SPIN GLASS ANALYSIS OF THE INVARIANT DISTRIBUTION OF A LOTKA-VOLTERRA SDE WITH A LARGE RANDOM INTERACTION MATRIX

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Abstract. The generalized Lotka-Volterra stochastic differential equation with a symmetric food interaction matrix is frequently used to model the dynamics of the abundances of the species living within an ecosystem when these interactions are mutualistic or competitive. In the relevant cases of interest, the Markov process described by this equation has an unique invariant distribution which has a Hamiltonian structure. Following an important trend in theoretical ecology, the interaction matrix is considered in this paper as a large random matrix. In this situation, the (conditional) invariant distribution takes the form of a random Gibbs measure that can be studied rigorously with the help of spin glass techniques issued from the field of physics of disordered systems. Considering that the interaction matrix is an additively deformed GOE matrix, which is a well-known model for this matrix in theoretical ecology, the free energy of the model is derived in the limit of the large number n of species, making rigorous some recent results from the literature. The free energy analysis made in this paper could be adapted to other situations where the Gibbs measure is non compactly supported.

Keywords : Free energy for a Gibbs measure, Lotka-Volterra stochastic differential equation, large random matrices, theoretical ecology

1. Introduction

The (generalized) Lotka-Volterra (LV) Stochastic Differential Equation (SDE) is a standard mathematical model for studying the population dynamics of biological ecosystems. Letting the integer $n > 0$ be the number of living species coexisting within an ecosystem, and writing $\mathbb{R}_+ = [0, \infty)$, the time evolution of the abundances of these species, *i.e.*, the biomasses or the numbers of individuals after an adequate normalization, is represented by the random function $x : \mathbb{R}_+ \to \mathbb{R}_+^n$ provided as the solution of the LV SDE

$$dx_t = x_t \left(1 + (\Sigma - I)\, x_t\right) dt + \phi dt + \sqrt{2Tx_t} dB_t, \tag{1}$$

with the following notational conventions: 1 is the $n \times 1$ vector of ones. Given two $\mathbb{R}^n$–valued vectors $x = [x_i]$ and $y = [y_i]$ and a function $f : \mathbb{R} \to \mathbb{R}$, we denote as xy the $\mathbb{R}^n$–valued vector $[x_i y_i]$ (similarly, $x/y = [x_i/y_i]$ in what follows), and we denote as $f(x)$ the $\mathbb{R}^n$–valued vector $[f(x_i)]$. When appropriate, a scalar s is understood as the vector $s1$.

In Equation (1), the $n \times n$ matrix Σ is called the food interaction matrix among the species, the scalar $\phi \geqslant 0$ is the immigration rate towards the ecosystem, $B_t \in \mathbb{R}^n$ is a standard multi-dimensional Brownian Motion (BM) representing a noise, and $T > 0$ is the noise temperature. We further assume that x_0 is a random variable supported by $\mathbb{R}_+^n$ and independent of the Brownian motion B. The model $\sqrt{2Tx_t}$ for the diffusion that we consider here is the so-called demographic noise model. Ecological justifications of Equation (1) can be found in, *e.g.*, [8, 27, 5, 2]. In all this paper, the interaction matrix Σ is assumed symmetric. This class of interaction matrices is frequently considered to model the mutualistic and the competitive interactions [3, 8, 2].

Recently, the SDE model has raised the interest of the physicists and the researchers working in the field of random disordered systems, focusing on the invariant distribution (when it exists) of the SDE (1) seen as a Markov process. The framework for this analysis can be described as follows. When the dimension of the system, *i.e.*, the number of species, is large, a large random matrix model is frequently advocated to represent the interaction matrix Σ. This follows a long tradition in theoretical ecology where the fine structure of the interaction matrix cannot be known, and is replaced by a random model. The more or less sophisticated random models used to represent this matrix aim to better understand the impact of the main ecological phenomena (competition for the resources, mutualism, predation, parasitism, ...) that govern the concrete dynamic behavior of these ecosystems in situations where they contain a large number of species. When Σ is random, the invariant measure of the Markov process (1) becomes a conditional (Gibbs) measure which is reminiscent of the Gibbs measure that appears in the celebrated Sherrington-Kirkpatrick (SK) model for the spin glasses in the mean field regime. In order to study the asymptotics of this Gibbs measure as $n \to \infty$, the first step is to compute the

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asymptotics of the free energy of the system, as was done by Biroli *et.al.* [8], and Altieri *et.al.* [5]. By using the replica method in the line of the celebrated papers [25, 26], they derived the limit free energy, which, similarly to the well-known SK case, involves a Parisi probability measure that captures the distribution of the overlaps between the replicas. These replica–based computations are used to characterize the structure of the energy landscape in terms of the temperature T and the parameters of the random matrix model for the interaction matrix.

