Diversity of MIMO multihop channels with linear relaying

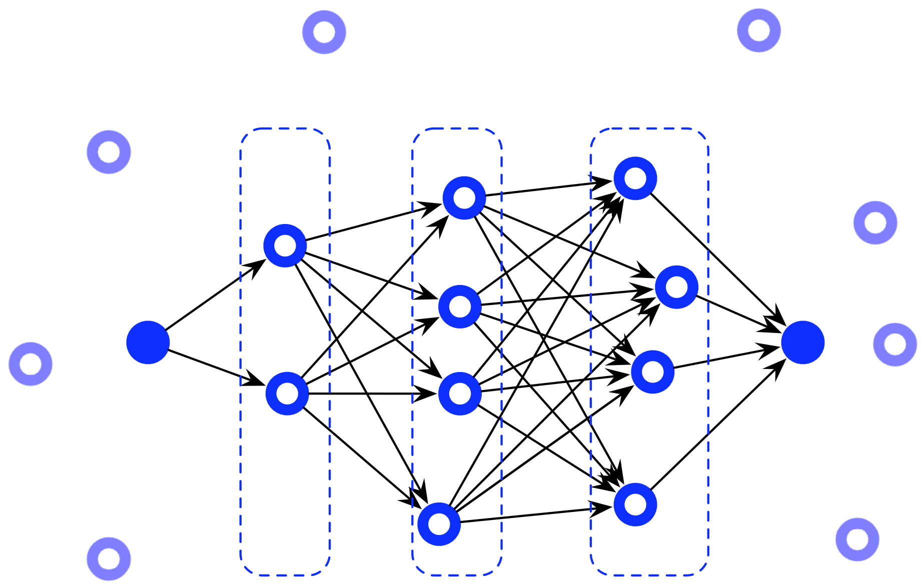
Sheng Yang

sheng.yang@supelec.fr
Department of Telecommunications - Supélec

Joint work with J.-C. Belfiore

December 4, 2009
UMLV-EPFL Workshop

Cooperation by relaying



How to cooperate, by relaying, in such a network?

Outline

- Introduction
- Diversity multiplexing tradeoff and random matrices
- Rayleigh product channel
- Linear relaying in multihop channels
- Conclusions

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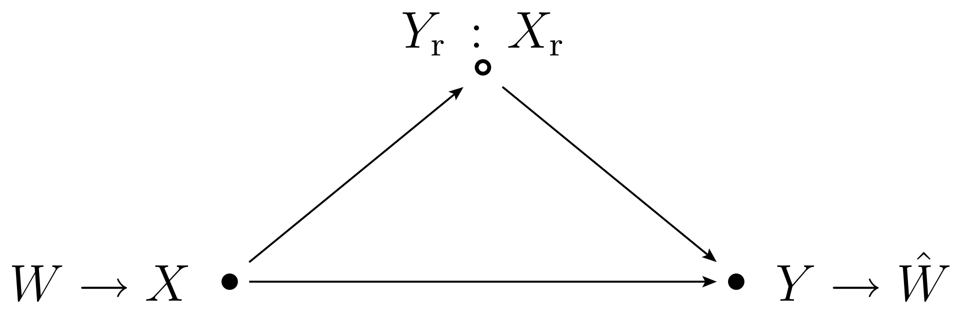
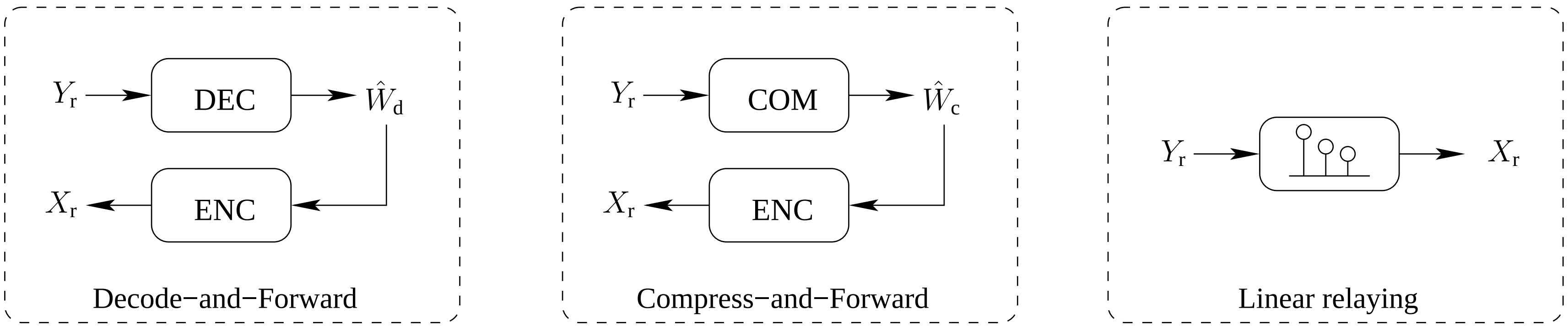
Relay Channel



[van der Meulen ’71]



Why Linear Relaying?

Cons

- Suboptimal in general
- Information lossy, noise amplification/cumulation
- ...

Pros

- No channel state information is needed at the relays
- Low relaying complexity: small terminals
- Low signaling complexity: codebook information, *ad hoc* setting
- Linearity: equivalent MIMO structure, code design
- ...

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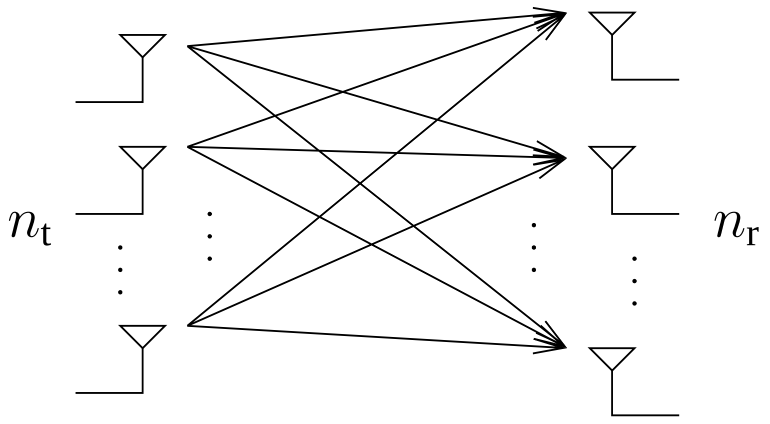
Diversity and Multiplexing



$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{z}, \quad \mathbf{z} \sim \mathcal{CN}(0, \mathbf{I})$$

multiplexing	diversity
$C \sim \min\{n_t, n_r\} \log \mathsf{SNR}$	$P_e(R) \sim \mathsf{SNR}^{-n_t n_r}$

