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Covariance, means and Fisher information: a quantum journey at the light of uncertainty relations

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Classical covariance

Classical covariance

$$\mathrm{Cov}_p(X,Y) := \mathbb{E}_p(XY) - \mathbb{E}_p(X)\mathbb{E}_p(Y) = \mathbb{E}_p(X_0 Y_0).$$

where $X_0 := X - \mathbb{E}_p(X)$.

To go "quantum" we consider:
- s.a. matrices A, B instead of r.v. X, Y and states ρ instead of densities p;
- $\mathrm{Tr}(\rho A)$ instead of the expectation $\mathbb{E}_p(X)$.

Quantum covariance

Quantum covariance

$$\mathrm{Cov}_\rho(A,B) := \frac{1}{2}\mathrm{Tr}(\rho(AB+BA)) - \mathrm{Tr}(\rho A)\cdot\mathrm{Tr}(\rho B) =$$

$$= \mathrm{Tr}\left[\left(\frac{L_\rho + R_\rho}{2}\right)(A_0)B_0\right].$$

where $A_0 := A - \mathrm{Tr}(\rho A)\cdot I$ and

$$L_\rho(A) := \rho A \quad R_\rho := A\rho$$

Different quantum covariances ...

Is the above definition “natural"?
Certainly it coincides with the classical covariance in a commutative setting.
It uses the "arithmetic mean" of the left and right multiplication operator

$$m_{arith}(L_\rho, R_\rho) := \frac{L_\rho + R_\rho}{2}$$

This suggest that we may consider other noncommutatitive "‘means".

... using different means?

If we consider the “harmonic" covariance

$$\mathrm{Cov}_\rho^{har}(A,B) := \mathrm{Tr}\left(\left(2(L_\rho^{-1}+R_\rho^{-1})^{-1}\right)(A_0)B_0\right),$$

also this coincides with the classical definition where there is no difference between L_ρ and R_ρ!

Is there a quantum criterion to prefere a certain covariance (mean)?

Operator means

Kubo-Ando 1980

Let $D_n := \{A \in M_n | A > 0\}$.

A *mean* is a function $m : \mathcal{D}_n \times \mathcal{D}_n \to \mathcal{D}_n$ such that

(i) $m(A, A) = A$,

(ii) $m(A, B) = m(B, A)$,

(iii) $A < B \implies A < m(A, B) < B$,

(vi) $A < A', B < B' \implies m(A, B) < m(A', B')$,

(v) m is continuous,

(vi) $Cm(A, B)C^* \leq m(CAC^*, CBC^*)$, for every $C \in M_n$.

Property (vi) is the *transformer inequality*.

Operator monotone functions

M_n = complex matrices

Definition

$f : (0, +\infty) \to R$ is operator monotone iff $\forall A, B \in M_n$ and $\forall n = 1, 2, ...$

$$0 \leq A \leq B \quad \Longrightarrow \quad 0 \leq f(A) \leq f(B).$$

Definition

φ is a Pick function if it is analytic in the upper half plane and map the latter into itself.

Löwner Theorem

Löwner 1932

Theorem

f is operator monotone iff it is the restriction of a Pick function.

Usually one consider o.m. functions that are:
i) normalized i. e. $f(1) = 1$;
ii) symmetric i.e. $tf(t^{-1}) = f(t)$.

$\mathcal{F}_{op}$:= family of normalized symmetric o. m. functions.

Examples

$$\frac{1+x}{2}, \quad \sqrt{x}, \quad \frac{2x}{1+x}.$$

Kubo–Ando theorem

$\mathcal{M}_{op}$:= family of matrix means.

Kubo and Ando (1980) proved the following, fundamental result.

