Central Limit Results for the Multiple User Interference at the SUMF Output for UWB Signals

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***Abstract*— In Ultra-Wide Band (UWB) communications based on Time Hopping (TH) Impulse Radio, one of the most frequently studied detectors is the correlation detector also called Single User Matched Filter (SUMF). Often the Multi-User Interference (MUI) at the output of this detector is modeled as a Gaussian random variable. In order to justify this assumption, the conditions of validity of the central limit theorem have to be studied in the asymptotic regime where the number of interferers and the spreading factor grow toward infinity at the same rate. An asymptotic study is made in this paper based on the so-called Lindeberg's condition. Non synchronized users sending their signals over independent multi-path channels and having possibly different powers are considered. In the situation where the limit distribution of the MUI is Gaussian, closed form expressions for the Signal to Interference plus Noise Ratio are given for TH Pulse Amplitude Modulation (Th-PAM) and Pulse Position Modulation (TH-PPM) UWB transmissions.**

I. Problem Formulation

Let us begin by considering a TH-PAM UWB system [1]. K being the number of users, we denote by $a_{k,m}^{(K)}$ the information symbol of user k at symbol interval m, having its values in the set $\{-1,1\}$. This symbol is repeated over N_s frames of duration $T_f = N_h T_c$ each, where T_c is the so-called chip time interval (N_s is therefore the repetition factor, and N_h is the frame length in chips). The time hopping code for this user is represented by the sequence $(c_{k,l}^{(K)})_{l\in\mathbb{Z}}$ which elements are discrete random variables equally distributed on $\{0,\ldots,N_h-1\}$. The random variables $\{c_{k,l}^{(K)}\}_{\substack{k=1,\ldots,K\\ l\in\mathbb{Z}}}$ are furthermore assumed independent. In the case the receiver is synchronized on user k, the contribution of this user to the received signal will be expressed as

$$y_k^{(K)}(t) = \sqrt{\frac{\mathcal{E}_k^{(K)}}{N_s}} \sum_m a_{k,m}^{(K)} \sum_{r=0}^{N_s-1} g_k^{(K)}(t - mN_sT_f - rT_f - c_{k,mN_s+r}^{(K)}T_c) \quad (1)$$

In this expression, $g_k^{(K)}(t)$ is the composite channel associated to user k. It is written $g_k^{(K)}(t) = \sum_{l=1}^{D} \gamma_{k,l}^{(K)} w(t-\tau_{k,l}^{(K)})$ where $w(t)$ is the unit-energy basic pulse waveform with a time support included in $[0,T_c)$, $\boldsymbol{\gamma}_k^{(K)} = [\gamma_{k,1}^{(K)},\ldots,\gamma_{k,D}^{(K)}]$ is the vector of random path amplitudes of the radio channel that carries the data of user k, $\boldsymbol{\tau}_k^{(K)} = [\tau_{k,1}^{(K)},\ldots,\tau_{k,D}^{(K)}]$ is the vector of the corresponding random path delays, and D is a uniform upper bound on the number of paths. We assume that for a given k, the zero-mean random variables $\{\gamma_{k,l}^{(K)}\}_{l=1,\ldots,D}$ are decorrelated and that $\sum_{l=1}^{D} E\left[{\gamma_{k,l}^{(K)}}^2\right] = 1$. In these conditions, it is easy to see that $\mathcal{E}_k^{(K)}$ is the energy per received symbol for user k. We further assume that $\mathcal{E}_k^{(K)}$ are uniformly bounded, and that the K vectors $\{\mathbf{h}_k^{(K)}\}_{k=1,\ldots,K}$ where $\mathbf{h}_k^{(K)} = [\boldsymbol{\gamma}_k^{(K)}, \boldsymbol{\tau}_k^{(K)}]$ represents the k^{th} radio channel, are independent but not necessarily identically distributed. The impulse responses $g_k^{(K)}(t)$ are also assumed causal and the delays $\tau_{k,l}^{(K)}$ are uniformly bounded with probability one. As a consequence, the time supports of these impulse responses lie in the interval $[0, LT_c)$ where L is a uniform upper bound with probability one on the lengths of these time supports in chip intervals.

Assuming that the receiver is perfectly synchronized on user 1, the received signal expresses as

$$y^{(K)}(t) = y_1^{(K)}(t) + \sum_{k=2}^{K} y_k^{(K)}(t-\Delta_k^{(K)}) + v(t) \quad (2)$$

where $v(t)$ is a Gaussian noise independent of all other random variables and having a spectral density of $N_0/2$ in the frequency band of $w(t)$. The delay $\Delta_k^{(K)}$ accounts for the absence of synchronization between user k and user 1. It will be assumed that the delays $\{\Delta_k^{(K)}\}_{k=2,\ldots,K}$ are independent and uniformly distributed over the interval $[0, N_sN_hT_c)$. In the sequel the superscript $^{(K)}$ will be often dropped when denoting the quantities associated to user 1.

