NON-HERMITIAN RANDOM MATRICES WITH A VARIANCE PROFILE (II): PROPERTIES AND EXAMPLES

NICHOLAS COOK, WALID HACHEM, JAMAL NAJIM, AND DAVID RENFREW*

Abstract. For each n, let $A_n = (\sigma_{ij})$ be an $n \times n$ deterministic matrix and let $X_n = (X_{ij})$ be an $n \times n$ random matrix with i.i.d. centered entries of unit variance. In the companion article [15], we considered the empirical spectral distribution μ_n^Y of the rescaled entry-wise product

$$Y_n = \frac{1}{\sqrt{n}} A_n \odot X_n = \left(\frac{1}{\sqrt{n}} \sigma_{ij} X_{ij} \right)$$

and provided a deterministic sequence of probability measures μ_n such that the difference $\mu_n^Y - \mu_n$ converges weakly in probability to the zero measure. A key feature in [15] was to allow some of the entries σ_{ij} to vanish, provided that the standard deviation profiles A_n satisfy a certain quantitative irreducibility property.

In the present article, we provide more information on the sequence (μ_n), described by a family of *Master Equations*. We consider these equations in important special cases such as sampled variance profiles $\sigma_{ij}^2 = \sigma^2\left(\frac{i}{n}, \frac{j}{n}\right)$ where $(x, y) \mapsto \sigma^2(x, y)$ is a given function on $[0, 1]^2$. Associated examples are provided where μ_n^Y converges to a genuine limit.

We study μ_n's behavior at zero. As a consequence, we identify the profiles that yield the circular law.

Finally, building upon recent results from Alt et al. [7, 8], we prove that, except possibly at the origin, μ_n admits a positive density on the centered disc of radius $\sqrt{\rho(V_n)}$, where $V_n = (\frac{1}{n}\sigma_{ij}^2)$ and $\rho(V_n)$ is its spectral radius.

1. Introduction

For an $n \times n$ matrix M with complex entries and eigenvalues $\lambda_i \in \mathbb{C}$ (counted with multiplicity and labeled in some arbitrary fashion), the *empirical spectral distribution (ESD)* is given by

$$\mu_n^M = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i} \, . \tag{1.1}$$

A seminal result in non-Hermitian random matrix theory is the *circular law*, which describes the asymptotic global distribution of the spectrum for matrices with i.i.d. entries of finite variance – see [15] for additional references and the survey [13] for a detailed historical account.

In the companion paper [15], we studied the limiting spectral distribution μ_n^Y for *random matrices with a variance profile* (see Definition 1.1). More precisely, we provided a deterministic sequence of probability measures μ_n each described by a family of *Master Equations* (see (2.3)), such that the difference $\mu_n^Y - \mu_n$ converges weakly in probability to the zero measure. Such master equations were introduced and studied by Girko; see, for example [18].

A key feature of this result was to allow a large proportion of the matrix entries to be zero, which is important for applications to the modeling of dynamical systems such as neural networks and food webs [3, 5]. This also presented challenges for the quantitative analysis of the Master Equations, for which we developed the *graphical bootstrapping* argument.

Date: October 21, 2021.

2010 *Mathematics Subject Classification.* Primary 15B52, Secondary 15A18, 60B20.

*Corresponding author.

We mention that in the appendix of [26] it was shown that the ESDs for two sequences of random matrices with the same mean and variance profile (but possibly different entry distributions) are asymptotically equivalent, assuming, among other (mild) conditions, that the variances are uniformly bounded away from zero. Our aims here and in the companion paper [15] are in an orthogonal direction: to establish asymptotic equivalence with a sequence of *deterministic* measures, and to study properties of these deterministic equivalents. Moreover, these tasks are far more challenging when one does not assume the variances are uniformly positive.

After the initial release of [15], a local law version of our main statement (Theorem 2.3) was proven in [7] under the restriction that the standard deviation profile σ_{ij} is uniformly strictly positive and that the distribution of the matrix entries possesses a bounded density and finite moments of every order. The results of [7] were extended in [9] to include random matrices with correlated entries and the behavior of the limiting density is investigated further. Early attempts to study non-Hermitian random matrices with dependent entries can be found in [12] (Markov entries), [21] (Doubly Stochastic matrices), and [2] (log-concave distributions). Predating the non-Hermitian setting, a large body of work exists for Hermitian random matrices with non-identically distributed entries and/or dependent entries, see for instance [24, 19, 4, 6, 22, 20].

In this article, we consider in more detail the measures (μ_n). In particular, we provide new conditions that ensure the positivity of the density of μ_n and study the behavior of μ_n at zero. This study allows us to deduce a necessary condition for the circular law. Additionally, we specialize to sampled standard deviation profiles, which are important from a modeling perspective and can yield genuine limits.

1.1. **The setting.** We study the following general class of random matrices with non-identically distributed entries.

Definition 1.1 (Random matrix with a variance profile)**.** *For each $n \geq 1$, let A_n be a (deterministic) $n \times n$ matrix with entries $\sigma_{ij}^{(n)} \geq 0$, let X_n be a random matrix with i.i.d. entries $X_{ij}^{(n)} \in \mathbb{C}$ satisfying*

$$\mathbb{E}X_{11}^{(n)} = 0\,, \quad \mathbb{E}|X_{11}^{(n)}|^2 = 1 \tag{1.2}$$

and set

$$Y_n = \frac{1}{\sqrt{n}} A_n \odot X_n \tag{1.3}$$

where $\odot$ is the matrix Hadamard product, i.e. Y_n has entries $Y_{ij}^{(n)} = \frac{1}{\sqrt{n}}\sigma_{ij}^{(n)} X_{ij}^{(n)}$. The empirical spectral distribution of Y_n is denoted by μ_n^Y. We refer to A_n as the standard deviation profile *and to $A_n \odot A_n = \big((\sigma_{ij}^{(n)})^2\big)$ as the* variance profile*. We additionally define the* normalized variance profile *as*

$$V_n = \frac{1}{n} A_n \odot A_n.$$

When no ambiguity occurs, we drop the index n and simply write σ_{ij}, X_{ij}, V, etc.

The main result of [15] states that under certain assumptions on the sequence of standard deviation profiles A_n and the distribution of the entries of X_n, there exists a tight sequence of deterministic probability measures μ_n that are *deterministic equivalents* of the spectral measures μ_n^Y, in the sense that for every continuous and bounded function $f : \mathbb{C} \to \mathbb{C}$,

$$\int f \, d\mu_n^Y - \int f \, d\mu_n \xrightarrow[n\to\infty]{} 0 \qquad \text{in probability.}$$

In other words, the signed measures $\mu_n^Y - \mu_n$ converge weakly in probability to zero. In the sequel this convergence will be simply denoted by

$$\mu_n^Y \sim \mu_n \qquad \text{in probability} \quad (n \to \infty).$$

The measures μ_n are described by a polynomial system of *Master Equations* that will be recalled in the next section. The main results of [15], see Theorems 2.2 and 2.3 below, establishes the existence and the uniqueness of the solution to these equations and establishes the connection to the deterministic equivalent μ_n. This probability law turns out to be a circularly symmetric law supported by the disk with center zero and radius $\sqrt{\rho(V_n)}$, where $\rho(V_n)$ is the spectral radius of V_n. Moreover, μ_n has a density on $\mathbb{C} \setminus \{0\}$.

1.2. Contributions of this paper.

In this article, we continue the study of the model initiated in [15], where we provided existence of a μ_n such that $\mu_n^Y \sim \mu_n$ for random matrices in Definition 1.1. In particular, we study properties of μ_n: positivity of its density and its behavior at zero, as well as identify variance profiles that yield the circular law. We also consider several special classes of variance profiles.

In Section 2, we recall the main results of [15]. Then, in Proposition 2.7 and Theorem 2.9 we provide sufficient conditions for which the density of μ_n is positive on the disc of radius $\sqrt{\rho(V_n)}$, with an emphasis on the behavior of this density near zero. In particular, a formula for the value of the density at zero is provided. In Corollary 2.8, we deduce from our formula at zero that the doubly stochastic normalized variance profiles, i.e. $V_n = \left(n^{-1}\sigma_{ij}^2\right)$ such that

$$\frac{1}{n}\sum_{i=1}^{n} \sigma_{ij}^2 = \mathcal{V} \quad \forall j \in [n] \qquad \text{and} \qquad \frac{1}{n}\sum_{j=1}^{n} \sigma_{ij}^2 = \mathcal{V} \quad \forall i \in [n]\,.$$

for some fixed $\mathcal{V} > 0$, are, up to conjugation by diagonal matrices, the only profiles that give the circular law.

In Section 3, we consider sampled variance profiles, where the profile is obtained by evaluating a fixed continuous function $\sigma(x, y)$ on the unit square at the grid points $\{(i/n, j/n) \colon 1 \le i, j \le n\}$. Here, in the large n limit the Master Equations (2.3) turn into an integral equation defining a genuine limit for the ESDs:

$$\int f\, d\mu_n^Y \xrightarrow[n\to\infty]{} \int f\, d\mu^\sigma \qquad \text{in probability;}$$

see Theorem 3.1.

Section 5 is devoted to the proof of the results in Section 2 concerning positivity and finiteness of the density of μ_n. Much of this analysis will build upon results developed by Alt et al. [7, 8] in combination with the regularity of the solutions to the Master Equations proven in [15].

Finally, in Section 4, we provide examples of variance profiles with vanishing entries. In particular, we study band matrices and give an example of a distribution with an atom and a vanishing density at zero (Proposition 4.2).

Acknowledgements. The work of NC was partially supported by NSF grants DMS-1266164 and DMS-1606310. The work of WH and JN was partially supported by the Labex BEZOUT from the Gustave Eiffel University. DR was partially supported by Austrian Science Fund (FWF): M2080-N35. DR would also like to thank Johannes Alt, László Erdős, and Torben Krüger for numerous enlightening conversations.

2. Limiting spectral distribution: a reminder and some complements

In this section, we recall the main results in Cook et al. [15] and then give theorems concerning the density of μ_n.

2.1. **Notational preliminaries.** Let $[n]$ be the set $\{1, \cdots, n\}$. The Lebesgue measure on $\mathbb{C}$ will be denoted as $\ell(\,dz)$. The cardinality of a finite set S is denoted by $|S|$. We denote by $\mathbf{1}_n$ the $n \times 1$ vector of 1's. Given two $n \times 1$ vectors $\boldsymbol{u}, \boldsymbol{v}$, we denote their scalar product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{i\in[n]} \bar{u}_i v_i$. Let $\boldsymbol{a} = (a_i)$ an $n \times 1$ vector. We denote by $\mathrm{diag}(\boldsymbol{a})$ the $n \times n$ diagonal matrix with the a_i's as its diagonal elements, by $\|\boldsymbol{a}\|_\infty = \max_{i\in[n]} |a_i|$. For a given matrix A, denote by A^{T} its transpose, by A^* its conjugate transpose, by $\|A\|$ its spectral norm, and by $\rho(A)$ its spectral radius, that is $\rho(A) = \max\{\lambda(A),\ \lambda(A) \text{ eigenvalue of } A\}$. Denote by I_n the $n \times n$ identity matrix. If clear from the context, we omit the dimension. For $a \in \mathbb{C}$ and when clear from the context, we sometimes write a instead of $a\,I$ and similarly write a^* instead of $(aI)^* = \bar{a}I$.

Notations $\succ$ and $\succcurlyeq$ refer to the element-wise inequalities for real matrices or vectors. Namely, if B and C are real matrices,

$$B \succ C \quad \Leftrightarrow \quad B_{ij} > C_{ij} \quad \forall i,j \qquad \text{and} \qquad B \succcurlyeq C \quad \Leftrightarrow \quad B_{ij} \ge C_{ij} \quad \forall i,j.$$

The notation $B \succcurlyeq_{\neq} 0$ stands for $B \succcurlyeq 0$ and $B \neq 0$.

2.2. **Model assumptions.** We will establish results concerning sequences of matrices Y_n as in Definition 1.1 under various additional assumptions on A_n and X_n, which we now summarize.

For our main result we will need the following additional assumption on the distribution of the entries of X_n.

A0 (Moments). We have $\mathbb{E}|X_{11}^{(n)}|^{4+\varepsilon} \le M_0$ for all $n \ge 1$ and some fixed $\varepsilon > 0$, $M_0 < \infty$.

We will also assume the entries of A_n are bounded uniformly in $i, j \in [n]$, $n \ge 1$:

A1 (Bounded variances). There exists $\sigma_{\max} \in (0, \infty)$ such that

$$\sup_n \max_{1\le i,j\le n} \sigma_{ij}^{(n)} \le \sigma_{\max}.$$

Assumption **A0** is necessary to obtain uniform integrability of the ESD of $(Y_n - z)^*(Y_n - z)$, a key step in Girko's Hermitization strategy for determining the spectrum of Y_n. In particular, this assumption is necessary in order to apply results from [14], which bound the least singular value of $Y_n - z$, as well as to quantitatively bound the difference between the Stieltjes transform of the ESD and the deterministic equivalents. Assumption **A1** was used in [15] to ensure the diagonal entries of the resolvent of a $2n \times 2n$ linearization of $(Y_n - z)^*(Y_n - z)f$ approximately solve the system of equations given in (2.1).

In order to express the next key assumption, we need to introduce the following *Regularized Master Equations* which are a specialization of the Schwinger–Dyson equations of Girko's so-called Hermitized model associated to Y_n (see [15] for more details about this subject).

Proposition 2.1 (Regularized Master Equations)**.** *Let $n \ge 1$ be fixed, let A_n be an $n \times n$ nonnegative matrix and write $V_n = \frac{1}{n} A_n \odot A_n$. Let $s, t > 0$ be fixed, and consider the following system of equations*

$$\begin{cases} r_i &=& \dfrac{(V_n^{\mathsf{T}} \boldsymbol{r})_i + t}{s^2 + ((V_n \widetilde{\boldsymbol{r}})_i + t)((V_n^{\mathsf{T}} \boldsymbol{r})_i + t)} \\[2ex] \widetilde{r}_i &=& \dfrac{(V_n \widetilde{\boldsymbol{r}})_i + t}{s^2 + ((V_n \widetilde{\boldsymbol{r}})_i + t)((V_n^{\mathsf{T}} \boldsymbol{r})_i + t)} \end{cases}, \tag{2.1}$$

where $\boldsymbol{r} = (r_i)$ and $\widetilde{\boldsymbol{r}} = (\widetilde{r}_i)$ are $n \times 1$ vectors. Denote by $\vec{\boldsymbol{r}} = \begin{pmatrix} \boldsymbol{r} \\ \widetilde{\boldsymbol{r}} \end{pmatrix}$. Then this system admits a unique solution $\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}(s,t) \succ 0$. This solution satisfies the identity

$$\sum_{i\in[n]} r_i = \sum_{i\in[n]} \widetilde{r}_i \,. \tag{2.2}$$

A2 (Admissible variance profile). Let $\vec{\boldsymbol{r}}(s,t) = \vec{\boldsymbol{r}}_n(s,t) \succ 0$ be the solution of the Regularized Master Equations for given $n \geq 1$. For all $s > 0$, there exists a constant $C = C(s) > 0$ such that

$$\sup_{n\geq 1} \sup_{t\in(0,1]} \frac{1}{n} \sum_{i\in[n]} r_i(s,t) \;\leq\; C\;.$$

A family of variance profiles (or corresponding standard deviation/normalized variance profiles) for which the previous estimate holds is called *admissible*.

Remark 2.1. After restating the main theorems we list concrete conditions under which we verify **A2**, namely **A3** (lower bound on V_n), **A4** (symmetric V_n) and **A5** (robust irreducibility for V_n), cf. section 2.4.

2.3. **Results from** [15]**.** The following system of Master Equations will be of central importance. Given a parameter $s \geq 0$, this is the system of $2n+1$ equations in $2n$ unknowns $q_1, \dots, q_n, \widetilde{q}_1, \dots, \widetilde{q}_n$ that reads:

$$\begin{cases} q_i &= \dfrac{(V_n^\mathsf{T} \boldsymbol{q})_i}{s^2 + (V_n \widetilde{\boldsymbol{q}})_i (V_n^\mathsf{T} \boldsymbol{q})_i} \\ \\ \widetilde{q}_i &= \dfrac{(V_n \widetilde{\boldsymbol{q}})_i}{s^2 + (V_n \widetilde{\boldsymbol{q}})_i (V_n^\mathsf{T} \boldsymbol{q})_i} \quad , \qquad q_i, \widetilde{q}_i \geq 0,\; i \in [n], \\ \\ & \sum_{i\in[n]} q_i = \sum_{i\in[n]} \widetilde{q}_i \end{cases} \tag{2.3}$$

where $\boldsymbol{q}, \widetilde{\boldsymbol{q}}$ are the $n \times 1$ column vectors with components $q_i, \widetilde{q}_i$, respectively. In the sequel, we shall write $\vec{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{q} \\ \widetilde{\boldsymbol{q}} \end{pmatrix}$. Observe that these equations are obtained from the Regularized Master Equations (2.1) by letting the parameter t go to zero. Notice however that condition $\sum q_i = \sum \widetilde{q}_i$ is required for uniqueness and not a consequence of the equations as in (2.1).

In what follows, we will always tacitly assume the standard deviation profile A_n is *irreducible*. This will cause no true loss of generality, as we can conjugate the matrix Y_n by an appropriate permutation matrix to put A_n in block-upper-triangular form with irreducible blocks on the diagonal. The spectrum of Y_n is then the union of the spectra of the corresponding block diagonal submatrices.

Theorem 2.2 (Cook et al. [15])**.** *Let $n \geq 1$ be fixed, let A_n be an $n \times n$ nonnegative matrix and write $V_n = \frac{1}{n} A_n \odot A_n$. Assume that A_n is irreducible. Then the following hold:*

(1) For $s \geq \sqrt{\rho(V_n)}$ the system (2.3) has the unique solution $\vec{\boldsymbol{q}}(s) = 0$.
(2) For $s \in (0, \sqrt{\rho(V)})$ the system (2.3) has a unique non-trivial solution $\vec{\boldsymbol{q}}(s) \succcurlyeq_{\neq} 0$. Moreover, this solution satisfies $\vec{\boldsymbol{q}}(s) \succ 0$.
(3) $\vec{\boldsymbol{q}}(s) = \lim_{t\downarrow 0} \vec{\boldsymbol{r}}(s,t)$ for $s \in (0,\infty)$.
(4) The function $s \mapsto \vec{\boldsymbol{q}}(s)$ defined in parts (1) and (2) is continuous on $(0,\infty)$ and is continuously differentiable on $(0, \sqrt{\rho(V)}) \cup (\sqrt{\rho(V)}, \infty)$.

Remark 2.2 (Convention). Above and in the sequel we abuse notation and write $\vec{\boldsymbol{q}} = \vec{\boldsymbol{q}}(s)$ to mean a solution of the equation (2.3), understood to be the nontrivial solution for $s \in (0, \sqrt{\rho(V)})$.