At high SNR, with i.i.d. Rayleigh fading [FGVP'99, Telatar'99]

Diversity Multiplexing Tradeoff

Defining multiplexing gain and diversity gain,

$$r \triangleq \lim_{\mathsf{SNR}\to\infty} \frac{R}{\log \mathsf{SNR}} \quad \text{and} \quad d \triangleq -\lim_{\mathsf{SNR}\to\infty} \frac{\log P_{\mathrm{e}}(R)}{\log \mathsf{SNR}},$$

there is a tradeoff between them in *slow fading* channels [ZT'03]

$$P_{\mathrm{e}}(r \log \mathsf{SNR}) \sim \mathsf{SNR}^{-d(r)}$$

Outage Formulation

We use outage probability to establish optimal DMT [ZT'03]

$$
\begin{aligned}
P_{\text{out}}(r\log\mathsf{SNR}) &\triangleq \inf_{p(\boldsymbol{x})} \mathsf{P}\left\{I(\mathbf{x};\mathbf{y}|\mathbf{H}=\boldsymbol{H}) < r\log\mathsf{SNR}\right\} \\
&= \inf_{\substack{\boldsymbol{Q}\succeq 0 \\ \mathrm{Tr}(\boldsymbol{Q})\le\mathsf{SNR}}} \mathsf{P}\left\{\log\det\left(\mathbf{I}+\boldsymbol{H}\boldsymbol{Q}\boldsymbol{H}^H\right) < r\log\mathsf{SNR}\right\} \\
&\doteq \mathsf{P}\left\{\log\det\left(\mathbf{I}+\mathsf{SNR}\boldsymbol{H}\boldsymbol{H}^H\right) < r\log\mathsf{SNR}\right\} \\
&= \mathsf{P}\left\{\sum_{i=1}^{\min\{n_{\mathrm{t}},n_{\mathrm{r}}\}} \log\left(1+\mathsf{SNR}\sigma_i^2\right) < r\log\mathsf{SNR}\right\} \\
&\doteq \mathsf{SNR}^{-d(r)},
\end{aligned}
$$

"$a \doteq b$" means $\lim_{\mathsf{SNR}\to\infty} \frac{\log a}{\log\mathsf{SNR}} = \lim_{\mathsf{SNR}\to\infty} \frac{\log b}{\log\mathsf{SNR}}$

Outage Formulation - MIMO Channel

Let $q \triangleq \min\{n_{\mathrm{t}}, n_{\mathrm{r}}\}$, $\alpha_i \triangleq -\frac{\log \sigma_i^2}{\log \mathsf{SNR}}$, $\sigma_i^2 = \mathsf{SNR}^{-\alpha_i}$,

$$p_{\alpha}(\alpha) \doteq \begin{cases} \mathsf{SNR}^{-E(\alpha)}, & \text{for } \alpha \in \mathcal{R}_{\alpha}, \\ \mathsf{SNR}^{-\infty}, & \text{otherwise} \end{cases}$$

$$\begin{aligned} \mathsf{P}\left\{ \sum_i \log\left(1 + \mathsf{SNR}\sigma_i^2\right) < r \log \mathsf{SNR} \right\} &= \mathsf{P}\left\{ \sum_i (1-\alpha_i)^+ < r \right\} \\ &= \int_{\mathcal{O}_{\alpha}(r)} p_{\alpha}(\alpha) d\alpha \\ &\doteq \int_{\mathcal{O}_{\alpha}(r) \cap \mathcal{R}_{\alpha}} \mathsf{SNR}^{-E(\alpha)} d\alpha \\ &\doteq \mathsf{SNR}^{-\inf_{\mathcal{O}_{\alpha}(r) \cap \mathcal{R}_{\alpha}} E(\alpha)} \end{aligned}$$

Outage Formulation - MIMO Channel

Example: i.i.d. Rayleigh fading paths

$$p_{\sigma^2}(\sigma^2) = K^{-1} \exp\left(-\sum_{i=1}^{q} \sigma_i^2\right) \prod_{i=1}^{q} \sigma_i^{-2(|n_\mathrm{t}-n_\mathrm{r}|)} \prod_{i<j} (\sigma_i^2 - \sigma_j^2)^2, \quad \sigma_1^2 > \cdots > \sigma_q^2$$

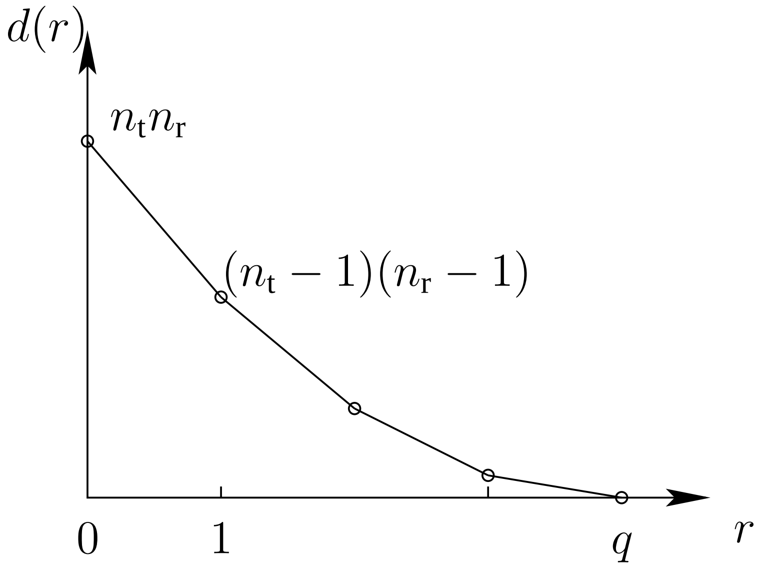
$$p_\alpha(\alpha) = K^{-1} (\log \mathsf{SNR})^n e^{-\sum_{i=1}^{q} \mathsf{SNR}^{-\alpha_i}} \cdot \underbrace{\prod_{i=1}^{q} \mathsf{SNR}^{-(|n_\mathrm{t}-n_\mathrm{r}|+1)\alpha_i}}_{=\mathsf{SNR}^{-\sum_i (|n_\mathrm{t}-n_\mathrm{r}|+1)\alpha_i}} \underbrace{\prod_{i<j} (\mathsf{SNR}^{-\alpha_i} - \mathsf{SNR}^{-\alpha_j})^2}_{\doteq \mathsf{SNR}^{-\sum_i 2(q-i)\alpha_i}}$$

Outage Formulation - MIMO Channel

$$E(\alpha) = \sum_i (n_t + n_r + 1 - 2i)\alpha_i, \quad 0 < \alpha_1 < \ldots < \alpha_q$$

$$\mathcal{O}_\alpha(r) = \left\{ \alpha \;\middle|\; \sum_i (1-\alpha_i)^+ < r \right\}$$

