Polarization codes and the rate of polarization

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Bilkent U., EPFL

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Channel Polarization

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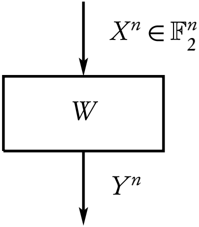

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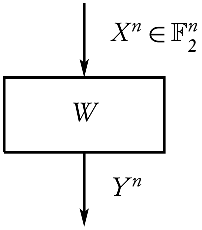

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Channel Polarization

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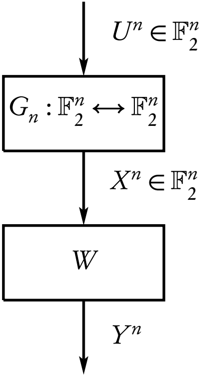

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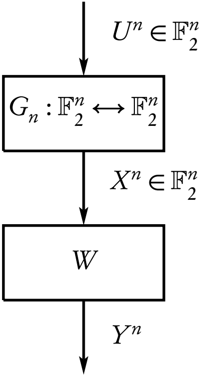
- Observe that U_i are i.i.d., uniform on $\{0,1\}$.



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Notation: $I(P)$ denotes the mutual information between the input and output of a channel P when input is uniformly distributed.

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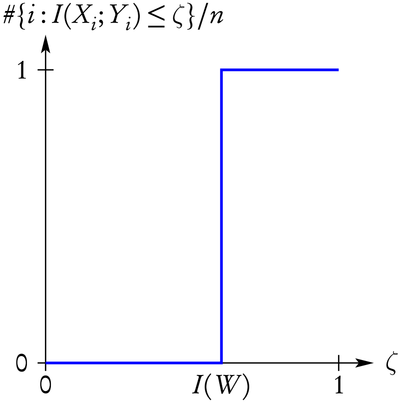

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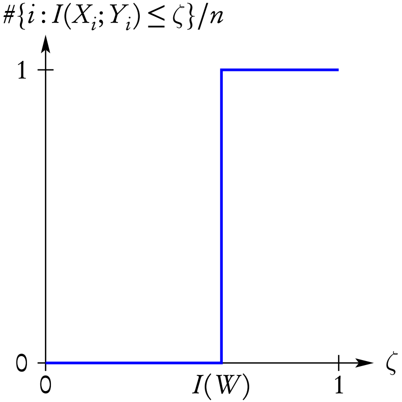

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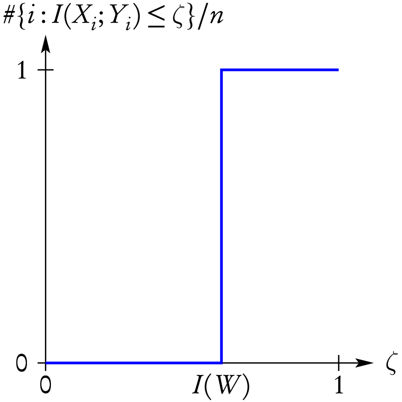

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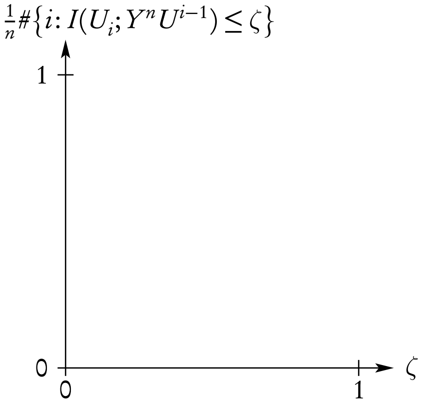

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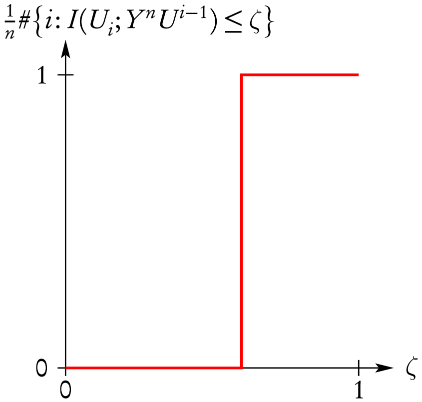