Theorem 2.3 (Cook et al. [15])**.** *Let $(Y_n)_{n\geq 1}$ be a sequence of random matrices as in Definition 1.1, and assume* **A0***,* **A1** *and* **A2** *hold. Assume moreover that A_n is irreducible for all $n \geq 1$.*

(1) There exists a sequence of deterministic measures $(\mu_n)_{n\geq 1}$ on $\mathbb{C}$ such that

$$\mu_n^Y \sim \mu_n \qquad \text{in probability.}$$

(2) *Let* $\boldsymbol{q}(s), \widetilde{\boldsymbol{q}}(s)$ *be as in Theorem 2.2, and for* $s \in (0,\infty)$ *let*

$$F_n(s) = 1 - \frac{1}{n}\langle \boldsymbol{q}(s), V_n \widetilde{\boldsymbol{q}}(s)\rangle. \tag{2.4}$$

Then F_n *extends to an absolutely continuous function on* $[0,\infty)$ *which is the cumulative distribution function (CDF) of a probability measure with support contained in* $[0, \sqrt{\rho(V_n)}]$ *and continuous density on* $(0, \sqrt{\rho(V_n)})$.

(3) *For each* $n \geq 1$ *the measure* μ_n *from part (1) is the unique radially symmetric probability measure on* $\mathbb{C}$ *with* $\mu_n(\{z : |z| \leq s\}) = F_n(s)$ *for all* $s \in (0,\infty)$.

It is natural to conjecture that the convergence in probability in the above theorem could be strengthened to almost-sure convergence. The main barrier to such an improvement is our use of a results from [14], where it shown the smallest singular value $Y_n - z$ is polynomially small except with probability at most n^{-c} for a small constant $c > 0$ depending on the parameter ε from Assumption **A0**. If this result were strengthened to a probability bound that is summable in n, it would immediately imply the above result with almost-sure convergence, via the Borel–Cantelli lemma.

This theorem calls for some comments. Using the fact that μ_n is radially symmetric along with the properties of $F_n(s) = \mu_n(\{z : |z| \leq s\})$, it is straightforward that μ_n has a density f_n on $\mathbb{C} \setminus \{0\}$ which is given by the formula

$$f_n(z) \;=\; \frac{1}{2\pi|z|}\frac{d}{ds}F_n(s)\Big|_{s=|z|} \;=\; -\frac{1}{2\pi n|z|}\frac{d}{ds}\langle \boldsymbol{q}(s), V\widetilde{\boldsymbol{q}}(s)\rangle\Big|_{s=|z|} \tag{2.5}$$

for $|z| \notin \{0, \sqrt{\rho(V_n)}\}$. We use the convention $f_n(z) = 0$ for $|z| = \sqrt{\rho(V_n)}$.

2.4. **Sufficient conditions for admissibility.** We now recall a series of assumptions that enforce **A2** and are directly checkable on the sequence (V_n) of variance profile matrices.

A3 (Lower bound on variances). There exists $\sigma_{\min} > 0$ such that $\inf_n \min_{1\leq i,j\leq n} \sigma_{ij}^{(n)} \geq \sigma_{\min}$.

A4 (Symmetric variance profile). For all $n \geq 1$, the normalized variance profile (or equivalently the standard deviation profile) is symmetric: $V_n = V_n^{\mathsf{T}}$.

The following assumption is a quantitative form of irreducibility that considerably generalizes **A3**, allowing a broad class of sparse variance profiles. We refer the reader to [15] for the definition.

A5 (Robust irreducibility). There exists constants $\sigma_0, \delta, \kappa \in (0,1)$ such that for all $n \geq 1$, the matrix $A_n(\sigma_0) = \left(\sigma_{ij}\, \mathbf{1}_{\sigma_{ij}\geq\sigma_0}\right)$ is (δ,κ)-robustly irreducible.

We gather in the following theorem some results from [15], namely Propositions 2.5 and 2.6, as well as Theorem 2.8.

Theorem 2.4 (Cook et al. [15])**.** *Let* (A_n) *be a family of standard deviation profiles for which* **A1** *holds. If either* **A3**, **A4**, *or* **A5** *holds then* **A2** *also holds: the family* (A_n) *is admissible.*

2.5. **Positivity of the density of** μ_n**.** In this section we consider the positivity of μ_n. In [7, Lemma 4.1], it is shown that under Assumption **A3**, the density of μ_n is strictly positive on the disk of radius $\sqrt{\rho(V)}$, centered at the origin. We will begin by giving a more general assumption, see **A6**, under which the density of μ_n, is uniformly bounded from below on its support.

Of particular interest is the behavior of μ_n near zero. By Theorem 2.3, F_n admits a limit as $s \downarrow 0$. Is this limit positive (atom) or equal to zero (no atom)? Is its derivative finite at $z = 0$ (finite density), zero (vanishing density), or does it blow up at $z = 0$? In Proposition 2.7, we will give an explicit formula for the density f_n at zero under Assumption **A6**. In Corollary 2.8, we use this formula to lower bound the density at zero and give a necessary condition for μ_n to be given by the circular law. Proposition 4.2 provides an example of a simple variance profile with large zero blocks where μ_n admits a closed-form expression with an atom and a vanishing density at $z = 0$. Section 4 gives further examples that shed additional light on these questions. Then in Theorem 2.9, we adapt an argument from [7] to bound the density of μ_n from below. Our bound does not require assumption **A3**, and in particular gives an effective bound even when the variance profile does not have a spectral gap, as in [7].

We recall the following definition used in [6]:

Definition 2.5. *A $K \times K$ matrix $T = (t_{ij})_{i,j=1}^K$ with nonnegative entries is called fully indecomposable if for any two subsets $I, J \subset \{1, \dots, K\}$ such that $|I| + |J| \ge K$, the submatrix $(t_{ij})_{i\in I, j\in J}$ contains a nonzero entry.*

See [11] for a detailed account on these matrices.

A6 (Block fully indecomposable) For all $n \ge 1$, the normalized variance profiles V_n are *block fully indecomposable*, i.e. there are constants $\phi > 0$, $K \in \mathbb{N}$ independent from $n \ge 1$, a fully indecomposable matrix $Z = (z_{ij})_{i,j\in[K]}$, with $z_{ij} \in \{0, 1\}$ and a partition $(I_j)_{j\in[K]}$ of $[n]$ such that

$$|I_i| = \frac{n}{K}, \qquad V_{xy} \ge \frac{\phi}{n} z_{ij}, \qquad x \in I_i \quad \text{and} \quad y \in I_j$$

for all $i, j \in [K]$.

Assumption **A6** can be seen as a robust version of the full indecomposability of the matrix V. It is well known that the full indecomposability implies the irreducibility of a matrix. Therefore, one can expect that the block full indecomposability implies the robust irreducibility. Indeed, the following is an immediate consequence of [15, Lemma 2.4].

Proposition 2.6. **A6** *implies* **A5**.

Remark 2.3. In [25] full indecomposability is shown to be equivalent to the existence and the uniqueness, up to scaling, of positive diagonal matrices D_1 and D_2 such that $D_1 V D_2$ is doubly stochastic. Below, in Proposition 2.7 and in particular (2.6), we see under Assumption **A6**, $\mathrm{diag}(\boldsymbol{q}) V \mathrm{diag}(\widetilde{\boldsymbol{q}})$ is doubly stochastic. Under Assumption **A6** an optimal local law for square Gram matrices was proven in [6]. The boundedness of the density near zero for Hermitian random matrices under the analogous conditions was proven in [4].

Proposition 2.7 (No atom and bounded density near zero)**.** *Consider a sequence (V_n) of normalized variance profiles and assume that* **A1** *and* **A6** *hold. Let $\vec{\boldsymbol{q}}(s)$ be as in Theorem 2.2, let μ_n be as in Theorem 2.3, and let $\vec{\boldsymbol{r}}(s,t) = \begin{pmatrix} \boldsymbol{r}(s,t) \\ \widetilde{\boldsymbol{r}}(s,t) \end{pmatrix}$ be as in Proposition 2.1. Then,*

(1) The limits $\lim_{t\downarrow 0} \vec{\boldsymbol{r}}(0,t)$ and $\lim_{s\downarrow 0} \vec{\boldsymbol{q}}(s)$ exist and are equal. Writing $\boldsymbol{q}(0) = (q_i(0)) = \lim_{s\downarrow 0} \boldsymbol{q}(s)$ and $\widetilde{\boldsymbol{q}}(0) = (\tilde{q}_i(0)) = \lim_{s\downarrow 0} \widetilde{\boldsymbol{q}}(s)$, it holds that

$$q_i(0)(V_n \widetilde{\boldsymbol{q}}(0))_i = 1 \quad \text{and} \quad \tilde{q}_i(0)(V_n^{\mathsf{T}} \boldsymbol{q}(0))_i = 1\,, \quad i \in [n]\,. \tag{2.6}$$

In particular, the probability measure μ_n has no atom at zero: $\mu_n(\{0\}) = 0$.

(2) The density f_n of μ_n on $\mathbb{C} \setminus \{0\}$ admits a limit as $z \to 0$. This limit $f_n(0)$ is given by

$$f_n(0) = \frac{1}{n} \sum_{i\in[n]} \frac{1}{(V_n^{\mathsf{T}} \boldsymbol{q}(0))_i (V_n \widetilde{\boldsymbol{q}}(0))_i} = \frac{1}{n} \sum_{i\in[n]} q_i(0)\tilde{q}_i(0)\,.$$

In particular, there exist finite constants κ, K independent of $n \ge 1$ such that

$$0 < \kappa \le f_n(0) \le K\,. \tag{2.7}$$

This proposition will be proven in Section 5.1.

Corollary 2.8. *Let V satisfy Assumptions* **A1** *and* **A6***. Then the density of μ_n at zero is greater than or equal to $1/(\pi\rho(V))$, with equality if and only if $V = D^{-1}SD$ for some diagonal matrix D and doubly stochastic matrix S. In the latter case, $\mu_n = \mu_{circ}$, the circular law.*

The proof of this corollary is given in Section 5.3.

Theorem 2.9. *Assume that* **A1** *holds true and that A_n is irreducible. Then,*

(1) Assuming **A2***, if $|z| \in (0, \sqrt{\rho(V)})$, then the density f_n of μ_n is bounded from below by a positive constant that depends on $|z|$ and is independent of n.*
(2) Assuming **A6***, then for $|z| \in [0, \sqrt{\rho(V)})$, the density f_n of μ_n (for which existence at zero is stated by Proposition 2.7) is bounded from below by a positive constant that depends on $|z|$ and is independent of n.*

The proof of Theorem 2.9 is postponed to Section 5.2. Part (2) will follow easily by noting that the proof of Proposition 2.7-(2) shows the lower bound in (5.21) is bounded away from zero. Finally, we remark the examples in Section 4 show that one cannot expect z independent lower bounds in general. We do note that our lower bounds only depend on the solution to (2.3).

3. Sampled variance profile

3.1. **Sampled variance profile.** Here, we are interested in the case where

$$\sigma_{ij}^2(n) = \sigma^2\left(\frac{i}{n}, \frac{j}{n}\right) ,$$

where σ is a continuous nonnegative function on $[0,1]^2$. In this situation, the deterministic equivalents will converge to a genuine limit as $n \to \infty$. Notice that **A1** holds and denote by

$$\sigma_{\max} = \max_{x,y\in[0,1]} \sigma(x,y) \quad \text{and} \quad \sigma_{\min} = \min_{x,y\in[0,1]} \sigma(x,y) .$$

For the sake of simplicity, we will restrict ourselves to the case where σ takes its values in $(0,\infty)$, i.e. where $\sigma_{\min} > 0$, which implies that **A3** holds.

We will use some results from the Krein–Rutman theory (see for instance [16]), which generalizes the spectral properties of nonnegative matrices to positive operators on Banach spaces. To the function σ^2 we associate the linear operator $\boldsymbol{V}$, defined on the Banach space $C([0,1])$ of continuous real-valued functions on $[0,1]$ as

$$(\boldsymbol{V}f)(x) = \int_0^1 \sigma^2(x,y) f(y)\, dy. \tag{3.1}$$

By the uniform continuity of σ^2 on $[0,1]^2$ and the Arzela–Ascoli theorem, it is a standard fact that this operator is compact [23, Ch. VI.5]. Let $C^+([0,1])$ be the convex cone of nonnegative elements of $C([0,1])$:

$$C^+([0,1]) = \{f \in C([0,1])\, ,\ f(x) \geq 0 \quad \text{for } x \in [0,1]\} .$$

Since $\sigma_{\min} > 0$, the operator $\boldsymbol{V}$ is strongly positive, i.e. it sends any element of $C^+([0,1]) \setminus \{0\}$ to the interior of $C^+([0,1])$, the set of continuous and positive functions on $[0,1]$. Under these conditions, it is well known that the spectral radius $\rho(\boldsymbol{V})$ of $\boldsymbol{V}$ is nonzero, and it coincides with the so-called Krein–Rutman eigenvalue of $\boldsymbol{V}$ [16, Theorem 19.2 and 19.3].

To be consistent with our notation for nonnegative finite dimensional vectors, we write $f \succcurlyeq_{\neq} 0$ when $f \in C^+([0,1]) \setminus \{0\}$, and $f \succ 0$ when $f(x) > 0$ for all $x \in [0,1]$.

Theorem 3.1 (Sampled variance profile). *Assume that there exists a continuous function $\sigma : [0,1]^2 \to (0,\infty)$ such that*

$$\sigma_{ij}^{(n)} = \sigma\left(\frac{i}{n}, \frac{j}{n}\right).$$

Let $(Y_n)_{n\geq 1}$ be a sequence of random matrices as in Definition 1.1 and assume that **A0** *holds. Then,*

(1) *The spectral radius $\rho(V_n)$ of the matrix $V_n = n^{-1}(\sigma_{ij}^2)$ converges to $\rho(\boldsymbol{V})$ as $n \to \infty$, where $\boldsymbol{V}$ is the operator on $C([0,1])$ defined by (3.1).*

(2) *Given $s > 0$, consider the system of equations:*

$$\begin{cases} Q_\infty(x,s) = \dfrac{\int_0^1 \sigma^2(y,x) Q_\infty(y,s)\, dy}{s^2 + \int_0^1 \sigma^2(y,x) Q_\infty(y,s)\, dy \int_0^1 \sigma^2(x,y) \widetilde{Q}_\infty(y,s)\, dy}\ , \\ \widetilde{Q}_\infty(x,s) = \dfrac{\int_0^1 \sigma^2(x,y) \widetilde{Q}_\infty(y,s)\, dy}{s^2 + \int_0^1 \sigma^2(y,x) Q_\infty(y,s)\, dy \int_0^1 \sigma^2(x,y) \widetilde{Q}_\infty(y,s)\, dy}\, , \\ \displaystyle\int_0^1 Q_\infty(y,s)\, dy = \int_0^1 \widetilde{Q}_\infty(y,s)\, dy. \end{cases} \tag{3.2}$$

with unknown parameters $Q_\infty(\cdot,s), \widetilde{Q}_\infty(\cdot,s) \in C^+([0,1])$. Then,

(a) *for $s \geq \sqrt{\rho(\boldsymbol{V})}$, $Q_\infty(\cdot,s) = \widetilde{Q}_\infty(\cdot,s) = 0$ is the unique solution of this system.*

(b) *for $s \in (0, \sqrt{\rho(\boldsymbol{V})})$, the system has a unique solution $Q_\infty(\cdot,s) + \widetilde{Q}_\infty(\cdot,s) \succcurlyeq_{\neq} 0$. This solution satisfies*

$$Q_\infty(\cdot,s), \widetilde{Q}_\infty(\cdot,s) \succ 0\ .$$

(c) *The functions $Q_\infty, \widetilde{Q}_\infty : [0,1]\times(0,\infty) \longrightarrow [0,\infty)$ are continuous, and continuously extended to $[0,1]\times[0,\infty)$, with*

$$Q_\infty(\cdot,0)\,,\, \widetilde{Q}_\infty(\cdot,0) \succ 0\,.$$

(3) *The function*

$$F_\infty(s) = 1 - \int_{[0,1]^2} Q_\infty(x,s)\, \widetilde{Q}_\infty(y,s)\, \sigma^2(x,y)\, dx\, dy\ , \quad s \in (0,\infty)$$

converges to zero as $s \downarrow 0$. Setting $F_\infty(0) = 0$, the function F_∞ is an absolutely continuous function on $[0,\infty)$ which is the CDF of a probability measure whose support is contained in $[0, \sqrt{\rho(\boldsymbol{V})}]$, and whose density is continuous on $[0, \sqrt{\rho(\boldsymbol{V})}]$.

(4) *Let μ_∞ be the rotationally invariant probability measure on $\mathbb{C}$ defined by the equation*

$$\mu_\infty(\{z\,:\, 0 \leq |z| \leq s\}) = F_\infty(s), \quad s \geq 0\,.$$

Then,

$$\mu_n^Y \xrightarrow[n\to\infty]{w} \mu_\infty \quad \textit{in probability}\ .$$

The proof of Theorem 3.1 is an adaptation of the proofs of Lemmas 4.3 and 4.4 from [15] to the context of Krein–Rutman's theory for positive operators in Banach spaces.

3.2. **Proof of Theorem 3.1.** Extending the maximum norm notation from vectors to functions, we also denote by $|||f|||_\infty = \sup_{x\in[0,1]} |f(x)|$ the norm on the Banach space $C([0,1])$. Given a positive integer n, the linear operator $\boldsymbol{V}_n$ defined on $C([0,1])$ as

$$\boldsymbol{V}_n f(x) = \frac{1}{n} \sum_{j=1}^{n} \sigma^2(x, j/n)\, f(j/n)$$

is a finite rank operator whose eigenvalues coincide with those of the matrix V_n. It is easy to check that $\boldsymbol{V}_n f \to \boldsymbol{V} f$ in $C([0,1])$ for all $f \in C([0,1])$, in other words, $\boldsymbol{V}_n$ converges strongly to $\boldsymbol{V}$ in $C([0,1])$, denoted by

$$\boldsymbol{V}_n \xrightarrow[n\to\infty]{str} \boldsymbol{V}$$

in the sequel. However, $\boldsymbol{V}_n$ does not converge to $\boldsymbol{V}$ in norm, in which case the convergence of $\rho(\boldsymbol{V}_n)$ to $\rho(\boldsymbol{V})$ would have been immediate. Nonetheless, the family of operators $\{\boldsymbol{V}_n\}$ satisfies the property that the set $\{\boldsymbol{V}_n f \ : \ n \geq 1, \, |||f|||_\infty \leq 1\}$ has a compact closure, being a set of equicontinuous and bounded functions thanks to the uniform continuity of σ^2 on $[0,1]^2$. Following [10], such a family is named *collectively compact*.