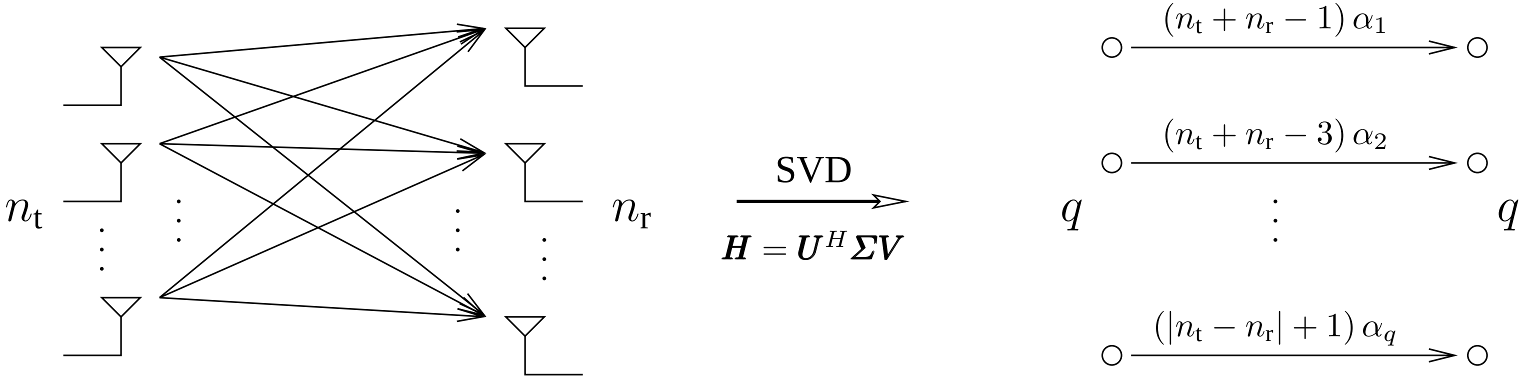
$$d(r) = \inf_{\substack{\mathcal{O}_\alpha(r) \\ 0<\alpha_1<\ldots<\alpha_q}} \sum_i (n_t + n_r + 1 - 2i)\alpha_i$$



Interpretation

$$d(r) = \inf_{\mathcal{O}_\alpha(r) \cap \mathcal{R}_\alpha} E(\alpha)$$

- $E(\alpha)$: "cost" function of α
- $\sum_i \alpha_i$: "singularity level" of the channel, $0 < \alpha_i < 1$
- $\mathcal{O}_\alpha(r)$: $\sum_i \alpha_i > q - r$
- $d(r)$: "mininum cost" to achieve "singularity level" $q - r$
  - $d(0)$ is the cost to "cut" all flows
  - $d(q-1)$ is the minimum cost to "cut" one flow: $|n_t - n_r| + 1$



- Introduction
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- **Rayleigh product channel**
- Linear relaying in multihop channels
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Rayleigh Product Channel

A $(n_0, \ldots, n_N)$ Rayleigh product channel is

$$\boldsymbol{y} = \boldsymbol{H}_N \boldsymbol{H}_{N-1} \cdots \boldsymbol{H}_1 \boldsymbol{x} + \boldsymbol{z}$$

- if $\boldsymbol{H}_i \in \mathbb{C}^{n_i \times n_{i-1}}$ has i.i.d. standard complex Gaussian entries
- $(\tilde{n}_0, \ldots, \tilde{n}_N)$, the inceasingly ordered version of $(n_0, \ldots, n_N)$, is called the ordered dimension
- Generally intractable, except in large dimensions [Müller'02, FD'08]
- We are only interested in the distribution of SNR exponents α_i

Our Result

Theorem 1. *Let us denote the non-zero ordered singular values of* $\boldsymbol{H}_N \cdots \boldsymbol{H}_1$ *by* $\sigma_1 > \cdots > \sigma_{n_{\min}} > 0$ *with* $n_{\min} \triangleq \min_{i=0,\ldots,N} n_i$*. Then, the joint pdf of* α *satisfies*

$$p(\alpha) \doteq \begin{cases} \mathsf{SNR}^{-E(\alpha)}, & \text{for } 0 < \alpha_1 < \ldots < \alpha_{n_{\min}} \\ \mathsf{SNR}^{-\infty}, & \text{otherwise} \end{cases}$$

where

$$E(\alpha) \triangleq \sum_{i=1}^{n_{\min}} c_i \alpha_i$$

$$c_i \triangleq 1 - i + \min_{k=1,\ldots,N} \left\lfloor \frac{\sum_{l=0}^{k} \tilde{n}_l - i}{k} \right\rfloor, \quad i = 1, \ldots, n_{\min}.$$

- The DMT can be easily deduced: $d(k) = \sum_{i \geq k+1} c_i$
- Only the ordered dimension $(\tilde{n}_0, \ldots, \tilde{n}_N)$ matters, e.g., $(1,2,1) \sim (1,1,2)$, $(2,4,3) \sim (2,3,4)$

Sketch of Proof

Our approach is by induction on N

- $N = 1$, Rayleigh channel, $E_1(\alpha) = \sum_i (n_t + n_r + 1 - 2i)\alpha_i$
- Given the SNR exponents β of $\mathbf{\Pi}_N \triangleq \boldsymbol{H}_N \cdots \boldsymbol{H}_1$, deduce the conditional distribution $p(\alpha|\beta)$ of the SNR exponents α of $\mathbf{\Pi}_{N+1}$ using

$$\mathbf{\Pi}_{N+1}\mathbf{\Pi}_{N+1}^H \mid \mathbf{\Pi}_N \quad \sim \quad \underbrace{\mathcal{W}_{n_0}(n_{N+1}, \mathbf{\Pi}_N \mathbf{\Pi}_N^H)}_{\text{correlated Wishart distribution}}$$