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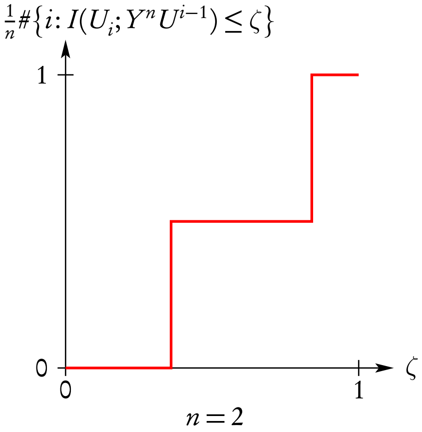

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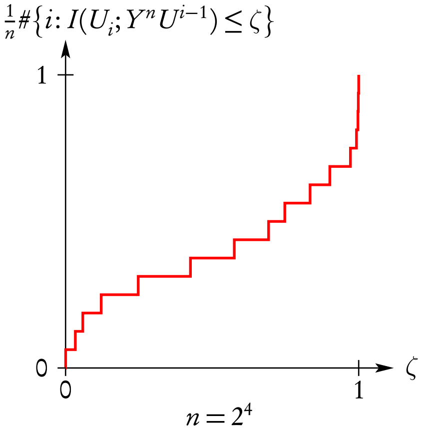

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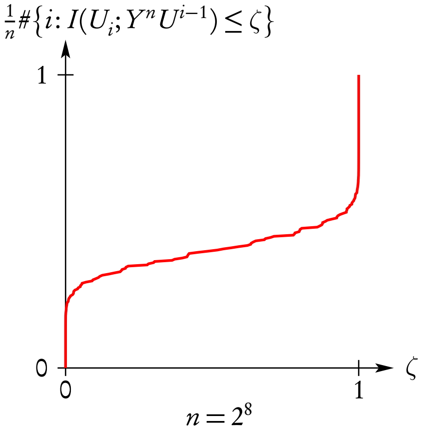

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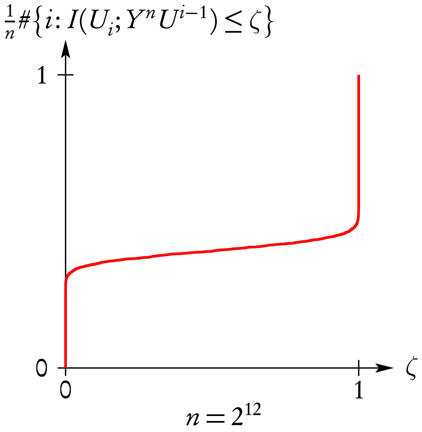

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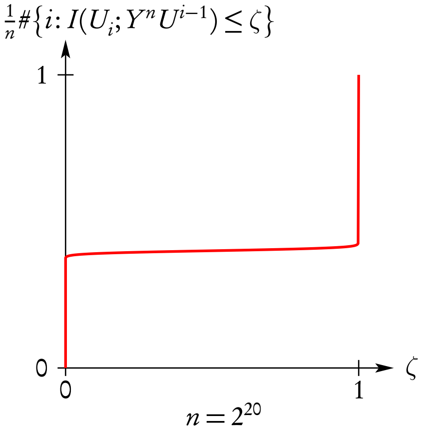

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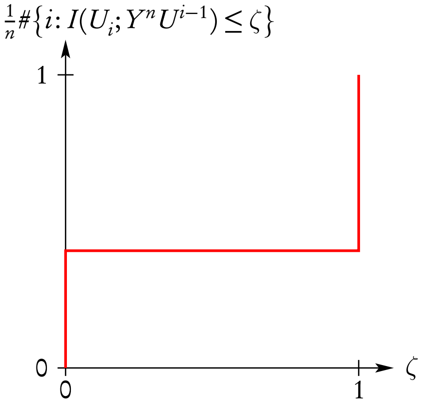