We recall the following important properties, cf. [10]. If a sequence $(\boldsymbol{T}_n)$ of collectively compact operators on a Banach space converges strongly to a bounded operator $\boldsymbol{T}$, then:

i) The spectrum of $\boldsymbol{T}_n$ is eventually contained in any neighborhood of the spectrum of $\boldsymbol{T}$. Furthermore, λ belongs to the spectrum of $\boldsymbol{T}$ if and only if there exist λ_n in the spectrum of $\boldsymbol{T}_n$ such that $\lambda_n \to \lambda$;
ii) $(\lambda - \boldsymbol{T}_n)^{-1} \xrightarrow[n\to\infty]{str} (\lambda - \boldsymbol{T})^{-1}$ for any λ in the resolvent set of $\boldsymbol{T}$.

The statement (1) of the theorem follows from *i)*. We now provide the main steps of the proof of the statement (2). Given $n \geq 1$ and $s > 0$, let $(\boldsymbol{q}^n(s)^\mathsf{T} \, \widetilde{\boldsymbol{q}}^n(s)^\mathsf{T})^\mathsf{T} \in \mathbb{R}^{2n}$ be the solution of the system (2.3) that is specified by Theorem 2.2. Denote by $\boldsymbol{q}^n(s) = (q_1^n(s), \ldots, q_n^n(s))$ and $\widetilde{\boldsymbol{q}}^n = (\widetilde{q}_1^n, \ldots, \widetilde{q}_n^n)$ and introduce the quantities

$$\Phi_n(x,s) = \frac{1}{n}\sum_{i=1}^n \sigma^2\left(x, \frac{i}{n}\right) \widetilde{q}_i^n(s) \quad \text{and} \quad \widetilde{\Phi}_n(x,s) = \frac{1}{n}\sum_{i=1}^n \sigma^2\left(\frac{i}{n}, x\right) q_i^n(s) \, . \tag{3.3}$$

By Proposition 2.5 of [15] (recall that **A3** holds), we know that the average

$$\langle \boldsymbol{q}^n(s) \rangle_n = \frac{1}{n}\sum_{i=1}^n q_i^n(s)$$

satisfies $\langle \boldsymbol{q}^n(s) \rangle_n \leq \sigma_{\min}^{-1}$. Therefore, we get from (2.3) that

$$\|\boldsymbol{q}^n(s)\|_\infty \ \leq \ \frac{\sigma_{\max}^2 \langle \boldsymbol{q}^n(s) \rangle_n}{s^2} \ \leq \ \frac{\sigma_{\max}^2}{\sigma_{\min} s^2} \, . \tag{3.4}$$

Consequently the family $\{\widetilde{\Phi}_n(\cdot, s)\}_{n\geq 1}$ is an equicontinuous and bounded subset of $C([0,1])$. Similarly, an identical conclusion holds for the family $\{\Phi_n(\cdot, s)\}_{n\geq 1}$. By Arzela–Ascoli's theorem, there exists a subsequence (still denoted by (n), with a small abuse of notation) along which $\widetilde{\Phi}_n(\cdot, s)$ and $\Phi_n(\cdot, s)$ respectively converge to given functions $\widetilde{\Phi}_\infty(\cdot, s)$ and $\Phi_\infty(\cdot, s)$ in $C([0,1])$. Denote

$$\Psi_n(x,s) = \frac{1}{s^2 + \Phi_n(x,s)\widetilde{\Phi}_n(x,s)} \quad \text{and} \quad \Psi_\infty(x,s) = \frac{1}{s^2 + \Phi_\infty(x,s)\widetilde{\Phi}_\infty(x,s)} \, .$$

and introduce the auxiliary quantities

$$Q_n(x,s) = \Psi_n(x,s)\widetilde{\Phi}_n(x,s) \qquad \text{and} \qquad \widetilde{Q}_n(x,s) = \Psi_n(x,s)\Phi_n(x,s) \, .$$

Then there exists $Q_\infty(x,s)$ and $\widetilde{Q}_\infty(x,s)$ such that $Q_n(\cdot,s) \to Q_\infty(\cdot,s)$ and $\widetilde{Q}_n(\cdot,s) \to \widetilde{Q}_\infty(\cdot,s)$ in $C([0,1])$. These limits satisfy

$$Q_\infty(x,s) \ = \ \frac{\widetilde{\Phi}_\infty(x,s)}{s^2 + \Phi_\infty(x,s)\widetilde{\Phi}_\infty(x,s)} \qquad \text{and} \qquad \widetilde{Q}_\infty(x,s) \ = \ \frac{\widetilde{\Phi}_\infty(x,s)}{s^2 + \Phi_\infty(x,s)\widetilde{\Phi}_\infty(x,s)} \, .$$

Moreover, the mere definition of $\boldsymbol{q}^n$ and $\widetilde{\boldsymbol{q}}^n$ as solutions of (2.3) yields that

$$\begin{cases} Q_n\left(\frac{i}{n}, s\right) = q_i^n(s) & 1 \le i \le n \\ \widetilde{Q}_n\left(\frac{i}{n}, s\right) = \widetilde{q}_i^n(s) & 1 \le i \le n. \end{cases} \tag{3.5}$$

Combining (3.3), (3.5) and the convergence of Q_n and $\widetilde{Q}_n$, we finally obtain the useful representation

$$\Phi_\infty(x,s) = \int_0^1 \sigma^2(x,y)\,\widetilde{Q}_\infty(y,s)\,dy \qquad \text{and} \qquad \widetilde{\Phi}_\infty(x,s) = \int_0^1 \sigma^2(y,x)\,Q_\infty(y,s)\,dy\ . \tag{3.6}$$

which yields that Q_∞ and $\widetilde{Q}_\infty$ satisfy the system (3.2).

To establish the first part of the statement (2), we show that these limits are zero if $s^2 \ge \rho(\boldsymbol{V})$ and positive if $s^2 < \rho(\boldsymbol{V})$, then we show that they are unique. It is known that $\rho(\boldsymbol{V})$ is a simple eigenvalue, it has a positive eigenvector, and there is no other eigenvalue with a positive eigenvector. If $\boldsymbol{T}$ is a bounded operator on $C([0,1])$ such that $\boldsymbol{T}f - \boldsymbol{V}f \succ 0$ for $f \succcurlyeq_{\neq} 0$, then $\rho(\boldsymbol{T}) > \rho(\boldsymbol{V})$ [16, Theorem 19.2 and 19.3].

We first establish (2)-(a). Fix $s^2 \ge \rho(\boldsymbol{V})$, and assume that $Q_\infty(\cdot,s) \succcurlyeq_{\neq} 0$. Since $Q_\infty(\cdot,s) = \Psi_\infty \boldsymbol{V} Q_\infty(\cdot,s)$, where $\Psi_\infty(\cdot,s)$ is the limit of $\Psi_n(\cdot,s)$ along the subsequence (n), it holds that $Q_\infty(\cdot,s) \succ 0$, and by the properties of the Krein–Rutman eigenvalue, that $\rho(\Psi_\infty \boldsymbol{V}) = 1$. From the identity $\int Q_\infty(x,s)\,dx = \int \widetilde{Q}_\infty(x,s)\,dx$, we get that $\widetilde{Q}_\infty(\cdot,s) \succcurlyeq_{\neq} 0$, hence $\widetilde{Q}_\infty(\cdot,s) \succ 0$ by the same argument. By consequence, $s^{-2}\boldsymbol{V}f - \Psi_\infty \boldsymbol{V} f \succ 0$ for all $f \succcurlyeq_{\neq} 0$. This leads to the contradiction $1 \ge \rho(s^{-2}\boldsymbol{V}) > \rho(\Psi_\infty \boldsymbol{V}) = 1$. Thus, $Q_\infty(\cdot,s) = \widetilde{Q}_\infty(\cdot,s) = 0$.

We now establish (2)-(b). Let $s^2 < \rho(\boldsymbol{V})$. By an argument based on collective compactness, it holds that

$$\rho(\Psi_n \boldsymbol{V}_n) \xrightarrow[n\to\infty]{} \rho(\Psi_\infty \boldsymbol{V})$$

and moreover, that $\rho(\Psi_n \boldsymbol{V}_n) = 1$ (see e.g. the proof of Lemma 4.3 of [15]). Thus, $Q_\infty(\cdot,s) \succcurlyeq_{\neq} 0$ and $\widetilde{Q}_\infty(\cdot,s) \succcurlyeq_{\neq} 0$, otherwise $\rho(\Psi_\infty \boldsymbol{V}) = \rho(s^{-2}\boldsymbol{V}) > 1$. Since $Q_\infty(\cdot,s) = \Psi_\infty \boldsymbol{V} Q_\infty(\cdot,s)$, we get that $Q_\infty(\cdot,s) \succ 0$ and similarly, that $\widetilde{Q}_\infty(\cdot,s) \succ 0$.

It remains to show that the accumulation point $(Q_\infty, \widetilde{Q}_\infty)$ is unique. The proof of this fact is similar to its finite dimensional analogue in the proof of Lemma 4.3 from [15]. In particular, the properties of the Perron–Frobenius eigenvalue and its eigenspace are replaced with their Krein–Rutman counterparts, and the matrices $K_{\vec{\boldsymbol{q}}}$ and $K_{\vec{\boldsymbol{q}},\vec{\boldsymbol{q}}'}$ in that proof are replaced with continuous and strongly positive integral operators. Note that the end of the proof is simpler in our context, thanks to the strong positivity assumption instead of the irreducibility assumption. We leave the details to the reader.

We now address (2)-(c) and first prove the continuity of Q_∞ and $\widetilde{Q}_\infty$ on $[0,1]\times(0,\infty)$. This is equivalent to proving the continuity of Φ_∞ and $\widetilde{\Phi}_\infty$ on this set. Let $(x_k,s_k) \to_k (x,s) \in [0,1]\times(0,\infty)$. The bound

$$0 \ \le\ \widetilde{Q}_\infty(y,s)\ \le\ \frac{\sigma^2_{\max}}{\sigma_{\min}\, s^2}$$

follows from (3.5) and the convergence of $\widetilde{Q}_n$ to $\widetilde{Q}_\infty$. As a consequence of (3.6), the family $\{\Phi_\infty(\cdot,s_k)\}_k$ is equicontinuous for k large. By Arzela–Ascoli's theorem and the uniqueness of the solution of the system, we get that $\Phi_\infty(\cdot,s_k) \to_k \Phi_\infty(\cdot,s)$ in $C([0,1])$. Therefore, writing

$$|\Phi_\infty(x_k,s_k) - \Phi_\infty(x,s)| \le \|\Phi_\infty(\cdot,s_k) - \Phi_\infty(\cdot,s)\|_\infty + |\Phi_\infty(x_k,s) - \Phi_\infty(x,s)|$$

and using the continuity of $\Phi_\infty(\cdot,s)$, we get that $\Phi_\infty(x_k,s_k) \to_k \Phi_\infty(x,s)$.

The main steps of the proof for extending the continuity of Q_∞ and $\widetilde{Q}_\infty$ from $[0,1]\times(0,\infty)$ to $[0,1]\times[0,\infty)$ are the following. Following the proof of Proposition 2.7, we can establish that

$$\liminf_{s\downarrow 0}\int_0^1 Q_\infty(x,s)\,dx \;>\; 0\;.$$

The details are omitted. Since

$$\frac{1}{\widetilde{Q}_\infty(x,s)}=\frac{s^2}{\Phi_\infty(x,s)}+\widetilde{\Phi}_\infty(x,s)>\sigma_{\min}\int_0^1 Q_\infty(y,s)\,dy\,,$$

we obtain that $\left|\!\left|\!\left|\widetilde{Q}_\infty(\cdot,s)\right|\!\right|\!\right|_\infty$ is bounded when $s\in(0,\varepsilon)$ for some $\varepsilon>0$. Thus, $\{\Phi_\infty(\cdot,s)\}_{s\in(0,\varepsilon)}$ is equicontinuous by (3.6), and it remains to prove that the accumulation point $\Phi_\infty(\cdot,0)$ is unique.

This can be done by working on the system (3.2) for $s=0$, along the lines of the proof of Lemma 4.3 of [15] and Proposition 2.7. Details are omitted.

Turning to Statement (3), the assertion $F(s)\to 0$ as $s\downarrow 0$ can be deduced from the proof of Proposition 2.7 and a passage to the limit, noting that the bounds in that proof are independent from n.

Consider the Banach space $\mathcal{B}=C([0,1];\mathbb{R}^2)$ of continuous functions

$$\vec{f}=(f,\widetilde{f})^{\mathsf{T}}:[0,1]\longrightarrow\mathbb{R}^2$$

endowed with the norm $\left|\!\left|\!\left|\vec{f}\right|\!\right|\!\right|_{\mathcal{B}}=\sup_{x\in[0,1]}\max(|f(x)|,|\widetilde{f}(x)|)$. In the remainder of the proof, we may use the notation shortcut Ψ_∞^s instead of $\Psi_\infty(\cdot,s)$ and corresponding shortcuts for quantities $\Phi_\infty(\cdot,s)$, $\widetilde{\Phi}_\infty(\cdot,s)$, $Q_\infty(\cdot,s)$ and $\widetilde{Q}_\infty(\cdot,s)$.

Given $s,s'\in(0,\sqrt{\rho(\boldsymbol{V})})$ with $s\neq s'$, consider the function

$$\Delta\vec{Q}_\infty^{s,s'}=\frac{\left(Q_\infty^s-Q_\infty^{s'},\widetilde{Q}_\infty^s-\widetilde{Q}_\infty^{s'}\right)^{\mathsf{T}}}{s^2-{s'}^2}\in\mathcal{B}.$$

Let $\boldsymbol{V}^{\mathsf{T}}$ be the linear operator associated to the kernel $(x,y)\mapsto\sigma^2(y,x)$, and defined as

$$\boldsymbol{V}^{\mathsf{T}}f(x)=\int_0^1\sigma^2(y,x)f(y)\,dy\;.$$

Then, mimicking the proof of Lemma 4.4 of [15], it is easy to prove that $\Delta\vec{Q}_\infty^{s,s'}$ satisfies the equation

$$\Delta\vec{Q}_\infty^{s,s'}=\boldsymbol{M}_\infty^{s,s'}\Delta\vec{Q}_\infty^{s,s'}+\boldsymbol{a}_\infty^{s,s'},$$

where $\boldsymbol{M}_\infty^{s,s'}$ is the operator acting on $\mathcal{B}$ and defined in a matrix form as

$$\boldsymbol{M}_\infty^{s,s'}=\begin{pmatrix} s^2\Psi_\infty^s\Psi_\infty^{s'}\boldsymbol{V}^{\mathsf{T}} & -\Psi_\infty^s\Psi_\infty^{s'}\widetilde{\Phi}_\infty^s\widetilde{\Phi}_\infty^{s'}\boldsymbol{V}\\ -\Psi_\infty^s\Psi_\infty^{s'}\Phi_\infty^s\Phi_\infty^{s'}\boldsymbol{V}^{\mathsf{T}} & s^2\Psi_\infty^s\Psi_\infty^{s'}\boldsymbol{V}\end{pmatrix},$$

and $\boldsymbol{a}_\infty^{s,s'}$ is a function $\mathcal{B}$ defined as

$$\boldsymbol{a}_\infty^{s,s'}=-\begin{pmatrix}\Psi_\infty^s\Psi_\infty^{s'}\boldsymbol{V}^{\mathsf{T}}Q_\infty^s\\ \Psi_\infty^s\Psi_\infty^{s'}\boldsymbol{V}\widetilde{Q}_\infty^s\end{pmatrix}.$$

To proceed, we rely on a regularized version of this equation. Denoting by $\mathbf{1}$ the constant function $\mathbf{1}(x)=1$ in $C([0,1])$, and letting $v=(\mathbf{1},-\mathbf{1})^{\mathsf{T}}\in\mathcal{B}$, the kernel operator vv^{T} on $\mathcal{B}$ is defined by the matrix

$$(vv^{\mathsf{T}})(x,y)=\begin{pmatrix}\mathbf{1}(x)\mathbf{1}(y) & -\mathbf{1}(x)\mathbf{1}(y)\\ -\mathbf{1}(x)\mathbf{1}(y) & \mathbf{1}(x)\mathbf{1}(y)\end{pmatrix}.$$

By the constraint $\int Q_\infty^s=\int\widetilde{Q}_\infty^s$, it holds that $(vv^{\mathsf{T}})\Delta\vec{Q}_\infty^{s,s'}=0$. Thus, $\Delta\vec{Q}_\infty^{s,s'}$ satisfies the identity

$$\left((I-(\boldsymbol{M}_\infty^{s,s'})^{\mathsf{T}})(I-\boldsymbol{M}_\infty^{s,s'})+vv^{\mathsf{T}}\right)\Delta\vec{Q}_\infty^{s,s'}=(I-(\boldsymbol{M}_\infty^{s,s'})^{\mathsf{T}})\boldsymbol{a}_\infty^{s,s'}\;. \tag{3.7}$$

We rewrite the left side of this identity as $(I - \boldsymbol{G}_\infty^{s,s'})\Delta\vec{Q}_\infty^{s,s'}$ where

$$\boldsymbol{G}_\infty^{s,s'} = \boldsymbol{M}_\infty^{s,s'} + (\boldsymbol{M}_\infty^{s,s'})^\mathsf{T} - (\boldsymbol{M}_\infty^{s,s'})^\mathsf{T}\boldsymbol{M}_\infty^{s,s'} - vv^\mathsf{T}\,,$$

and we study the behavior of $\boldsymbol{M}_\infty^{s,s'}$ and $\boldsymbol{G}_\infty^{s,s'}$ as $s' \to s$.

Let $s \in (0, \sqrt{\rho(\boldsymbol{V})})$ and s' belong to a small compact neighborhood $\mathcal{K}$ of s. Then the first component of $\boldsymbol{M}_\infty^{s,s'}\vec{f}(x)$ has the form

$$\int \big(h_{11}(x,y,s')f(y) + h_{12}(x,y,s')\widetilde{f}(y)\big)\,dy\,,$$

where h_{11} and h_{12} are continuous on the compact set $[0,1]^2 \times \mathcal{K}$ by the previous results. A similar argument holds for the other component of $\boldsymbol{M}_\infty^{s,s'}\vec{f}(x)$. By the uniform continuity of these functions on this set, we get that the family

$$\Big\{\boldsymbol{M}_\infty^{s,s'}\vec{f}\;:\;s' \in \mathcal{K},\;\Big|\Big|\Big|\vec{f}\Big|\Big|\Big|_{\mathcal{B}} \le 1\Big\}$$

is equicontinuous, and by the Arzela–Ascoli theorem, the family $\{\boldsymbol{M}_\infty^{s,s'}\;:\;s' \in \mathcal{K}\}$ is collectively compact. Moreover,

$$\boldsymbol{M}_\infty^{s,s'} \xrightarrow[s'\to s]{str} M_\infty^s = \begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix} N_\infty^s \begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}\,,$$

where

$$N_\infty^s = \begin{pmatrix} s^2\Psi_\infty^2(\cdot,s)\boldsymbol{V}^\mathsf{T} & \Psi_\infty^2(\cdot,s)\widetilde{\Phi}_\infty^2(\cdot,s)\boldsymbol{V} \\ \Psi_\infty^2(\cdot,s)\Phi_\infty^2(\cdot,s)\boldsymbol{V}^\mathsf{T} & s^2\Psi_\infty^2(\cdot,s)\boldsymbol{V}\end{pmatrix}\,.$$

By a similar argument, $\{\boldsymbol{G}_\infty^{s,s'}\;:\;s' \in \mathcal{K}\}$ is collectively compact, and $\boldsymbol{G}_\infty^{s,s'} \xrightarrow[s'\to s]{str} G_\infty^s$, where

$$G_\infty^s = M_\infty^s + (M_\infty^s)^\mathsf{T} - (M_\infty^s)^\mathsf{T} M_\infty^s - vv^\mathsf{T}\,.$$

We now claim that 1 belongs to the resolvent set of the compact operator G_∞^s.