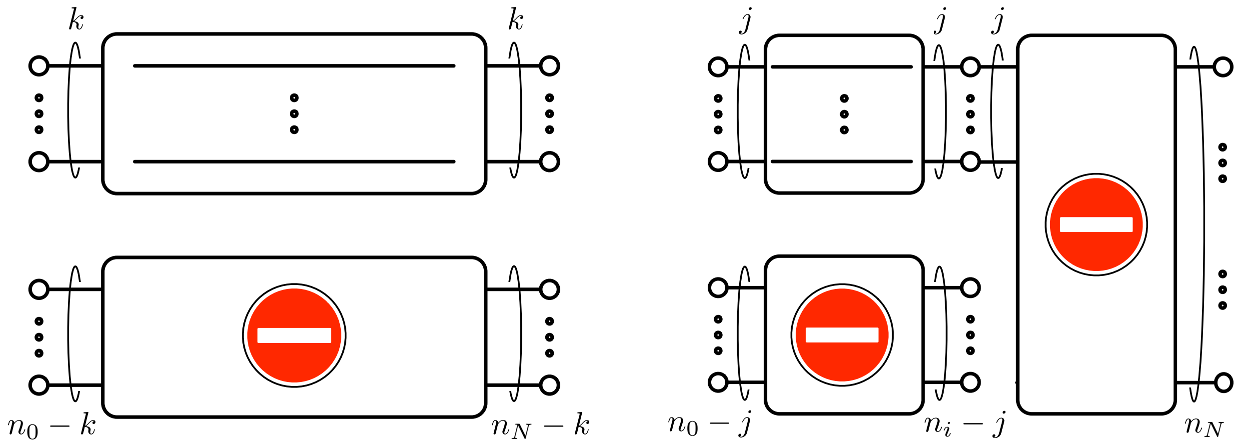
$$\begin{aligned} p_\alpha(\alpha) &= \int_{\mathbb{R}^{n_{\min}}} p_{\alpha|\beta}(\alpha|\beta) p_\beta(\beta) d\beta \\ &\doteq \int_{\mathcal{R}_{\alpha|\beta} \cap \mathcal{R}_\beta} \mathsf{SNR}^{-E_N(\alpha|\beta)} \mathsf{SNR}^{-E_N(\beta)} d\beta \\ &\doteq \mathsf{SNR}^{-E_{N+1}(\alpha)} \\ E_{N+1}(\alpha) &= \inf_{\beta \in \mathcal{R}_{\alpha|\beta} \cap \mathcal{R}_\beta} \{E_N(\alpha|\beta) + E_N(\beta)\} \end{aligned}$$

Recursive Characterization

Theorem 2. *The DMT of a Rayleigh product channel satisfies*

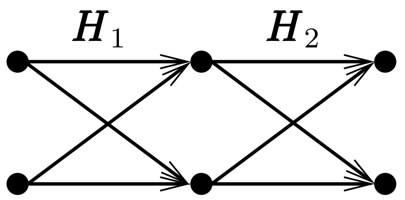
$$d_{(n_0,\ldots,n_N)}(k) = d_{(n_0-k,\ldots,n_N-k)}(0)$$
$$d_{(n_0,\ldots,n_N)}(0) = \min_j \left\{ d_{(n_0,\ldots,n_i)}(j) + d_{(j,\ldots,n_N)}(0) \right\}, \quad i = 1,\ldots,N-1$$

- k: "network flow"
- $d(k)$: minimum "cost" to limit the network flow to k
- $d(0)$: "disconnection cost"



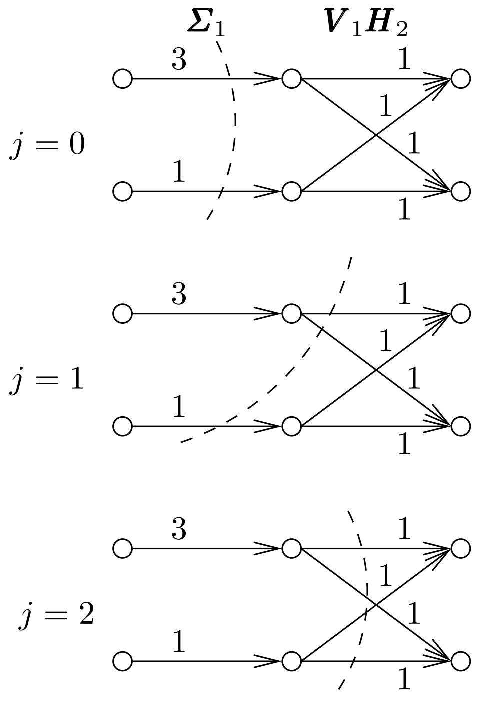
flow-k/singularity-$(n_{\min}-k)$ disconnection/singularity-$n_{\min}$

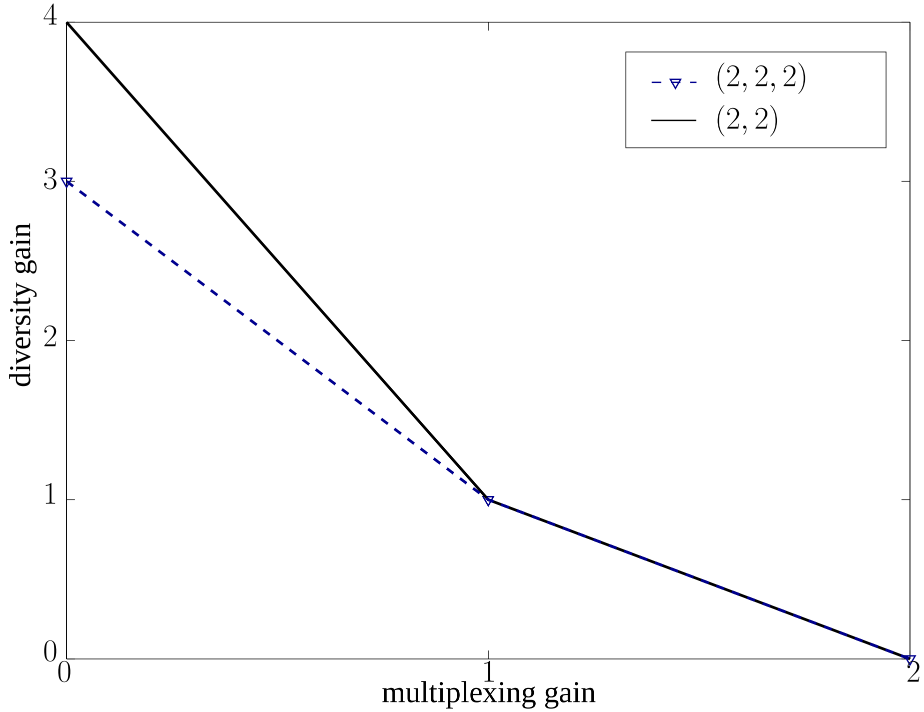
Recursive Characterization - $(2,2,2)$



$$d_{(2,2,2)}(2) = d_{(0,0,0)}(0) = 0, \quad d_{(2,2,2)}(1) = d_{(1,1,1)}(0) = 1$$

$$d_{(2,2,2)}(0) = \min_j \{ d_{(2,2)}(j) + d_{(j,2)}(0) \}$$





Recaps

- Rayleigh product channel is completely characterized in the high SNR regime
- Only ordered dimension matters