Theorem

There exists a bijection between $\mathcal{M}_{op}$ and $\mathcal{F}_{op}$ given by the formula

$$m_f(A, B) := A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

Kubo–Ando inequality

Examples of operator means

$$\frac{A+B}{2}$$

$$A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$$

$$2(A^{-1}+B^{-1})^{-1}$$

Fundamental inequality

$$2(A^{-1}+B^{-1})^{-1} \le m_f(A,B) \le \frac{A+B}{2} \qquad \forall f \in \mathcal{F}_{op}$$

g-Covariance

To each operator monotone $g \in \mathcal{F}_{op}$ one associate the means $m_g(\cdot,\cdot)$.

Define the g-covariance as

$$\mathrm{Cov}_\rho^g(A,B) := \mathrm{Tr}(m_g(L_\rho, R_\rho)(A_0)B_0)$$

Main question

The standard quantum covariance

$$\mathrm{Cov}_\rho(A,B) := \frac{1}{2}\mathrm{Tr}(\rho(AB+BA)) - \mathrm{Tr}(\rho A)\cdot\mathrm{Tr}(\rho B) =$$

$$= \mathrm{Tr}\left[\left(\frac{L_\rho + R_\rho}{2}\right)(A_0)B_0\right].$$

plays a fundamental role with respect to uncertainty relations.

Can one find in this field some criteria to select one (or more) specific covariance?

Heisenberg uncertainty principles

Let $A, B \in \mathcal{M}_{n,sa}(\mathbb{C})$.

$$\mathrm{Cov}_\rho(A,B) := \left[\mathrm{Tr}\rho\left(\frac{AB+BA}{2}\right)\right] - \mathrm{Tr}(\rho A)\cdot\mathrm{Tr}(\rho B),$$

$$\mathrm{Var}_\rho(A) := \mathrm{Cov}_\rho(A,A).$$

Heisenberg uncertainty principle (1927) reads as

$$\mathrm{Var}_\rho(\mathrm{A})\cdot\mathrm{Var}_\rho(\mathrm{B}) \geq \frac{1}{4}|\mathrm{Tr}(\rho[\mathrm{A},\mathrm{B}])|^2.$$

Schrödinger – Robertson UP

Schrödinger and Robertson (1929-1930) improved UP

$$\mathrm{Var}_\rho(A) \cdot \mathrm{Var}_\rho(B) - \mathrm{Cov}_\rho(A, B)^2 \geq \frac{1}{4} |\mathrm{Tr}(\rho[A, B])|^2.$$

The standard uncertainty principles are non-trivial whenever A, B are not compatible, that is, $[A, B] \neq 0$.

Robertson general UP (1934)

Let $A_1 \ldots, A_N \in \mathcal{M}_{n,sa}(\mathbb{C})$.

$$\det \{\mathrm{Cov}_\rho(A_h, A_j)\} \geq \det \left\{ -\frac{i}{2} \mathrm{Tr}(\rho[A_h, A_j]) \right\},$$

for $h, j = 1, \ldots, N$
The l.e.s. is the *generalized variance* of the random vector $(A_1, ..., A_N)$.

g-version of Robertson UP

Let $A_1 \ldots, A_N \in \mathcal{M}_{n,sa}(\mathbb{C})$.

$$\det\left\{\mathrm{Cov}_\rho^g(A_h, A_j)\right\} \geq \det\left\{-i \cdot g(0) \cdot \mathrm{Tr}(\rho[A_h, A_j])\right\},$$

for $h, j = 1, \ldots, N$,
for all $g \in \mathcal{F}_{op}$.

Remark: $g(0)$ is the best constant in the above inequality.

Conclusion

Quantum g-covariances coming from regular g (constant $g(0) \neq 0$) do have uncertainty relations.

Quantum g-covariances coming from nonregular g (constant $g(0) = 0$) do NOT have uncertainty relations.

The usual quantum covariance has the most demanding one (since $g(0) = \frac{1}{2}$ only for the arithmetic mean).

After all Schrödinger and Robertson were right ...

