

*Nonlinear reaction diffusion  
equations connected with  
stochastic control problems*

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*(J. Optimization Theory and Appl. 137, (2008), 497-505.)*



# History and motivation

## Physically relevant nonlinear PDE

- Burgers' eq. (B)

$$v_t(x, t) = v_{x,x}(x, t) - 2v(x, t)v_x(x, t).$$

- Boltzmann type eq. (RW),

(Th. W. Ruijgrok and T. T. Wu *Physica* **113 A**, (1982), 401-16.)

$$(\partial_t + \partial_x) f_+(x, t) = +f_+(x, t)f_-(x, t) - \alpha f_+(x, t) + \beta f_-(x, t),$$

$$(\partial_t - \partial_x) f_-(x, t) = -f_+(x, t)f_-(x, t) - \beta f_-(x, t) + \alpha f_+(x, t).$$

# History and motivation

- These PDEs are explicitly soluble (linearizable).

- Hopf-Cole transform *Comm. Pure and Appl. Math.*, **3**, (1950), 201-30.
- Th. W. Ruijgrok and T.T Wu *Physica* **113 A**, (1982), 401-16.

(can be linearized to Heat eq. for B and to Telegraphist eq. for RW)

- RW generalizes B.

- L. Streit and M.-O H. *Europhys.lett.*, **12**, (1990), 193-97.
- E. Gabetta and B. Perthame *Math. Meth. Appl. Sci.*, **24**, (2001), 949-67.

# Objectives of this lecture

- Exhibit connections between B and RW and Stochastic Optimal Control problems.
- Construct a class of soluble nonlinear reaction diffusion (RD) eq.:

$$(\partial_t - \nu \partial_x) f_- = \frac{\sigma^2}{2} (\partial_{x,x}^2 f_-) - K_\sigma(f_+, f_-, \partial_x f_+, \partial_x f_-),$$

$$(\partial_t + \nu \partial_x) f_+ = \frac{\sigma^2}{2} (\partial_{x,x}^2 f_+) + K_\sigma(f_+, f_-, \partial_x f_+, \partial_x f_-).$$



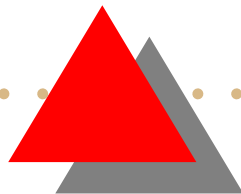
# *Logarithmic transform, (L-T).*

Let  $\phi(t, x) \geq 0$  a solution to the (linear) eq.:

$$\mathbf{A} \phi(t, x) = 0, \quad (*)$$

A backward op. of a Markov process  $X_t$ .

(L - T), defined as  $V = -\ln(\phi)$ , transforms (\*) into a nonlinear eq. for  $V$ :





# *Logarithmic transform, (L-T).*

Nonlinear eq. for  $V$  can be viewed as the dynamic programming (DP) eq. :

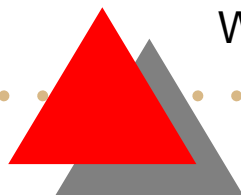
$$0 = \min_{v \in \mathcal{C}} [\mathbf{A}^v V + L].$$

(A generator of a Markov process,  $\mathcal{C}$  set of admissible controls)



- Stochastic control problem with cost function  $L$ .

W. H Fleming *Appl. Math. Optim.* **4**, (1978), 329-46.



# Example of the L-T

Take e.g.,

$$\mathbf{A}\phi \equiv \frac{\partial}{\partial t}\phi + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x,x}\phi + b \frac{\partial}{\partial x}\phi,$$

with underlying Markov diffusion in  $\mathbb{R}$ :

$$dX_s = b(s, X_s)ds + \sigma(s, X_s)dB_s, \quad s \in [0, t_1].$$

Let  $\phi(t, x) \geq 0$  with  $\mathbf{A}\phi(t, x) = 0$   
and terminal data:  $\phi(t_1, x) = \phi_1(x)$ .