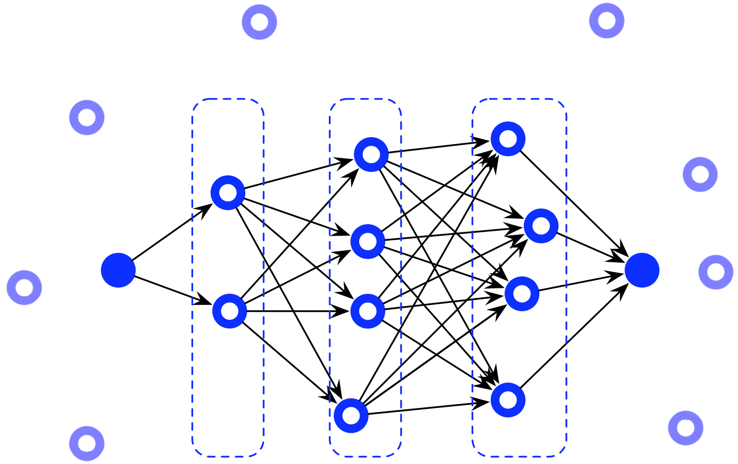
$$d_{(n_0,\dots,n_N)}(r) = d_{(\tilde{n}_0,\dots,\tilde{n}_N)}(r)$$

- Matrices multiplication reduces diversity

$$d_{(n_0,\dots,n_N)}(r) \leq d_{(\tilde{n}_0,\tilde{n}_1)}(r)$$

  with equality iff $\tilde{n}_2 + 1 \geq \tilde{n}_0 + \tilde{n}_1$
- Nice intuitive recursive characterization: outage events identified

Layered Multihop Networks

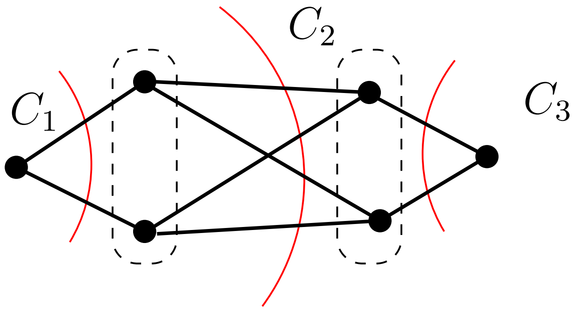


- Single-source single-destination $(n_0, \ldots, n_N)$ multihop network
- Full duplex relaying
- Layer $i+1$ only hears layer i (e.g., directional antennas, scheduling, etc.)

$$\boldsymbol{y}_{i+1}[t] = \boldsymbol{H}_{i+1}[t]\boldsymbol{x}_i[t] + \boldsymbol{z}_{i+1}[t], \quad i = 0, \ldots, N-1$$

DMT: Limits? Achievability?

Cut-Set Bound



$$C \le \min\{C_1, C_2, C_3\}$$

$$P_{\text{out}}(R) \ge \max\left\{P_{\text{out}}^{(1)}(R), P_{\text{out}}^{(2)}(R), P_{\text{out}}^{(3)}(R)\right\}$$

$$\mathsf{SNR}^{-d(r)} \ge \max\left\{\mathsf{SNR}^{-d_1(r)}, \mathsf{SNR}^{-d_2(r)}, \mathsf{SNR}^{-d_3(r)}\right\}$$

$$= \mathsf{SNR}^{-\min\{d_1(r), d_2(r), d_3(r)\}},$$

Theorem 3 (Cut-set bound)**.** *For any relaying strategy $\mathcal{T}$, we have*

$$d^{\mathcal{T}}(r) \le \min_{i=1,\dots,N} d_{(n_{i-1},n_i)}(r)$$

where $d_{(n_{i-1},n_i)}(r)$ is the DMT of the point-to-point channel between layer $i-1$ and layer i.

Supélec

Linear Relaying

- We consider antenna-wise linear relaying

$$\boldsymbol{x}_i[t] = \boldsymbol{D}_i[t]\boldsymbol{y}_i[t-1]$$

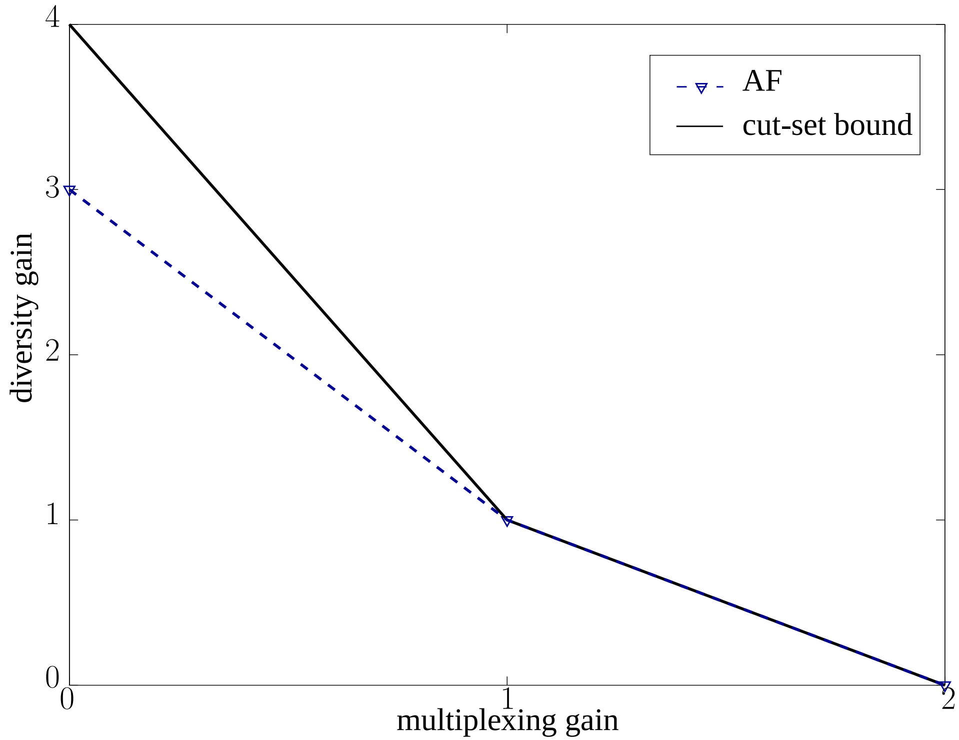
- The received signal at the destination, for $N = 2$, is

$$\begin{aligned}\boldsymbol{y}_2[t+1] &= \boldsymbol{H}_2[t+1]\boldsymbol{D}_1[t+1]\boldsymbol{H}_1[t]\boldsymbol{x}_0[t] + \boldsymbol{H}_2[t+1]\boldsymbol{D}_1[t+1]\boldsymbol{z}_1[t] + \boldsymbol{z}_2[t+1] \\ &= \underbrace{\boldsymbol{H}_2\boldsymbol{D}_1[t+1]\boldsymbol{H}_1}_{\text{eq. channel}}\boldsymbol{x}_0[t] + \underbrace{\boldsymbol{H}_2\boldsymbol{D}_1[t+1]\boldsymbol{z}_1[t] + \boldsymbol{z}_2[t+1]}_{\text{eq. noise}}\end{aligned}$$

