Random Correlation Matrices, Top Eigenvalue with Heavy Tails and Financial Applications

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Portfolio theory: Basics

- Portfolio weights w_i, Asset returns X_i^t
- If predicted gains are g_i then the expected gain of the portfolio is $G = \sum w_i g_i$.
- Risk: variance of the portfolio returns

$$R^2 = \sum_{ij} w_i \sigma_i C_{ij} \sigma_j w_j$$

where σ_i^2 is the variance of asset i and C_{ij} is the correlation matrix.

Empirical Correlation Matrix

- Large set of Assets N and comparable set of data points T

- Empirical Variance

$$\sigma_i^2 = \frac{1}{T} \sum_t \left(X_i^t \right)^2$$

can be assumed to be know (or predicted) with enough precision – note: returns have fat tails.

- Empirical Equal-Time Correlation Matrix

$$E_{ij} = \frac{1}{T} \sum_t \frac{X_i^t X_j^t}{\sigma_i \sigma_j}$$

order N^2 quantities estimated with NT datapoints. If $T < N$ $\mathbf{E}$ is not even invertible.

Markowitz Optimization

- Find the portfolio with maximum expected return for a given risk or equivalently, minimum risk for a given return (G)

- In matrix notation:

$$\mathbf{w}_C = G \frac{\mathbf{C}^{-1}\mathbf{g}}{\mathbf{g}^T\mathbf{C}^{-1}\mathbf{g}}$$

- Where all returns are measured with respect to the risk-free rate and $\sigma_i = 1$ (absorbed in g_i).

- Non-linear problem: $\sum_i |w_i| \leq A$ – a "spin-glass" problem!

- Related problem: find the "irreducible" idiosyncratic part of a stock

Risk of Optimized Portfolios

- Let $\mathbf{E}$ be an noisy estimator of $\mathbf{C}$ such that $\langle \mathbf{E} \rangle = \mathbf{C}$

- "In-sample" risk

$$R^2_{\text{in}} = \mathbf{w}_E^T \mathbf{E} \mathbf{w}_E = \frac{G^2}{\mathbf{g}^T \mathbf{E}^{-1} \mathbf{g}}$$

- True minimal risk

$$R^2_{\text{true}} = \mathbf{w}_C^T \mathbf{C} \mathbf{w}_C = \frac{G^2}{\mathbf{g}^T \mathbf{C}^{-1} \mathbf{g}}$$

- "Out-of-sample" risk

$$R^2_{\text{out}} = \mathbf{w}_E^T \mathbf{C} \mathbf{w}_E = \frac{G^2 \mathbf{g}^T \mathbf{E}^{-1} \mathbf{C} \mathbf{E}^{-1} \mathbf{g}}{(\mathbf{g}^T \mathbf{E}^{-1} \mathbf{g})^2}$$

Risk of Optimized Portfolios

- Using convexity arguments, and for large matrices:

$$R^2_{\mathrm{in}} \leq R^2_{\mathrm{true}} \leq R^2_{\mathrm{out}}$$

- Importance of eigenvalue cleaning:

$$w_i \propto \sum_{kj} \lambda_k^{-1} V_i^k V_j^k g_j = g_i + \sum_{kj} (\lambda_k^{-1} - 1) V_i^k V_j^k g_j$$

  - Eigenvectors with $\lambda > 1$ are suppressed,
  - Eigenvectors with $\lambda < 1$ are enhanced. Potentially very large weight on small eigenvalues.
  - Must determine which eigenvalues to keep and which one to correct to avoid over-allocation on pseudo-low risk modes

Possible Ensembles

- Null hypothesis Wishart ensemble:

$$\langle X_i^t X_j^s \rangle = \sigma_i \sigma_j \delta_{ij} \delta_{ts}$$

with constant volatilities, and X Gaussian – or at least with a finite second moment

- General Wishart ensemble:

$$\langle X_i^t X_j^s \rangle = \sigma_i \sigma_j C_{ij} \delta_{ts}$$

with constant volatilities and X with a finite second moment

- Elliptic Ensemble

$$\langle X_i^t X_j^s \rangle = \Sigma^{t2} \sigma_i \sigma_j C_{ij} \delta_{ts}$$

with a random common vol. with a certain $P(\Sigma)$ – example: Student

Green function (Stieljes transform)