Repeating an argument of the proof of Lemma 4.4 from [15], we can prove that the Krein–Rutman eigenvalue of the strongly positive operator N_∞^s is equal to one, and its eigenspace is generated by the vector $\vec{Q}_\infty^s = \big(Q_\infty^s, \widetilde{Q}_\infty^s\big)^\mathsf{T}$. From the expression of M_∞^s, we then obtain that the spectrum of this compact operator contains the simple eigenvalue 1, and its eigenspace is generated by the vector $\big(Q_\infty^s, -\widetilde{Q}_\infty^s\big)$.

We now proceed by contradiction. If 1 were an eigenvalue of G_∞^s, there would exist a non zero vector $\vec{f} \in \mathcal{B}$ such that $(I - G_\infty^s)\vec{f} = 0$, or, equivalently,

$$(I - (M_\infty^s)^\mathsf{T})(I - M_\infty^s)\vec{f} + vv^\mathsf{T}\vec{f} = 0\,.$$

Left-multiplying the left hand side of this expression by $\vec{f}^\mathsf{T}$ and integrating on $[0,1]$, we get that $(I - M_\infty^s)\vec{f} = 0$ and $\int f = \int \widetilde{f}$, which contradicts the fact the $\vec{f}$ is collinear with $\big(Q_\infty(\cdot,s), -\widetilde{Q}_\infty(\cdot,s)\big)$.

Returning to (3.7) and observing that $\{\boldsymbol{M}_\infty^{s,s'}\;:\;s' \in \mathcal{K}\}$ is bounded, we get from the convergence $(\boldsymbol{M}_\infty^{s,s'})^\mathsf{T} \xrightarrow[s'\to s]{str} (M_\infty^s)^\mathsf{T}$ that

$$(I - (\boldsymbol{M}_\infty^{s,s'})^\mathsf{T})\boldsymbol{a}_\infty^{s,s'} \xrightarrow[s'\to s]{} (I - (M_\infty^s)^\mathsf{T})a_\infty^s\,,$$

where

$$a_\infty^s(\cdot) = -\begin{pmatrix} \Psi_\infty(\cdot,s)^2\boldsymbol{V}^\mathsf{T} Q_\infty(\cdot,s) \\ \Psi_\infty(\cdot,s)^2\boldsymbol{V}\widetilde{Q}_\infty(\cdot,s)\end{pmatrix}\,.$$

From the aforementioned results on the collectively compact operators, it holds that there is a neighborhood of 1 where $\boldsymbol{G}_\infty^{s,s'}$ has no eigenvalue for all s' close enough to s (recall that 0 is the only possible accumulation point of the spectrum of G_∞^s). Moreover,

$$(I - \boldsymbol{G}_\infty^{s,s'})^{-1} \xrightarrow[s'\to s]{str} (I - G_\infty^s)^{-1} .$$

In particular, for s' close enough to s, the family $\{(I - \boldsymbol{G}_\infty^{s,s'})^{-1}\}$ is bounded by the Banach-Steinhaus theorem. Thus,

$$\begin{aligned} \Delta \vec{Q}_\infty^{s,s'} \quad &\xrightarrow[s'\to s]{} \quad \left((I - (M_\infty^s)^\mathsf{T})(I - M_\infty^s) + vv^\mathsf{T}\right)^{-1}(I - (M_\infty^s)^\mathsf{T})a_\infty^s \\ &= (\partial_{s^2} Q_\infty^s, \partial_{s^2} \widetilde{Q}_\infty^s)^\mathsf{T} . \end{aligned}$$

Using this result, we straightforwardly obtain from the expression of F_∞ that this function is differentiable on $(0, \sqrt{\rho(\boldsymbol{V})})$. The continuity of the derivative as well as the existence of a right limit as $s \downarrow 0$ and a left limit as $s \uparrow \sqrt{\rho(\boldsymbol{V})}$ can be shown by similar arguments involving the behaviors of the operators M_∞^s and G_∞^s as s varies. The details are skipped.

Since $\mu_n^Y \sim \mu_n$ in probability and since we have the straightforward convergence $\mu_n \xrightarrow[n\to\infty]{w} \mu_\infty$, the statement (4) of the theorem follows.

4. Examples and simulations

In this section, we provide simulations for band matrix models in Section 4.1 and exhibit a model with vanishing density and an atom at zero in Section 4.2.

4.1. **Band matrix models.** We illustrate Theorem 2.3 with simulations. In the case of band matrices, closed-form expressions for the density seem out of reach but plots can be obtained by numerics. We consider two probabilistic matrix models with complex entries (with independent Bernoulli real and imaginary parts) and sampled variance profiles associated to the following functions:

Model A	Model B
$\sigma^2(x,y) = \mathbb{1}_{\left\{\lvert x-y\rvert \le \frac{1}{20}\right\}}$	$\sigma^2(x,y) = (x+2y)^2\, \mathbb{1}_{\left\{\lvert x-y\rvert \le \frac{1}{10}\right\}}$

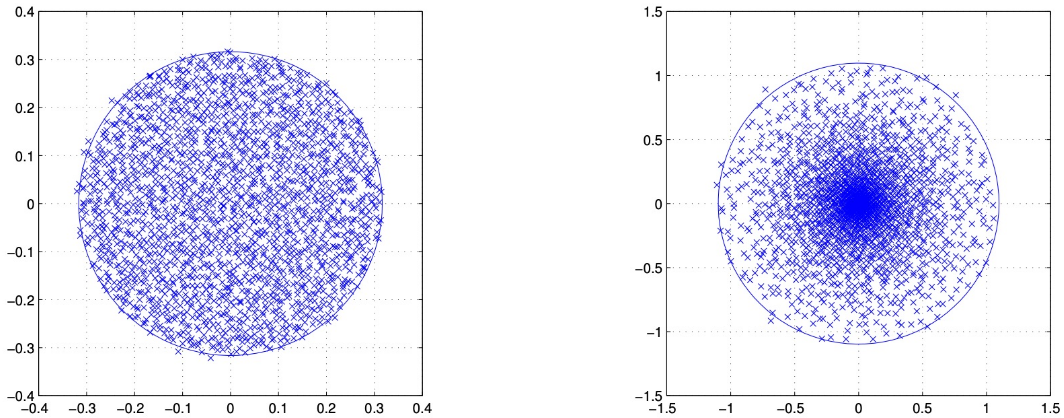
Clearly, the function associated to Model A yields a symmetric variance profile, admissible by Theorem 2.4. Model B satisfies the broad connectivity hypothesis (see [15, Remark 2.8]), hence **A5** (which is weaker than the broad connectivity assumption).

Lemma 4.1. *Given $\alpha \in (0,1)$ and $a > 0$, consider the standard deviation profile matrix $A_n = (\sigma(i/n, j/n))_{i,j=1}^n$ where $\sigma^2(x,y) = (x+ay)^2\, \mathbb{1}_{\lvert x-y\rvert \le \alpha}$. Then, there exists a cutoff $\boldsymbol{\sigma}_0 \in (0,1)$ such that for all n large enough, the matrix $A_n(\boldsymbol{\sigma}_0)$ satisfies the broad connectivity hypothesis with $\delta = \kappa = c\alpha$ for a suitable absolute constant $c > 0$.*

Proof. One can take the cutoff parameter σ_0 sufficiently small that the entries $\sigma_{ij} < \sigma_0$ within the band are confined to the top-left corner of A of dimension $n/100$, say, at which point the argument of [14, Corollary 1.17] applies with minor modification. ☐

Eigenvalue realizations for models A and B are shown on Figure 1.

Up to the "corner effects", the variance profile for Model A is a scaled version of the doubly stochastic variance profile considered in Section 4.3. It is therefore expected that the density for Model A is "close" to the density of the circular law.



(a) Model A **(b)** Model B

Figure 1. Eigenvalues realizations. Setting: $n = 2000$; the circles' radii are $\sqrt{\rho(V)}$.

Due to the form of the variance profile of Model B, a good proportion of the rows and columns of the matrix Y_n have small Euclidean norms. We can therefore expect that many of the eigenvalues of Y_n will concentrate towards zero. This phenomenon is particularly visible in Figure 1b.

4.2. **A limiting distribution with an atom at $z = 0$.** The following Proposition gives an example of a variance profile with a deterministic equivalent that has an atom at zero.

Proposition 4.2 (Example with an atom and vanishing density at zero)**.** *Denote by J_m the $m \times m$ matrix whose elements are all equal to one. Let $k \geq 1$ be a fixed integer, assume that $n = km$ ($m \geq 1$) and consider the $n \times n$ matrix*

$$A_n = \begin{pmatrix} 0 & J_m & \cdots & J_m \\ J_m & 0 & \cdots & 0 \\ \vdots & & & \\ J_m & 0 & \cdots & 0 \end{pmatrix} . \tag{4.1}$$

Associated to matrix A_n is the sequence of normalized variance profiles $V_n = \frac{1}{n} A_n \odot A_n$ with spectral radius $\rho(V_n) = \frac{\sqrt{k-1}}{k}$. Denote by $\rho^* = \sqrt{\rho(V_n)} = \frac{\sqrt[4]{k-1}}{\sqrt{k}}$. Then

1. *Assumptions* **A1** *and* **A2** *hold true.*
2. *The function F_n defined in Theorem 2.3 does not depend on n and is given by*

$$F_n(s) = F_\infty(s) = \frac{1}{k}\sqrt{(k-2)^2 + 4k^2 s^4} \quad \text{if} \quad 0 \leq s \leq \rho^* ,$$

and $F_\infty(s) = 1$ if $s > \rho^*$. In particular, $F_\infty(0) = 1 - \frac{2}{k}$ and $\lim_{s \uparrow \rho^*} F_\infty(s) = 1$.

3. *The density $f_n (= f_\infty)$ and the measure $\mu_n (= \mu_\infty)$ do not depend on n and are given by*

$$\begin{aligned} f_\infty(z) &= \frac{4k}{\pi} \frac{|z|^2}{\sqrt{(k-2)^2 + 4k^2|z|^4}} \mathbf{1}_{\{|z| \leq \rho^*\}} , \\ \mu_\infty(\,dz) &= \left(1 - \frac{2}{k}\right) \delta_0(\,dz) + \frac{4k}{\pi} \frac{|z|^2}{\sqrt{(k-2)^2 + 4k^2|z|^4}} \mathbf{1}_{\{|z| \leq \rho^*\}} \ell(dz) . \end{aligned}$$

In particular, $f_\infty(0) = 0$.

Proof of Proposition 4.2 is left to the reader.

The definition of F_n readily implies that measure μ_n admits an atom at zero of weight $1 - \frac{2}{k}$ since $\mu_n(\{0\}) = F_n(0) = 1 - \frac{2}{k}$. This result can (almost) be obtained by simple linear algebra: Note that $\mathrm{rank}(Y_n) = \mathrm{rank}(n^{-1/2} A_n \odot X_n) \leq (m-2)k$ for any X_n. Indeed, since the top-right $m \times (k-1)m$ submatrix of Y_n has row-rank at most m, its kernel, and hence the kernel of Y_n, has dimension at least $m(k-2)$. Therefore, μ_n^Y has an atom at zero with the weight $\frac{m(k-2)}{mk} = 1 - \frac{2}{k}$ (at least) when n is a multiple of k.

Remark 4.1 (Typical spacing for the random eigenvalues near zero). We heuristically evaluate the typical spacing for the random eigenvalues in a small disk centered at zero.

$$\mu_n^Y(B(0,\varepsilon)) \simeq \left(1 - \frac{2}{k}\right) + \int_{B(0,\varepsilon)} f_\infty(z)\ell(dz)$$

If we remove the $n\left(1 - \frac{2}{k}\right) = km\left(1 - \frac{2}{k}\right) = (k-2)m$ deterministic zero eigenvalues, the typical number of random eigenvalues in $B(0,\varepsilon)$ is

$$\#\{\lambda_i \text{ random } \in B(0,\varepsilon)\} \;=\; n \times \int_{B(0,\varepsilon)} f_\infty(z)\ell(dz) \;=\; 2\pi n \int_0^\varepsilon s h(s)\, ds \;\propto\; n\varepsilon^4,$$

with $h(|z|) = f_\infty(z)$. Hence, if we want the number of random eigenvalues in $B(0,\varepsilon)$ to be of order $\mathcal{O}(1)$, we need to tune $\varepsilon = n^{-1/4}$ and the typical spacing should be $n^{-1/4}$ near zero. On the other hand, the typical spacing at any point z where $f_\infty(z) > 0$ is $n^{-1/2}$. Notice that $n^{-1/4} \gg n^{-1/2}$. This is confirmed by the simulations which show some repulsion phenomenon at zero, cf. Figure 2.

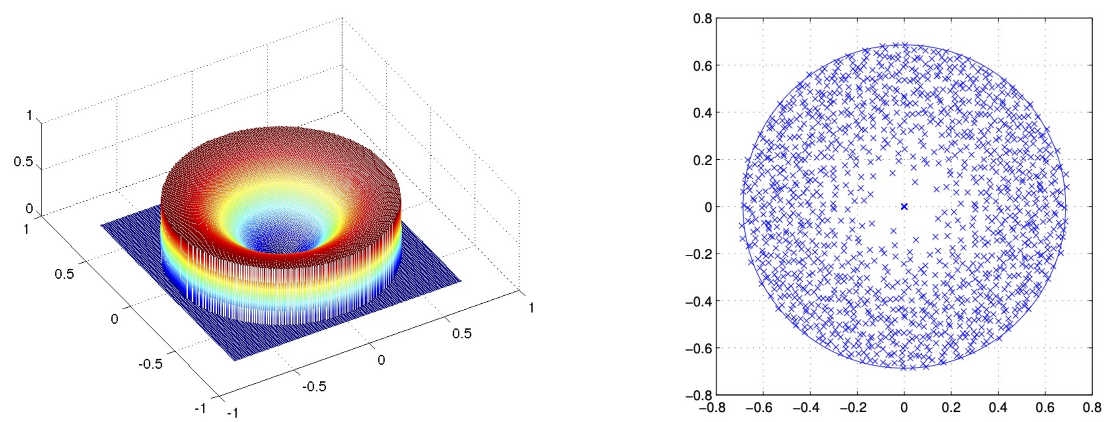


Figure 2. Density f_∞ and eigenvalue realizations of a 2001×2001 matrix for the model studied in Proposition 4.2 in the case $k = 3$. A repulsion phenomenon can be observed near zero.

4.3. **Revisiting the circular law.** Example 2.1 in [15] uses Theorem 2.3 to rederive the classical circular law. In [15, Example 2.2] and [15, Theorem 2.4] the circular law is shown to also hold for any doubly stochastic variance profile that satisfies Assumption **A1**. In both these cases the master equations (2.1), (2.3) simplify to:

$$r_i \equiv r = \frac{r+t}{s^2 + (r+t)^2}\ , \quad r > 0 \qquad \text{and} \qquad q_i \equiv q = \frac{q}{s^2 + q^2}\ , \quad q \geq 0\ . \tag{4.2}$$

Remark 4.2. Beyond doubly stochastic variance profiles, it is not hard to see that the circular law also holds for any variance profile of the form DSD^{-1}, where D is a diagonal, positive matrix and S is a doubly stochastic matrix. Indeed, a random matrix with such a variance profile can be represented as DCD^{-1}, where C is a random matrix with a doubly stochastic variance profile. As the matrices DCD^{-1} and C have the same eigenvalues, we see the circular law is the deterministic equivalent for both.

We illustrate this observation by recovering a result by Aagaard and Haagerup [1, Section 4].

Example 4.1. Let $\epsilon > 0$ and consider the variance profile $\widetilde{C}$ with entries:

$$\sigma_{ij}^2 = \begin{cases} \epsilon & \text{if } i \geq j \\ \epsilon + 1 & \text{if } i < j \end{cases}.$$

Let A be the associated standard deviation profile and consider the random matrix model $n^{-1/2} A \odot X$. Then its deterministic equivalent is given by μ_n, the uniform measure on the disk of radius square root of $\frac{\epsilon}{n}\sum_{i=0}^{n-1}\left(\frac{1+\epsilon}{\epsilon}\right)^{\frac{i}{n}}$. In the limit $n \to \infty$, the expression for the radius converges to $(1/\log(1+1/\epsilon))^{1/2}$.

To prove this, we begin by conjugating the variance profile by D, the diagonal matrix with diagonal element $D_{ii} = \left(\frac{1+\epsilon}{\epsilon}\right)^{\frac{i-1}{n}}$. Matrix $n^{-1} D\widetilde{C}D^{-1}$ is a circulant matrix with positive entries. Since the row and column sums of a circulant matrix are all equal it follows immediately from Theorem 2.3 and Section 4.3 that the deterministic equivalent for the ESD is uniform on a disk. The radius of this disk follows from computing the first eigenvalue of the circulant variance profile.

Recall that by Corollary 2.8 the variance profiles given in Remark 4.2 are the only ones that yield the circular law.

4.4. Approximately doubly stochastic variance profile.

We show that a variance profile matrix which is approximately doubly stochastic as in the statement of the following proposition leads to an approximation of the circular law[1].

Proposition 4.3. *Assume that the family (V_n) of variance profile matrices satisfies Assumptions* **A1** *and* **A2**, *and that*

$$\|V_n \mathbf{1}_n - \mathbf{1}_n\|_\infty \xrightarrow[n\to\infty]{} 0 \quad \textit{and} \quad \|V_n^\mathsf{T} \mathbf{1}_n - \mathbf{1}_n\|_\infty \xrightarrow[n\to\infty]{} 0\,.$$

Then, μ_n converges weakly to μ_{circ} in probability as $n \to \infty$.

Sketch of proof. For arbitrary $s \in \mathbb{R}$ and $t > 0$, let $a = a(t)$ be the unique positive solution of the equation

$$a = \frac{a+t}{s^2 + (a+t)^2} \tag{4.3}$$

in the real variable a. As stated in Section 4.3, the system of equations (2.1) collapses into this scalar equation, when the variance profile matrix is doubly stochastic. In fact, $g(\imath t) = \imath a(t)$ coincides with the Stieltjes transform of the limiting probability measure given by the Schwinger-Dyson equations associated to the Hermitized doubly stochastic model, when restricted to the positive imaginary axis.

Getting back to the system of equations (2.1), recall the definitions of the vectors $\boldsymbol{r} = (r_i)$ and $\widetilde{\boldsymbol{r}} = (\widetilde{r}_i)$, and write

$$m_n(t) = \frac{1}{n}\sum_{i=1}^n r_i, \quad t > 0.$$

[1]The authors thank the reviewer for having pointed out this approximation problem.