- We call it naive amplify-and-forward scheme, if $\boldsymbol{D}_i[t]$ does not depend on t
- If $\boldsymbol{D}_i$ are full rank and constant, *it can be shown that the naive AF scheme is DMT-equivalent to a $(n_0, \ldots, n_N)$ Rayleigh product channel*

AF vs. Cut-Set Bound

	AF	cut-set bound
DMT	$d_{(\tilde{n}_0,\ldots,\tilde{n}_N)}(r)$	$\min_i d_{(n_i,n_{i+1})}(r)$
$r_{\max}$	$n_{\min}$	$n_{\min}$
$d_{\max}$	$\leq \min_{i\neq j}\{n_i n_j\}$	$\min_i\{n_i n_{i+1}\}$



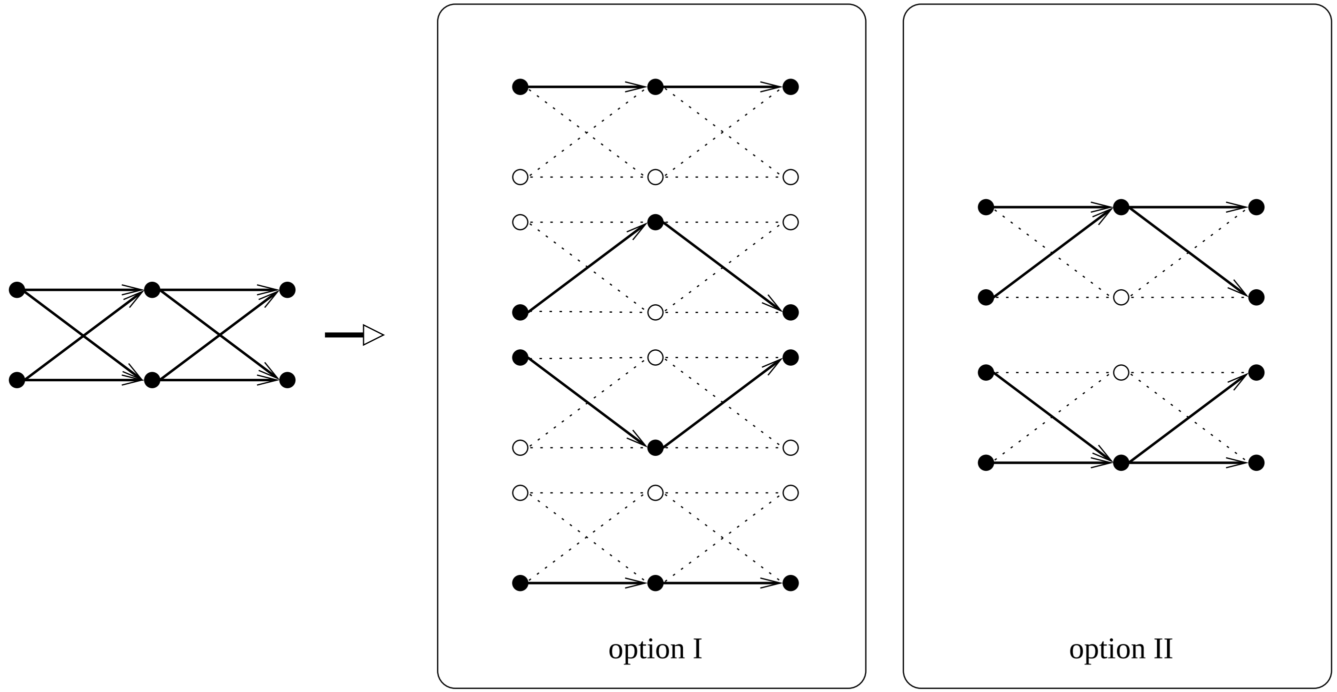
Recaps

- Naive AF induces multiplication of channel matrices
- Diversity is degraded with AF operations with multiple hops
- The cut-set bound suggests a better performance in general.

But we can do better, at least for maximum diversity!

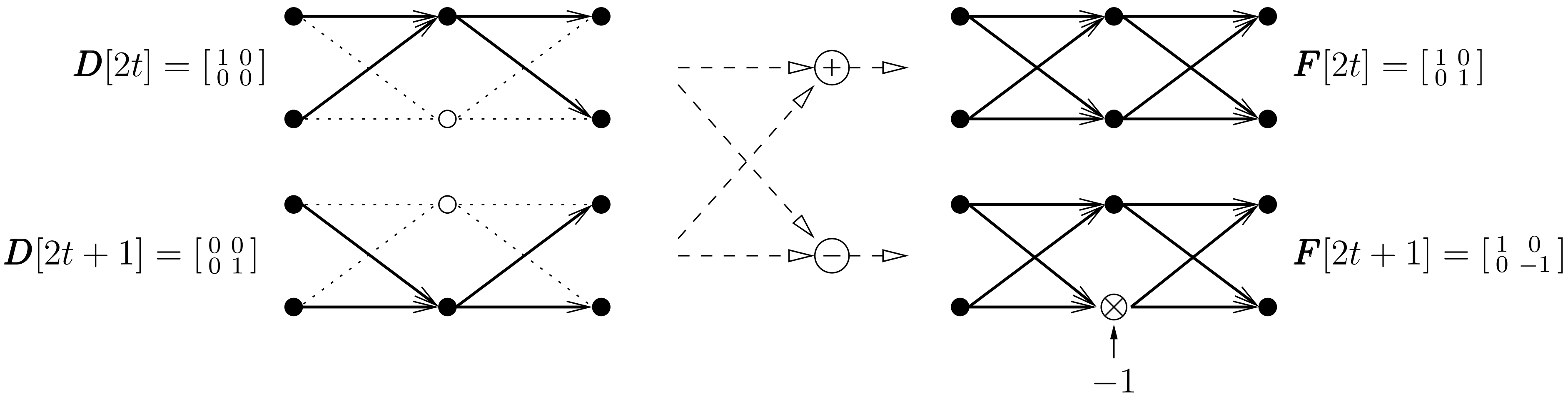
Parallel Partitions

- Find independent "AF paths", exactly $\min_i \{n_i n_{i+1}\}$ in total
- Take the paths in an orthogonal way, e.g., in different time slots
- Let each codeword go through all paths, then end-to-end diversity is the sum diversity, $\min_i \{n_i n_{i+1}\}$



But multiplexing severely suboptimal!

