AN ENTROPY POWER INEQUALITY FOR THE BINOMIAL FAMILY

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ABSTRACT. In this paper, we prove that the classical Entropy Power Inequality, as derived in the continuous case, can be extended to the discrete family of binomial random variables with parameter 1/2.

1. INTRODUCTION

The continuous Entropy Power Inequality

$$e^{2h(X)} + e^{2h(Y)} \leq e^{2h(X+Y)} \tag{1.1}$$

was first stated by Shannon [1] and later proved by Stam [2] and Blachman [3]. Later, several related inequalities for continuous variables have been proved [4], [5] and [6]. There has been several attempts to provide discrete versions of the Entropy Power Inequality: in the case of Bernoulli sources with addition modulo 2, results have been obtained in a series of papers [7], [8], [9] and [10].

In general, inequality (1.1) does not hold when X and Y are discrete random variables and the differential entropy is replaced by the discrete entropy: a simple counterexample is provided when X and Y are deterministic.

In what follows, $X_n \sim B(n, 1/2)$ denotes a binomial random variable with parameters n and $1/2$, and we prove our main theorem:

Theorem 1.1. *The sequence X_n satisfies the following Entropy Power Inequality*

$$\forall m, n \geq 1 \quad e^{2H(X_n)} + e^{2H(X_m)} \leq e^{2H(X_n+X_m)}.$$

In this aim, we use a characterization of the superadditivity of a function, together with an entropic inequality.

2. SUPERADDITIVITY

Definition 2.1. A function $n \curvearrowright Y_n$ is superadditive if

$$\forall m, n \quad Y_{m+n} \geq Y_m + Y_n.$$

A sufficient condition for superadditivity is given by the following result.

Proposition 2.1. *If $\frac{Y_n}{n}$ is increasing, then Y_n is superadditive*

Proof. Take m and n and suppose $m \geq n$. Then by assumption

$$\frac{Y_{m+n}}{m+n} \geq \frac{Y_m}{m}$$

or

$$Y_{m+n} \geq Y_m + \frac{n}{m} Y_m,$$

but by the hypothesis $m \geq n$

$$\frac{Y_m}{m} \geq \frac{Y_n}{n}$$

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so that

$$Y_{m+n} \geq Y_m + Y_n.$$

□

In order to prove that the function

$$Y_n = e^{2H(X_n)} \tag{2.1}$$

is superadditive, it suffices then to show that function $n \curvearrowright \frac{Y_n}{n}$ is increasing.

3. An information theoretic inequality

Denote as $B \sim Ber\,(1/2)$ a Bernoulli random variable so that

$$X_{n+1} = X_n + B \tag{3.1}$$

and

$$P_{X_{n+1}} = P_{X_n} * P_B = \frac{1}{2}\left(P_{X_n} + P_{X_n+1}\right), \tag{3.2}$$

where $P_{X_n} = \{p_k^n\}$ denotes the probability law of X_n with

$$p_k^n = 2^{-n}\binom{n}{k}. \tag{3.3}$$

A direct application of an equality by Topsøe [11] yields

$$H\left(P_{X_{n+1}}\right) = \frac{1}{2}H\left(P_{X_n+1}\right) + \frac{1}{2}H\left(P_{X_n}\right) + \frac{1}{2}D\left(P_{X_n+1}||P_{X_{n+1}}\right) + \frac{1}{2}D\left(P_{X_n}||P_{X_{n+1}}\right). \tag{3.4}$$

Introduce the Jensen-Shannon divergence

$$\mathcal{JSD}\left(P,Q\right) = \frac{1}{2}D\left(P||\frac{P+Q}{2}\right) + \frac{1}{2}D\left(Q||\frac{P+Q}{2}\right) \tag{3.5}$$

and remark that

$$H\left(P_{X_n}\right) = H\left(P_{X_n+1}\right), \tag{3.6}$$

since each distribution is a shifted version of the other. We conclude thus that

$$H\left(P_{X_{n+1}}\right) = H\left(P_{X_n}\right) + \mathcal{JSD}\left(P_{X_n+1}, P_{X_n}\right), \tag{3.7}$$

showing that then entropy of a binomial law is an increasing function of n. Now we need the stronger result that $\frac{Y_n}{n}$ is an increasing sequence, or equivalently that

$$\log\frac{Y_{n+1}}{n+1} \geq \log\frac{Y_n}{n} \tag{3.8}$$

or

$$\mathcal{JSD}\left(P_{X_n+1}, P_{X_n}\right) \geq \frac{1}{2}\log\frac{n+1}{n}. \tag{3.9}$$

In this aim, we use the following expansion of the Jensen-Shannon divergence, due to B. Y. Ryabko and reported in [12].

Lemma 3.1. *The Jensen-Shannon divergence can be expanded as follows*

$$\mathcal{JSD}\left(P,Q\right) = \frac{1}{2}\sum_{\nu=1}^{\infty}\frac{1}{2\nu\left(2\nu-1\right)}\Delta_\nu\left(P,Q\right)$$

with

$$\Delta_\nu\left(P,Q\right) = \sum_{i=1}^{n}\frac{\left|p_i - q_i\right|^{2\nu}}{\left(p_i+q_i\right)^{2\nu-1}}.$$

This lemma, applied in the particular case where $P = P_{X_n}$ and $Q = P_{X_n+1}$ yields the following result.