Channel Polarization

We say channel polarization takes place if it is the case that almost all of the numbers $I(U_i; Y^n U^{i-1})$ are near the extremal values,

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Equivalently, if

$$\frac{1}{n}\#\{i: I(U_i; Y^n U^{i-1}) \approx 1\} \to I(W)$$

and

$$\frac{1}{n}\#\{i: I(U_i; Y^n U^{i-1}) \approx 0\} \to 1 - I(W).$$

Channel Polarization

If polarization takes place and we wish to communicate at rate R:

- Pick n, and $k = nR$ good indices i such that $I(U_i; Y^n U^{i-1})$ is high,
- let the transmitter set U_i to be uncoded binary data for good indices, and set U_i to random but publicly known values for the rest,
- let the receiver decode the U_i successively: U_1 from Y^n; U_i from $Y^n \hat{U}^{i-1}$.
- One would expect this scheme to do well as long as there are k good indices, i.e., if $R < I(W)$.

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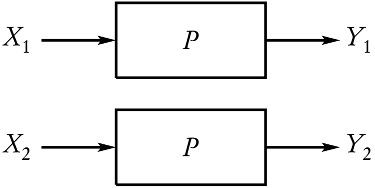
Polarization – HowTo

Given two copies of a binary input channel $P\colon \mathbb{F}_2 \to \mathscr{Y}$



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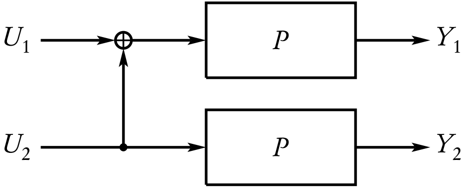

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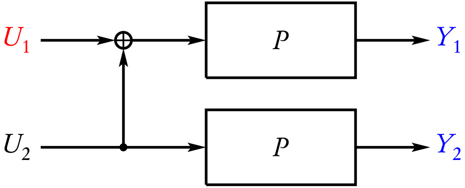

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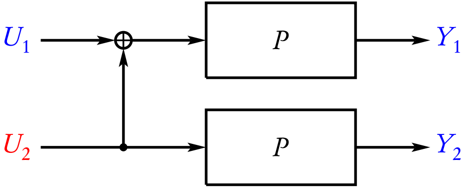

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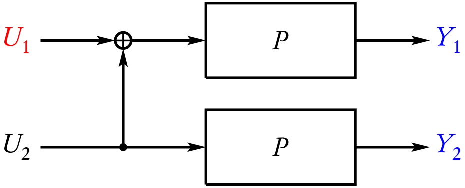

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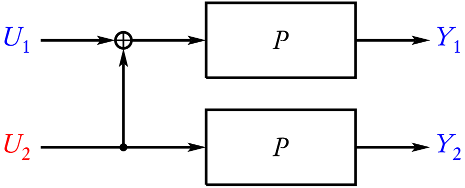

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- and $I(P^-) \leq I(P) \leq I(P^+)$.

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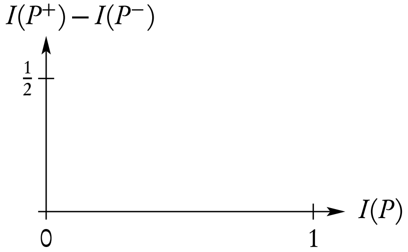
Polarization – Howto

How far apart are $I(P^-)$ and $I(P^+)$?



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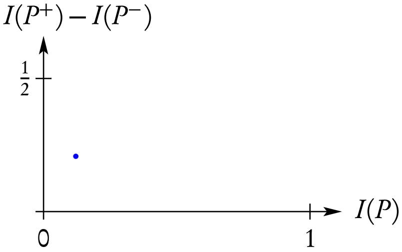

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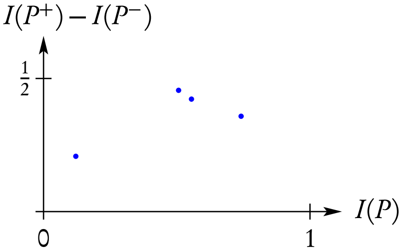