Flip-and-Forward



- Preserve optimal diversity, due to orthogonal transformation

$$\|\boldsymbol{H}_2\boldsymbol{D}[2t]\boldsymbol{H}_1\|^2 + \|\boldsymbol{H}_2\boldsymbol{D}[2t+1]\boldsymbol{H}_1\|^2 \sim \|\boldsymbol{H}_2\boldsymbol{F}[2t]\boldsymbol{H}_1\|^2 + \|\boldsymbol{H}_2\boldsymbol{F}[2t+1]\boldsymbol{H}_1\|^2$$

- Full rank channel matrix: multiplexing optimal

Flip-and-Forward

- Flip-and-forward introduces *full rank space-time* processing in a *distributed* manner
- Time-varying equivalent channel is created

$$\boldsymbol{G}[t] = \boldsymbol{H}_2 \boldsymbol{F}[t] \boldsymbol{H}_1$$

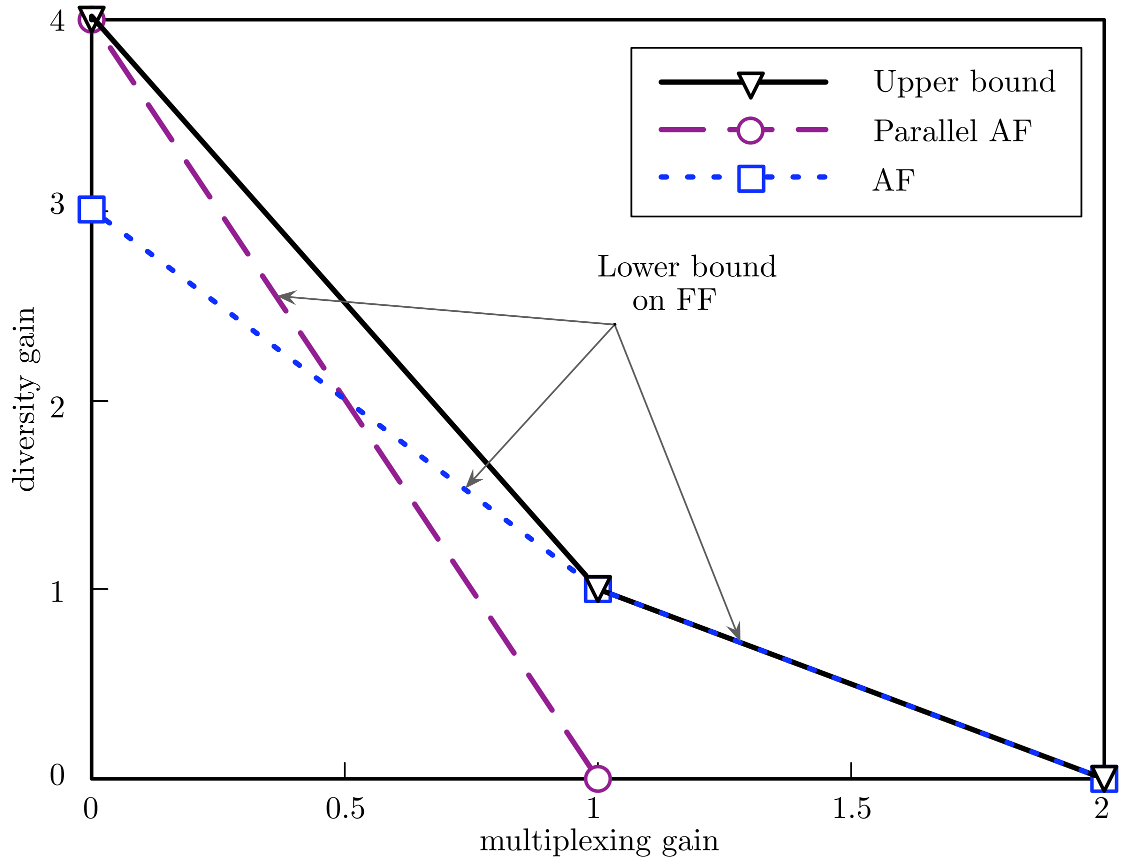
  with

$$\boldsymbol{F}[t] \in \mathcal{F} \triangleq \left\{ \mathrm{diag}\left\{ e^{j\theta_1}, \ldots, e^{j\theta_{n_1}} \right\} \middle| \theta_k \in \{0, \pi\} \right\}$$

- Let $\boldsymbol{F}[i] \neq \boldsymbol{F}[j]$, $\forall i \neq j \in [0, |\mathcal{F}| - 1]$. The equivalent mutual information is