- We need to find the trace of the resolvent or Stieljes transform:

$$G(z) = \frac{1}{N}\mathrm{Tr}\left[(z\mathbf{I} - \mathbf{E})^{-1}\right]$$

$$\rho(\lambda) = \lim_{\epsilon \to 0} \frac{1}{\pi} \Im\left(G(\lambda - \mathrm{i}\epsilon)\right).$$

Null hypothesis $\mathbf{C} = \mathbf{I}$

- E_{ij} is a sum of (rotationally invariant) matrices $E^t_{ij} = (X^t_i X^t_j)/T$

- Free random matrix theory: Find the additive R-transform $R(x) = B(x) - 1/x$; $B(G(z)) = z$)

$$G_t(z) = \frac{1}{N}\left(\frac{1}{z-q} + \frac{N-1}{z}\right)$$

- defining $q = N/T$, inverting $G_t(z)$ to first order in $1/N$,

$$R_t(x) = \frac{1}{T(1-qx)} \quad \text{by additivity} \quad R_E(x) = \frac{1}{(1-qx)}$$

$$G_E(z) = \frac{(z+q-1) - \sqrt{(z+q-1)^2 - 4zq}}{2zq}$$

Null hypothesis $\mathbf{C} = \mathbf{I}$

$$\rho_E(\lambda) = \frac{\sqrt{4\lambda q - (\lambda + q - 1)^2}}{2\pi\lambda q} \qquad \lambda \in [(1-\sqrt{q})^2, (1+\sqrt{q})^2]$$

Marcenko-Pastur (1967) (and many rediscoveries)

- Any eigenvalue beyond the Marcenko-Pastur band can be deemed to contain some information (but see below)

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Null hypothesis $\mathbf{C} = \mathbf{I}$

- Remark 1: Non-Gaussian corrections vanish as $(2 + \kappa)/N$

- Remark 2: $-G_E(0) = \langle \lambda^{-1} \rangle_E = (1-q)^{-1}$, allowing to compute the different risks:

$$R_{\text{in}} = R_{\text{true}}\sqrt{1-q} = R_{\text{out}}(1-q)$$

General C Case

- The general case for $\mathbf{C}$ cannot be directly written as a sum of "Blue" functions.

- Solution using different techniques (replicas, diagrams, S-transform:

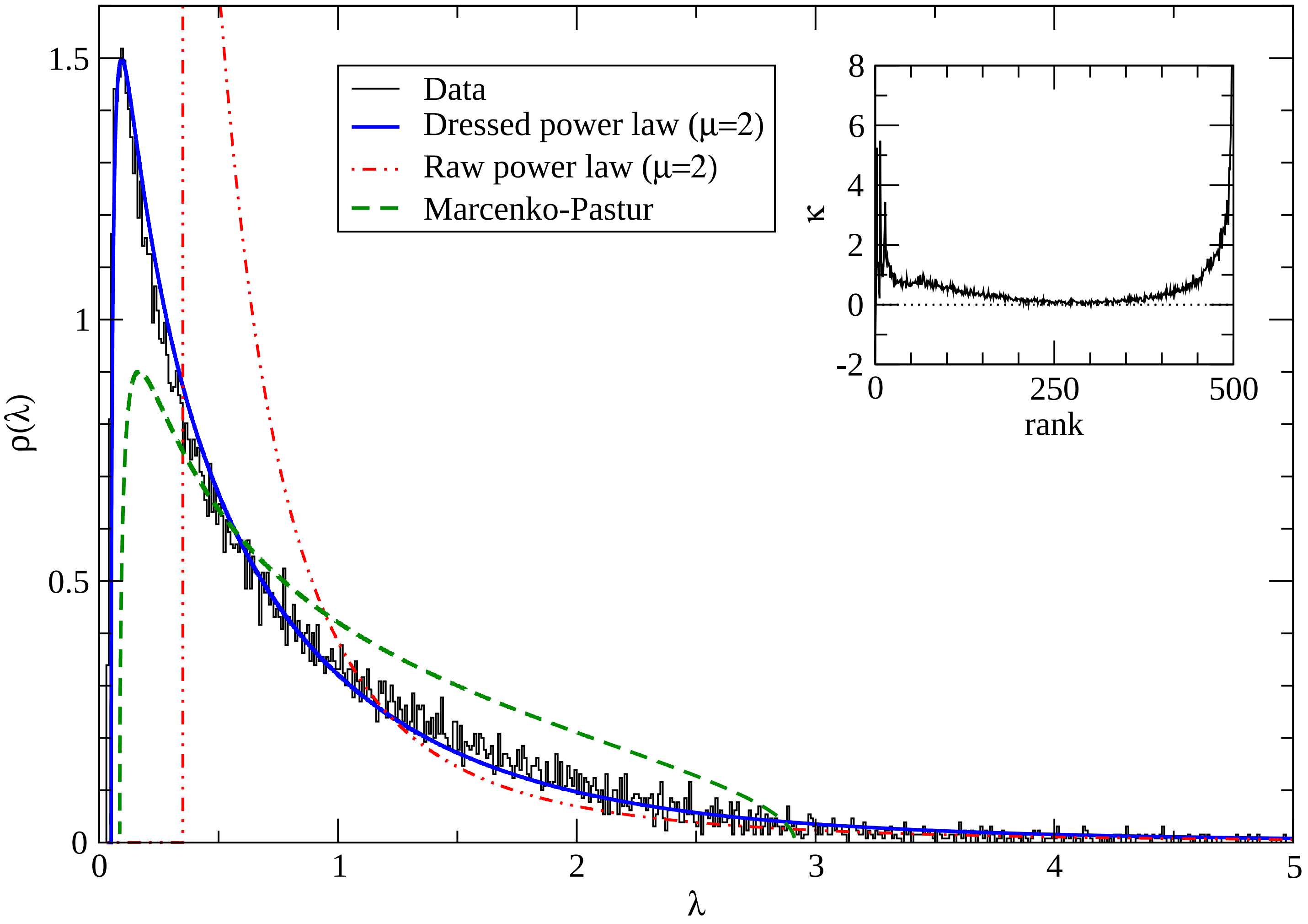
$$G_E(z) = \int d\lambda\, \rho_C(\lambda) \frac{1}{z - \lambda(1 - q + qzG_E(z))},$$

