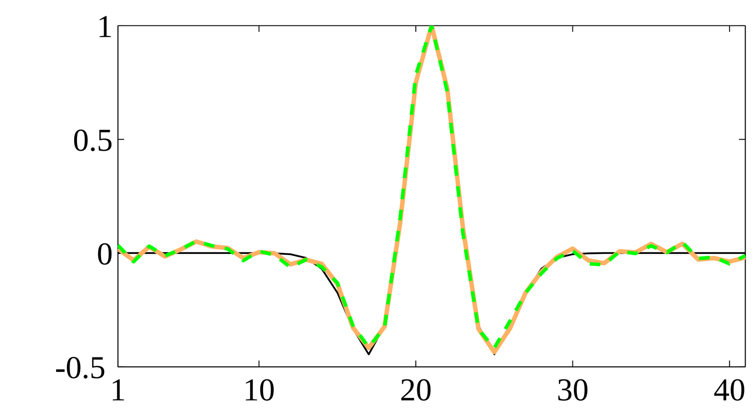
Unknown seismic signal $\overline{x}$ (top) and blurred/noisy observation y (bottom)



Signal estimation error $\overline{x} - \widehat{x}$ with estimates $\widehat{x}$ given by (a) [Krishnan *et al.*, 2011] and (b) SOOT



Reconstruction time for different numbers of inner-loops J (averaged over 30 noise realizations)



Original blur $\overline{h}$ (continuous thin), estimated blur $\widehat{h}$ with [Krishnan *et al.*, 2011] (dashed thick) and SOOT (continuous thick)

Toolbox: http://lc.cx/soot