How multi-agent interactions can beat a central controller - A solvable model based on Kullback-Leibler relative entropy.

M.-O. Hongler, O. Gallay,
R. Colmorn, Ph. Cordes and M. Hülsmann

Ecole Polytechnique Fédérale de Lausanne (EPFL), (CH)
Faculty of Engineering Sciences (STI)

Jacobs University - Bremen (D)
Systems Management - International Logistics.

Workshop on *Random Matrices, Information Theory and Applications*
Paris - 03/04 December 2009

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Highly complex decision issues $\Rightarrow$ tendency to decentralize the management

- Huge number of control parameters
- Feedback (i.e. non-linearity) in the underlying dynamics
- Ubiquitous presence of randomness in the dynamics
- ...

$\Downarrow$

Decisions based on limited rationality $\Rightarrow$ Rigid pre-planning offers poor performance

mutual interactions $\Downarrow$ self-organization

Autonomous agents might better perform than a central controller

$\Downarrow$ Goal of today's lecture

Exhibit a solvable model showing performance of decentralized control

A simple model for competitive dynamics

$$\dot{X}_k(t) = \underbrace{v_k(t)}_{\text{velocity}} - \gamma_k \underbrace{\mathbb{I}_k(\vec{X}(t), X_k(t))}_{\text{multi}-\text{agent interactions}} + \underbrace{q_k(\vec{X}(t)) dB_{k,t}}_{\text{noise sources}}, \qquad k = 1, 2, ..., N.$$

Multi-agent interactions:

$$\mathbb{I}_k(\vec{X}(t), X_k(t)) = \frac{1}{\mathcal{N}_k} \sum_{j \neq k}^{\mathcal{N}_k} \mathcal{I}(X_j(t)), \qquad \mathcal{N}_k := \text{neighbourhood of agent } k,$$

$$\mathcal{I}_k(X_j(t)) = \begin{cases} 0 & \text{if} \quad 0 \leq X_j(t) < X_k(t), \quad (\underline{\text{velocity unchanged}}), \\ 1 & \text{if} \quad X_k(t) \leq X_j(t) < X_k(t) + U, \quad (U > 0), \quad (\underline{\text{accelerate}}), \\ 0 & \text{if} \quad X_j(t) > X_k(t) + U, \quad (\underline{\text{velocity unchanged}}). \end{cases}$$

($U :=$ *"mutual influence"* interval).

Homogeneous population of agents

$$dX_k(t) = \underbrace{\left[v(t) - \gamma\mathbb{I}(\vec{X}(t), X_k(t))\right]}_{:=\ \text{drift field}\ \mathcal{D}_{k,v(x,t)}} dt \quad + \quad \underbrace{q\ dB_{k,t}}_{k\ \text{indep. White Gaussian Noise}}.$$

$\Downarrow$ diffusion process

Fokker - Planck diffusion equation

$$\frac{\partial}{\partial t}P(\vec{x},t) = -\sum_k \frac{\partial}{\partial x_k}\left[\mathcal{D}_{k,v(\vec{x},t)}P(\vec{x},t)\right] + \frac{1}{2}q^2 \sum_k \frac{\partial^2}{\partial x_k^2}\left[P(\vec{x},t)\right],$$

$P(\vec{x},t)$:= conditional probability density.

Mean Field Dynamics, (MFD) for homogeneous population of agents

$$\mathcal{N}_k \equiv \mathcal{N} \to \infty \quad \Rightarrow \quad \text{Mean Field Dynamics, (MFD).}$$

$\Downarrow$ dynamics for a representative effective agent

trajectories point of view — **probabilistic point of view**

$$\underbrace{\frac{1}{\mathcal{N}} \sum_{j \neq k}^{\mathcal{N}} \mathcal{I}(X_j(t))}_{\text{proportion of influencing agents acting on } k} \approx \underbrace{\int_x^{x+U} P(x,t)\, dx}_{\text{proportion of representative agents located in } [x, x+U]}$$