- Remark 1: $-G_E(0) = (1-q)^{-1}$ independently of $\mathbf{C}$

- Remark 2: One should postulate a parametric form for $\rho_C(\lambda)$, for example:

$$\rho_C(\lambda) = \frac{\mu A}{(\lambda - \lambda_0)^{1+\mu}} \Theta(\lambda - \lambda_{\min})$$

Empirical Correlation Matrix



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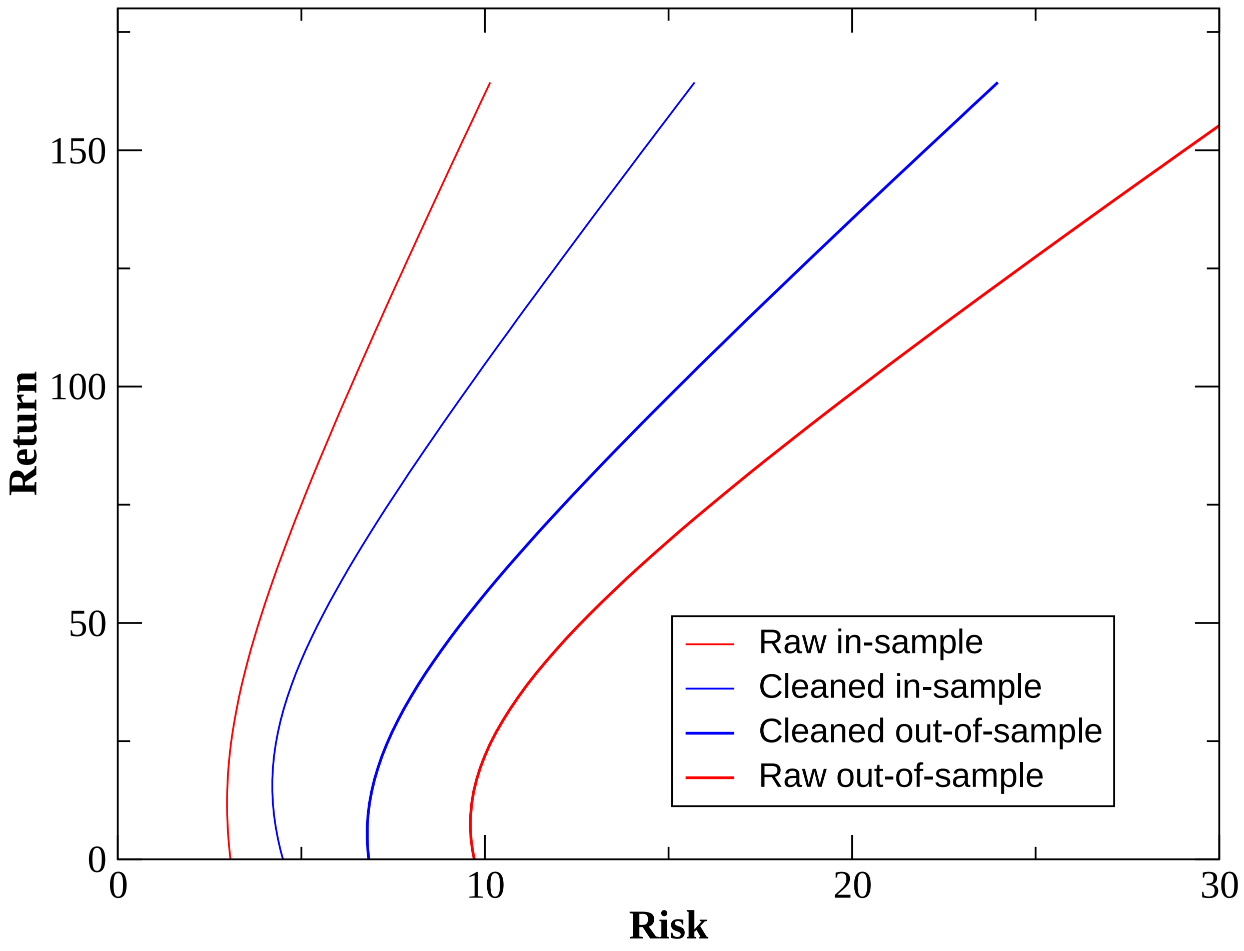
Matrix Cleaning



