Nonlinear reaction diffusion equations connected with stochastic control problems

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History and motivation

Physically relevant nonlinear PDE

- Burgers' eq. (B)

$$v_t(x,t) = v_{x,x}(x,t) - 2v(x,t)v_x(x,t).$$

- Boltzmann type eq. (RW),

(Th. W. Ruijgrok and T. T. Wu *Physica* **113 A**,*(1982),401-16.)*

$$(\partial_t + \partial_x)\, f_+(x,t) = +f_+(x,t)f_-(x,t) - \alpha f_+(x,t) + \beta f_-(x,t),$$

$$(\partial_t - \partial_x)\, f_-(x,t) = -f_+(x,t)f_-(x,t) - \beta f_-(x,t) + \alpha f_+(x,t).$$

History and motivation

- These PDEs are explicitly soluble (linearizable).
  - Hopf-Cole transform *Comm. Pure an Appl. Math.*, **3**, *(1950), 201-30.*
  - Th. W. Ruijgrok and T.T Wu *Physica* **113 A**, *(1982), 401-16.*

(can be linearized to Heat eq. for B and to Telegraphist eq. for RW)

- RW generalizes B.
  - L. Streit and M.-O H. *Europhys.lett.*, **12**, *(1990), 193-97.*
  - E. Gabetta and B. Perthame *Math. Meth. Appl. Sci.*, **24**, *(2001), 949-67*.

Objectives of this lecture

- Exhibit connections between B and RW and Stochastic Optimal Control problems.

- Construct a class of soluble nonlinear reaction diffusion (RD) eq.:

$$(\partial_t - \nu\partial_x)f_- = \frac{\sigma^2}{2}\left(\partial^2_{x,x} f_-\right) - K_\sigma(f_+, f_-, \partial_x f_+, \partial_x f_-),$$
$$(\partial_t + \nu\partial_x)f_+ = \frac{\sigma^2}{2}\left(\partial^2_{x,x} f_+\right) + K_\sigma(f_+, f_-, \partial_x f_+, \partial_x f_-).$$

Logarithmic transform, (L-T).

Let $\phi(t,x) \geq 0$ a solution to the (linear) eq.:

$$\mathbf{A}\,\phi(t,x) = 0, \qquad (*)$$

$\mathbf{A}$ backward op. of a Markov process X_t.

(L - T), defined as $V = -\ln(\phi)$, transforms $(*)$ into a nonlinear eq. for V:

Logarithmic transform, (L-T).

Nonlinear eq. for V can be viewed as the dynamic programming (DP) eq. :

$$0 = \min_{v \in \mathcal{C}} \left[\mathbf{A}^v V + L \right].$$

$\Big($$\mathbf{A}$ generator of a Markov process, $\mathcal{C}$ set of admissible controls$\Big)$

$$\Downarrow$$

- Stochastic control problem with cost function L.

W. H Fleming *Appl. Math. Optim.* **4**, *(1978), 329-46.*

Example of the L-T

Take e.g.,

$$\mathbf{A}\phi \equiv \frac{\partial}{\partial_t}\phi + \frac{\sigma^2}{2}\frac{\partial^2}{\partial_{x,x}}\phi + b\frac{\partial}{\partial_x}\phi,$$

with underlying Markov diffusion in $\mathbb{R}$:

$$dX_s = b(s, X_s)ds + \sigma(s, X_s)dB_s, \quad s \in [0, t_1].$$

Let $\phi(t,x) \geq 0$ with $\mathbf{A}\,\phi(t,x) = 0$
and terminal data: $\phi(t_1, x) = \phi_1(x)$.

Example of the L-T

$V = -\ln(\phi)$ solves:

$$V_t + \frac{\sigma^2}{2} V_{x,x} = H(t, x, V_x), \qquad (*)$$

$$H(t, x, p) = -b(t, x)p + \frac{\sigma^2}{2} p^2,$$

$$V(t_1, x) = -\ln(\phi_1(x)).$$

Construct a cost function L by a duality relation:

$$L(t, x, v) = \max_{p \in \mathbb{R}} \Big(-vp - H(t, x, p) \Big) =$$

$$= (b - v)^2 / (2\sigma^2).$$

Example of the L-T

Controlled dynamics :

$$dX_s^v = v(s, X_s^v)ds + \sigma(s, X_s^v)dB_s,$$

with generator $\mathbf{A}^v\phi = \frac{\partial}{\partial_t}\phi + \frac{\sigma^2}{2}\frac{\partial^2}{\partial_{x,x}}\phi + v\frac{\partial}{\partial_x}\phi$.