The impulse response $g_1(t)$ being perfectly known at the receiver, the output of the SUMF for symbol $a_{1,0}$ is

$$x = \sqrt{\frac{\mathcal{E}}{N_s}} \sum_{r=0}^{N_s-1} \int y^{(K)}(t) g_1(t - rN_hT_c - c_{1,r}T_c)dt \ , \quad (3)$$

and the decided symbol is $\hat{a}_{1,0} = \text{sign}(x)$. By plugging (1) and (2) into (3), the SUMF output can be written $x = x_{\text{u}} + x_{\text{ISI}} + x_{\text{MUI}} + x_{\text{AWGN}}$, where x_{u}, x_{ISI}, x_{MUI} and x_{AWGN} are the "useful" term, the Inter-Symbol Interference term, the Multi-User Interference term, and the AWGN term respectively. In particular, $x_{\text{MUI}} = \sum_{k=2}^{K} \xi_k^{(K)}$ where $\xi_k^{(K)}$ is the contribution of $y_k^{(K)}(t)$ to the SUMF output.

In the sequel, we consider the asymptotic regime where the spreading factor $N = N_hN_s$ and the number of users K grow toward infinity in such a way that $K/N \to \alpha$, a quantity that we designate by the system load. Notice that the uniform upper bound L on channel lengths measured in chip intervals remains constant. This means in practice that our analysis is suited to situations where the Inter-Symbol Interference is negligible thanks to the choice of a large spreading factor.

II. Asymptotic Analysis

In the remainder, all probabilities are conditioned on $a_{1,0}$ and $\mathbf{h}_1$. It can first be shown that as $N_h \to \infty$, $x_{\rm u}$ converges in probability toward $\mathcal{E}_1 a_{1,0} \int |G_1(f)|^2 df$ where $G_1(f)$ is the Fourier transform of $g_1(t)$, and furthermore, that $x_{\rm ISI}$ converges in probability to 0. Concerning the AWGN term $x_{\rm AWGN}$, one can easily notice that conditionally to the code vector $[c_{1,0}, \ldots, c_{1,N_s-1}]$ associated to user 1 for his data symbol $a_{1,0}$, this term is Gaussian. It can also be proved that in the asymptotic regime as $N_h \to \infty$, the unconditional distribution of $x_{\rm AWGN}$ is Gaussian with zero mean and variance $\sigma^2_{\rm AWGN} = \frac{N_0}{2}\mathcal{E}_1 \int |G_1(f)|^2 df$.

Let us consider now the MUI term. We shall study two modes for the asymptotic regime where $N = N_s N_h \to \infty$ while $K/N \to \alpha$. We first consider the case where N_s is kept constant while $N_h \to \infty$, and then the case where both these parameters grow toward infinity in such a way that $\frac{N_s}{N_h} \to \rho$ where ρ is a constant $\rho > 0$. The first case is treated by the following proposition :

Proposition 1: Assume $N_h \to \infty$ while N_s is kept constant. Then as $K \to \infty$ and $K/N \to \alpha$, $x_{\rm MUI}$ does not converge in distribution toward a Gaussian law.

In our setting, the so-called Lindeberg's condition

$$\forall \varepsilon > 0, \ \lim_{K\to\infty} \sum_{k=2}^{K} E\left[{\xi_k^{(K)}}^2 \mathbf{1}_{|\xi_k^{(K)}| \geq \varepsilon} \right] = 0 \qquad (4)$$

appears to be a necessary and sufficient condition for $x_{\rm MUI}$ to converge in distribution toward a Gaussian law. It can indeed be shown that this condition is not satisfied under the assumptions of proposition 1.