The Student ensemble

- Exact calculation can be done again for a general $\mathbf{C}$

- For $\mathbf{C} = \mathbf{1}$,

$$\lambda = \frac{G_R}{G_R^2 + \pi^2 \rho_E^2} + \int ds P(s) \frac{\mu(s - q\mu G_R)}{(s - q\mu G_R)^2 + \pi^2 \rho_E^2}$$

$$0 = \rho \left(-\frac{1}{G_R^2 \pi^2 \rho_E^2} + \int ds P(s) \frac{q\mu^2}{(s - q\mu G_R)^2 + \pi^2 \rho_E^2} \right),$$

where G_R is the real part of the resolvent, and $P(s) = s^{\mu/2-1} e^{-s} / \Gamma(\mu/2)$

- Appears to give a very good fit of $\rho(\lambda)$ too !

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The Student ensemble

- However, the maximum likelihood estimator of $\mathbf{C}$ is in that case given by:

$$\widehat{E}_{ij} = \frac{N+\mu}{T} \sum_{t=1}^{T} \frac{X_i^t X_j^t}{\mu + \sum_{mn} X_m^t (\widehat{C}^{-1})_{mn} X_n^t}.$$

- But the spectrum of $\widehat{\mathbf{E}}$ is Marcenko-Pastur again !!

- ...whereas the actual empirical $\widehat{\mathbf{E}}$ is nearly identical to that of $\mathbf{E}$

More General Correlation matrices

- Non equal time correlation matrices

$$E_{ij}^{\tau} = \frac{1}{T} \sum_t \frac{X_i^t X_j^{t+\tau}}{\sigma_i \sigma_j}$$

$N \times N$ but not symmetrical: ‘leader-lagger’ relations

- General rectangular correlation matrices

$$G_{\alpha i} = \frac{1}{T} \sum_{t=1}^{T} Y_\alpha^t X_i^t$$

N ‘input’ factors X; M ‘output’ factors Y

– Example: $Y_\alpha^t = X_j^{t+\tau}$, $N = M$

Singular values and relevant correlations

- Singular values: Square root of the non zero eigenvalues of GG^T or G^TG, with associated eigenvectors u_α^k and $v_i^k \rightarrow 1 \geq s_1 > s_2 > ...s_{(M,N)^-} \geq 0$

- Interpretation: $k = 1$: best linear combination of input variables with weights v_i^1, to optimally predict the linear combination of output variables with weights u_α^1, with a cross-correlation $= s_1$.

- s_1: measure of the predictive power of the set of Xs with respect to Ys

- Other singular values: orthogonal, less predictive, linear combinations

Benchmark: no cross-correlations

- Null hypothesis: No correlations between Xs and Ys $- \langle G \rangle = \mathbf{0}$

- But arbitrary correlations *among* Xs, C_X, and Ys, C_Y, are possible

- Consider exact normalized principal components for the sample variables Xs and Ys:

$$\hat{X}_i^t = \frac{1}{\sqrt{\lambda_i}} \sum_j U_{ij} X_j^t; \qquad \hat{Y}_\alpha^t = \ldots$$

and define $\hat{G} = \hat{Y}\hat{X}^T$.