# Example of the L-T

$V = -\ln(\phi)$  solves:

$$V_t + \frac{\sigma^2}{2} V_{x,x} = H(t, x, V_x), \quad (*)$$

$$H(t, x, p) = -b(t, x)p + \frac{\sigma^2}{2} p^2,$$

$$V(t_1, x) = -\ln(\phi_1(x)).$$

Construct a cost function  $L$  by a duality relation:

$$\begin{aligned} L(t, x, v) &= \max_{p \in \mathbb{R}} \left( -vp - H(t, x, p) \right) = \\ &= (b - v)^2 / (2\sigma^2). \end{aligned}$$



# Example of the L-T

Controlled dynamics :

$$dX_s^v = v(s, X_s^v)ds + \sigma(s, X_s^v)dB_s,$$

with generator  $\mathbf{A}^v \phi = \frac{\partial}{\partial t} \phi + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x, x} \phi + v \frac{\partial}{\partial x} \phi$ .

Criterion to be minimized:  $V(t, x) = \inf_v J(t, x, v)$ :

$$J(t, x, v) = \mathbb{E}_{t,x} \left( \int_t^{t_1} (b - v)^2 / (2\sigma^2) ds + \psi(t_1, X(t_1)) \right),$$

with  $\exp(-\psi)$  as boundary data for  $\phi = \exp(-V)$ .

# Example of L-T example

- $V$  solves the (DP) eq.

$$0 = \min_v [\mathbf{A}^v V + L].$$

- Optimal control  $v^*$  given by:

$$v^* = \arg \min [v V_x + L] = b + \sigma^2 \partial_x [\ln(\phi)].$$

- Simplest case :  $b = 0$  (i.e. no drift) and  $\sigma = 1$ ,

$$2v_t^*(x, t) + v_{x,x}^*(x, t) - 2v^*(x, t)v_x^*(x, t) = 0. \quad (\mathbf{B})$$

# Construction summary

<i>I</i> Markov dyn. $A\phi = 0.$	$\xrightarrow{L-T}$ $V = -\ln(\phi)$	<i>II</i> DP for $V$ cost $L$	$\xrightarrow{[\mathcal{O}]V}$	<i>III</i> nonlinear physics
$[\partial_t + \partial_{xx}] \phi = 0.$	$\xrightarrow{L-T}$ $V = -\ln(\phi)$	DP for $V$ $L_D = v^2$	$\xrightarrow{v = -\partial_x V}$	$2v_t + v_{xx} - 2vv_x = 0.$ Burgers' eq.



DP for  $V$  is here:

$$V_t + \frac{1}{2} [V_{xx} - (V_x)^2] = 0.$$



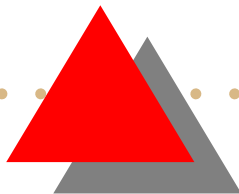
# The $L$ - $T$ and the $RW$ dynamics

Random evolution (two-velocity model)

$$\text{(TEL)} \quad \begin{cases} (\partial_t + \partial_x) P(x, t) = -\alpha P(x, t) + \beta Q(x, t) \\ (\partial_t - \partial_x) Q(x, t) = -\beta Q(x, t) + \alpha P(x, t). \end{cases}$$

$dX_t = I(t)dt$ ,  $I(t)$  Markov Alternating Renewal

$(\alpha^{-1}, \beta^{-1}$  average sojourn times)



# The $L$ - $T$ and the $RW$ dynamics

$$(RW) \quad \begin{cases} (\partial_t + \partial_x) f_+ = +f_+ f_- - \alpha f_+ + \beta f_-, \\ (\partial_t - \partial_x) f_- = -f_+ f_- - \beta f_- + \alpha f_+. \end{cases}$$

$$(L - T) \quad \begin{cases} f_+ = [\partial_t - \partial_x] \ln(P + Q) \\ f_- = [\partial_t + \partial_x] \ln(P + Q) \end{cases}$$



$$(TEL) \quad \begin{cases} (\partial_t + \partial_x) P = -\alpha P + \beta Q, \\ (\partial_t - \partial_x) Q = -\beta Q - \alpha P. \end{cases}$$

# The $L$ - $T$ and the $RW$ dynamics

RW-model  $\xrightarrow{L-T}$  Telegraphist equation

↓ [S-H], [G-P]

↓ CLT

Burgers' equation  $\xrightarrow{H-C}$  Heat equation

L-T: logarithmic transform

[S-H]: L. Streit and M.-O. H (1990)