Criterion to be minimized: $V(t,x) = \inf_v J(t,x,v)$:

$$J(t,x,v) = \mathbb{E}_{t,x}\Big(\int_t^{t_1} (b-v)^2/(2\sigma^2)ds + \psi(t_1, X(t_1))\Big),$$

with $\exp(-\psi)$ as boundary data for $\phi = \exp(-V)$.

Example of L-T example

- V solves the (DP) eq.

$$0 = \min_v \left[\mathbf{A}^v V + L\right].$$

- Optimal control v^* given by:

$$v^* = \arg\min \left[vV_x + L\right] = b + \sigma^2 \partial_x \left[\ln(\phi)\right].$$

- Simplest case : $b = 0$ (i.e. no drift) and $\sigma = 1$,

$$2v_t^*(x,t) + v_{x,x}^*(x,t) - 2v^*(x,t)v_x^*(x,t) = 0. \quad \text{(B)}$$

Construction summary

I		II		III
Markov dyn. $A\phi = 0.$	$\xrightarrow[V=-\ln(\phi)]{L-T}$	DP for V cost L	$\xrightarrow{[\mathcal{O}]\ V}$	nonlinear physics
$[\partial_t + \partial_{xx}]\,\phi = 0.$	$\xrightarrow[V=-\ln(\phi)]{L-T}$	DP for V $L_D = v^2$	$\xrightarrow{v=-\partial_x V}$	$2v_t + v_{xx} - 2vv_x = 0.$ Burgers' eq.

$$\downarrow$$

DP for V is here:

$$V_t + \frac{1}{2}\left[V_{xx} - (V_x)^2\right] = 0.$$

The L-T and the RW dynamics

Random evolution (two-velocity model)

$$(\mathrm{TEL}) \begin{cases} (\partial_t + \partial_x)\, P(x,t) = -\alpha P(x,t) + \beta Q(x,t) \\ \\ (\partial_t - \partial_x)\, Q(x,t) = -\beta Q(x,t) + \alpha P(x,t). \end{cases}$$

$dX_t = I(t)dt, \;\; I(t)$ Markov Alternating Renewal

(α^{-1}, β^{-1} average sojourn times)

The L-T and the RW dynamics

$$\text{(RW)} \quad \begin{cases} (\partial_t + \partial_x)\, f_+ = +f_+ f_- - \alpha f_+ + \beta f_-, \\ (\partial_t - \partial_x)\, f_- = -f_+ f_- - \beta f_- + \alpha f_+. \end{cases}$$

$$\text{(L} - \text{T)} \quad \begin{cases} f_+ = [\partial_t - \partial_x] \ln(P+Q) \\ f_- = [\partial_t + \partial_x] \ln(P+Q) \end{cases}$$

$$\Downarrow$$

$$\text{(TEL)} \quad \begin{cases} (\partial_t + \partial_x)\, P = -\alpha P + \beta Q, \\ (\partial_t - \partial_x)\, Q = -\beta Q - \alpha P. \end{cases}$$

The L-T and the RW dynamics

RW-model $\xrightarrow{L-T}$ Telegraphist equation

$\downarrow$ [S-H], [G-P] $\qquad\qquad$ $\downarrow$ CLT

Burgers' equation $\xrightarrow{H-C}$ Heat equation

L-T: logarithmic transform

H-C: Hopf-Cole transform

CLT: central limit theorem

[S-H]: L. Streit and M.-O. H (1990)

[G-P] : E. Gabetta and B. Perthame (2001)

the RW-model converges...

RW:

$$\begin{aligned}(\partial_t + v_1\partial_x)f_+ &= +\mu f_+ f_- - \alpha f_+ + \beta f_- \\ (\partial_t + v_2\partial_x)f_- &= -\mu f_+ f_- + \alpha f_+ - \beta f_-,\end{aligned}\ .$$

Diffusive re-scaling:

$$(x,t) \mapsto (y,\tau) := \left[c(x - \frac{v_1+v_2}{2}t),\ c^2 t\right],$$

with c a dimensionless parameter.

… to the Burger' eq.

Set:

$$\rho(y,\tau) = \frac{1}{c} f_{+}(y,\tau) + \frac{1}{c} f_{-}(y,\tau).$$

For $c \to 0$, RW converges to B dynamics:

$$\rho_t = D\,\rho_{yy} - 2\rho\rho_y$$

with: $D = \frac{(v_2 - v_1)^2}{4(\alpha+\beta)}$.

Construction summary

I		II		III
$A\phi = 0.$ Markov dyn.	$\xrightarrow{L-T}$	DP for V cost L	$\xrightarrow{[\mathcal{O}]\ V}$	nonlinear physics
Teleg. eq. (X_t, Z_t) Markov	$\longrightarrow$	DP for V $L_{RE} = u\ln(u) - u + 1$	$\xrightarrow{[\partial_t \pm \partial_x]V}$	RW's dyn.
Heat eq. X_t Brown. mvt.	$\longrightarrow$	DP for V $L_D = v^2$	$\xrightarrow{-\partial_x V}$	Burgers' dyn.