We now turn to the second case :

Proposition 2: Assume that $\frac{N_s}{N_h} \to \rho > 0$, that the empirical mean of the energies $\overline{\mathcal{E}}^{(K)} = \frac{1}{K}\sum_{k=1}^{K} \mathcal{E}_k^{(K)}$ converges to a limit $\overline{\mathcal{E}}$ as $K \to \infty$, and that the random variables $\|\boldsymbol{\gamma}_k^{(K)}\|^2$ are uniformly integrable, *i.e.*, that

$$\lim_{a\to\infty} \sup_{K} \max_{k=1,\ldots,K} E\left[\|\boldsymbol{\gamma}_k^{(K)}\|^2 \mathbf{1}_{\|\boldsymbol{\gamma}_k^{(K)}\| > a} \right] = 0 \ . \qquad (5)$$

Then $x_{\rm MUI}$ converges in distribution toward a Gaussian random variable. The Signal to Interference plus Noise Ratio (SINR) at the output of the SUMF detector converges in these conditions to

$$\beta_{\rm PAM} = \frac{\mathcal{E}_1 \int |G_1(f)|^2 df}{\frac{N_0}{2} + \alpha \overline{\mathcal{E}} \eta_{\rm PAM}} \qquad (6)$$

where $\eta_{\rm PAM}$ is given by equation (7) in which $W(f)$ is the Fourier transform of $w(t)$.

Here also, the proof relies on Lindeberg's condition. It appears from this proposition that at high spreading factors, the Gaussian character is obtained through repetition. However, a large value of the repetition factor N_s results in a large value of ρ, and thus in a high MUI variance, as shown by the expression of the numerator of $\eta_{\rm PAM}$. In other words, a large repetition factor reduces the multiplexing ability of the UWB access technique.

Notice that the technical assumption (5) is not restrictive in practice. In particular, it is satisfied if the vectors $\boldsymbol{\gamma}_k^{(K)}$ are identically distributed.

III. The TH-PPM Case

In the Time Hopping - Pulse Position Modulation (TH-PPM) case (see [2], [3], [4]), equation (1) is replaced by

$$y_k^{(K)}(t) = \sqrt{\frac{\mathcal{E}_k^{(K)}}{N_s}} \sum_m \sum_{r=0}^{N_s-1} g_k^{(K)}(t - mN_sT_f - rT_f - c_{k,mN_s+r}^{(K)} T_c - d a_{k,m}^{(K)}) \ ,$$

where the symbols $\{a_{k,m}^{(K)}\}$ have their values in $\{0,1\}$ and d is the time shift used for position modulation. The description of the received signal is otherwise unchanged. The output of the SUMF for the symbol $a_{1,0}$ expresses here as

$$x = \sqrt{\frac{\mathcal{E}}{N_s}} \sum_{r=0}^{N_s-1} \int y^{(K)}(t) p_1(t - rN_hT_c - c_{1,r}T_c) dt \ .$$

where $p_1(t) = g_1(t) - g_1(t-d)$, and the decision rule is $\hat{a}_{1,0} = 0$ if $x > 0$ and $\hat{a}_{1,0} = 1$ otherwise. Here also we have $x = x_{\rm u} + x_{\rm ISI} + x_{\rm MUI} + x_{\rm AWGN}$ where the terms of the RHS member play the same role as in section I.

We shall just give the main results concerning the TH-PPM case. As for the TH-PAM case, when $N_h \to \infty$ and N_s is kept constant, the MUI term $x_{\rm MUI}$ does not converge in distribution to a Gaussian random variable. On the other hand, it does if $N \to \infty$, $N_s/N_h \to \rho > 0$, and $K/N \to \alpha$. The expression of the limit SINR $\beta_{\rm PPM}$ at the output of the SUMF is then

$$\beta_{\rm PPM} = \frac{\mathcal{E}_1 \int |G_1(f)|^2 (1 - \cos 2\pi f d) df}{N_0 + 2\alpha \overline{\mathcal{E}} \eta_{\rm PPM}}$$

where $\eta_{\rm PPM}$ is given by (8).

IV. Simulations

Simulations are carried out in the context of TH-PAM transmissions. In figures 1 and 3, the solid line plots indicate the Bit Error Rate (BER) versus $2E_{\rm b}/N_0$ that result from the Gaussian approximation in the asymptotic regime. More precisely, $\beta_{\rm PAM}$ being the limit SINR predicted by equation (6), the BER will be $Q(\sqrt{\beta})$ if the Gaussian approximation is valid. Here $Q(.)$ is the Gaussian tail function. The dashed curves are the ones obtained by simulation.