H-C: Hopf-Cole transform

[G-P] : E. Gabetta and B. Perthame (2001)

CLT: central limit theorem

# *the RW-model converges...*

RW:

$$(\partial_t + v_1 \partial_x) f_+ = +\mu f_+ f_- - \alpha f_+ + \beta f_-$$

$$(\partial_t + v_2 \partial_x) f_- = -\mu f_+ f_- + \alpha f_+ - \beta f_-,$$

Diffusive re-scaling:

$$(x, t) \mapsto (y, \tau) := \left[ c \left( x - \frac{v_1 + v_2}{2} t \right), c^2 t \right],$$

with  $c$  a dimensionless parameter.



... to the Burger' eq.

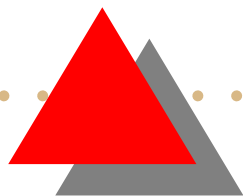
Set:

$$\rho(y, \tau) = \frac{1}{c} f_+(y, \tau) + \frac{1}{c} f_-(y, \tau).$$

For  $c \rightarrow 0$ , RW converges to B dynamics:

$$\rho_t = D \rho_{yy} - 2\rho\rho_y$$

with:  $D = \frac{(v_2 - v_1)^2}{4(\alpha + \beta)}$ .





# Construction summary

<p><i>I</i></p> <p><math>A\phi = 0.</math></p> <p>Markov dyn.</p>	$\xrightarrow{L-T}$	<p><i>II</i></p> <p>DP for <math>V</math></p> <p>cost <math>L</math></p>	$\xrightarrow{[\mathcal{O}]V}$	<p><i>III</i></p> <p>nonlinear physics</p>
<p>Teleg. eq.</p> <p><math>(X_t, Z_t)</math> Markov</p>	$\longrightarrow$	<p>DP for <math>V</math></p> <p><math>L_{RE} = u \ln(u) - u + 1</math></p>	$\xrightarrow{[\partial_t \pm \partial_x]V}$	<p>RW's dyn.</p>
<p>Heat eq.</p> <p><math>X_t</math> Brown. mvt.</p>	$\longrightarrow$	<p>DP for <math>V</math></p> <p><math>L_D = v^2</math></p>	$\xrightarrow{-\partial_x V}$	<p>Burgers' dyn.</p>

## Construction summary, (continue)

DP eq. for  $V(x, z, t)$ ,  $(z = \{-1, +1\})$ .

$$\partial_t V(x, -1, t) + \partial_x V(x, -1, t) - e^{[V(x, -1, t) - V(x, +1, t)]} + 1 = 0,$$

$$\partial_t V(x, +1, t) + \partial_x V(x, +1, t) - e^{[V(x, +1, t) - V(x, -1, t)]} + 1 = 0.$$

$$\Downarrow \quad (f(x, \pm, t) = [\partial_t \mp \partial_x] V(x, \pm, t)) \quad \Downarrow$$

$$\partial_t f(x, +1, t) + \partial_x f(x, +1, t) = -2f(x, +1, t)f(x, -1, t) - f(x, +1, t) + f(x, -1, t),$$

$$\partial_t f(x, -1, t) + \partial_x f(x, -1, t) = +2f(x, +1, t)f(x, -1, t) - f(x, -1, t) + f(x, +1, t).$$

(M. Soner, L. Streit and M.-O. H. *Appl. Math. & Opt.* **49**, (2004), 113-21. )

# Construction of a RD model

<p><i>I</i></p> <p><math>X_t</math> Markov dyn.</p> <p><math>\mathbf{A}\phi = 0</math></p>	$\xrightarrow{LT}$	<p><i>II</i></p> <p>DP for <math>V</math></p> <p>cost funct. <math>L</math></p>	$\xrightarrow{[\mathcal{O}]V}$	<p><i>III</i></p> <p>nonlinear physics</p>
<p><math>dX_s = Z_s ds</math></p> <p>Teleg. eq.</p>	$\xrightarrow{LT}$	<p>DP for <math>V</math></p> <p><math>L_{RE} = u \ln(u) - u + 1</math></p>	$\xrightarrow{[\partial_t \pm \partial_x]V}$	<p>RW's dyn.</p>
<p><math>dX_s = \sigma dB_s</math></p> <p>Heat eq.</p>	$\xrightarrow{LT}$	<p>DP for <math>V</math></p> <p><math>L_D = v^2</math></p>	$\xrightarrow{[\partial_x]V}$	<p>B dyn.</p>
<p><math>dX_s = Z_s ds + dB_s</math></p>	$\xrightarrow{LT}$	<p>DP for <math>V</math></p> <p><math>L = L_{RE} + L_D</math></p>	$\xrightarrow{[\mathcal{O}^\pm]V}$	<p>RD's dyn.</p>