It suffices to prove that for all t large enough, $m_n(t) \to_n a(t)$. Indeed, $g_n(\imath t) = \imath m_n(t)$ coincides with the restriction to the positive imaginary axis of the finite-n deterministic approximation of the spectral measure of the Hermitized model associated with V_n. Standard results pertaining to the convergence of Stieltjes transforms of the spectral measures of the Hermitized models, coupled with the admissibility of the family (V_n) lead to the result of the proposition, see [15].

Given two vectors $\boldsymbol{b}$, $\tilde{\boldsymbol{b}} \in \mathbb{R}^n_+$, write

$$\Delta_i(\boldsymbol{b}) = t + (V_n^\top \boldsymbol{b})_i, \quad \text{and} \quad \widetilde{\Delta}_i(\tilde{\boldsymbol{b}}) = t + (V_n \tilde{\boldsymbol{b}})_i.$$

After some simple manipulations, (4.3) rearranges to

$$a(t) = \frac{\Delta_i(a\mathbf{1}_n)}{s^2 + \Delta_i(a\mathbf{1}_n)\widetilde{\Delta}_i(a\mathbf{1}_n)} + \varepsilon_i = \frac{\widetilde{\Delta}_i(a\mathbf{1}_n)}{s^2 + \Delta_i(a\mathbf{1}_n)\widetilde{\Delta}_i(a\mathbf{1}_n)} + \widetilde{\varepsilon}_i, \quad i = 1, \ldots, n,$$

where the vector $\vec{\boldsymbol{\varepsilon}} = (\varepsilon_1, \ldots, \varepsilon_n, \widetilde{\varepsilon}_1, \ldots, \widetilde{\varepsilon}_n)^\top$ satisfies $\|\vec{\boldsymbol{\varepsilon}}\|_\infty \to_n 0$ by the assumptions on (V_n). Subtracting (2.1) from the first equality leads to:

$$a(t) - r_i = \frac{s^2(V^\top(a\mathbf{1} - \boldsymbol{r})_i) + \Delta_i(a\mathbf{1})\Delta_i(\boldsymbol{r})V(a\mathbf{1} - \widetilde{\boldsymbol{r}}))_i}{(s^2 + \Delta_i(a\mathbf{1}_n)\widetilde{\Delta}_i(a\mathbf{1}_n))(s^2 + \Delta_i(\boldsymbol{r})\widetilde{\Delta}_i(\widetilde{\boldsymbol{r}}))} + \varepsilon_i,$$

with a similar equation for $a(t) - \widetilde{r}_i$. The denominator of this expression is lower bounded by t^4. Moreover, $0 \le \Delta_i(a\mathbf{1}), \Delta_i(\boldsymbol{r}) \le t + \sigma^2_{\max}/t$. Writing $\vec{\boldsymbol{r}} = \begin{pmatrix} \boldsymbol{r} \\ \widetilde{\boldsymbol{r}} \end{pmatrix}$ gives the bound:

$$\|a\mathbf{1}_{2n} - \vec{\boldsymbol{r}}\|_\infty \le K\Big(\frac{t^2 + s^2}{t^4} + \frac{1}{t^6}\Big)\|a\mathbf{1}_{2n} - \vec{\boldsymbol{r}}\|_\infty + \|\vec{\boldsymbol{\varepsilon}}\|_\infty,$$

where K is an absolute constant that depends on $\sigma^2_{\max}$. Choosing t large enough that

$$K((t^2 + s^2)t^{-4} + t^{-6}) < 1$$

holds, we obtain $\|a\mathbf{1}_{2n} - \vec{\boldsymbol{r}}\|_\infty \to_n 0$. Which in turn implies that $m_n(t) \to_n a(t)$ for these values of t, as desired. □

5. Positivity of the density

In this section we prove Proposition 2.7, Theorem 2.9 and Corollary 2.8.

In the remainder, the following notation will be useful. For two $n \times 1$ nonnegative vectors $\boldsymbol{a}$ and $\widetilde{\boldsymbol{a}}$ and two parameters $s, t \ge 0$, we shall write $\vec{\boldsymbol{a}}^\top = \big(\boldsymbol{a}^\top \ \widetilde{\boldsymbol{a}}^\top\big)$, and

$$\Psi(\vec{\boldsymbol{a}}, s, t) = \operatorname{diag}\left(\frac{1}{s^2 + [(V_n\widetilde{\boldsymbol{a}})_i + t][(V_n^\top \boldsymbol{a})_i + t]}; \, i \in [n]\right). \tag{5.1}$$

With these notations, the reals r_i and q_i in the systems (2.1) and (2.3) respectively can be written as

$$r_i = (\Psi(\vec{\boldsymbol{r}}, s, t))^{-1}_{ii}\,((V_n^\top \boldsymbol{r})_i + t), \quad \text{and} \quad q_i = (\Psi(\vec{\boldsymbol{q}}, s, 0))^{-1}_{ii}\,(V_n^\top \boldsymbol{q})_i,$$

with similar expressions for $\tilde{r}_i$ and $\tilde{q}_i$.

5.1. **Proof of Proposition 2.7.** Most of the work will go into showing that the limits $\lim_{t\downarrow 0} \vec{\boldsymbol{r}}(0,t)$ and $\lim_{s\downarrow 0} \vec{\boldsymbol{q}}(s)$ exist and are equal. To that end, we rely on some of the results of [6], from which we start by borrowing some notations. We emphasize that the following notation is only used in this subsection. Given to sequences (a_n) and (b_n) of real numbers, $a_n \lesssim b_n$ refers to the fact that there exists a constant $\kappa > 0$ independent of $n \ge 1$ such that $a_n \le \kappa b_n$. The notation $a_n \sim b_n$ stands for $a_n \lesssim b_n$ and $b_n \lesssim a_n$. Given a real vector $\boldsymbol{x}$, the notation $\min \boldsymbol{x}$ refers to the smallest element of $\boldsymbol{x}$.

Lemma 5.1 (Lemmas 3.11, 3.13 and Eq. (3.56) of [6]). *Let* **A1** *and* **A6** *hold true, and recall that* $\vec{\boldsymbol{r}}(0,t)$ *is the unique positive solution of* (2.1) *for* $s=0$ *and* $t>0$. *Then,*

$$1 \quad \lesssim \quad \inf_{t\in(0,10]} \min \vec{\boldsymbol{r}}(0,t) \quad \leq \quad \sup_{t>0} \| \vec{\boldsymbol{r}}(0,t)\|_\infty \quad \lesssim \quad 1\,.$$

The limit $\vec{\boldsymbol{r}}_0 = \begin{pmatrix} \boldsymbol{r}_0 \\ \widetilde{\boldsymbol{r}}_0 \end{pmatrix} = \lim_{t\downarrow 0} \vec{\boldsymbol{r}}(0,t)$ *exists and satisfies* $1 \lesssim \min \vec{\boldsymbol{r}}_0 \leq \|\vec{\boldsymbol{r}}_0\|_\infty \lesssim 1$. *Moreover, writing* $\boldsymbol{r}_0 = (r_{0,i})$ *and* $\widetilde{\boldsymbol{r}}_0 = (\tilde{r}_{0,i})$, *it holds that*

$$r_{0,i}(V_n \widetilde{\boldsymbol{r}}_0)_i = 1, \quad \textit{and} \quad \tilde{r}_{0,i}(V_n^{\mathsf{T}} \widetilde{\boldsymbol{r}}_0)_i = 1\,, \quad i\in[n]\;. \tag{5.2}$$

Proposition 5.2 (Proposition 3.10 (ii) of [6]). *Let* **A1** *and* **A6** *hold. Suppose the functions*

$$\vec{\boldsymbol{d}} = \begin{pmatrix} \boldsymbol{d} \\ \widetilde{\boldsymbol{d}} \end{pmatrix} = \begin{pmatrix} (d_i)_{i\in[n]} \\ (\tilde{d}_i)_{i\in[n]} \end{pmatrix} : \mathbb{R}^+ \to \mathbb{C}^{2n}, \qquad \textit{and} \qquad \vec{\boldsymbol{g}} = \begin{pmatrix} \boldsymbol{g} \\ \widetilde{\boldsymbol{g}} \end{pmatrix} = \begin{pmatrix} (g_i)_{i\in[n]} \\ (\tilde{g}_i)_{i\in[n]} \end{pmatrix} : \mathbb{R}^+ \to (\mathbb{C}\setminus\{0\})^{2n}$$

satisfy

$$\frac{1}{g_i(t)} = (V_n \widetilde{\boldsymbol{g}}(t))_i + t + d_i(t)\ , \quad \frac{1}{\widetilde{g}_i(t)} = (V_n^{\mathsf{T}} \boldsymbol{g}(t))_i + t + \widetilde{d}_i(t) \quad \textit{and} \quad \sum_{i\in[n]} g_i(t) \;=\; \sum_{i\in[n]} \widetilde{g}_i(t) \tag{5.3}$$

for all $t\in\mathbb{R}^+$. *Then, there exist* $\lambda^*>0$ *and* $C>0$, *depending on* V, *such that*

$$\|\vec{\boldsymbol{g}}(t) - \vec{\boldsymbol{r}}(0,t)\|_\infty\, \mathbf{1}_{\{\|\vec{\boldsymbol{g}}(t)-\vec{\boldsymbol{r}}(0,t)\|_\infty \leq \lambda^*\}} \quad \leq \quad C\|\vec{\boldsymbol{d}}(t)\|_\infty \qquad \textit{for all} \quad |t|<10\,.$$

Let us outline the proof of Proposition 2.7–(1). Lemma 5.1 shows that $\vec{\boldsymbol{r}}(0,t)$ converges as $t\downarrow 0$. In parallel, we know from Theorem 2.2–(3) that for each $s>0$, it holds that $\vec{\boldsymbol{r}}(s,t) \to_{t\downarrow 0} \vec{\boldsymbol{q}}(s)$ under the irreducibility assumption, which is implied by **A6**. To prove that $\vec{\boldsymbol{q}}(s) \to_{s\downarrow 0} \vec{\boldsymbol{r}}_0$, we fix $s>0$ small enough and find a sequence $t_k \downarrow 0$ such that $\|\vec{\boldsymbol{r}}(s,t_k) - \vec{\boldsymbol{r}}(0,t_k)\|_\infty \leq \text{Constant} \times s^2$. This inequality will be established iteratively on k. Specifically, we start with a t_0 large enough so that the inequality is satisfied, then we apply a bootstrap procedure on k, controlling $\|\vec{\boldsymbol{r}}(s,t_k) - \vec{\boldsymbol{r}}(0,t_k)\|_\infty$ at each step with the help of Proposition 5.2 with $\vec{\boldsymbol{g}}(t) = \vec{\boldsymbol{r}}(s,t)$. We now begin the proof.

Proof of Proposition 2.7. Letting $\vec{\boldsymbol{g}}(t) = \vec{\boldsymbol{r}}(s,t)$, we get from (2.1) that $\vec{\boldsymbol{g}}(t)$ satisfies (5.3) with

$$d_i(s,t) = \frac{s^2}{((V_n^{\mathsf{T}} \boldsymbol{r}(s,t))_i + t} \quad \text{and} \quad \widetilde{d}_i(s,t) = \frac{s^2}{((V_n \widetilde{\boldsymbol{r}}(s,t))_i + t}\,.$$

We now start our iterative procedure by choosing properly the initial value t_0. Using the bound $\|\vec{\boldsymbol{r}}(0,t)\|_\infty \leq t^{-1}$ and $\|\vec{\boldsymbol{r}}(s,t)\|_\infty \leq t^{-1}$ from (2.1), and $\|\vec{\boldsymbol{d}}(s,t)\|_\infty \leq s^2 t^{-1}$ we get that for t_0 sufficiently large, $\|\vec{\boldsymbol{r}}(s,t_0) - \vec{\boldsymbol{r}}(0,t_0)\|_\infty \leq \lambda^*$ and thus Proposition 5.2 gives the bound

$$\|\vec{\boldsymbol{r}}(s,t_0) - \vec{\boldsymbol{r}}(0,t_0)\|_\infty \leq C s^2 t_0^{-1}. \tag{5.4}$$

We now fix this t_0 and let $K = \sup_{0<t<t_0} \|\vec{\boldsymbol{r}}(0,t)\|_\infty$, which is finite by Lemma 5.1. We also introduce $\ell^*, s^* > 0$ such that

$$\ell^* \quad \leq \quad \min\left(\lambda^*, \frac{1}{2\sigma_{\max}^2 K}\right) \qquad \text{and} \qquad (s^*)^2 \quad \leq \quad \min\left(\frac{\ell^*}{8CK}, \frac{t_0 \ell^*}{4C}\right)\,. \tag{5.5}$$

Fix s such that $0<s<s^*$. From the choice of s^* and (5.4), we get that

$$\|\vec{\boldsymbol{r}}(s,t_0) - \vec{\boldsymbol{r}}(0,t_0)\|_\infty \quad \leq \quad \frac{\ell^*}{4}\,.$$

By Lemma 5.1 and Theorem 2.2–(3), the functions $t\mapsto \vec{\boldsymbol{r}}(0,t)$ and $t\mapsto \vec{\boldsymbol{r}}(s,t)$ extend continuously to $t=0$ and hence are uniformly continuous on the compact interval $[0,t_0]$. Thus, there exists $\eta>0$ such that for $0\leq t,t' \leq t_0$ and $|t-t'|\leq \eta$, we have

$$\|\vec{\boldsymbol{r}}(0,t) - \vec{\boldsymbol{r}}(0,t')\|_\infty \leq \frac{\ell^*}{4}\,,\ \|\vec{\boldsymbol{r}}(s,t) - \vec{\boldsymbol{r}}(s,t')\|_\infty \leq \frac{\ell^*}{4}\ ,\ \left|(V^{\mathsf{T}}\boldsymbol{r}(s,t))_i + t - (V^{\mathsf{T}}\boldsymbol{r}(s,t'))_i - t'\right| \leq \frac{1}{4K}\,.$$

Consider a sequence of real numbers $(t_k)_{k\geq 0}$ such that $t_k \downarrow 0$ and $|t_{k+1}-t_k| < \eta$ for $k \geq 0$. We shall prove inductively that

$$\|\vec{\boldsymbol{r}}(s,t_k) - \vec{\boldsymbol{r}}(0,t_k)\|_\infty \quad \leq \quad \frac{\ell^*}{4} \,. \tag{5.6}$$

Using the uniform continuity and the inductive assumption, we obtain

$$\begin{aligned} &\|\vec{\boldsymbol{r}}(s,t_{k+1}) - \vec{\boldsymbol{r}}(0,t_{k+1})\|_\infty \\ &\quad \leq \quad \|\vec{\boldsymbol{r}}(s,t_{k+1}) - \vec{\boldsymbol{r}}(s,t_k)\|_\infty + \|\vec{\boldsymbol{r}}(s,t_k) - \vec{\boldsymbol{r}}(0,t_k)\|_\infty + \|\vec{\boldsymbol{r}}(0,t_k) - \vec{\boldsymbol{r}}(0,t_{k+1})\|_\infty \;, \\ &\quad \leq \quad \frac{\ell^*}{4} + \frac{\ell^*}{4} + \frac{\ell^*}{4} \quad < \quad \ell^* \quad < \quad \lambda^* \;, \end{aligned} \tag{5.7}$$

thus, Proposition 5.2 leads to the bound

$$\|\vec{\boldsymbol{r}}(s,t_{k+1}) - \vec{\boldsymbol{r}}(0,t_{k+1})\|_\infty \quad \leq \quad C\|\vec{\boldsymbol{d}}(s,t_{k+1})\|_\infty \,.$$

We now upper bound $\|\vec{\boldsymbol{d}}(s,t_{k+1})\|_\infty$. We have:

$$\begin{aligned} (V_n^\mathsf{T}\boldsymbol{r}(s,t_{k+1}))_i + t_{k+1} \quad &\geq \quad (V_n^\mathsf{T}\boldsymbol{r}(0,t_{k+1}))_i + t_{k+1} - \Big(((V_n^\mathsf{T}\boldsymbol{r}(0,t_{k+1}))_i - (V_n^\mathsf{T}\boldsymbol{r}(s,t_{k+1}))_i \Big) \;, \\ &\overset{(a)}{\geq} \quad (V_n^\mathsf{T}\boldsymbol{r}(0,t_{k+1}))_i + t_{k+1} - \sigma_{\max}^2\ell^* \;, \\ &\overset{(b)}{=} \quad \frac{1}{r_i(0,t_{k+1})} - \sigma_{\max}^2\ell^* \quad \geq \quad \frac{1}{K} - \sigma_{\max}^2\ell^* \quad \overset{(c)}{\geq} \quad \frac{1}{2K} \;, \end{aligned}$$

where (a) follows from (5.7), (b) from the system satisfied by $\vec{\boldsymbol{r}}(0,t_{k+1})$ and (c) from the constraint (5.5) of ℓ^*. We finally end up with the estimation $\|\vec{\boldsymbol{d}}(s,t_{k+1})\|_\infty \leq 2Ks^2$. Applying Proposition 5.2 together with (5.7), we obtain

$$\|\vec{\boldsymbol{r}}(s,t_{k+1}) - \vec{\boldsymbol{r}}(0,t_{k+1})\|_\infty \quad \leq \quad C\|\vec{\boldsymbol{d}}(s,t_{k+1})\|_\infty \quad \leq \quad 2CKs^2 \quad \overset{(a)}{\leq} \quad \frac{\ell^*}{4} \,,$$

where (a) follows from the fact that $s < s^*$ and the constraint (5.5) on s^*. Hence the induction step is verified. As a byproduct of the induction, we have, after taking $t_k \downarrow 0$,

$$\forall s \in (0,s^*)\,, \quad \|\vec{\boldsymbol{q}}(s) - \vec{\boldsymbol{r}}_0\|_\infty \leq 2CK\, s^2 \tag{5.8}$$

and in particular, $\vec{\boldsymbol{q}}(s)$ converges to $\vec{\boldsymbol{q}}(0) = \vec{\boldsymbol{r}}_0$ as $s \downarrow 0$.

Combining $q_i(0)(V\widetilde{\boldsymbol{q}}(0))_i = 1$ and $\widetilde{q}_i(0)(V^\mathsf{T}\boldsymbol{q}(0))_i = 1$ with the definition of μ_n, we obtain

$$\mu_n(\{0\}) = 1 - \lim_{s\downarrow 0} \frac{1}{n}\langle \boldsymbol{q}(s), V\widetilde{\boldsymbol{q}}(s)\rangle \;= 1 - \frac{1}{n}\sum_{i\in[n]} q_i(0)(V\widetilde{\boldsymbol{q}}(0))_i = 0 \;.$$

Proposition 2.7-(1) is proven.