Robertson general UP (2nd version)

The matrix $\{-\frac{i}{2}\mathrm{Tr}(\rho[A_h, A_j])\}$ is anti-symmetric. Therefore, the Robertson UP reads as

$$\det\{\mathrm{Cov}_\rho(A_h, A_j)\} \geq \begin{cases} 0, & N \text{ odd} \\ \det\{-\frac{i}{2}\mathrm{Tr}(\rho[A_h, A_j])\}, & N \text{ ever} \end{cases}$$

If $N = 2m + 1$, UP says (classically!) that the *generalized variance* is non-negative.

Where to look for an UP for N odd?

- Robertson UP is based on the commutator $[A_h, A_j]$. If $N = 1$ this structure becomes meaningless !
- Intuitively, an UP for N odd should be based on a structure which involves $[\rho, A]$.
- This commutator appears in quantum dynamics.

Fisher information

$X : \Omega \to \mathbb{R}$ real r. v. with diff. density ρ
The *score* is

$$J_\rho := \frac{\rho'}{\rho} \qquad \mathbb{E}_\rho(J_\rho) = 0$$

The *Fisher information* is

$$I_X := I_\rho = \mathrm{Var}_\rho(J_\rho) = \int_{\mathbb{R}} \frac{(\rho')^2}{\rho}$$

FI as a Riemannian metric

- The ρ-centered variables ($\mathbb{E}_\rho(U) = 0$) should be considered as “tangent vectors" at the “point" ρ.
- On this “tangent space" Fisher information (covariance) gives a Riemannian metric.
- To understand this costruction in the quantum setting one needs to understand the links among means, monotone functions and Fisher information(s).
- We restrict to the simplex of probability vectors.

Classical Fisher information

$(\Omega, \mathcal{G}, \rho)$ probability space
ρ-scores = random variable s.t. $E_\rho(U) = 0$
on ρ-scores U, V the Fisher information is defined as

$$g_\rho(U, V) := \mathrm{Cov}_\rho(U, V) = E_\rho(UV)$$

The ρ–scores are "tangent vectors".
We restrict on the simplex

$$\mathcal{P}_n^1 := \{\rho \in R^n | \sum_i \rho_i = 1, \quad \rho_i > 0\}.$$

Properties of Fisher information

Look at Fisher information in different ways:
i) Hessian of Kullback-Leibler relative entropy

$$K(\rho, \sigma) := \sum_i \rho_i (\log \rho_i - \log \sigma_i);$$

ii) pull-back of the map $\rho \to \sqrt{\rho}$;
iii) get the scores using the (Symmetric) Logarithmic Derivative

$$\frac{\partial \rho(\theta)}{\partial \theta} = \frac{1}{2}\left(\frac{\partial}{\partial \theta} \log(\rho(\theta)) \cdot \rho(\theta) + \rho(\theta) \cdot \frac{\partial}{\partial \theta} \log(\rho(\theta)) \right)$$

Examples of QFI

Examples of quantum Fisher informations

Hessian of Umegaki relative entropy
$\mathrm{Tr}(\rho(\log\rho - \log\sigma))$
$\longrightarrow$ BKM metric

Pull-back of the map $\rho \to \sqrt{\rho}$
$\longrightarrow$ WY metric

Symmetric logarithmic derivative
$\longrightarrow$ Bures-Uhlmann metric (SLD)

Chentsov Theorem

Can we have a unified quantum approach?
Yes using the classical **Chentsov theorem**.
On the simplex $\mathcal{P}_n^1$ the Fisher information is the only Riemannian metric contracting under an arbitrary coarse graining T, namely for any tangent vector X at the point ρ we have

$$g^m_{T(\rho)}(TX, TX) \leq g^n_\rho(X, X)$$

Remark

Coarse graining = stochastic map = linear, positive, trace preserving.

Monotone metrics (or QFI)

$$D_n^1 := \{\rho \in M_n | \mathrm{Tr}(\rho) = 1 \quad \rho > 0\}$$ = faithful states

Definition

A quantum Fisher information is a Riemaniann metric on D_n^1 contracting under an arbitrary coarse graining T, namely

$$g^m_{T(\rho)}(TA, TA) \leq g^n_\rho(A, A).$$

(quantum) coarse graining = linear, (completely) positive, trace preserving map.