Benchmark: no cross-correlations

- Tricks:
  - Non zero eigenvalues of $\widehat{G}\widehat{G}^T$ are the same as those of $\widehat{X}^T\widehat{X}\widehat{Y}^T\widehat{Y}$
  - $A = \widehat{X}^T\widehat{X}$ and $B = \widehat{Y}^T\widehat{Y}$ are mutually free, with n (m) eigenvalues equal to 1 and $1-n$ $(1-m)$ equal to 0
  - "S-transforms" are multiplicative

Technicalities

- $$\eta_A(y) \equiv \frac{1}{T}\mathrm{Tr}\frac{1}{1+yA}.$$

- $$\Sigma_A(x) \equiv -\frac{1+x}{x}\eta_A^{-1}(1+x).$$

- $$\eta_A(y) = 1 - n + \frac{n}{1+y}, \qquad \eta_B(y) = 1 - m + \frac{m}{1+y}.$$

- $$\Sigma_{GG}(x) = \Sigma_A(x)\Sigma_B(x) = \frac{(1+x)^2}{(x+n)(x+m)}.$$

Benchmark: Random SVD

- Final result:([LL,MAM,MP,JPB])

$$\rho(s) = (m+n-1)^+\delta(s-1) + \frac{\sqrt{(s^2-\gamma_-)(\gamma_+-s^2)}}{\pi s(1-s^2)}$$

with

$$\gamma_\pm = n + m - 2mn \pm 2\sqrt{mn(1-n)(1-m)}, \quad 0 \le \gamma_\pm \le 1$$

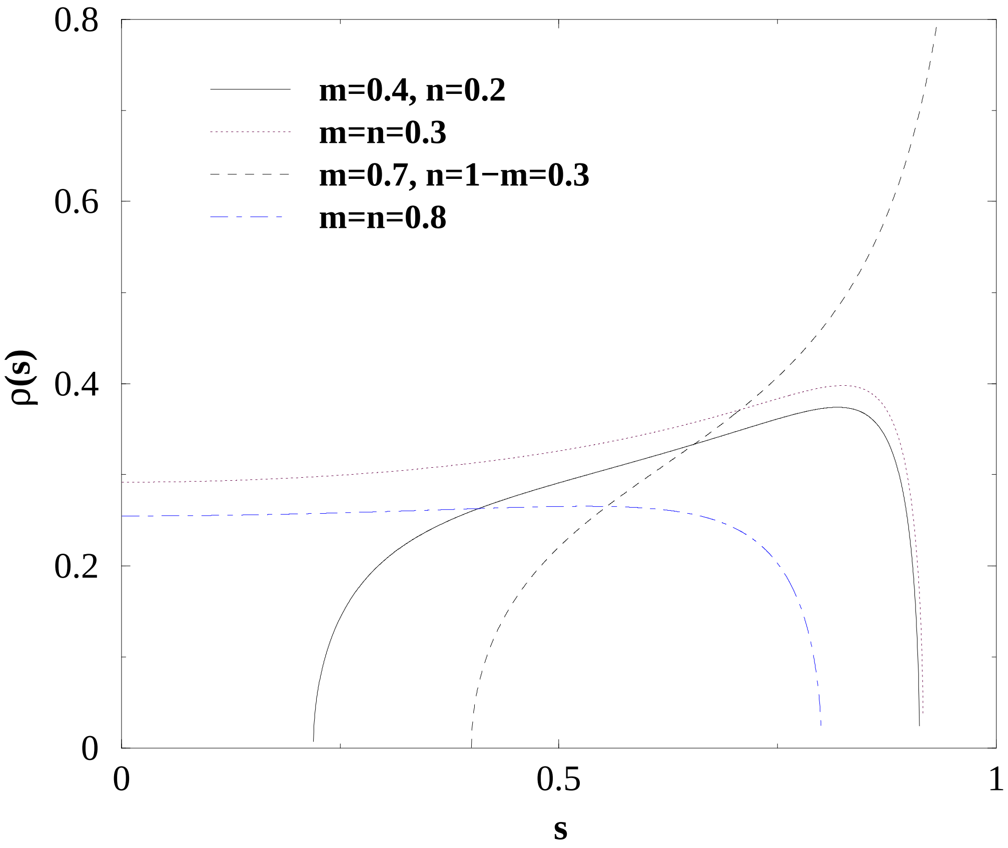
- Analogue of the Marcenko-Pastur result for rectangular correlation matrices – first derived by Wachter

- Many applications; finance, econometrics ('large' models), genomics, etc.