We now turn to Proposition 2.7-(2). To establish the existence of the limit of $f(z)$ as $z \to 0$, we first show that $\partial_{s^2}\vec{\boldsymbol{q}}(s)$ can be continuously extended to $s = 0$ as $s \downarrow 0$. This can be done by considering [15, Lemma 4.4]. Using the shorthand notation $\Psi(s) = \Psi(\vec{\boldsymbol{q}}, s, 0)$ from (5.1), let us define

$$M(s) = \begin{pmatrix} s^2\Psi(s)^2V^\mathsf{T} & -\mathrm{diag}(\boldsymbol{q}(s))^2V \\ -\mathrm{diag}(\widetilde{\boldsymbol{q}}(s))^2V^\mathsf{T} & s^2\Psi(s)^2V \end{pmatrix},$$

$$A(s) = \begin{pmatrix} I - M(s) \\ (\mathbf{1}_n^\mathsf{T} \quad -\mathbf{1}_n^\mathsf{T}) \end{pmatrix} \in \mathbb{R}^{(2n+1)\times 2n}, \quad \text{and} \quad b(s) = -\begin{pmatrix} \Psi(s)\boldsymbol{q}(s) \\ \Psi(s)\widetilde{\boldsymbol{q}}(s) \\ 0 \end{pmatrix} \in \mathbb{R}^{2n+1}.$$

Then, it is shown in [15, Lemma 4.4] that $A(s)$ is a full column-rank matrix for $s \in (0, \sqrt{\rho(V)})$, and that $\partial_{s^2}\vec{\boldsymbol{q}}(s) = A(s)^{-\mathrm{L}}b(s)$, where $A(s)^{-\mathrm{L}}$ is the left inverse of $A(s)$. Now, the important observation here is that if we take $s \downarrow 0$, then $A(s)$ converges to the full column-rank matrix

$$A(0) = \begin{pmatrix} I - M(0) \\ (\mathbf{1}_n^\top \quad -\mathbf{1}_n^\top) \end{pmatrix}, \quad \text{with} \quad M(0) = \begin{pmatrix} 0 & -\mathrm{diag}(\boldsymbol{q}(0))^2 V \\ -\mathrm{diag}(\widetilde{\boldsymbol{q}}(0))^2 V^\top & 0 \end{pmatrix}.$$

The convergence to $A(0)$ is an immediate consequence of the convergence of $\vec{\boldsymbol{q}}(s)$ that we just established, and of Lemma 5.1. To show that $A(0)$ is full column-rank, consider the matrix non-negative matrix $N = -M(0)$. We show that $\vec{\boldsymbol{q}}(0)$ is the unique eigenvector of N, up to scaling, such that $N\vec{\boldsymbol{q}}(0) = \vec{\boldsymbol{q}}(0)$.

For any non zero vector $\vec{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{x} \\ \tilde{\boldsymbol{x}} \end{pmatrix}$ such that $\vec{\boldsymbol{x}} = N\vec{\boldsymbol{x}}$, we have

$$\begin{aligned} \mathrm{diag}(\boldsymbol{q}(0)) V \mathrm{diag}(\widetilde{\boldsymbol{q}}(0)) \mathrm{diag}(\widetilde{\boldsymbol{q}}(0))^{-1} \tilde{\boldsymbol{x}} &= \mathrm{diag}(\boldsymbol{q}(0))^{-1} \boldsymbol{x}, \quad \text{and} \\ \mathrm{diag}(\widetilde{\boldsymbol{q}}(0)) V^\top \mathrm{diag}(\boldsymbol{q}(0)) \mathrm{diag}(\boldsymbol{q}(0))^{-1} \boldsymbol{x} &= \mathrm{diag}(\widetilde{\boldsymbol{q}}(0))^{-1} \tilde{\boldsymbol{x}}, \end{aligned} \tag{5.9}$$

thus, writing $Q = \mathrm{diag}(\boldsymbol{q}(0)) V \mathrm{diag}(\widetilde{\boldsymbol{q}}(0))^2 V^\top \mathrm{diag}(\boldsymbol{q}(0))$, we get that

$$Q \mathrm{diag}(\boldsymbol{q}(0))^{-1} \boldsymbol{x} = \mathrm{diag}(\boldsymbol{q}(0))^{-1} \boldsymbol{x}. \tag{5.10}$$

We know from Proposition 2.7–(1) that Q is doubly stochastic (see also Remark 2.3). Moreover, since V is fully indecomposable, Q is also fully indecomposable, see, *e.g.* [11, Theorem 2.2.2]. Thus, it is irreducible, which implies that the only non-zero vectors $\boldsymbol{x}$ that satisfy (5.10) take the form $\boldsymbol{x} = \alpha \boldsymbol{q}(0)$ for $\alpha \neq 0$. Plugging this identity into (5.9), we also get that $\tilde{\boldsymbol{x}} = \alpha \widetilde{\boldsymbol{q}}(0)$, which shows that $\vec{\boldsymbol{x}}$ exists and is equal to $\alpha \vec{\boldsymbol{q}}(0)$.

As a consequence, the right null space of the matrix $I - M(0)$ is spanned by the vector $\begin{pmatrix} \boldsymbol{q}(0) \\ -\widetilde{\boldsymbol{q}}(0) \end{pmatrix}$. Since the inner product of the last row of $A(0)$ with this vector is non zero, $A(0)$ is full column-rank. By the right continuity of $A(s)$ and $b(s)$ at zero and the fact that $A(s)$ is full column-rank on $[0, \sqrt{\rho(V)})$, we conclude that $\partial_{s^2} \vec{\boldsymbol{q}}(s)$ can be continuously extended to $s = 0$ as $s \downarrow 0$.

Now, from the expression (2.5) of the density and Equations (2.3), we have for $|z|$ near zero

$$\begin{aligned} f_n(z) &= -\frac{1}{2\pi n |z|} \frac{d}{ds} \langle \boldsymbol{q}(s), V \widetilde{\boldsymbol{q}}(s) \rangle \Big|_{s=|z|} = -\frac{1}{\pi n} \frac{d}{ds^2} \langle \boldsymbol{q}(s), V \widetilde{\boldsymbol{q}}(s) \rangle \Big|_{s=|z|} \\ &= -\frac{1}{\pi n} \sum_{i \in [n]} \partial_{s^2} \frac{(V_n \widetilde{\boldsymbol{q}}(s))_i (V_n^\top \boldsymbol{q}(s))_i}{s^2 + (V_n \widetilde{\boldsymbol{q}}(s))_i (V_n^\top \boldsymbol{q}(s))_i} \Big|_{s=|z|} \\ &= \frac{1}{\pi n} \sum_{i \in [n]} \frac{(V_n \widetilde{\boldsymbol{q}}(|z|))_i (V_n^\top \boldsymbol{q}(|z|))_i - |z|^2 \partial_{s^2} \left((V_n \widetilde{\boldsymbol{q}}(s))_i (V_n^\top \boldsymbol{q}(s))_i \right) |_{s=|z|}}{\left(|z|^2 + (V_n \widetilde{\boldsymbol{q}}(|z|))_i (V_n^\top \boldsymbol{q}(|z|))_i \right)^2} \end{aligned} \tag{5.11}$$

Since $\|\partial_{s^2} \vec{\boldsymbol{q}}(s)\|_\infty$ is bounded near zero by what we have just shown, it is easily seen that

$$|z|^2 \partial_{s^2} \left((V_n \widetilde{\boldsymbol{q}}(s))_i (V_n^\top \boldsymbol{q}(s))_i \right) |_{s=|z|} \xrightarrow[z \to 0]{} 0.$$

We therefore get that

$$f_n(z) \xrightarrow[z \to 0]{} \frac{1}{\pi n} \sum_{i \in [n]} \frac{1}{(V_n \widetilde{\boldsymbol{q}}(0))_i (V_n^\top \boldsymbol{q}(0))_i}$$

as well as the inequalities (2.7) by using Lemma 5.1 again, which completes the proof of Proposition 2.7-(2). □

5.2. **Proof of Theorem 2.9.** The positivity of the density has been established under Assumptions **A1** and **A3** in [7, Lemma 4.1]. We will follow a similar strategy. The proof of [7, Lemma 4.1] relies on two crucial steps: the existence and regularity of solutions to the master equations (2.3), and an expression for the density (2.5) in terms of a certain operators whose spectrum can be controlled. In [15, Section 5], the first step is established, as long as $|z|$ is away from 0, under the more general Assumption **A5**. Following the calculations from [7], we now carry out the second step, occasionally referring the reader to [7] for details. We note that while the calculations can be closely followed, the weaker assumptions on the variance profile V introduces new complications.

In all this section, we follow the notational convention of [7] stating that if $\boldsymbol{u} = (u_i)$ and $\boldsymbol{v} = (v_i)$ are $n \times 1$ vectors, then $\frac{1}{\boldsymbol{u}}$ is the vector $(\frac{1}{u_i})_{i\in[n]}$, $\sqrt{\boldsymbol{u}} = (\sqrt{u_i})_{i\in[n]}$, $\boldsymbol{uv} = (u_i v_i)_{i\in[n]}$, and so on.

In what follows, $\mathcal{O}(t)$ refers to error terms that are bounded in magnitude by Ct for small t, where the constant C can depend on n or on $|z|$. We use the notation $a(t) \lesssim b(t)$ if there exists a constant C that might depend on n or on $|z|$, such that $a(t) \le Cb(t)$. The notation $a(t) \sim b(t)$ refers to $a(t) \lesssim b(t) \lesssim a(t)$. Note that in this subsection the relation $\sim$ refers to the limit $t \to 0$, rather than $n \to \infty$, as in the previous subsection.

Proof of Theorem 2.9. We now prove part (1), in particular in this section we will always assume Assumption **A2** holds and that $s = |z|^2$ is in the interval $(0, \sqrt{\rho(V)})$. As mentioned in the introduction, we will prove a lower bound that depends on $\boldsymbol{q}$ and $\widetilde{\boldsymbol{q}}$. By Proposition 2.7, we have that under Assumption **A6** these vectors are continuous in a neighborhood of 0, therefore can continuously extend our lower bound to zero and match it with the bound in the previous section, ensuring the lower bound stays away from 0 for all z in the support, verifying part (2).

We start with the expression of the density in (2.5). In what follows it will be more convenient to work on the regularized master equations provided by the system (2.1) rather than those given by the system (2.3), recalling from Theorem 2.2–(3) that $\vec{\boldsymbol{q}}(s) = \lim_{t\downarrow 0} \vec{\boldsymbol{r}}(s,t)$ for $s > 0$. In [15, Section 7], it is indeed proven that we can switch d/ds^2 and $\lim_{t\downarrow 0}$, and write

$$f_n(z) = -\frac{1}{\pi n}\frac{d}{ds^2}\left(\lim_{t\downarrow 0}\langle \boldsymbol{r}(s,t), V\widetilde{\boldsymbol{r}}(s,t)\rangle\right)\Big|_{s=|z|} = -\frac{1}{\pi n}\lim_{t\downarrow 0}\frac{d}{ds^2}\langle \boldsymbol{r}(s,t), V\widetilde{\boldsymbol{r}}(s,t)\rangle\Big|_{s=|z|}.$$

Introducing the notation

$$\boldsymbol{\varphi}(s,t) = V\widetilde{\boldsymbol{r}}(s,t) + t\ , \quad \widetilde{\boldsymbol{\varphi}}(s,t) = V^{\mathsf{T}}\boldsymbol{r}(s,t) + t, \qquad \text{and} \qquad \vec{\boldsymbol{\varphi}}(s,t) = \begin{pmatrix}\boldsymbol{\varphi}(s,t)\\ \widetilde{\boldsymbol{\varphi}}(s,t)\end{pmatrix},$$

we can rewrite the expression of the density as

$$f_n(z) = -\frac{1}{\pi n}\lim_{t\downarrow 0}\langle \vec{\boldsymbol{\varphi}}(s,t), \frac{d}{ds^2}\vec{\boldsymbol{r}}(s,t)\rangle\Big|_{s=|z|}.$$

We now use the shorthand $\Psi(s,t) = \Psi(\vec{\boldsymbol{r}}(s,t), s, t)$ from (5.1) and let

$$\boldsymbol{\Psi}(s,t) = \begin{pmatrix}\Psi(s,t) & \\ & \Psi(s,t)\end{pmatrix}, \quad \vec{\widetilde{\boldsymbol{r}}}(s,t) = \begin{pmatrix}\widetilde{\boldsymbol{r}}(s,t)\\ \boldsymbol{r}(s,t)\end{pmatrix}.$$

In what follows we will often drop the dependence on s and t. In expressions with t taken to zero we will use $\boldsymbol{q}$ instead of $\boldsymbol{r}$. With this notation, we reformulate (2.1) as

$$\vec{\boldsymbol{\varphi}}(s,t) = \boldsymbol{\Psi}(s,t)^{-1}\vec{\widetilde{\boldsymbol{r}}}(s,t). \tag{5.12}$$

We now turn to the derivative $d\vec{\boldsymbol{r}}(s,t)/ds^2$. A straightforward adaption of [15, Lemma 4.4] with $\vec{\boldsymbol{q}}(s)$ replaced by $\vec{\boldsymbol{r}}(s,t)$ yields:

$$\frac{d}{ds^2}\vec{\boldsymbol{r}}(s,t) = \boldsymbol{A}(s,t)^{-1}\boldsymbol{b}(s,t). \tag{5.13}$$

where

$$\boldsymbol{M}(s,t)=\begin{pmatrix} s^2\Psi(s,t)^2V^\top & -\operatorname{diag}(\boldsymbol{r}(s,t)^2)V \\ -\operatorname{diag}(\widetilde{\boldsymbol{r}}(s,t)^2)V^\top & s^2\Psi(s,t)^2V\end{pmatrix},$$

$$\boldsymbol{A}(s,t)=I-\boldsymbol{M}(s,t)\in\mathbb{R}^{2n\times 2n},\quad \text{and}\quad \boldsymbol{b}(s,t)=-\boldsymbol{\Psi}(s,t)\vec{\boldsymbol{r}}(s,t)\in\mathbb{R}^{2n}.$$

We note that from [15], $\boldsymbol{A}(s,t)$ is invertible.

In [7], a fine analysis of the spectrum of $\boldsymbol{A}(s,t)$ is done for the purpose of establishing an optimal local law on the eigenvalues of Y_n. Here we borrow some of the results of [7] in order to control the inverse of this matrix. Following the proof of [7, Lemma 4.1], the matrix $\boldsymbol{A}(s,t)$ can be factored as

$$\boldsymbol{A}(s,t)=\boldsymbol{W}(I-\boldsymbol{T}\boldsymbol{F})\boldsymbol{W}^{-1}, \tag{5.14}$$

where $\boldsymbol{W},\boldsymbol{T}$ and $\boldsymbol{F}$ are the $2n\times 2n$ symmetric matrices given as

$$\boldsymbol{T}=\boldsymbol{\Psi}^{-1}\begin{pmatrix}-\operatorname{diag}(\boldsymbol{r}\widetilde{\boldsymbol{r}}) & s^2\Psi^2\\ s^2\Psi^2 & -\operatorname{diag}(\boldsymbol{r}\widetilde{\boldsymbol{r}})\end{pmatrix},\ \boldsymbol{W}=\begin{pmatrix}W & \\ & \widetilde{W}\end{pmatrix},\ \boldsymbol{F}=\begin{pmatrix} & WV\widetilde{W}\\ \widetilde{W}V^\top W & \end{pmatrix}=\begin{pmatrix} & F\\ F^\top & \end{pmatrix},$$

$$W=\sqrt{\operatorname{diag}\left(\frac{\boldsymbol{r}}{\widetilde{\boldsymbol{r}}}\right)\Psi},\qquad\text{and}\qquad \widetilde{W}=\sqrt{\operatorname{diag}\left(\frac{\widetilde{\boldsymbol{r}}}{\boldsymbol{r}}\right)\Psi}.$$

We note that $\boldsymbol{T},\boldsymbol{F},\boldsymbol{W}$ each depend on s,t but we omit the notation for readability. From Equations (5.12)–(5.14), we have

$$f_n(z)=\lim_{t\to 0}\frac{1}{\pi n}\left\langle \boldsymbol{\Psi}^{-1}\vec{\widetilde{\boldsymbol{r}}},\boldsymbol{W}(I-\boldsymbol{T}\boldsymbol{F})^{-1}\boldsymbol{W}^{-1}\boldsymbol{\Psi}\vec{\boldsymbol{r}}\right\rangle\ =\lim_{t\to 0}\frac{1}{\pi n}\left\langle\sqrt{\vec{\boldsymbol{r}}\vec{\widetilde{\boldsymbol{r}}}},\boldsymbol{\Psi}^{-1/2}(I-\boldsymbol{T}\boldsymbol{F})^{-1}\boldsymbol{\Psi}^{1/2}\sqrt{\vec{\boldsymbol{r}}\vec{\widetilde{\boldsymbol{r}}}}\right\rangle. \tag{5.15}$$

In order to exploit this decomposition, the will need the following lemmas, which all hold under the assumptions of Theorem 2.9–(1).

Lemma 5.3. *$r_i(s,t)\sim 1$ and $\widetilde{r}_i(s,t)\sim 1$ uniformly in $i\in[n]$.*

Proof. Under **A2**, the average of $\boldsymbol{r}$ is bounded. Since each term is positive, we trivially have each term is bounded by an (n-dependent) constant. For the (n-dependent) lower bounds on r_i and $\widetilde{r}_i$, we refer to [15, Eq. (5.17) and (5.31)]. □

The following two lemmas provide control on the spectrum of the symmetric operators $\boldsymbol{T}$ and $\boldsymbol{F}$. While the proofs appeal to arguments from [7], we point out that we only use the parts of their theorems that hold without that work's assumption of **A3**.

Lemma 5.4. *Let $s>0$ and $t\in(0,1)$. Then, there exists a constant $\varepsilon>0$ such that the spectrum $\operatorname{spec}(\boldsymbol{T})$ of $\boldsymbol{T}$ satisfies*

$$\min(\operatorname{spec}(\boldsymbol{T}))=-1\qquad\text{and}\qquad \operatorname{spec}(\boldsymbol{T})\subset\{-1\}\cup(-1+\varepsilon,1-\varepsilon)$$

Moreover, the eigenspace for the eigenvalue -1 is the span of all vectors of the form $(-\boldsymbol{y}^\top,\boldsymbol{y}^\top)^\top$.

This lemma follows from the definition of $\boldsymbol{T}$, (2.1), and the bound in Lemma 5.3, see [7, Lemma 3.6] for details.

The following lemma gives bounds on the spectrum of $\boldsymbol{F}$. Unlike in [7], our assumptions on V do not imply the matrix $\boldsymbol{F}$ is irreducible, but we will not need its Perron-Frobenius subspace to be one-dimensional. Although we will use that the vector $\boldsymbol{\Psi}^{-1/2}\sqrt{\vec{\boldsymbol{r}}\vec{\widetilde{\boldsymbol{r}}}}$ is near this Perron-Frobenius subspace. In particular in the following lemma, we compute the "correction" term.