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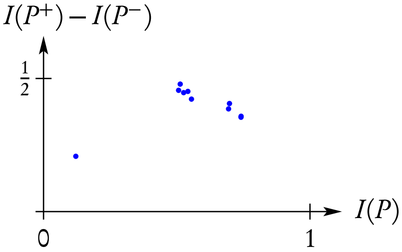

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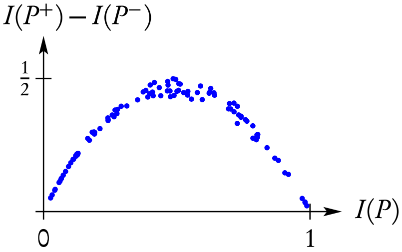

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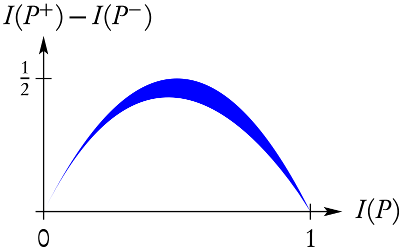

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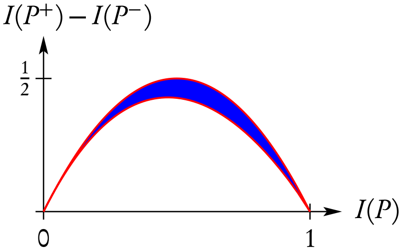

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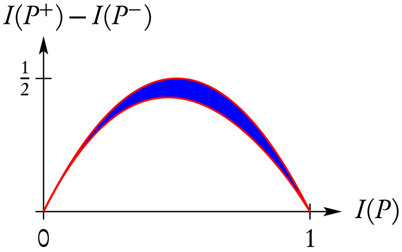

$I(P^+) - I(P^-) < \epsilon$ implies that $I(P) \notin (\delta, 1-\delta)$.

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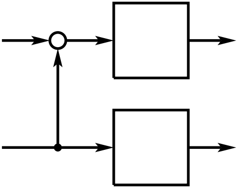

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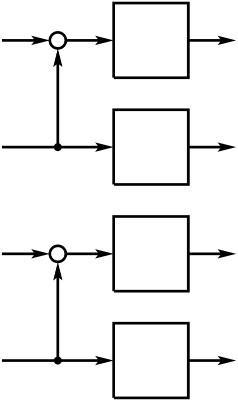

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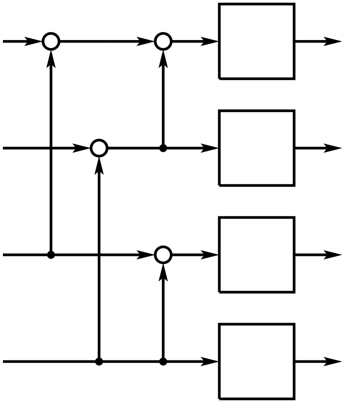

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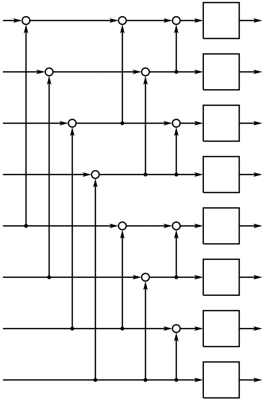
- Duplicate W^{--} (W^{-+}, W^{+-}, W^{++}) and obtain W^{---} and W^{--+} (W^{-+-}, W^{-++}, W^{+--}, W^{+-+}, W^{++-}, W^{+++}).



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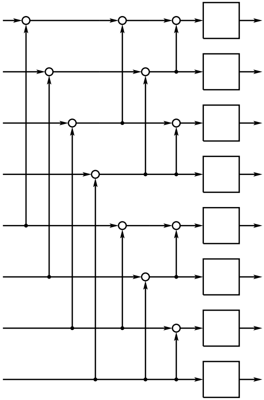

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- We now have a one-to-one transformation G_n between $U_1, \ldots, U_n$ and $X_1, \ldots, X_n$ for $n = 2^\ell$.