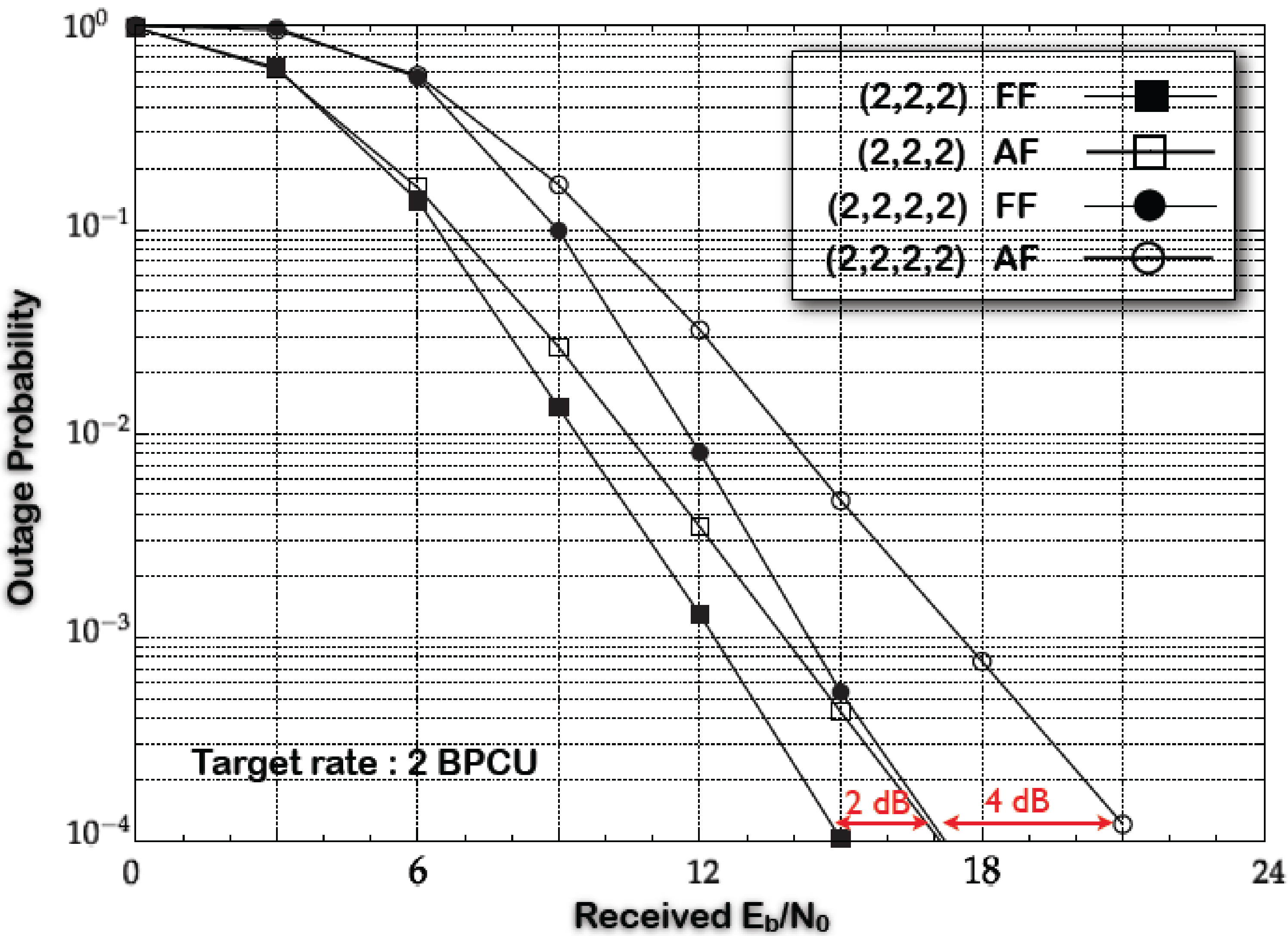
$$\frac{1}{|\mathcal{F}|} \sum_{t=0}^{|\mathcal{F}|-1} \log\det\left(\mathbf{I} + \mathsf{SNR}\, \boldsymbol{G}[t] \boldsymbol{G}[t]^H\right)$$

Flip-and-Forward



- Flip-and-forward is diversity and multiplexing optimal in general cases

Numerical Example



Generalizing the FF Scheme, $N = 2$

Is it possible to achieve the optimal DMT with linear relaying?

- Relaxing the constraint

$$\mathcal{F} \triangleq \left\{ \operatorname{diag}\left\{ e^{j\theta_1}, \ldots, e^{j\theta_{n_1}} \right\} \middle| \theta_k \in \left\{ 0, \frac{2\pi}{K}, \ldots, \frac{2(K-1)\pi}{K} \right\} \right\}, K \geq 1$$

- $K \to \infty$, $\theta_k \in [0, 2\pi)$

$$\frac{1}{|\mathcal{F}|} \sum_{t=0}^{|\mathcal{F}|-1} \log\det\left(\mathbf{I} + \mathsf{SNR}\,\boldsymbol{G}[t]\boldsymbol{G}[t]^H\right)$$

$$\xrightarrow{K\to\infty} \mathbb{E}_\theta \left\{ \log\det\left(\mathbf{I} + \mathsf{SNR}\,\boldsymbol{H}_2\boldsymbol{F}_\theta\boldsymbol{H}_1\boldsymbol{H}_1^H\boldsymbol{F}_\theta^H\boldsymbol{H}_2^H\right) \right\}, \ \theta_k \in \mathrm{Uniform}([0, 2\pi))$$

- New outage probability, setting $\boldsymbol{W}_i \triangleq \boldsymbol{H}_i^H\boldsymbol{H}_i$

$$\mathsf{P}\left\{ \mathbb{E}_\theta \left\{ \log\det\left(\mathbf{I} + \mathsf{SNR}\,\boldsymbol{W}_1^H\boldsymbol{F}_\theta^H\boldsymbol{W}_2\boldsymbol{F}_\theta\right) \right\} < r \log \mathsf{SNR} \right\}$$

Generalizing the FF Scheme - $(2,2,2)$ Revisited

Let us define

$$\boldsymbol{H}_1 = \begin{bmatrix} \boldsymbol{h}_{11} & \boldsymbol{h}_{12} \end{bmatrix}^{\mathsf{T}}, \quad \boldsymbol{H}_2 = \begin{bmatrix} \boldsymbol{h}_{21} & \boldsymbol{h}_{22} \end{bmatrix}, \quad u_i = \frac{\boldsymbol{h}_{i1}^H \boldsymbol{h}_{i2}}{\|\boldsymbol{h}_{i1}\| \|\boldsymbol{h}_{i2}\|}, \quad \boldsymbol{G} = \boldsymbol{H}_2 \begin{bmatrix} e^{j\theta_1} & 0 \\ 0 & e^{j\theta_2} \end{bmatrix} \boldsymbol{H}_1$$

$$\begin{aligned} &\mathbb{E}_\theta \left\{ \log\det\left(\mathbf{I} + \mathsf{SNR}\boldsymbol{G}\boldsymbol{G}^H\right) \right\} \\ &\qquad = \log\left(1 + \mathsf{SNR}\left(\|\boldsymbol{h}_{11}\|^2 \|\boldsymbol{h}_{21}\|^2 + \|\boldsymbol{h}_{12}\|^2 \|\boldsymbol{h}_{22}\|^2\right)\right. \\ &\qquad\qquad \left. + \mathsf{SNR}^2 (\|\boldsymbol{h}_{11}\|^2 \|\boldsymbol{h}_{21}\|^2 \|\boldsymbol{h}_{12}\|^2 \|\boldsymbol{h}_{22}\|^2)(1 - |u_1|^2)(1 - |u_2|^2)\right) + O(1) \end{aligned}$$

For $\boldsymbol{H}_i$ i.i.d. Gaussian, we can show that

$$\mathsf{P}\left\{\mathbb{E}_\theta \left\{ \log\det\left(\mathbf{I} + \mathsf{SNR}\boldsymbol{G}\boldsymbol{G}^H\right) \right\} < r \log \mathsf{SNR} \right\} \doteq \mathsf{SNR}^{-d_{(2,2)}(r)}.$$

The cut-set bound is achieved for $(2,2,2)$ channel with linear relaying!

Conclusions

- Rayleigh product channel is studied in terms of near-zero behavior
- Linear relaying with space-only processing is suboptimal in diversity
- Space-time processing is needed to achieve optimal diversity without sacreficing the multiplexing gain
- Distributed linear relaying can be optimal in some non-trivial scenarios
- How about general cases?
- Limited feedback?

Thank you!