Lemma 5.5. *Let $s > 0$ and $t \in (0,1)$. There exists a $c_t \sim t$ such that $\|\boldsymbol{F}\| = 1 - c_t$. Let $\mathcal{V}$ be the subspace spanned by all eigenvalues with magnitude greater than $1 - Ct$ for some $C > 0$. Then for all t sufficiently small, $\|\boldsymbol{F}|_{\mathcal{V}^\perp}\| \leq 1 - \varepsilon$, for some small ε. Moreover, there exists an eigenvector f_- such that*

$$\boldsymbol{F} f_- = -\|\boldsymbol{F}\| f_-, \quad \text{and} \quad f_- = \boldsymbol{\Psi}^{-1/2}\sqrt{\vec{\boldsymbol{r}}\vec{\tilde{\boldsymbol{r}}}}\boldsymbol{e}_- + \boldsymbol{\varepsilon}(t), \tag{5.16}$$

where $\boldsymbol{e}_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, *and* $\|\boldsymbol{\varepsilon}(t)\| = \mathcal{O}(t)$. *Finally, it holds that*

$$(I + \boldsymbol{F})^{-1}\left(\boldsymbol{\Psi}^{-1/2}\sqrt{\vec{\boldsymbol{r}}\vec{\tilde{\boldsymbol{r}}}} - \frac{t}{2}\boldsymbol{W}\boldsymbol{1}\right) = \frac{1}{2}\boldsymbol{\Psi}^{-1/2}\sqrt{\vec{\boldsymbol{r}}\vec{\tilde{\boldsymbol{r}}}}. \tag{5.17}$$

Proof. The bound on the norm and the spectral gap can be obtained by combining Lemma 5.3 with the proof of [7, Lemma 3.4], in particular (5.16) follows from (3.45) and (3.46) in [7]. Let us verify (5.17). By direct calculation, using Equation (5.12) along with the expression of $\boldsymbol{W}$, we have

$$\boldsymbol{F}\boldsymbol{\Psi}^{-1/2}\sqrt{\vec{\boldsymbol{r}}\vec{\tilde{\boldsymbol{r}}}} = \boldsymbol{W}\begin{pmatrix} V\tilde{\boldsymbol{r}} \\ V^{\mathsf{T}}\boldsymbol{r}\end{pmatrix} = \boldsymbol{W}(\vec{\boldsymbol{\varphi}} - t\boldsymbol{1}) = \boldsymbol{W}\left(\boldsymbol{\Psi}^{-1}\vec{\tilde{\boldsymbol{r}}} - t\boldsymbol{1}\right) = \boldsymbol{\Psi}^{-1/2}\sqrt{\vec{\boldsymbol{r}}\vec{\tilde{\boldsymbol{r}}}} - t\boldsymbol{W}\boldsymbol{1}. \tag{5.18}$$

Thus,

$$(I + \boldsymbol{F})\boldsymbol{\Psi}^{-1/2}\sqrt{\vec{\boldsymbol{r}}\vec{\tilde{\boldsymbol{r}}}} = 2\boldsymbol{\Psi}^{-1/2}\sqrt{\vec{\boldsymbol{r}}\vec{\tilde{\boldsymbol{r}}}} - t\boldsymbol{W}\boldsymbol{1},$$

and applying $(I + \boldsymbol{F})^{-1}$ to both sides of this equation, we obtain (5.17). □

We can now manipulate (5.15), the expression for the density. Following [7], the technique is based on a factorization of the term $I - \boldsymbol{\Psi}^{-1/2}\boldsymbol{T}\boldsymbol{F}\boldsymbol{\Psi}^{1/2}$. One of the factors will be dealt with by means of the identity (5.17). In order to be able to use this identity, we shall have to inject the "correction" term $0.5t\boldsymbol{W}\boldsymbol{1}$ into the expression (5.15) of the density. The following lemma shows that this can be done safely.

Lemma 5.6. $\left|\left\langle \boldsymbol{\Psi}^{1/2}\boldsymbol{W}\boldsymbol{1}\,,\, \boldsymbol{\Psi}^{-1/2}(I - \boldsymbol{T}\boldsymbol{F})^{-1}\boldsymbol{\Psi}^{1/2}\sqrt{\vec{\boldsymbol{r}}\vec{\tilde{\boldsymbol{r}}}}\right\rangle\right| \lesssim 1.$

We prove this technical lemma in Appendix A.

Now, writing $\boldsymbol{E} = \begin{pmatrix} I & I \\ I & I\end{pmatrix} \in \mathbb{R}^{2n\times 2n}$, we factor the matrix $\boldsymbol{\Psi}^{-1/2}(I - \boldsymbol{T}\boldsymbol{F})\boldsymbol{\Psi}^{1/2}$ as in [7, Equation 4.16], namely

$$\boldsymbol{\Psi}^{-1/2}(I - \boldsymbol{T}\boldsymbol{F})\boldsymbol{\Psi}^{1/2} = (I - s^2\boldsymbol{\Psi}^{1/2}\boldsymbol{E}\boldsymbol{F}(I + \boldsymbol{F})^{-1}\boldsymbol{\Psi}^{1/2})(I + \boldsymbol{\Psi}^{-1/2}\boldsymbol{F}\boldsymbol{\Psi}^{1/2}).$$

Using Lemma 5.6 to add a correction term and then substituting this relationship gives:

$$\begin{aligned} f_n(z) &= \lim_{t\to 0}\frac{1}{\pi n}\left\langle \sqrt{\vec{\boldsymbol{r}}\vec{\tilde{\boldsymbol{r}}}} - 0.5t\boldsymbol{\Psi}^{1/2}\boldsymbol{W}\boldsymbol{1}, \boldsymbol{\Psi}^{-1/2}(I - \boldsymbol{T}\boldsymbol{F})^{-1}\boldsymbol{\Psi}^{1/2}\sqrt{\vec{\boldsymbol{r}}\vec{\tilde{\boldsymbol{r}}}}\right\rangle \\ &= \lim_{t\to 0}\frac{1}{\pi n}\Big\langle (I + \boldsymbol{\Psi}^{1/2}\boldsymbol{F}\boldsymbol{\Psi}^{-1/2})^{-1}(\sqrt{\vec{\boldsymbol{r}}\vec{\tilde{\boldsymbol{r}}}} - 0.5t\boldsymbol{\Psi}^{1/2}\boldsymbol{W}\boldsymbol{1}), \\ &\qquad\qquad (I - s^2\boldsymbol{\Psi}^{1/2}\boldsymbol{E}\boldsymbol{F}(I + \boldsymbol{F})^{-1}\boldsymbol{\Psi}^{1/2})^{-1}\sqrt{\vec{\boldsymbol{r}}\vec{\tilde{\boldsymbol{r}}}}\Big\rangle \\ &= \lim_{t\to 0}\frac{1}{2\pi n}\left\langle \sqrt{\vec{\boldsymbol{r}}\vec{\tilde{\boldsymbol{r}}}}, (I - s^2\boldsymbol{\Psi}^{1/2}\boldsymbol{E}\boldsymbol{F}(I + \boldsymbol{F})^{-1}\boldsymbol{\Psi}^{1/2})^{-1}\sqrt{\vec{\boldsymbol{r}}\vec{\tilde{\boldsymbol{r}}}}\right\rangle, \end{aligned}$$

where the final equality uses (5.17). After some algebraic manipulations, it is shown in [7] that

$$(I - s^2\boldsymbol{\Psi}^{1/2}\boldsymbol{E}\boldsymbol{F}(I + \boldsymbol{F})^{-1}\boldsymbol{\Psi}^{1/2})^{-1}\begin{pmatrix} x \\ x\end{pmatrix} = \begin{pmatrix} (I - s^2\Psi^{1/2}B\Psi^{1/2})^{-1}x \\ (I - s^2\Psi^{1/2}B\Psi^{1/2})^{-1}x\end{pmatrix},$$

where

$$Bx = \begin{pmatrix} I & I \end{pmatrix} \left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} I & F \\ F^{\mathsf{T}} & I \end{pmatrix}^{-1} \right) \begin{pmatrix} x \\ x \end{pmatrix}.$$

We thus obtain that

$$f_n(z) \quad = \quad \lim_{t\to 0} \frac{1}{\pi n} \left\langle \sqrt{\boldsymbol{r}\widetilde{\boldsymbol{r}}}, (I - s^2\Psi^{1/2}B\Psi^{1/2})^{-1}\sqrt{\boldsymbol{r}\widetilde{\boldsymbol{r}}} \right\rangle. \tag{5.19}$$

The matrix B is symmetric. Furthermore, because the spectrum of F is contained in $[-1,1]$ and the vector $s^2\Psi$ has entries strictly less than 1 we have the eigenvalues of $s^2\Psi^{1/2}B\Psi^{1/2}$ are bounded away from 1, uniformly in t; see [7, Eq. (4.20) - (4.22)] for details (note the matrix B is labeled A there). To lower bound this expression we begin by noting that if $\begin{pmatrix} x \\ x \end{pmatrix}$ is an eigenvector of $\boldsymbol{F}$, with eigenvalue λ, then

$$Bx = \frac{2\lambda}{1+\lambda} x. \tag{5.20}$$

From Lemma 5.5 we have that $\begin{pmatrix} \Psi^{-1/2}\sqrt{\boldsymbol{r}\widetilde{\boldsymbol{r}}} \\ \Psi^{-1/2}\sqrt{\boldsymbol{r}\widetilde{\boldsymbol{r}}} \end{pmatrix}$ is $O(t)$ from an eigenvector of $\boldsymbol{F}$ with eigenvalue 1. Let f_+ be this eigenvector. Since the operator $(I - s^2\Psi^{1/2}B\Psi^{1/2})^{-1}$ has uniformly bounded norm, we can replace $\sqrt{\boldsymbol{r}\widetilde{\boldsymbol{r}}}$ with $\Psi^{1/2}f_+$, at the cost of an error that goes to zero as $t \to 0$. We now have all the elements to provide a lower bound on the density. Using the Cauchy-Schwarz inequality (with respect to the inner product $\langle \cdot, (s^{-2}\Psi^{-1} - B)^{-1} \cdot \rangle$) along with (5.20), we have

$$\begin{aligned}
\lim_{t\to 0} \langle \sqrt{\boldsymbol{r}\widetilde{\boldsymbol{r}}}, (I - s^2\Psi^{1/2}B\Psi^{1/2})^{-1}\sqrt{\boldsymbol{r}\widetilde{\boldsymbol{r}}} \rangle &= \lim_{t\to 0} \langle \Psi^{-1/2}f_+, (I - s^2\Psi^{1/2}B\Psi^{1/2})^{-1}\Psi^{-1/2}f_+ \rangle \\
&= \lim_{t\to 0} s^{-2} \langle f_+, (s^{-2}\Psi^{-1} - B)^{-1} f_+ \rangle \\
&\geq \lim_{t\to 0} \frac{\|f_+\|^2}{s^2\langle f_+, (s^{-2}\Psi^{-1} - B) f_+ \rangle} \\
&= \lim_{t\to 0} \frac{\|f_+\|^2}{s^2\langle f_+, (s^{-2}\Psi^{-1} - I) f_+ \rangle}.
\end{aligned}$$

Taking the limit $t \to 0$ and using that $f_+ \to \Psi^{-1/2}\sqrt{\boldsymbol{q}\widetilde{\boldsymbol{q}}}$ as $t \to 0$ gives

$$\lim_{t\to 0} \frac{\|f_+\|^2}{s^2\langle f_+, (s^{-2}\Psi^{-1} - I) f_+ \rangle} = \frac{\|\Psi^{-1/2}\sqrt{\boldsymbol{q}\widetilde{\boldsymbol{q}}}\|^2}{s^2\langle \Psi^{-1}\boldsymbol{q}\widetilde{\boldsymbol{q}}, (s^{-2}\Psi^{-1}\mathbf{1} - \mathbf{1}) \rangle}.$$

Then using the equalities

$$\Psi^{-1}(s^{-2}\Psi^{-1}\mathbf{1} - \mathbf{1}) = \Psi^{-1}\frac{\boldsymbol{\varphi}\widetilde{\boldsymbol{\varphi}}}{s^2} = \frac{\Psi\boldsymbol{q}\widetilde{\boldsymbol{q}}}{s^2}$$

gives

$$f_n(z) \quad \geq \quad \frac{\sum_{i=1}^n \Psi_i^{-1} q_i \widetilde{q}_i}{\sum_{i=1}^n \Psi_i q_i^2 \widetilde{q}_i^2}. \tag{5.21}$$

From the uniformity in t in Lemma 5.3, $q_i, \widetilde{q}_i$ are upper and lower-bounded and hence Theorem 2.9–(1) is proven. □

5.3. **Proof of Corollary 2.8.** The proof relies on the following theorem by Friedland and Karlin:

Theorem 5.7 (Theorem 3.1, Equation (1.9) in [17]). *Let M be an irreducible non-negative matrix with Perron-Frobenius left and right eigenvectors $\boldsymbol{u}, \boldsymbol{v}$ normalized so that $\sum_{i\in[n]} u_i v_i = 1$ and $\rho(M) = 1$. Let D be a diagonal matrix with positive entries. Then*

$$\rho(MD) \geq \prod_{i}^{n} d_i^{u_i v_i} \tag{5.22}$$

Proof of Corollary 2.8. Without loss of generality we consider V such that $\rho(V) = 1$. Proposition 2.7, μ_n gives the formula for the density at 0. By (2.6), matrix $S := \mathrm{diag}(\boldsymbol{q})V\mathrm{diag}(\widetilde{\boldsymbol{q}})$ is doubly stochastic hence with spectral radius 1 and any left or right Perron-Frobenius eigenvector $\boldsymbol{u}$ or $\boldsymbol{v}$ is proportional to $\mathbf{1}_n$. In particular, the normalization $\sum_{i\in[n]} u_i v_i = 1$ implies $u_i v_i = n^{-1}$. We now apply Theorem 5.7 with $M = S$ and $D = (\mathrm{diag}(\widetilde{\boldsymbol{q}})\mathrm{diag}(\boldsymbol{q}))^{-1}$ to get

$$\rho(S\,(\mathrm{diag}(\widetilde{\boldsymbol{q}})\mathrm{diag}(\boldsymbol{q}))^{-1}) \quad \geq \quad \prod_{i\in[n]} \left(\frac{1}{q_i(0)\widetilde{q}_i(0)}\right)^{\frac{1}{n}}.$$

Since $\rho(SD) = \rho((\mathrm{diag}(\boldsymbol{q}))^{-1}\,S\,(\mathrm{diag}(\widetilde{\boldsymbol{q}}))^{-1}) = \rho(V) = 1$, we arrive at

$$1 \quad \leq \quad \prod_{i\in[n]} [q_i(0)\widetilde{q}_i(0)]^{\frac{1}{n}} \quad \leq \quad \frac{1}{n}\sum_{i\in[n]} q_i(0)\widetilde{q}_i(0)\,,$$

where the second inequality is the the AM-GM inequality. We note that equality in the final inequality only occurs if $q_i(0)\widetilde{q}_i(0) = 1$ for all $i \in [n]$. This condition can be rewritten as $\mathrm{diag}(\boldsymbol{q})^{-1} = \mathrm{diag}(\widetilde{\boldsymbol{q}})$, which, by Remark 4.2, implies the desired form $V = \mathrm{diag}(\boldsymbol{q})^{-1}\,S\,\mathrm{diag}(\boldsymbol{q})$. □

Appendix A. Proof of Lemma 5.6

Before completing the proof, we state several technical lemmas, from which the Lemma 5.6 will immediately follow. The first step is to define the subspace on which the inverse $(I - \boldsymbol{T}\boldsymbol{F})^{-1}$ is not bounded.

Lemma A.1. *Let V_{-1} be spanned by eigenvectors of $\boldsymbol{F}$ with eigenvalues in $(-1, -1 + Ct]$, that are additionally of the form $\begin{pmatrix}\boldsymbol{x}\\ -\boldsymbol{x}\end{pmatrix} + \vec{\boldsymbol{w}}$, where $\|\vec{\boldsymbol{w}}\| < 2\|\boldsymbol{\varepsilon}(t)\|$ and C and $\boldsymbol{\varepsilon}(t)$ are from in Lemma 5.5. Then the subspace V_{-1} is spanned by f_-.*

Proof. From Lemma 5.5, we have that f_- is an eigenvector of $\boldsymbol{F}$, within an $\|\boldsymbol{\varepsilon}(t)\|$ distance of $\boldsymbol{\Psi}^{-1/2}\sqrt{\vec{\boldsymbol{r}}\vec{\tilde{\boldsymbol{r}}}}\boldsymbol{e}_-$. Now we show f_- spans V_{-1}. Let $\vec{\boldsymbol{y}} = \begin{pmatrix}\boldsymbol{y}\\ -\boldsymbol{y}\end{pmatrix} + \begin{pmatrix}\boldsymbol{w}\\ \tilde{\boldsymbol{w}}\end{pmatrix} \in V_{-1}$ be a unit vector. The block structure of $\boldsymbol{F}$, then implies $F\boldsymbol{y} = \boldsymbol{y} + \boldsymbol{w} + F\tilde{\boldsymbol{w}}$. The irreducible matrix F has non-negative entries, with norm $1 - c_t$ and and spectral radius also tending to 1 as $t \to 0$. Additionally $\boldsymbol{y}$, up to an $4\|\boldsymbol{\varepsilon}(t)\|$ error, saturates this norm bound, so we must have that $\boldsymbol{y} = \boldsymbol{y}_1 + \boldsymbol{y}_2$, where the entries of $\boldsymbol{y}_1$ have the same sign and $\|\boldsymbol{y}_2\| = C_1\|\boldsymbol{\varepsilon}(t)\|$. Otherwise, setting the entries equal to their absolute values would give a bigger norm. Finally, as the vectors f_- and $\vec{\boldsymbol{y}}$ are both $C_1\|\boldsymbol{\varepsilon}(t)\|$ away from vectors who each have the same sign, we conclude they cannot be orthogonal for all small t, and therefore f_- spans V_{-1}. □

To prove Lemma 5.6, we will use the following identity to bound $(I - \boldsymbol{F}\boldsymbol{T})^{-1}\boldsymbol{W}\mathbf{1}$:

$$(I - \boldsymbol{F}\boldsymbol{T})^{-1}\vec{\boldsymbol{x}} = \frac{1}{2}\vec{\boldsymbol{x}} + (I - \boldsymbol{F}\boldsymbol{T})^{-1}\left(\frac{\boldsymbol{F}\boldsymbol{T}\vec{\boldsymbol{x}} + \vec{\boldsymbol{x}}}{2}\right) \tag{A.1}$$

or any vector $\vec{\boldsymbol{x}}$. We will apply this identity with $\vec{\boldsymbol{x}} = \left(\frac{\boldsymbol{F}\boldsymbol{T}+I}{2}\right)^k \boldsymbol{W}\mathbf{1}$, for k a non-negative integer. We now bound the inner product of the final term and f_-. Afterwards, we show this is an effective bound.