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- Furthermore, the $n = 2^\ell$ quantities

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- are exactly the n quantities

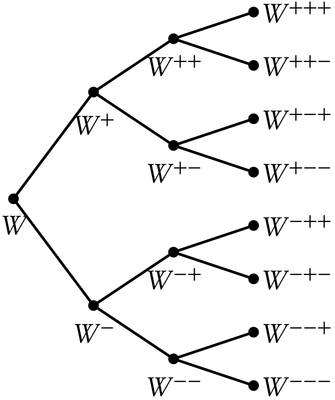
$$I(U_i; Y^n U^{i-1}), \quad i = 1, \ldots, n$$

whose empirical distribution we seek.

Polarization – HowTo

This suggests the following:

- Let $B_1, B_2, \ldots$ be i.i.d., equally likely to be $\{+, -\}$, $W_0 = W$, $W_\ell = W_{\ell-1}^{B_\ell}$.



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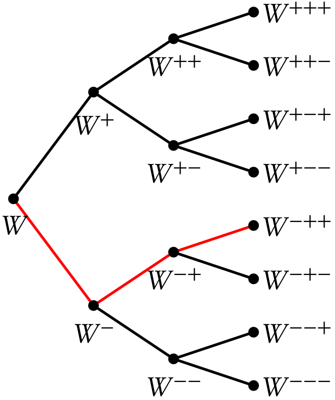

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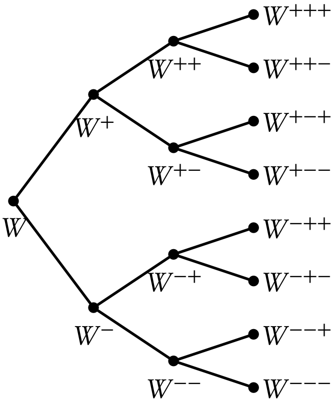

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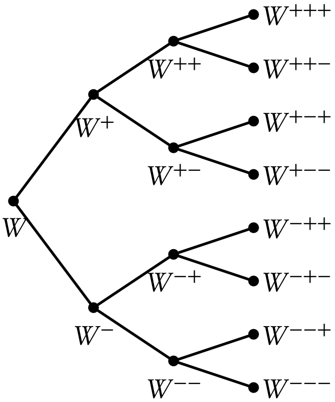
- $I_\ell = I(W_\ell)$ is distributed as

$$\Pr(I_\ell \in E) = \frac{1}{n} \#\{i : I(U_i; Y^n U^{i-1}) \in E\}.$$



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  and we see that $\{I_\ell\}$ is a martingale.
- Recall that what we are trying to show is that I_ℓ converges weakly to a $\{0,1\}$-valued random variable.

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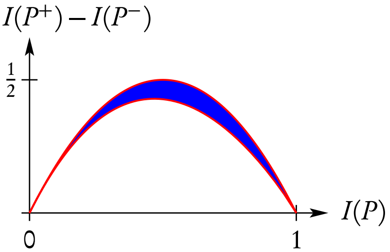

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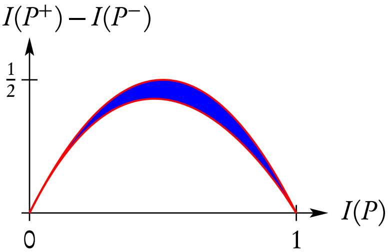
- Thus I_∞ is $\{0, 1\}$ valued!



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- But how fast? Fast enough to arrest error propagation?
- Introduce an auxiliary quantity

$$Z(P) = \sum_{y} \sqrt{P(y|0)P(y|1)}$$

as a companion to $I(P)$. Note that this is the Bhattacharyya upper bound on probability of error for uncoded transmission over P.