Petz theorem

$$L_\rho(A) := \rho A \qquad R_\rho(A) := A\rho$$

Petz theorem

There is bijection among quantum Fisher information and operator monotone functions given by the formula

$$\langle A, B \rangle_{\rho,f} := \mathrm{Tr}(A \cdot m_f(L_\rho, R_\rho)^{-1}(B)).$$

Summary

Löwner-Kubo-Ando-Petz

$$f$$

$$\Updownarrow$$

$$m_f(A,B) := A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

$$\Updownarrow$$

$$\langle A,B \rangle_{\rho,f} := \mathrm{Tr}(A \cdot m_f(L_\rho, R_\rho)^{-1}(B)).$$

Regular and non-regular QFI

$$\mathcal{F}_{op} := \{f \text{ op. mon.} | f(1) = 1, \quad t f(t^{-1}) = f(t)\}$$

$$\mathcal{F}_{op}^{r} := \{f \in \mathcal{F}_{op} | f(0) := \lim_{t \to 0} f(t) > 0\}$$

$$\mathcal{F}_{op}^{n} := \{f \in \mathcal{F}_{op} | f(0) = 0\}$$

$$\mathcal{F}_{op} = \mathcal{F}_{op}^{r} \cup \mathcal{F}_{op}^{n}$$

The function $\tilde{f}$

$$\tilde{f}(x) := \frac{1}{2}\left[(x+1) - (x-1)^2 \frac{f(0)}{f(x)}\right]$$

Theorem

$f \in \mathcal{F}_{op}^{r}$ (f is a regular n. s. o. m. function)

$$\Downarrow$$

$\tilde{f} \in \mathcal{F}_{op}^{n}$ ($\tilde{f}$ is a non-regular n. s. o. m. function)

Regular and non-regular means

$$f \;\rightarrow\; \tilde{f}$$

$$m_f \;\rightarrow\; m_{\tilde{f}}$$

Examples

$$\frac{x+y}{2} \;\rightarrow\; \frac{2}{\frac{1}{x}+\frac{1}{y}}$$

$$\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 \;\rightarrow\; \sqrt{xy}$$

Fundamental formula

Theorem

If f is regular then

$$\frac{f(0)}{2}\langle i[\rho,A],i[\rho,B]\rangle_{\rho,f}=\mathrm{Cov}_\rho(A,B)-\mathrm{Cov}_\rho^{\tilde{f}}(A,B).$$

The dynamical UP

Let $A_1 \ldots, A_N \in \mathcal{M}_{n,sa}(\mathbb{C})$.

$$\det\left\{\mathrm{Cov}_\rho(A_h, A_j)\right\} \geq \det\left\{f(0)\langle i[\rho, A_h], i[\rho, A_j]\rangle_{\rho,f}\right\}$$

for $h, j = 1, \ldots, N$,
for all $f \in \mathcal{F}_{op}$.

Nontrivial bound also if N is odd!

The dynamical UP (g-version)

Let $A_1 \ldots, A_N \in \mathcal{M}_{n,sa}(\mathbb{C})$.

$$\det\left\{\mathrm{Cov}^g_\rho(A_h, A_j)\right\} \geq \det\left\{g(0)f(0)\langle i[\rho, A_h], i[\rho, A_j]\rangle\right\}$$

for $h, j = 1, \ldots, N$,
for all $g, f \in \mathcal{F}_{op}$.

WYD information

$$I_\rho(\beta, A) = -\frac{1}{2}\mathrm{Tr}([\rho^\beta, A]\cdot[\rho^{1-\beta}, A])$$

plays a role in ....