# Mixed diffusive-random evol.

$$dX_s = Z_s ds + \sigma dB_s, \quad Z_s \in \{\nu, -\nu\}.$$

$(X_s, Z_s)$  Markov on state space  $\mathbb{R} \times \{\pm\nu\}$ .

$\mathbf{A} = (A_{-\nu}, A_{\nu})$  acts as:

$$A_z \phi^z = \frac{\partial}{\partial t} \phi^z + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x, x} \phi^z + z \frac{\partial}{\partial x} \phi^z + (\phi^{-z} - \phi^z),$$

$z \in \{-\nu, +\nu\},$

$$\phi := (\phi^{-\nu}(x, t), \phi^{\nu}(x, t)) \in (\mathcal{C}_0^{2,1}(\mathbb{R} \times [t, t_1]); \mathbb{R})^2.$$



# *Mixed diffusive-random evol.*

Seek  $\phi^{\pm\nu} \in \mathcal{C}_0^{4,1} \times \mathcal{C}_0^{4,1}$  solving 4<sup>th</sup>-ord. (lin.) PDE:

$$\det \begin{pmatrix} -(\mathcal{O}^- + 1) & 1 \\ 1 & -(\mathcal{O}^+ + 1) \end{pmatrix} \phi^\nu(x, t) = 0,$$

with  $\mathcal{O}^\pm = \partial_t \mp \nu \partial_x - \frac{\sigma^2}{2} \partial_{x,x}^2$ .

# Diffusive random evolution

$$\phi^\nu(x, t) = \int_{\mathbb{R}} G(y, t) T(x - y, t) dy,$$

- $\partial_t G - \frac{\sigma^2}{2} \partial_{x,x}^2 G = 0,$  (i.e. Gaussian)
- $\partial_{t,t}^2 T + 2\partial_t T - \nu^2 \partial_{x,x}^2 T = 0,$  (i.e. solutions via Bessel funct.)

(Ph. Blanchard and M.-O. H. *Phys. Lett. A*, **180**, 225, (1993), 225.)

# Associated control problem

(L-T):  $V = -\ln(\phi)$  solution of the DP eq.

$$\min_{u,v} [\mathbf{A}^{u,v}V + L] = 0$$

with running cost  $L = L_{RE} + L_D$ ,

$$L_{RE} = u \ln(u) - u + 1 \quad \text{and} \quad L_D = \frac{v^2}{2\sigma^2}.$$

$$\mathbb{E}_{t,x,z} \left( \int_t^{t_1} L_{RE}(X_s^{u,v}, u) + L_D(X_s^{u,v}, v) ds + \psi(t_1, X(t_1)) \right)$$

# The resulting RD eq.

$$\text{Apply } \mathcal{O}^\pm V = \left[ \partial_t \mp \nu \partial_x - \frac{\sigma^2}{2} \partial_{x,x}^2 \right] V$$

⇓

soluble (nonlinear) reaction diffusion (RD) eq.

<i>I</i> diffusive Rand. evol. $dX_s = Z_s ds + \sigma dB_s$	$\xrightarrow{LT}$	<i>II</i> nonlin. 4th. ord. eq. for $V$ $L = L_{RE} + L_D$	$\xrightarrow{[\mathcal{O}^\pm]}$	<i>III</i> RD equation
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(R. Lima and M.-O. H. *Phys. Let.A* **198**, (1995), 100.)