Benchmark: Random SVD

- Simple cases:
  - $n = m, \; s \in [0, 2\sqrt{n(1-n)}]$
  - $n, m \to 0, \; s \in [|\sqrt{m} - \sqrt{n}|, \sqrt{m} + \sqrt{n}]$
  - $m = 1, \; s \to \sqrt{1-n}$
  - $m \to 0, \; s \to \sqrt{n}$

RSVD: Numerical illustration



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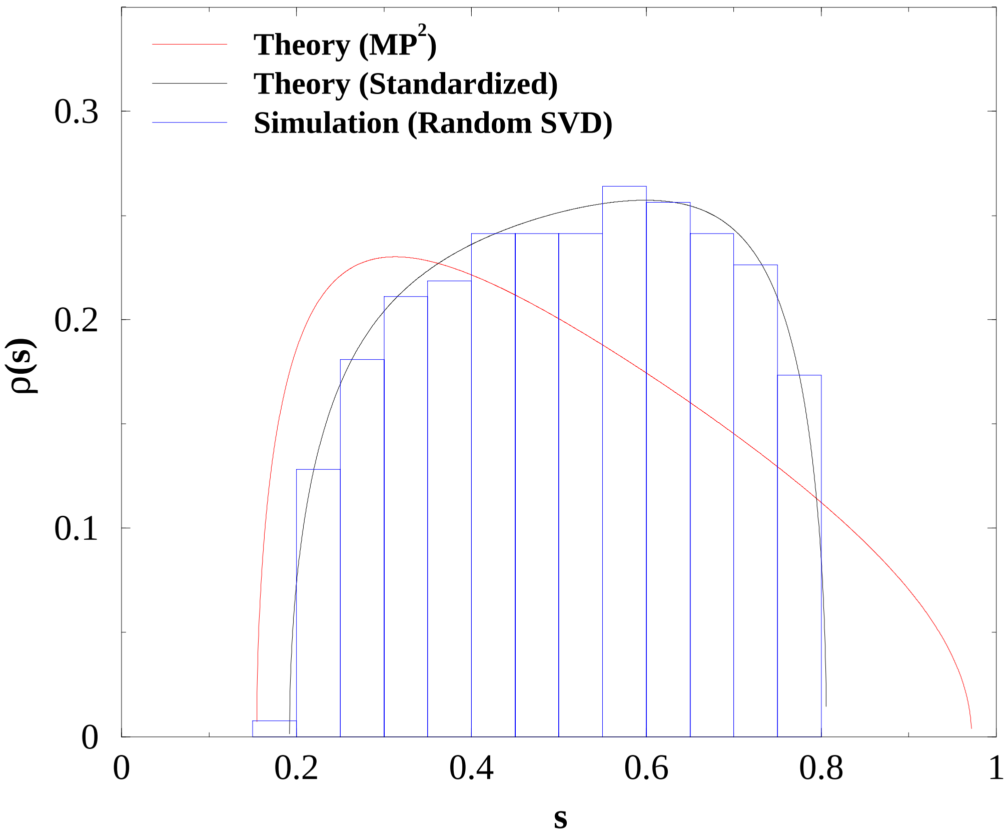
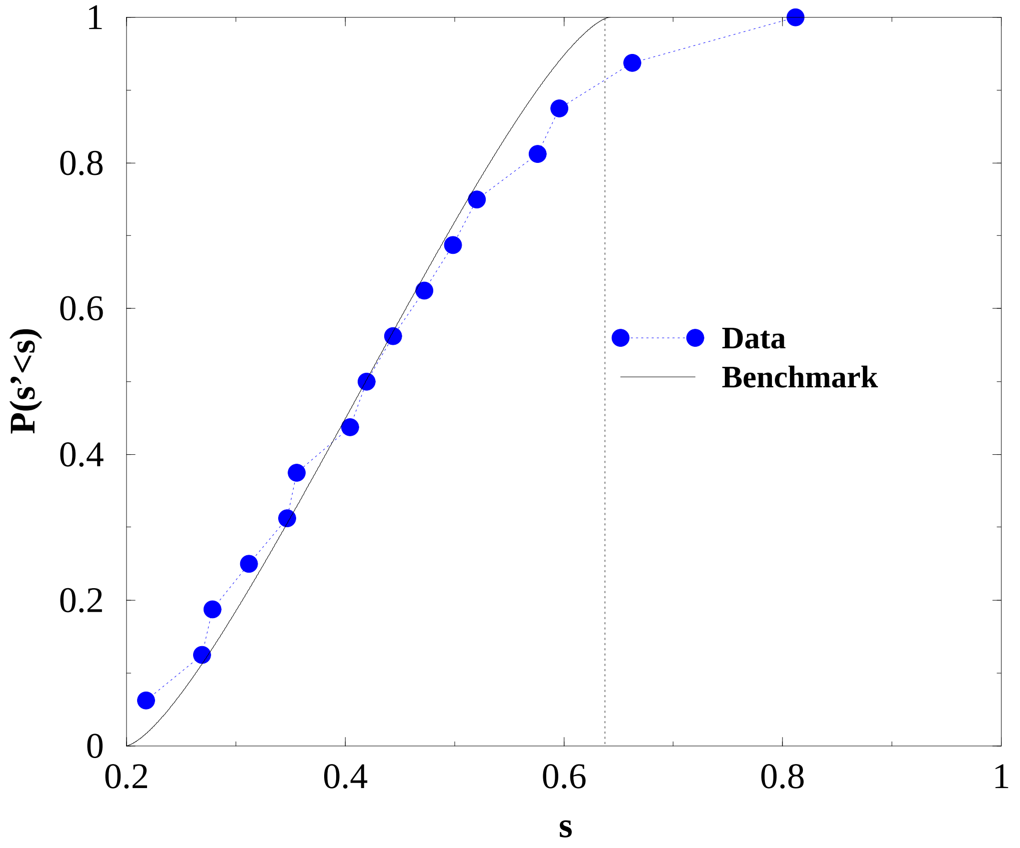
RSVD: Numerical illustration