Lemma 3.2. *The Jensen-Shannon divergence between P_{X_n+1} and P_{X_n} writes*

$$\mathcal{JSD}\left(P_{X_n+1}, P_{X_n}\right) = \sum_{\nu=1}^{\infty} \frac{1}{\nu\left(2\nu-1\right)} \frac{2^{2\nu-1}}{\left(n+1\right)^{2\nu}} m_{2\nu}\left(B\left(n+1, 1/2\right)\right)$$

where $m_{2\nu}\left(B\left(n+1,1/2\right)\right)$ denotes the order 2ν central moment of a binomial random variable $B\left(n+1, \frac{1}{2}\right)$.

Proof. Denote $P = p_i, Q = p_i^+$ and $\bar{p}_i = (p_i + p_i^+)/2$, The term $\Delta_\nu\left(P_{X_n+1}, P_{X_n}\right)$ writes

$$\begin{aligned} \Delta_\nu\left(P_{X_n+1}, P_{X_n}\right) &= \sum_{i=1}^{n} \frac{\left|p_i^+ - p_i\right|^{2\nu}}{\left(p_i^+ + p_i\right)^{2\nu-1}} \\ &= 2\sum_{i=1}^{n} \left(\frac{p_i^+ - p_i}{p_i^+ + p_i}\right)^{2\nu} \bar{p}_i \end{aligned}$$

and

$$\begin{aligned} \frac{p_i^+ - p_i}{p_i^+ + p_i} &= \frac{2^{-n}\binom{n}{i-1} - 2^{-n}\binom{n}{i}}{2^{-n}\binom{n}{i-1} + 2^{-n}\binom{n}{i}} \\ &= \frac{2i-n-1}{n+1} \end{aligned}$$

so that

$$\begin{aligned} \Delta_\nu\left(P_{X_n+1}, P_{X_n}\right) &= 2\sum_{i=1}^{n} \left(\frac{2i-n-1}{n+1}\right)^{2\nu} \bar{p}_i \\ &= 2\left(\frac{2}{n+1}\right)^{2\nu} \sum_{i=1}^{n} \left(i - \frac{n+1}{2}\right)^{2\nu} \bar{p}_i \\ &= \frac{2^{2\nu+1}}{\left(n+1\right)^{2\nu}} m_{2\nu}\left(B\left(n+1, \frac{1}{2}\right)\right). \end{aligned}$$

Finally, the Jensen-Shannon divergence writes

$$\begin{aligned} \mathcal{JSD}\left(P_{X_n+1}, P_{X_n}\right) &= \frac{1}{4} \sum_{\nu=1}^{+\infty} \frac{1}{\nu\left(2\nu-1\right)} \Delta_\nu\left(P_{X_n+1}, P_{X_n}\right) \\ &= \sum_{\nu=1}^{+\infty} \frac{1}{\nu\left(2\nu-1\right)} \frac{2^{2\nu-1}}{\left(n+1\right)^{2\nu}} m_{2\nu}\left(B\left(n+1, \frac{1}{2}\right)\right). \end{aligned}$$

□

4. Proof of the main theorem

We are now in position to show that the function $n \curvearrowright \frac{Y_n}{n}$ is increasing, or equivalently that inequality (3.9) holds.

Proof. Remark that it suffices to prove the following inequality

$$\sum_{\nu=1}^{3} \frac{1}{\nu\left(2\nu-1\right)} \frac{2^{2\nu-1}}{\left(n+1\right)^{2\nu}} m_{2\nu}\left(B\left(n+1, \frac{1}{2}\right)\right) \geq \frac{1}{2} \log\left(1 + \frac{1}{n}\right) \tag{4.1}$$

since the terms $\nu > 3$ in the expansion of the Jensen-Shannon divergence are all non-negative. Now an explicit computation of the three first even central moments of a binomial random variable with parameters $n+1$ and $\frac{1}{2}$ yields

$$m_2 = \frac{n+1}{4}, \quad m_4 = \frac{(n+1)(3n+1)}{16} \quad \text{and} \quad m_6 = \frac{(n+1)(15n^2+1)}{64},$$

so that inequality (4.1) writes

$$\frac{1}{60}\frac{30n^4 + 135n^3 + 245n^2 + 145n + 37}{(n+1)^5} \geq \frac{1}{2}\log\left(1+\frac{1}{n}\right).$$

Let us now upper-bound the right hand side as follows

$$\log\left(1+\frac{1}{n}\right) \leq \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}$$

so that it suffices to prove that

$$\frac{1}{60}\frac{30n^4 + 135n^3 + 245n^2 + 145n + 37}{(n+1)^5} - \frac{1}{2}\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}\right) \geq 0.$$

Rearranging the terms yields the equivalent inequality

$$\frac{1}{60}\frac{10n^5 - 55n^4 - 63n^3 - 55n^2 - 35n - 10}{(n+1)^5 n^3} \geq 0$$

which is equivalent to the positivity of polynomial

$$P(n) = 10n^5 - 55n^4 - 63n^3 - 55n^2 - 35n - 10.$$

Assuming first $n \geq 7$, we remark that

$$P(n) \geq 10n^5 - n^4\left(55 + \frac{63}{6} + \frac{55}{6^2} + \frac{35}{6^3} + \frac{10}{6^4}\right) = \left(10n - \frac{5443}{81}\right)n^4$$

whose positivity is ensured as soon as $n \geq 7$.

This result can be extended to the values $1 \leq n \leq 6$ by a direct inspection at the values of function $n \curvearrowright \frac{Y_n}{n}$ as given in the following table.

n	1	2	3	4	5	6
$\frac{e^{2H(X_n)}}{n}$	4	4	4.105	4.173	4.212	4.233

Table 4.1: Values of the function $n \curvearrowright \frac{Y_n}{n}$ for $1 \leq n \leq 6$.

□

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