Lemma A.2. *For any positive integer k,*

$$\left|\left\langle f_-, \left(\frac{\boldsymbol{FT}+I}{2}\right)^k \boldsymbol{W1}\right\rangle\right| \leq \left|\left\langle f_-, \left(\frac{\boldsymbol{FT}+I}{2}\right)^{k-1} \boldsymbol{W1}\right\rangle\right| + \|\boldsymbol{\varepsilon}(t)\| \left\|\left(\frac{\boldsymbol{FT}+I}{2}\right)^{k-1} \boldsymbol{W1}\right\| \quad \text{(A.2)}$$
$$\leq |\langle f_-, \boldsymbol{W1}\rangle| + \|\boldsymbol{\varepsilon}(t)\| \sum_{j=0}^{k-1} \left\|\left(\frac{\boldsymbol{FT}+I}{2}\right)^{j} \boldsymbol{W1}\right\|.$$

Furthermore,

$$|\langle f_-, \boldsymbol{W1}\rangle| \leq \|\boldsymbol{\varepsilon}(t)\|\|\boldsymbol{W}\|.$$

Proof. We will prove the inequality in the first line of (A.2), the second line follows by inductively applying the first line.

$$\left\langle f_-, \left(\frac{\boldsymbol{FT}+I}{2}\right)^k \boldsymbol{W1}\right\rangle = \left\langle \left(\frac{\boldsymbol{TF}+I}{2}\right) f_-, \left(\frac{\boldsymbol{FT}+I}{2}\right)^{k-1} \boldsymbol{W1}\right\rangle$$
$$= \|\boldsymbol{F}\| \left\langle f_-, \left(\frac{\boldsymbol{FT}+I}{2}\right)^{k-1} \boldsymbol{W1}\right\rangle + \|\boldsymbol{F}\| \left\langle \left(\frac{I-\boldsymbol{T}}{2}\right)\boldsymbol{\varepsilon}(t), \left(\frac{\boldsymbol{FT}+I}{2}\right)^{k-1} \boldsymbol{W1}\right\rangle$$

where we use that

$$\boldsymbol{TF}f_- = -\|\boldsymbol{F}\|\boldsymbol{T}f_- = \|\boldsymbol{F}\|f_- + \|\boldsymbol{F}\|(I-\boldsymbol{T})\boldsymbol{\varepsilon}(t)$$

then the desired inequality follows by applying the Cauchy-Schwarz inequality to the second term. The inner product between $\boldsymbol{W1}$ and f_- is bounded using (5.16) along with the identity $\sum r_i = \sum \tilde{r}_i$:

$$|\langle \boldsymbol{W1}, f_-\rangle| = |\langle \boldsymbol{r}, 1\rangle - \langle \widetilde{\boldsymbol{r}}, 1\rangle + \langle \boldsymbol{W1}, \boldsymbol{\varepsilon}(t)\rangle| \leq \|\boldsymbol{\varepsilon}(t)\|\|\boldsymbol{W}\|.$$

□

We now show that final term in the identity (A.1) will have smaller norm than vector on the left side.

Lemma A.3. *There exist a constant $c > 0$ such that, for each non-negative integer k, we have*

$$\left\|\left(\frac{\boldsymbol{FT}+I}{2}\right)^k \boldsymbol{W1}\right\| \leq (1-c\epsilon)^k \|\boldsymbol{W}\|.$$

Proof. We prove this lemma by induction. If $k = 0$ the lemma is trivial. Let $k > 0$ and let $\vec{\boldsymbol{x}} = \left(\frac{\boldsymbol{FT}+I}{2}\right)^{k-1} \boldsymbol{W1}$. By the induction hypothesis we have

$$\left\|\left(\frac{\boldsymbol{FT}+I}{2}\right)^j \boldsymbol{W1}\right\| \leq (1-c\epsilon)^j \|\boldsymbol{W}\|$$

for all $0 \leq j \leq k-1$.

$$\left\|\left(\frac{\boldsymbol{FT}+I}{2}\right)\vec{\boldsymbol{x}}\right\|^2 = \frac{1}{4}\left(\|\vec{\boldsymbol{x}}\|^2 + \|\boldsymbol{FT}\vec{\boldsymbol{x}}\|^2 + 2\langle \boldsymbol{FT}\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}\rangle\right). \quad \text{(A.3)}$$

We bound the second term by $\|\boldsymbol{FT}\vec{\boldsymbol{x}}\| \leq \|\boldsymbol{F}\|\|\boldsymbol{T}\|\|\vec{\boldsymbol{x}}\| \leq \|\vec{\boldsymbol{x}}\|$. Let $\vec{\boldsymbol{x}} = f_-\langle f_-, \vec{\boldsymbol{x}}\rangle + \vec{\boldsymbol{x}}'$ be the orthogonal decomposition of $\vec{\boldsymbol{x}}$ onto f_- and its orthogonal complement. Then we expand the final term as

$$\langle \boldsymbol{FT}\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}\rangle = \langle \boldsymbol{FT}\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}'\rangle + \langle \boldsymbol{FT}\vec{\boldsymbol{x}}, f_-\rangle\langle \vec{\boldsymbol{x}}, f_-\rangle = \langle \boldsymbol{FT}\vec{\boldsymbol{x}}', \vec{\boldsymbol{x}}'\rangle + \langle \boldsymbol{FT}f_-, \vec{\boldsymbol{x}}'\rangle\langle \vec{\boldsymbol{x}}, f_-\rangle + \langle \boldsymbol{FT}\vec{\boldsymbol{x}}, f_-\rangle\langle \vec{\boldsymbol{x}}, f_-\rangle.$$

which we bound by

$$-\|\vec{\boldsymbol{x}}\|^2 \leq \langle \boldsymbol{F}\boldsymbol{T}\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}\rangle \leq \langle \boldsymbol{T}\vec{\boldsymbol{x}}', \boldsymbol{F}\vec{\boldsymbol{x}}'\rangle + 2\|\vec{\boldsymbol{x}}\|\|f_-\|\langle \vec{\boldsymbol{x}}, f_-\rangle \,. \tag{A.4}$$

From the induction hypothesis along with Lemma A.2 we have

$$|\langle f_-, \vec{\boldsymbol{x}}\rangle| \leq 2\|\boldsymbol{\varepsilon}(t)\| \sum_{j=0}^{k-2}(1-c\epsilon)^j \,\|\boldsymbol{W}\mathbf{1}\| \leq \frac{2}{c\,\varepsilon}\|\boldsymbol{\varepsilon}(t)\|\|\boldsymbol{W}\mathbf{1}\| \,. \tag{A.5}$$

To bound $\langle \boldsymbol{T}\vec{\boldsymbol{x}}', \boldsymbol{F}\vec{\boldsymbol{x}}'\rangle$, let $\vec{\boldsymbol{x}}' = \vec{\boldsymbol{x}}_1 + \vec{\boldsymbol{x}}_2$ where $\vec{\boldsymbol{x}}_1$ is the projection onto the eigenspace of $\boldsymbol{T}$ corresponding to the eigenvalue -1, and $\vec{\boldsymbol{x}}_2$ is the projection onto the remaining eigenspaces. We now consider two cases based on the size of $\|\vec{\boldsymbol{x}}_2\|$ compared to $\|\vec{\boldsymbol{x}}\|$. In what follows c_1 will be an appropriately chosen small constant depending only on ε. Case I. If $\|\vec{\boldsymbol{x}}_2\| \leq c_1\|\vec{\boldsymbol{x}}'\|$ then we begin by expanding:

$$\langle \boldsymbol{T}\vec{\boldsymbol{x}}', \boldsymbol{F}\vec{\boldsymbol{x}}'\rangle = -\langle \vec{\boldsymbol{x}}_1, \boldsymbol{F}\vec{\boldsymbol{x}}_1\rangle + \langle \boldsymbol{T}\vec{\boldsymbol{x}}_2, \boldsymbol{F}\vec{\boldsymbol{x}}_1\rangle + \langle \boldsymbol{T}\vec{\boldsymbol{x}}', \boldsymbol{F}\vec{\boldsymbol{x}}_2\rangle \,. \tag{A.6}$$

To bound $-\langle \vec{\boldsymbol{x}}_1, \boldsymbol{F}\vec{\boldsymbol{x}}_1\rangle$ from above we project $\vec{\boldsymbol{x}}_1$ onto f_- and its orthogonal complement. By choice of c_1, we will make the projection onto f_- small. We will bound the orthogonal term by using that it is of the form $\begin{pmatrix} \boldsymbol{x} \\ -\boldsymbol{x} \end{pmatrix} + \vec{\boldsymbol{w}}$ and thus not in V_{-1}. Indeed, for c_1 is chosen sufficiently small (compared to ε)

$$|\langle \vec{\boldsymbol{x}}_1, f_-\rangle| = |\langle \vec{\boldsymbol{x}}', f_-\rangle - \langle \vec{\boldsymbol{x}}_2, f_-\rangle| \leq 0 + c_1\|\vec{\boldsymbol{x}}\|\|f_-\|$$

and then

$$-\langle \vec{\boldsymbol{x}}_1, \boldsymbol{F}\vec{\boldsymbol{x}}_1\rangle = -\langle \vec{\boldsymbol{x}}_1, \boldsymbol{F}f_-\rangle\langle \vec{\boldsymbol{x}}_1, f_-\rangle - \langle \vec{\boldsymbol{x}}_1, \boldsymbol{F}(\vec{\boldsymbol{x}}_1 - \langle \vec{\boldsymbol{x}}_1, f_-\rangle f_-)\rangle \leq c_1\|\vec{\boldsymbol{x}}'\|^2\|f_-\|^2 + (1-\varepsilon)\|\vec{\boldsymbol{x}}'\|^2.$$

So we have that there exist a constant c_2 such that

$$-\langle \vec{\boldsymbol{x}}_1, \boldsymbol{F}\vec{\boldsymbol{x}}_1\rangle \leq (1 - c_2\,\varepsilon)\|\vec{\boldsymbol{x}}'\|$$

and if c_1 is chosen smaller, then c_2 can be chosen closer to 1. Then continuing from (A.6) gives:

$$\langle \boldsymbol{T}\vec{\boldsymbol{x}}', \boldsymbol{F}\vec{\boldsymbol{x}}'\rangle \leq (1 - c_2\,\varepsilon)\|\vec{\boldsymbol{x}}'\|^2 + 2\|\vec{\boldsymbol{x}}'\|\|\vec{\boldsymbol{x}}_2\|.$$

Thus, for a sufficiently small choice of c_1, there is a c_3 such that

$$\langle \boldsymbol{T}\vec{\boldsymbol{x}}', \boldsymbol{F}\vec{\boldsymbol{x}}'\rangle \leq (1 - c_3\,\varepsilon)\|\boldsymbol{x}\|^2. \tag{A.7}$$

Case II: If $\|\vec{\boldsymbol{x}}_2\| > c_1\|\vec{\boldsymbol{x}}\|$ From the bound $\|\boldsymbol{T}\vec{\boldsymbol{x}}_2\| \leq (1-\varepsilon)\|\vec{\boldsymbol{x}}_2\|$, we have that

$$\langle \boldsymbol{T}\vec{\boldsymbol{x}}', \boldsymbol{F}\vec{\boldsymbol{x}}'\rangle \leq \sqrt{\|\boldsymbol{T}\vec{\boldsymbol{x}}_1\|^2 + \|\boldsymbol{T}\vec{\boldsymbol{x}}_2\|^2}\|\vec{\boldsymbol{x}}'\| \leq \sqrt{\|\vec{\boldsymbol{x}}_1\|^2 + (1-\varepsilon)\|\vec{\boldsymbol{x}}_2\|^2}\|\vec{\boldsymbol{x}}'\| \leq \sqrt{1 - c_1^2\,\varepsilon}\|\vec{\boldsymbol{x}}'\|^2.$$

Choosing c' to be the smaller of the bounds between the two cases, we have for any possible $\vec{\boldsymbol{x}}'$

$$\langle \boldsymbol{T}\vec{\boldsymbol{x}}', \boldsymbol{F}\vec{\boldsymbol{x}}'\rangle \leq (1 - c'\,\varepsilon)\|\boldsymbol{x}\|^2. \tag{A.8}$$

So for all t sufficiently small, combining (A.4), (A.5), and (A.8) gives for some constant c_4:

$$-\|\vec{\boldsymbol{x}}\|^2 \leq \langle \boldsymbol{F}\boldsymbol{T}\vec{\boldsymbol{x}}, \vec{\boldsymbol{x}}\rangle \leq (1 - c_4\,\varepsilon)\|\vec{\boldsymbol{x}}\|^2.$$

Substituting these estimates into (A.3) gives, that there exist a c such that

$$\left\|\left(\frac{\boldsymbol{F}\boldsymbol{T} + I}{2}\right)\vec{\boldsymbol{x}}'\right\| \leq (1 - c\,\epsilon)\|\vec{\boldsymbol{x}}'\|.$$

as desired. ∎

Proof of Lemma 5.6. By taking the adjoint and then applying the Cauchy-Schwarz inequality we have

$$\left|\left\langle \boldsymbol{\Psi}^{1/2}\boldsymbol{W}\mathbf{1}\,,\, \boldsymbol{\Psi}^{-1/2}(I-\boldsymbol{T}\boldsymbol{F})^{-1}\boldsymbol{\Psi}^{1/2}\sqrt{\vec{\boldsymbol{r}}\vec{\tilde{\boldsymbol{r}}}}\right\rangle\right| \le \|(I-\boldsymbol{F}\boldsymbol{T})^{-1}\boldsymbol{W}\mathbf{1}\| \left\|\boldsymbol{\Psi}^{1/2}\sqrt{\vec{\boldsymbol{r}}\vec{\tilde{\boldsymbol{r}}}}\right\|.$$

Then applying (A.1) iteratively gives:

$$(I-\boldsymbol{F}\boldsymbol{T})^{-1}\boldsymbol{W}\mathbf{1} = \sum_{k=0}^{\infty}\left(\frac{I+\boldsymbol{F}\boldsymbol{T}}{2}\right)^k \frac{1}{2}\boldsymbol{W}\mathbf{1}\,.$$

Then applying Lemma A.3 we have

$$\|(I-\boldsymbol{F}\boldsymbol{T})^{-1}\boldsymbol{W}\mathbf{1}\| \le \|\boldsymbol{W}\mathbf{1}\| \sum_{k=0}^{\infty}(1-c\,\varepsilon)^k\,.$$

The desired inequality then follows. □

References

[1] L. Aagaard and U. Haagerup. Moment formulas for the quasi-nilpotent DT-operator. *Internat. J. Math.*, 15(6):581–628, 2004.

[2] R. Adamczak. On the Marchenko-Pastur and circular laws for some classes of random matrices with dependent entries. *Electron. J. Probab.*, 16:no. 37, 1068–1095, 2011.

[3] Y. Ahmadian, F. Fumarola, and K. D. Miller. Properties of networks with partially structured and partially random connectivity. *Physical Review E*, 91(1):012820, 2015.

[4] O. Ajanki, L. Erdos, and T. Krüger. Quadratic vector equations on complex upper half-plane. *Mem. Amer. Math. Soc.*, 261(1261):v+133, 2019.

[5] S. Allesina and S. Tang. The stability–complexity relationship at age 40: a random matrix perspective. *Population Ecology*, 57(1):63–75, 2015.

[6] J. Alt, L. Erdős, and T. Krüger. Local law for random Gram matrices. *Electron. J. Probab.*, 22:Paper No. 25, 41, 2017.

[7] J. Alt, L. Erdős, and T. Krüger. Local inhomogeneous circular law. *Ann. Appl. Probab.*, 28(1):148–203, 2018.

[8] J. Alt, L. Erdős, T. Krüger, and Y. Nemish. Location of the spectrum of Kronecker random matrices. *Ann. Inst. Henri Poincaré Probab. Stat.*, 55(2):661–696, 2019.

[9] J. Alt and T. Krüger. Inhomogeneous circular law for correlated matrices. *J. Funct. Anal.*, 281(7):Paper No. 109120, 73, 2021.

[10] P. M. Anselone and T. W. Palmer. Spectral analysis of collectively compact, strongly convergent operator sequences. *Pacific J. Math.*, 25:423–431, 1968.

[11] R. B. Bapat and T. E. S. Raghavan. *Nonnegative matrices and applications*, volume 64 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1997.

[12] C. Bordenave, P. Caputo, and D. Chafaï. Circular law theorem for random Markov matrices. *Probab. Theory Related Fields*, 152(3-4):751–779, 2012.

[13] C. Bordenave and D. Chafaï. Around the circular law. *Probab. Surv.*, 9:1–89, 2012.

[14] N. A. Cook. Lower bounds for the smallest singular value of structured random matrices. *Ann. Probab.*, 46(6):3442–3500, 2018.

[15] N. A. Cook, W. Hachem, J. Najim, and D. Renfrew. Non-Hermitian random matrices with a variance profile (I): deterministic equivalents and limiting ESDs. *Electron. J. Probab.*, 23:Paper No. 110, 61, 2018.

[16] K. Deimling. *Nonlinear functional analysis*. Springer-Verlag, Berlin, 1985.

[17] S. Friedland and S. Karlin. Some inequalities for the spectral radius of non-negative matrices and applications. *Duke Math. J.*, 42(3):459–490, 1975.

[18] V. L. Girko. *Theory of stochastic canonical equations. Vol. I*, volume 535 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2001.

[19] F. Götze, A. A. Naumov, and A. N. Tikhomirov. Limit theorems for two classes of random matrices with dependent entries. *Theory Probab. Appl.*, 59(1):23–39, 2015.

[20] W. Hachem, P. Loubaton, and J. Najim. The empirical eigenvalue distribution of a Gram matrix: from independence to stationarity. *Markov Process. Related Fields*, 11(4):629–648, 2005.

[21] H. H. Nguyen. Random doubly stochastic matrices: the circular law. *Ann. Probab.*, 42(3):1161–1196, 2014.

[22] S. O'Rourke. A note on the marchenko-pastur law for a class of random matrices with dependent entries. *Electronic Communications in Probability*, 17, 2012.

[23] M. Reed and B. Simon. *Methods of modern mathematical physics. I.* Academic Press Inc., New York, second edition, 1980. Functional analysis.
[24] D. Shlyakhtenko. Random Gaussian band matrices and freeness with amalgamation. *Internat. Math. Res. Notices*, (20):1013–1025, 1996.
[25] R. Sinkhorn and P. Knopp. Concerning nonnegative matrices and doubly stochastic matrices. *Pacific J. Math.*, 21:343–348, 1967.
[26] T. Tao, V. Vu, and M. Krishnapur. Random matrices: universality of esds and the circular law. *The Annals of Probability*, 38(5):2023–2065, 2010.

Nicholas Cook
Department of Mathematics
Duke University
Durham, NC 27708
e-mail: nickcook@math.duke.edu

Walid Hachem, Jamal Najim,
CNRS / Laboratoire d'Informatique Gaspard Monge,
Université Gustave Eiffel, ESIEE
5, Boulevard Descartes,
Champs sur Marne, F-77454 Marne-la-Vallée, France
e-mail: {walid.hachem,jamal.najim}@univ-eiffel.fr

David Renfrew
Department of Mathematical Sciences
Binghamton University (SUNY)
Binghamton, NY 3902-6000
e-mail: renfrew@math.binghamton.edu