Inflation vs. Economic indicators



$N = 50, M = 16, T = 265$

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Statistics of the Top Eigenvalue

- All previous results are true when $N, M, T \to \infty$ with fixed n, m

- How far is the top eigenvalue expected to leak out at finite N?

- Precise answer when matrix elements are iid Gaussian: Tracy-Widom statistics

- Width of the smoothed edge: $N^{-2/3}$

Statistics of the Top Eigenvalue

- Exceptions
  - 'Strong' Rank One Perturbation $\rightarrow$ emergence of an isolated eigenvalue with *Gaussian*, $N^{-1/2}$ fluctuations (Baik, Ben-Arous, Péché)
  - E.g.: $E_{ij} \rightarrow E_{ij} + \rho(1 - \delta_{ij})$ leads to a *market mode* $\lambda_{\mathrm{max}} \approx N\rho$
  - Fat tailed distribution of matrix elements

Fat tails and Top Eigenvalue: Wigner Case

- Eigenvalue statistics of large real symmetric matrices with iid elements X_{ij}, $P(x) \sim |x|^{-1-\mu}$

- Eigenvalue density:

  - $\mu > 2 \rightarrow$ Wigner semi-circle in $[-2, 2]$

  - $\mu < 2 \rightarrow$ unbounded density with tails $\rho(\lambda) \sim \lambda^{-1-\mu}$ (Cizeau,JPB)

- Note: $\mu < 2$ non trivial statistics of eigenvectors (localized/delocalized) (Cizeau,JPB)

Fat tails and Top Eigenvalue: Wigner Case

- A little lemma: Take a Wigner Matrix and add a finite rank perturbation matrix with largest eigenvalue S (Péché):

- Then:

$$|S| < 1 \to \lambda_{\text{max}} = 2 \qquad |S| \geq 1 \to \lambda_{\text{max}} = S + \frac{1}{S}$$

- Condensation/evaporation phenomenon

- Example: $X_{ij} \to X_{ij} + S$, $X_{ji} \to X_{ji} + S$

Fat tails and Top Eigenvalue: Wigner Case

- Largest Eigenvalue statistics ([GB,MP,JPB])
  - $\mu > 4$: $\lambda_{\text{max}} - 2 \sim N^{-2/3}$ with a Tracy-Widom distribution (max of strongly correlated variables)
  - $2 < \mu < 4$: $\lambda_{\text{max}} \sim N^{\frac{2}{\mu}-\frac{1}{2}}$ with a *Fréchet* distribution (although the density goes to zero when $\lambda > 2$!!)
  - $\mu = 4$: $\lambda_{\text{max}} \geq 2$ but remains $O(1)$, with a new distribution:

$$P_>(\lambda_{\text{max}}) = w\theta(\lambda_{\text{max}} - 2) + (1 - w)F(s) \quad \lambda_{\text{max}} = s + \frac{1}{s}$$