The present paper is a first step towards making rigorous the analyses made in [8, 5], and the subsequent papers. Before delving into the statistical physics, the first part of this paper consists in a complete analysis of the SDE in terms of existence, uniqueness, and non-explosion of its solution when Σ is deterministic. The existence of an unique invariant distribution for the Markov process determined by (1), and the structure of this distribution are also studied. For a symmetric $n \times n$ matrix A, define $\lambda_+^{\max}(A)$ as

$$\lambda_+^{\max}(A) = \max_{u \in \mathbb{S}_+^{n-1}} u^\top A u,$$

where $\mathbb{S}_+^{n-1} = \{u \in \mathbb{R}_+^n, \ \|u\| = 1\}$ with $\| \cdot \|$ being the Euclidean norm. We show that when the condition

$$\lambda_+^{\max}(\Sigma) < 1 \tag{2}$$

is satisfied, then, the LV SDE (1) has an unique strong solution which is well-defined on $\mathbb{R}_+$. Furthermore, when $T < \phi$, the Markov process issued from this SDE has an unique invariant measure, and this invariant measure is given as $G(x) \sim \exp(\mathscr{H}(x)/T)$ where the Hamiltonian $\mathscr{H}$ is given in Equation (5) below. In the context of our SDE (1), much of these results are scattered in the literature, however, mostly without a rigorous proof up to our knowledge. This will be the object of Section 2.

In Section 3 and in all the remainder of the paper, we assume that our interaction matrix Σ is a symmetric random matrix which is independent of the BM B and of the initial value x_0. Recall that a random matrix $W_n \in \mathbb{R}^{n\times n}$ is said to belong to the Gaussian Orthogonal Ensemble (notation $W_n \sim \mathrm{GOE}_n$) if W_n is equal in law to $(M + M^\top)/\sqrt{2}$ where M is a real random $n \times n$ matrix with independent standard Gaussian elements. In Section 3 and following, we redenote Σ as Σ_n when useful, and assume that

$$\Sigma_n = \frac{\kappa}{\sqrt{n}} W_n + \alpha \frac{1 1^\top}{n}, \tag{3}$$

where $W_n \sim \mathrm{GOE}_n$, and $\kappa > 0$ and $\alpha \in \mathbb{R}$ are constants. This random matrix model has been frequently considered in the field of theoretical ecology as an academic model for a mutualistic or weakly competitive interaction matrix, where α and κ represent respectively the normalized mean and standard deviation of the interaction among two species [3, 9, 8].

For this model, $\lambda_+^{\max}(\Sigma_n)$ becomes of course random. However, it has been shown by Montanari and Richard [19] in another context that $\lambda_+^{\max}(\Sigma_n)$ converges almost surely in the large dimensional regime where $n \to \infty$ to a quantity that can be identified as the solution of a system of equations in (κ, α). In Section 3, we extend the results of [19] to the situation where α can be negative. By doing so, we recover the realizability bound in the large dimensional regime that is shown in [8] by building on a former result of Bunin in [9].

When Σ_n is random, the invariant measure of the Markov process defined by (1) becomes a conditional probability measure. In Section 4, we provide an asymptotic analysis of the free energy associated to this Gibbs measure by using the tools of that are now available in the mathematical physics literature, leading to Theorem 7 which is the most important result of this paper. In most of this literature, the "spins" are valued on the set $\{-1, 1\}^n$ [30, 31, 22], on the unit-sphere $\mathbb{S}^{n-1}$ [28], or on the rectangle $\mathcal{K}^n$ where $\mathcal{K} \subset \mathbb{R}$ is a compact set [23]. One difficulty of our analysis lies in the fact that our Gibbs measure is supported by the non-compact set $\mathbb{R}_+^n$. We hope that our approach can be adapted to other situations where the support of a Gibbs measure is non-compact, and that our results results open the door to some further mathematical research on the nature of the Parisi measure that underlies the limit free energy, leading towards the study of the Hamiltonian landscape as can be found in the physics literature [8, 5, 4].