Construction summary, (continue)

DP eq. for $V(x, z, t)$, $(z = \{-1, +1\})$.

$$\partial_t V(x,-1,t) + \partial_x V(x,-1,t) - e^{[V(x,-1,t)-V(x,+1,t)]} + 1 = 0,$$

$$\partial_t V(x,+1,t) + \partial_x V(x,+1,t) - e^{[V(x,+1,t)-V(x,-1,t)]} + 1 = 0.$$

$$\Downarrow \qquad (f(x,\pm,t) = [\partial_t \mp \partial_x]\, V(x,\pm,t)) \qquad \Downarrow$$

$$\partial_t f(x,+1,t) + \partial_x f(x,+1,t) = -2 f(x,+1,t) f(x,-1,t) - f(x,+1,t) + f(x,-1,t),$$

$$\partial_t f(x,-1,t) + \partial_x f(x,-1,t) = +2 f(x,+1,t) f(x,-1,t) - f(x,-1,t) + f(x,+1,t).$$

(M. Soner, L. Streit and M.-O. H. *Appl. Math. & Opt.* **49**, *(2004),113-21.)*

Construction of a RD model

I		II		III
X_t Markov dyn. $\mathbf{A}\phi = 0$	$\xrightarrow{LT}$	DP for V cost funct. L	$\xrightarrow{[\mathcal{O}]\,V}$	nonlinear physics
$dX_s = Z_s ds$ Teleg. eq.	$\xrightarrow{LT}$	DP for V $L_{RE} = u\ln(u) - u + 1$	$\xrightarrow{[\partial_t \pm \partial_x]V}$	RW's dyn.
$dX_s = \sigma dB_s$ Heat eq.	$\xrightarrow{LT}$	DP for V $L_D = v^2$	$\xrightarrow{[\partial_x]V}$	B dyn.
$dX_s = Z_s ds + dB_s$	$\xrightarrow{LT}$	DP for V $L = L_{RE} + L_D$	$\xrightarrow{[\mathcal{O}^{\pm}]V}$	RD's dyn.

Mixed diffusive-random evol.

$$dX_s = Z_s ds + \sigma dB_s, \quad Z_s \in \{\nu, -\nu\}.$$

(X_s, Z_s) Markov on state space $\mathbb{R} \times \{\pm\nu\}$.

$\mathbf{A} = (A_{-\nu}, A_\nu)$ acts as:

$$A_z \phi^z = \frac{\partial}{\partial_t}\phi^z + \frac{\sigma^2}{2}\frac{\partial^2}{\partial_{x,x}}\phi^z + z\frac{\partial}{\partial_x}\phi^z + (\phi^{-z} - \phi^z),$$

$$z \in \{-\nu, +\nu\},$$

$$\phi := (\phi^{-\nu}(x,t), \phi^\nu(x,t)) \in \left(\mathcal{C}_0^{2,1}(\mathbb{R} \times [t, t_1]); \mathbb{R}\right)^2.$$

Mixed diffusive-random evol.

Seek $\phi^{\pm\nu} \in \mathcal{C}_0^{4,1} \times \mathcal{C}_0^{4,1}$ solving 4^{th}-ord. (lin.) PDE:

$$\det \begin{pmatrix} -(\mathcal{O}^- + 1) & 1 \\ 1 & -(\mathcal{O}^+ + 1) \end{pmatrix} \phi^\nu(x,t) = 0,$$

with $\mathcal{O}^\pm = \partial_t \mp \nu\partial_x - \frac{\sigma^2}{2}\partial^2_{x,x}$.

Diffusive random evolution

$$\phi^{\nu}(x,t) = \int_{\mathbb{R}} G(y,t)T(x-y,t)dy,$$

- $\partial_t G - \frac{\sigma^2}{2}\partial^2_{x,x} G = 0,$ (i.e. Gaussian)
- $\partial^2_{t,t} T + 2\partial_t T - \nu^2 \partial^2_{x,x} T = 0,$ (i.e. solutions via Bessel funct.)

(Ph. Blanchard and M.-O. H. *Phys. Lett. A*, **180**, *225, (1993), 225.)*

Associated control problem

(L-T): $V = -\ln(\phi)$ solution of the DP eq.

$$\min_{u,v}\left[\mathbf{A}^{u,v}V + L\right] = 0$$

with running cost $L = L_{RE} + L_D$,

$$L_{RE} = u\ln(u) - u + 1 \qquad \text{and} \qquad L_D = \frac{v^2}{2\sigma^2}.$$

$$\mathbb{E}_{t,x,z}\Big(\int_t^{t_1} L_{RE}(X_s^{u,v}, u) + L_D(X_s^{u,v}, v)ds + \psi(t_1, X(t_1))\Big)$$

The resulting RD eq.