- Note: The case $\mu > 4$ still has a power-law tail for finite N, of amplitude $N^{2-\mu/2}$

Fat tails and Correlation Matrices

- $$E_{ij} = \frac{1}{T} \sum_t X_i^t X_j^t$$

- $\mu > 4$: $\lambda_{\max} - (1+\sqrt{n})^2 \sim N^{-2/3}$ (but with a power-law tail as above)

- $\mu < 4$: $\lambda_{\max} \sim N^{\frac{4}{\mu}-1} n^{1-2/\mu}$

- Fat tails induce fictitious 'strong' correlations – important for applications in finance where $\mu \approx 3-5$.

Dynamics of the top eigenvector – Non stationarity

- Specific dynamics of large top eigenvalue and eigenvector: Ornstein-Uhlenbeck processes (on the unit sphere for $\mathbf{V}^1$)

- The angle obeys the following SDE:

$$d\theta \approx -\frac{\epsilon}{2} \sin 2\theta dt + \zeta_t \, dW_t$$

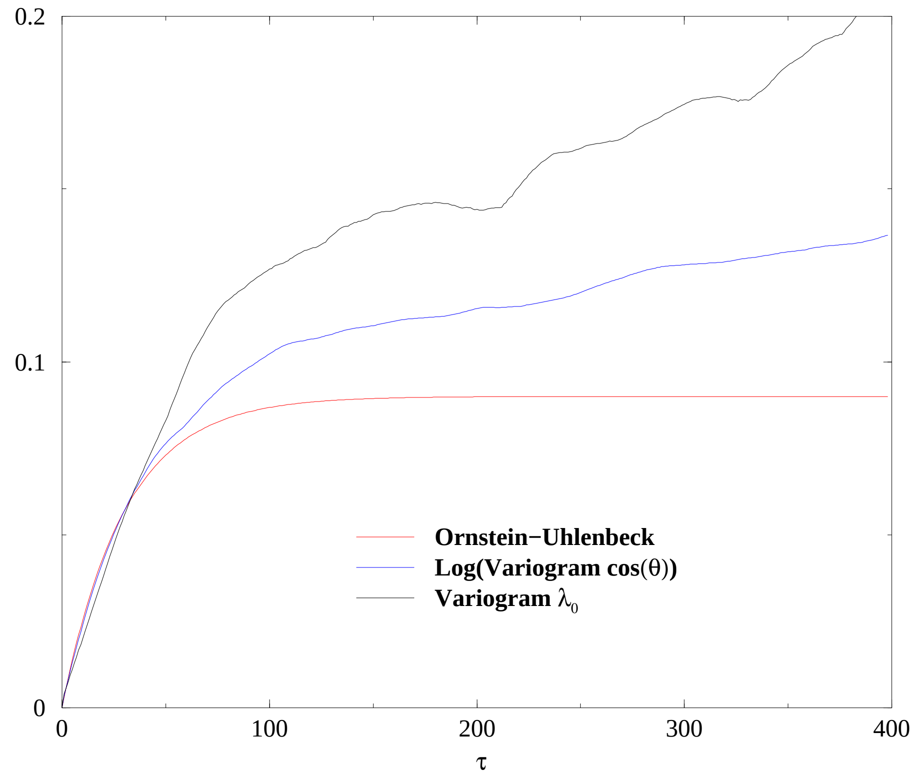
with

$$\zeta_t^2 \approx \epsilon^2 \left[\frac{1}{2} \sin^2 2\theta_t + \frac{\Lambda_1}{\Lambda_0} \cos^2 2\theta_t \right]$$

- Eigenvector dynamics:

$$\Big\langle \langle \psi_{0t+\tau} | \psi_{0t} \rangle \Big\rangle \approx E(\cos(\theta_t - \theta_{t+\tau})) \approx 1 - \epsilon \frac{\Lambda_1}{\Lambda_0} (1 - \exp(-\epsilon\tau))$$

The variogram of the top eigenvector



Clear signal for a true time evolution of the correlation matrix

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