$\Downarrow$ effective Fokker-Planck equation

$$\frac{\partial}{\partial t} P(x,t) = -\frac{\partial}{\partial x} \underbrace{\left\{ \left[v(t) - \gamma \left(\int_x^{x+U} P(x,t) dx \right) \right] P(x,t) \right\}}_{non-linear \;\; and \;\; non-local \;\; field \;\; equation} + \frac{1}{2} q^2 \frac{\partial^2}{\partial x^2} \left[P(x,t) \right],$$

Small influence region - Burgers equation dynamics

Small values of $U \Rightarrow$ Taylor expand up to 1st order in U.

$$\Downarrow \qquad \int_x^{x+U} P(x,t)dx \simeq U\,P(x,t).$$

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}\underbrace{\{[v(t) - \gamma\, U\, P(x,t)]\, P(x,t)\}}_{nonlinear\ \ but\ \ local\ \ drift\ \ field} + \frac{1}{2}q^2\frac{\partial^2}{\partial x^2}[P(x,t)]\,.$$

$$t \mapsto \tau = \gamma t. \qquad \Downarrow \qquad x \mapsto z = \frac{\overbrace{x - \int_0^t v(s)\, ds}^{:=y}}{2U}.$$

Burgers Equation - (to be solved with initial condition $P(z,t) = \delta(z_-)$).

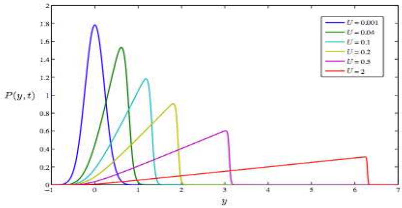
$$\boxed{\dot{P}(z,t) = \frac{1}{2}\frac{\partial}{\partial z}\left[P(z,t)^2\right] + \left[\frac{q^2}{8U^2\gamma}\right]\frac{\partial^2}{\partial z^2}[P(z,t)]}$$

Burgers Eq. $\longleftarrow$ $\text{logarithmic transformation, (Hopf} - \text{Cole)}$ $\longrightarrow$ Heat Eq.

Initial density: $P(y, t = 0) = \delta(y_-)$ $\Downarrow$ Exact integration

$$P(y,t) = -\frac{q^2}{4\gamma U^2}\frac{\partial}{\partial y}\ln\left[1 + \frac{(e^R - 1)}{2}\text{Erfc}\left(\frac{y}{q\sqrt{t}}\right)\right] =$$

$$= \frac{1}{R}\left[\frac{(e^R-1)\frac{1}{\sqrt{\pi q^2 t}}e^{-\frac{y^2}{q^2 t}}}{1 + \frac{(e^R-1)}{2}\text{Erfc}\left(\frac{y}{q\sqrt{t}}\right)}\right] := \frac{1}{R}\frac{(e^R-1)\mathbb{G}(y,t)}{\mathbb{E}(y,t)}. \qquad (1)$$

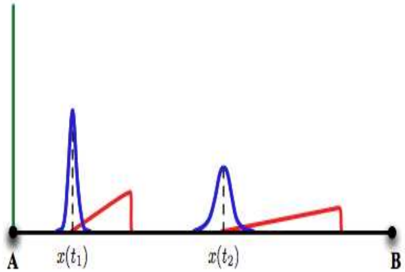


Typical shape of $P(y,t)$ for various R factors.

(view from the relative moving frame)

Normalization and positivity are visually manifest !!

The explicit benefit of competition - noise induced transport enhancement.



Position probability distribution: without interaction, with interactions.

- Additional traveled distance when $R = \frac{4\gamma U^2}{q^2} \to \infty$: $\langle X(t) \rangle_{t\to\infty} \simeq \frac{4U}{3}\sqrt{2\gamma t}$,
- Additional traveled distance when $R = \frac{4\gamma U^2}{q^2} \to 0$: $\langle X(t) \rangle_{t\to\infty} \simeq 0$.