The pertinence of the asymptotic regimes described by propositions 1 and 2 are first tested on transmissions over single path channels. The spreading factor is $N = 200$ and the number of users is $K = 100$, resulting in a load of $\alpha = 0.5$. It is further assumed that $\mathcal{E}_1^{(K)} = \cdots = \mathcal{E}_K^{(K)} = \overline{\mathcal{E}}$. Figure 1 shows that when $N_h = 200$ and $N_s = 1$, then the transmission

$$\eta_{\rm PAM} = \frac{\frac{2\rho}{3}\frac{1}{T_c^2}\left(\sum_l \left|W\left(l/T_c\right)\right|^2 \left|G_1\left(l/T_c\right)\right|^2\right) + \frac{1}{T_c}\int \left|W\left(f\right)\right|^2 \left|G_1\left(f\right)\right|^2 df}{\int |G_1(f)|^2 df} \quad (7)$$

$$\eta_{\rm PPM} = \frac{\frac{2\rho}{3}\frac{1}{T_c^2}\left(\sum_l \left|W\left(l/T_c\right)\right|^2 \left|G_1\left(l/T_c\right)\right|^2 \left(1-\cos\frac{2\pi l d}{T_c}\right)\right) + \frac{1}{T_c}\int \left|W\left(f\right)\right|^2 \left|G_1\left(f\right)\right|^2 \left(1-\cos 2\pi f d\right) df}{\int \left|G_1\left(f\right)\right|^2 \left(1-\cos 2\pi f d\right) df} \quad (8)$$

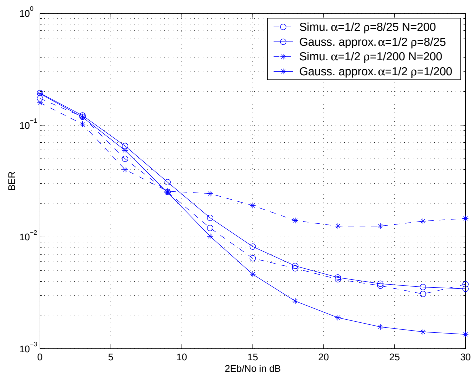

Fig. 1. BER for different values of N_s, N_h. Single path channels

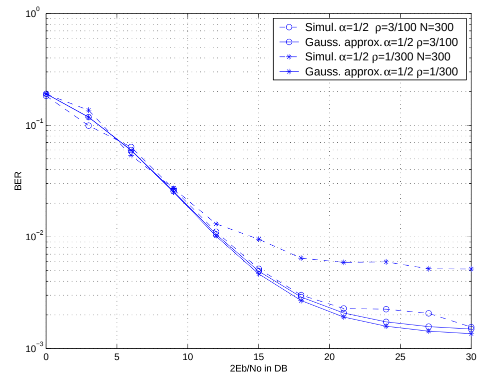

Fig. 3. BER for different values of N_s, N_h. Multi-path channels

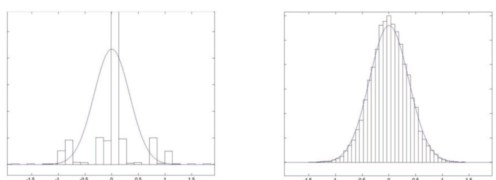

Fig. 2. MUI histograms, Single path. Left : $N_s = 1, N_h = 200$, right : $N_s = 8, N_h = 25$.

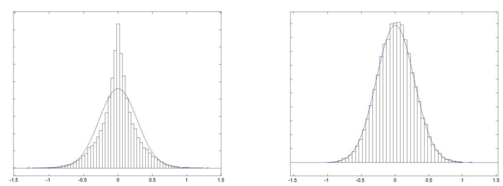

Fig. 4. MUI histograms, multi-path. Left : $N_s = 1, N_h = 300$, right : $N_s = 3, N_h = 150$

conditions meet practically the assumptions of proposition 1, and therefore the Gaussian approximation is not valid. However, when $N_s = 8$ and $N_h = 25$, a situation modeled in equation (6) by $\rho = 0.32$, then the asymptotic regime of proposition 2 is practically attained. The empirical histograms of the random variable $x_{\rm MUI}$ corresponding to these two cases are shown on figure 2. The centered Gaussian densities with variances $E\left[x_{\rm MUI}^2\right]$ are also shown on this figure.

Figure 3 represents simulation results for transmissions over multi-path channels. The channel model is the so-called modified Saleh-Valenzuela model described in [5]. Channels with a RMS delay spread of 5ns are considered. The basic pulse waveform is the second derivative of a Gaussian pulse with a pulse shape parameter of 0.4ns [6]. The spreading factor $N = 300$ and the chip period $T_c = 2$ns are chosen, resulting in a data rate of 1.67 Mbit/s per user. The figure shows that when $N_h = 300$ and $N_s = 1$, then the Gaussian approximation is not valid. However, when $N_s = 3$ and $N_h = 100$, then the detector performance can be predicted reliably by the result of proposition 2. Like for single path channels, the histograms of $x_{\rm MUI}$ are also shown.

These experiments show that the condition $N_s \to \infty$ implicitly required by proposition 2 is somehow theoretical. In most practical situations, the asymptotic regime of this proposition is attained for small values of N_s.

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