WYD II

- strong subadditivity of entropy (Lieb-Ruskai,1973)
- homogeneity of the state space of factors of type III_1 (Connes-Stormer,1978;
- measures for quantum entanglement (Chen,2005; Klyachko-Oztop-Shumovsky,2006;
- uncertainty relations ;
- quantum hypothesis testing (Calsamiglia et al., 2008

Explanation

Indeed WYD information is a quantum Fisher information. To prove it one has to prove that the function

$$f_\beta(x) = \beta(1-\beta)\frac{(x-1)^2}{(x^\beta - 1)(x^{1-\beta} - 1)} \qquad 0 < \beta < 1,$$

is operator monotone. The original proof is quite complicated.

The inversion formula

For $g \in \mathcal{F}_{op}^{n}$ set

$$\check{g}(x) = g''(1) \cdot \frac{(x-1)^2}{2g(x) - (x+1)}$$

Then

$$\check{\check{\tilde{f}}} = f$$

WYD as QFI: a simple proof

The function $f_\beta \in \mathcal{F}_{op}^r$ for $0 < \beta < 1$.

Proof

The function

$$g_\beta(x) = \frac{x^\beta + x^{1-\beta}}{2} \qquad 0 < \beta < 1$$

is operator monotone. It easily follows that $g_\beta \in \mathcal{F}_{op}$ and that g_β is non-regular. Since $\tilde{f}_\beta = g_\beta$ we get the desired conclusion.

History of the results – I

- Luo (2000), *Lett. Math. Phys.*:
  N=1; proof for the SLD metric.
- Luo (2003), *Phys. Rev. Lett*:
  N=1; proof for the WY metric.
- Luo-Zhang Z.(2004.), *J. Statist. Phys.*:
  N=2; conjecture for the WY metric.
- Luo-Zhang Q. (2004), *IEEE Trans. Inform. Theory*:
  N=2; proof for the WY metric.

History of the results – II

- Kosaki (2005), *Internat. J. Math.*, **N=2; WYD(β) metric.**(Monotonicity for WYD and condition for equality)
- Yanagi *et alii* (2005), *IEEE Trans. Inform. Theory*:
  N=2; WYD(β) metric.
- Gibilisco-Isola (2007) *Ann. Ins. Stat. Math.*: **N=2; conjecture f arbitrary**
- Hansen (2008), *Proc. Nat. Acad. Sci. USA*, **N=1; proof f arbitrary.**

History of the results – III

- Gibilisco-Imparato-Isola (2007), *J. Math. Phys.*
  N=2; proof f arbitrary
- Gibilisco-Imparato-Isola (2008), *J. Stat. Phys.*
  conjecture N and f arbitrary
- Gibilisco-Imparato-Isola (2008), *Lin. Alg. Appl.*
  proof N and f arbitrary
- Andai (2008), *J. Math. Phys.*,
  proof N and f arbitrary

History of the results – IV

- Gibilisco-Isola (2008) *Inf. Dim. Anal. Quant. Prob.*
  N=2; WYD(β) metric.; s.f. Von Neumann alg.
- Gibilisco-Isola (2008) *Int. J. Math.*
  N=2; f arbitrary; Von Neumann alg.
- Gibilisco-Isola (2008) *J. Stat. Phys.*
  N arbitrary; f arbitrary; Von Neumann alg.
- Petz-Szabo (2009) *to appear on Int. J. Math.*
  N arbitrary; f arbitrary; Von Neumann alg.

History of the results – V

- Gibilisco-Petz-Hiai (2009) *IEEE Trans. Inf. Theor.*
  Dynamical UP for arbitrary g-covariance
- Audenaert-Cai-Hansen (2009) *Lett. Math. Phys.*
  New simple proof of Dynamical UP
- Gibilisco-Hansen-Isola (2009), *Lin. Alg. Appl.*
  Correspondence $f \longleftrightarrow \tilde{f}$
- Gibilisco-Isola (2009) *Preprint*
  Standard and dynamical UP for arbitrary g-covariance on Von Neumann alg.