Centralized optimal control

Controlled diffusion process

$$dY_t = \underbrace{c(Y,t)\,dt}_{\text{central controller}} + q\,dB_t, \qquad \underbrace{Y_0 = 0}_{\text{initial condition}}, \qquad (0 \le t \le T),$$

$\Downarrow$ (Fokker-Planck equation)

$$\frac{\partial}{\partial t}P_c(y,t) = -\frac{\partial}{\partial y}\left[c(y,t)P_c(y,t)\right] + \frac{q^2}{2}\frac{\partial^2}{\partial y^2}P_c(y,t).$$

Construct a drift controller $c(Y,t)$ which, for time T, fulfills

$$\underbrace{P_c(y,T)}_{\text{Prob. density with central controller}} = \underbrace{P(y,T)}_{\text{Prob. density due to agents interactions}}$$

Burgers' exact solution

Centralized optimal control - (continued)

Introduce an utility function $J_{\text{central},T}\left[c(y,t;T)\right]$ defined as:

$$J_{\text{central},T}\left[c(y,t;T)\right] = \left\langle \int_0^T \underbrace{\frac{c^2(y,s;T)}{2q^2}}_{\text{cost rate } \rho(y,s)} ds \right\rangle,$$

($\langle\cdot\rangle :=$ average over the realization of underlying stochastic process).

Optimal controller problem

Construct an $\underbrace{\text{optimal drift}}_{\text{i.e. yielding minimal cost}}$ $c^*(y,t;T)$ such that:

$$J_{\text{central},T}\left[c^*(y,t;T)\right] \leq J_{\text{central},T}\left[c(y,t;T)\right].$$

The **Paolo Dai Pra** solution of the optimal control problem

Optimal drift controller

$$c^*(y,t;T) = \frac{\partial}{\partial y} \ln\left[h(y,t)\right],$$

$$h(y,t) = \int_{\mathbb{R}} \mathbb{G}\left[(z-y),(T-t)\right] \frac{P(z,T)}{\mathbb{G}(z,t)} dz.$$

Paolo Dai Pra. *"A Stochastic Control Approach to Reciprocal Diffusion Processes"*. Appl. Math, Optim. **23**, (1991), 313-329.

Minimal cost

$$J_{\mathrm{central},T}\left[c^*(y,t;T)\right] = \underbrace{\mathcal{D}(P|\mathbb{G})}_{\mathrm{Kullback-Leibler}} = \begin{cases} 0 & \text{for} \quad t=0, \\ \frac{R}{2} + \ln\left[\frac{(e^R-1)}{R}\right] & \text{for} \quad t>0. \end{cases}$$

Decentralized agents - cost estimation

Cost $J_{\text{agents}}(T)$ for decentralized evolution during time horizon T :

$$J_{\text{agents}}(T) := \underbrace{\mathcal{N}}_{\sharp\ \text{population}} \cdot \underbrace{\frac{\rho}{\mathcal{N}}}_{\text{scaled cost}} \cdot \int_0^T ds \underbrace{\Phi(s)}_{\text{interacting agents}},$$

- $\rho = \dfrac{\overbrace{\gamma^2 U^2/2}^{\text{kinetic energy}}}{\underbrace{q^2}_{\text{diffusion rate}}} :=$ individual cost rate function,
- $\Phi(t) \in [0, 1] :=$ proportion of interacting agents at time t.

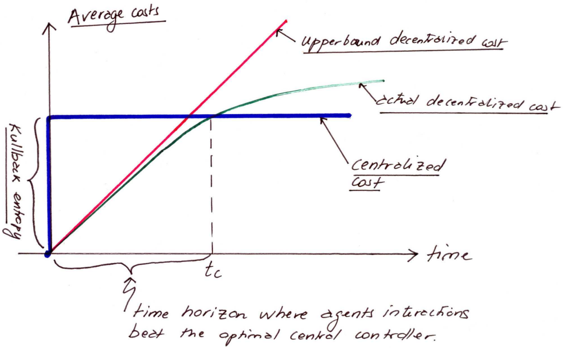
****

Cost upper-bound - reached when $\Phi(t) \equiv 1$

$$\Downarrow$$

$$\boxed{J_{\text{agent}}(T) \leq \rho T}$$

Costs comparison for our specific model



Heuristic interpretation :

For times $t < t_c$, the huge number of interactions favors the decentralized control.

To summarize and to somehow "philosophically" conclude !

The stylized model cartoons basic and somehow "universal" features:

- Agents' mimetic interactions produce an emergent structure - (here a "shock"- like wave).
- Competition enhances global transport flow - (here a $\sqrt{t}$-increase of the traveled distance).
- Self-organization via autonomous agents interactions can reduce costs.