# Resulting RD evolution

$$(\partial_t \pm \nu \partial_x) f_{\pm} = \frac{\sigma^2}{2} (\partial_{x,x}^2 f_{\pm}) \pm B(f_+, f_-) \pm K_{\sigma}(f_+, f_-, \partial_x f_+, \partial_x f_-),$$

$$K_{\sigma}(f_+, f_-, \partial_x f_+, \partial_x f_-) = \frac{\sigma^2}{2\nu} \left[ \frac{1}{2\nu} S^2 - S D_x + \frac{1}{4\nu} D S^2 \right] + \\ + \frac{\sigma^4}{4\nu^2} \left[ -\frac{1}{16\nu^2} S^4 - \frac{1}{2} (S_x)^2 + \frac{1}{2\nu} S^2 S_x \right],$$

$$B(f_+, f_-) = f_+ f_- - f_+ + f_-,$$

$$S = f_+ + f_- \text{ and } D = f_+ - f_-.$$



# *Interpretation of $L = L_D + L_{RE}$*

$L_D = \frac{v^2}{2\sigma^2} \dots$  is an Onsager-Machlup functional.

View  $v \equiv \dot{\phi}$  derivative of a smooth curve  $\phi$ .



$$\lim_{\epsilon \searrow 0} \lim_{\sigma^2 \searrow 0} \sigma^2 \ln P \{ |X_t - \phi(t)| < \epsilon, \text{ for all } t \in [0, t_1] \} =$$

$$= - \int_0^{t_1} L_D(\dot{\phi}(s)) ds.$$



# The cost function $L_D$

OM-funct.  $L_D$  follows from the Legendre trans.

$$L(z, u) = \sup_{\xi \in \mathbb{R}} [u\xi - H(z, \xi)]$$

with  $H$  defined on  $\mathbb{R} \times \mathbb{R}$  as:

$$H(z, \xi) = \lim_{t \searrow 0} t^{-1} \ln E_x \left( \exp(\xi(X_t - z)) \right),$$

( $E_x$  conditional expectation with  $X_0 = z$ .)

(H. Ito *J. Phys. A. Math. Gen.* **14**, (1981), L385-L388.)



# *The cost function $L_D$*

**Observation.** Uncontrolled diffusion

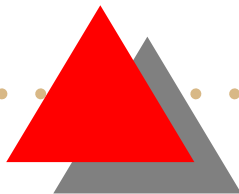
$$dX_s = \sigma dB_s$$

and opt. control. diff. (with resp. to  $L_D$ )

$$dX_s = \sigma^2 \partial_x \ln(\phi) ds + \sigma dB_s,$$

share same extremal trajectories.

(P. Dai Pra *Appl. Math. Opt.*, **23**, (1991), 313-29.)





# The cost function $L_{RE}$

Along the same lines, define:

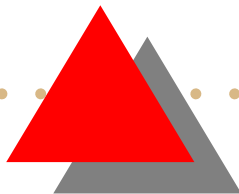
$$H_Z(z, \xi) = \lim_{t \searrow 0} t^{-1} \ln E_z(\exp(\xi(Z_t - z))), \quad z \in \{\pm\nu\}, \quad \xi \in \mathbb{R}.$$

**Here:**  $H_Z(z, \xi) = e^{-2\xi z} - 1.$



$$L_Z(z, u) \equiv \sup_{\xi \in \mathbb{R}} \left\{ \xi u - H_Z(z, \xi) \right\} = \frac{u}{2z} \ln \left( \frac{u}{2z} \right) - \frac{u}{2z} + 1,$$

(with  $u$  such that  $uz > 0$ ).





# Large deviations principle (LDP)

(P. Dupuis and R.S. Ellis *T.A.M.S.*, **347**, (1995), 2689-2751.)

(N. Privault and J.-C. Zambrini *Ann. Inst H. Poincaré* **40**, (2004), 599-633.)



integr. Lagrangian = rate of LDP for  $Z_s$ .

For every continuous and piecewise linear  $\psi$

$\psi : [0, t_1] \rightarrow \mathbb{R}$  with  $\psi(0) = z$ .

$$\lim_{\epsilon \searrow 0} \ln P \{ |Z_t - \psi(t)| < \epsilon; \forall t \in [0, t_1] \} = - \int_0^{t_1} L_Z(z, \dot{\psi}(s)) ds.$$