Apply $\mathcal{O}^{\pm} V = \left[\partial_t \mp \nu \partial_x - \frac{\sigma^2}{2} \partial^2_{x,x} \right] V$

$$\Downarrow$$

soluble (nonlinear) reaction diffusion (RD) eq.

I		II		III
diffusive Rand. evol. $dX_s = Z_s ds + \sigma dB_s$	$\xrightarrow{LT}$	nonlin. 4th. ord. eq. for V $L = L_{RE} + L_D$	$\xrightarrow{[\mathcal{O}^{\pm}]\ V}$	RD equation

(R. Lima and M.-O. H. *Phys. Let.A* **198**, *(1995), 100.)*

Resulting RD evolution

$$(\partial_t \pm \nu \partial_x) f_\pm = \frac{\sigma^2}{2}\left(\partial^2_{x,x} f_\pm\right) \pm B(f_+, f_-) \pm K_\sigma(f_+, f_-, \partial_x f_+, \partial_x f_-),$$

$$K_\sigma(f_+, f_-, \partial_x f_+, \partial_x f_-) = \frac{\sigma^2}{2\nu}\left[\frac{1}{2\nu} S^2 - S D_x + \frac{1}{4\nu} D S^2\right] + \\ + \frac{\sigma^4}{4\nu^2}\left[-\frac{1}{16\nu^2} S^4 - \frac{1}{2}(S_x)^2 + \frac{1}{2\nu} S^2 S_x\right],$$

$$B(f_+, f_-) = f_+ f_- - f_+ + f_-,$$

$S = f_+ + f_-$ and $D = f_+ - f_-$.

Interpretation of $L = L_D + L_{RE}$

$L_D = \frac{v^2}{2\sigma^2}$... is an Onsager-Machlup functional.

View $v \equiv \dot{\phi}$ derivative of a smooth curve ϕ.

$$\Downarrow$$

$$\lim_{\epsilon \searrow 0} \lim_{\sigma^2 \searrow 0} \sigma^2 \ln P \left\{ |X_t - \phi(t)| < \epsilon, \text{ for all } t \in [0, t_1] \right\} =$$

$$= - \int_0^{t_1} L_D(\dot{\phi}(s)) ds.$$

The cost function L_D

OM-funct. L_D follows from the Legendre trans.

$$L(z,u) = \sup_{\xi \in \mathbb{R}} [u\xi - H(z,\xi)]$$

with H defined on $\mathbb{R} \times \mathbb{R}$ as:

$$H(z,\xi) = \lim_{t \searrow 0} t^{-1} \ln E_x \Big(\exp(\xi(X_t - z)) \Big),$$

(E_x conditional expectation with $X_0 = z$.)

(H. Ito *J. Phys. A. Math. Gen.* **14**, *(1981), L385-L388.)*

The cost function L_D

Observation. Uncontrolled diffusion

$$dX_s = \sigma dB_s$$

and opt. control. diff. (with resp. to L_D)

$$dX_s = \sigma^2 \partial_x \ln(\phi) ds + \sigma dB_s,$$

share same extremal trajectories.

(P. Dai Pra *Appl. Math. Opt.*, **23**, *(1991), 313-29.)*

The cost function L_{RE}

Along the same lines, define:

$$H_Z(z,\xi) = \lim_{t\searrow 0} t^{-1} \ln E_z\big(\exp(\xi(Z_t - z))\big),\ z \in \{\pm\nu\},\ \xi \in \mathbb{R}.$$

Here: $H_Z(z,\xi) = e^{-2\xi z} - 1.$

$$\Downarrow$$

$$L_Z(z,u) \equiv \sup_{\xi :\in\mathbb{R}} \Big\{\xi u - H_Z(z,\xi)\Big\} = \frac{u}{2z} \ln\left(\frac{u}{2z}\right) - \frac{u}{2z} + 1,$$

(with u such that $uz > 0$).

Large deviations principle (LDP)

(P. Dupuis and R.S. Ellis *T.A.M.S.*, **347**, *(1995), 2689-2751.)*

(N. Privault and J.-C. Zambrini *Ann. Inst H. Poicaré* **40**, *(2004), 599-633.)*

$$\Downarrow$$

integr. Lagrangian = rate of LDP for Z_s.

For every continuous and piecewise linear ψ

$\psi : [0, t_1] \to \mathbb{R}$ with $\psi(0) = z$.

$$\lim_{\epsilon \searrow 0} \ln P \{ |Z_t - \psi(t)| < \epsilon; \forall\, t \in [0, t_1] \} = - \int_0^{t_1} L_Z(z, \dot{\psi}(s)) ds.$$