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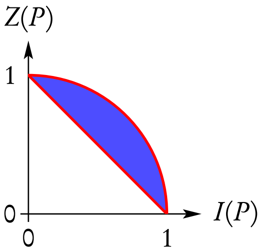

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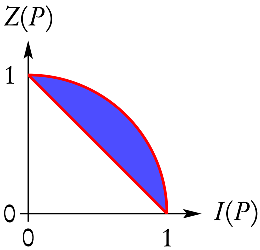

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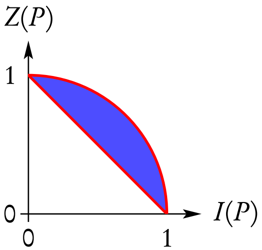

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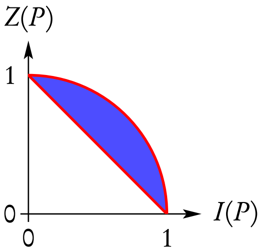

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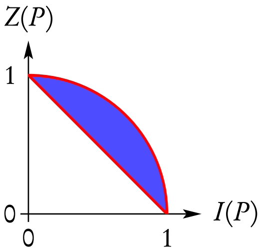

Since $Z(P)$ an upper bound on probability of error for uncoded transmission over P, we can choose the good indices on the basis of $Z(P)$. The sum of the Z's of the chosen channels will upper bound the block error probability. This suggests studying the polarization rate of Z.

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- This means that for any $\beta < 1/2$, as long as $R < I(W)$ the error probability of polarization codes decays to 0 faster than 2^{-n^β}.

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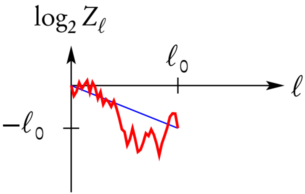
- Intersect this set with high probability event: that '$B_i = +$' and '$B_i = -$' occur with almost equal frequency for $i \le \ell$. It is then easy to see that $\lim_{\ell\to\infty} P(\log_2 Z_\ell \le -\ell) = I(W)$.

Polarization Rate

Proof idea – cont.



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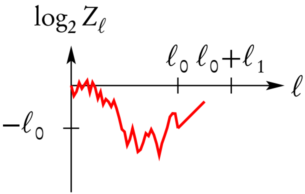

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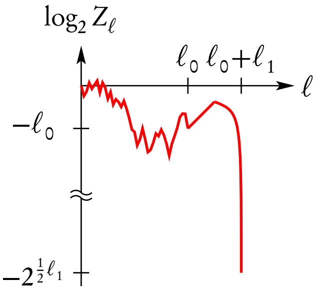

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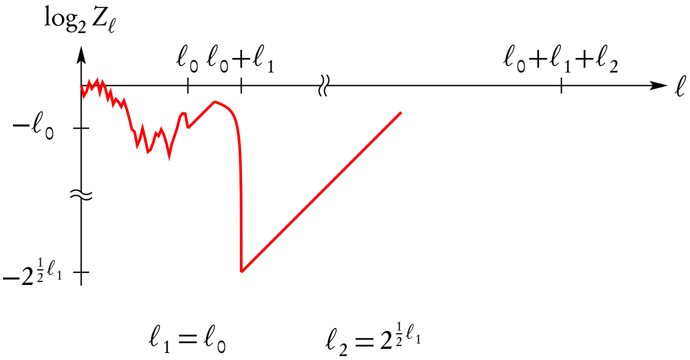

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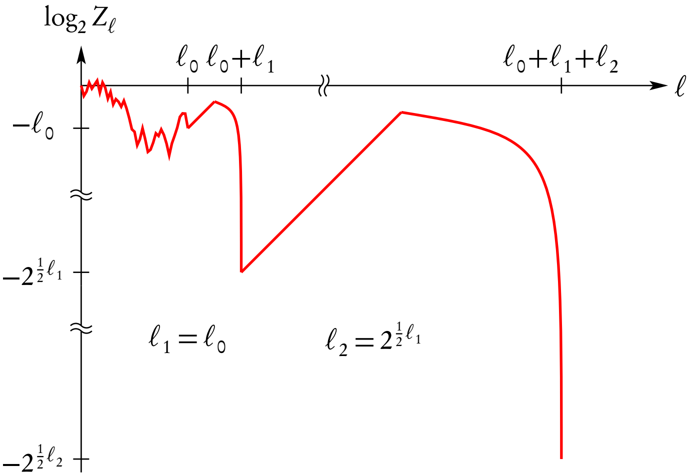

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- Dual constructions give quantizers.

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