A whiteness test based on the spectral measure of large non-Hermitian random matrices

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Problem

Observed N-dimensional multivariate time series:

$$y_k = B_0 x_k + B_1 x_{k-1} \in \mathbb{C}^N$$

$B_0, B_1 \in \mathbb{C}^{N \times N}$: deterministic matrices
$(x_k)_{k \in \mathbb{Z}}$: spatially and temporally $\mathbb{C}^N$–valued white noise

We observe n samples $y_0, \ldots, y_{n-1}$

Whiteness test on Y: $\begin{cases} \textbf{H0} : B_0 = I_N, B_1 = 0, \\ \quad\quad\quad \text{against} \\ \textbf{H1} : B_0 = I_N, B_1 \neq 0 \text{ unknown} \end{cases}$

Context

Known tests:

- Box-Pierce & Ljung-Box ($N = 1$) or Li-McLeod ($N > 1$) tests: built on sample autocovariance matrices

$$\widehat{R}_\ell = \frac{1}{n}\sum_{k=0}^{n-1} y_k y_{k-\ell}^* \quad \text{for } \ell = 0, 1$$

  Well known in classical regime where N fixed and $n \to \infty$.

- **Large Random Matrix (LRM)** regime:

$$N, n \to \infty, \; N/n \to \gamma > 0$$

  Tests in [Bose *et.al.*'18, Li *et.al.*'18, ...] are based on spectra of **Hermitized matrices**, such as $\widehat{R}_1 \widehat{R}_1^*$.

Our idea: A test based on **eigenvalues** of $\widehat{R}_1$ instead of its singular values in LRM regime.
Intuition: Eigenvalues are more sensitive to perturbations.

LSM of $\widehat{R}_1$ under **H0**

Under **H0**, $\widehat{R}_1 = \frac{1}{n}\sum_{k=0}^{n-1} y_k y^*_{(k-1)\mathrm{mod}\ n} = XJX^*$ with

$$X = \frac{1}{\sqrt{n}}\left[x_0, \ldots, x_{n-1}\right] \in \mathbb{C}^{N\times n} \text{ (iid entries)},\ J = \begin{bmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

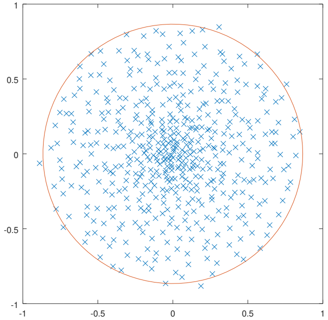
Our purpose: Show that the random spectral measure

$$\mu_n = \frac{1}{N}\sum_{i=0}^{N-1} \delta_{\lambda_i}, \quad \{\lambda_i\}_{i=0}^{N-1} \text{ eigenvalues of } \widehat{R}_1$$

converges narrowly to a deterministic probability measure $\boldsymbol{\mu}$, the **Limit Spectral Measure** (LSM) in the LRM regime.

LSM of $\widehat{R}_1$ under **H0**

Theorem 1. In LRM regime, μ_n converges narrowly in probability towards a rotationnally invariant LSM $\boldsymbol{\mu}$. Expression in paper.



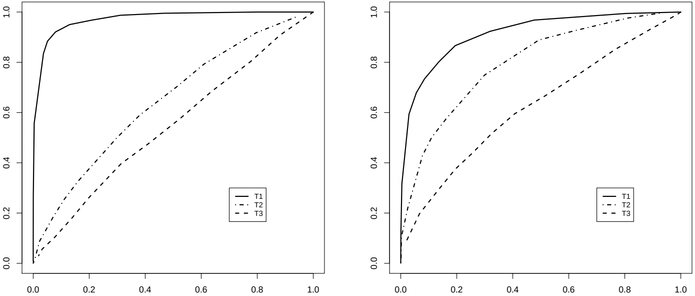
A realization of μ_n and support of $\boldsymbol{\mu}$ for $(N, n) = (500, 1000)$

Whiteness test

- Our test (T1): compare the Wasserstein-2 distance between μ_n and $\boldsymbol{\mu}$ with a threshold.

T1 *vs* singular-value based tests

- **T2**: Compare tr $\widehat{R}_1 \widehat{R}_1^*$ with a threshold.
- **T3**: Wasserstein-2 distance between spectral measure of $\frac{1}{n}\sum_{k=0}^{n-1} \begin{bmatrix} y_k \\ y_{k-1} \end{bmatrix} \begin{bmatrix} y_k^* & y_{k-1}^* \end{bmatrix}$ and Marchenko-Pastur.



ROC curves, $(N, n) = (50, 100)$. Left: $B_1 = \alpha I, \alpha^2 = 10^{-2.5}$. Right: B_1 Toeplitz, tr $B_1 B_1^* / N = 10^{-2}$.

Proof of Theorem 1

Following a now well-known approach in non-Hermitian LRM theory, for almost each $z \in \mathbb{C}$,

1. Using Girko's "Hermitization technique", show that $XJX^* - zI$ has a **limit singular value distribution**
2. Control the **smallest singular value** of $XJX^* - zI$

We prove a general result that includes Item 2. Proof follows the technique of [Vershynin'14].

Theorem 2. Let $A \in \mathbb{C}^{n \times n}$ be a deterministic matrix such that

$$\sup_n \|A\| < \infty \text{ and } \sup_n \|A^{-1}\| < \infty.$$

Then, there exist $\alpha, \beta > 0$ such that for each $C > 0$, $t > 0$, and $z \in \mathbb{C} \setminus \{0\}$, the smallest singular value s_{N-1} of $XAX^* - zI$ satisfies

$$\mathbb{P}\left[s_{N-1} \leq t,\ \|X\| \leq C\right] \leq c\left(n^{\alpha} t^{1/2} + n^{-\beta}\right).$$

Work in progress

- LSM of $\widehat{R}_1$ under **H1** and test consistency

- Proof of Theorem 2 in for real matrices

- **Future work**: Rigorous performance comparison with respect to known singular-value based tests to lay our initial intuition on mathematical grounds