In all what follows, $C > 0$ is a generic constant that can change from a line to another. This constant can depend on the model dimension n in the next section but not in the following ones. We denote as $(x \cdot y)$ the inner product of the vectors $x, y \in \mathbb{R}^n$.

2. The LV SDE analysis

In all this section, n is fixed, and Σ is a deterministic symmetric $n \times n$ matrix. We are concerned here with the well-definiteness of the SDE (1) as a SDE on $\mathbb{R}_+ = [0, \infty)$ and by the existence of an unique invariant measure for the continuous time homogeneous Markov process $(x_t)_{t\geqslant 0}$ defined by this SDE.

Given a symmetric $n \times n$ matrix A, similarly to the number $\lambda_+^{\max}(A)$ defined above, we define $\lambda_+^{\min}(A)$ as

$$\lambda_+^{\min}(A) = \min_{u \in \mathbb{S}_+^{n-1}} u^\top A u.$$

With this definition, Condition (2) is equivalent to $\lambda_+^{\min}(I - \Sigma) > 0$. In all the remainder of the paper, we denote as $\lambda_+^{\min}$ this quantity.

We start with the following proposition, which is proven in Appendix A.

Proposition 1. Assume that Condition (2) is satisfied. Then, for each initial probability measure μ such that $x_0 \sim \mu$, the SDE (1) admits an unique strong solution on $\mathbb{R}_+$.

We shall denote hereinafter as $x_t^{x_0}$ the solution of the SDE (1) that starts from x_0. We have the following proposition:

Proposition 2. Assume that Condition (2) is satisfied. Assume furthermore that

$$T < \phi. \tag{4}$$

Then the Markov process (x_t) has an unique invariant distribution $G(dx) \in \mathcal{P}(\mathbb{R}_+^n)$ given as

$$G(dx) = \frac{e^{\mathscr{H}(x)/T}}{\mathcal{Z}} dx,$$

where $\mathscr{H} : (0, \infty)^n \to \mathbb{R}$ is the Hamiltonian

$$\mathscr{H}(x) = \frac{1}{2} x^\top (\Sigma - I) x + (1 \cdot x) + (\phi - T)(1 \cdot \log x) \tag{5}$$

and $\mathcal{Z} = \int_{\mathbb{R}_+^n} \exp(\mathscr{H}(x)/T)\, dx < \infty$. Furthermore, for each function φ integrable with respect to $G(dx)$ and each initial value $x_0 \in \mathbb{R}_+^n$, it holds that

$$\frac{1}{T} \int_0^T \varphi(x_t^{x_0})\, dt \xrightarrow[T \to \infty]{\text{a.s.}} \int_{\mathbb{R}_+^n} \varphi(y) G(dy)$$

To prove this result, we rely on a recurrence result that appears in the book of Khasminskii [16], see also, *e.g.*, [17]:

Proposition 3. [application of Th. 4.1 and 4.2 and Cor. 4.4 of [16]] The Markov process (x_t) has an unique invariant distribution that we denote as G if there exists a bounded open domain $\mathcal{D} \subset (0, \infty)^n$, with a regular boundary and a closure $\bar{\mathcal{D}} \subset (0, \infty)^n$, that satisfies the following property. For each deterministic $x_0 \in \mathbb{R}_+^n \backslash \mathcal{D}$, let

$$\tau_{\mathcal{D}}^{x_0} = \inf\{t \geqslant 0 \,:\, x_t^{x_0} \in \mathcal{D}\}$$

be the entry time of x^{x_0} in $\mathcal{D}$ with $\inf \varnothing = \infty$. For each compact set $\mathcal{K} \subset \mathbb{R}_+^n$, it holds that $\sup_{x_0 \in \mathcal{K}} \mathbb{E} \tau_{\mathcal{D}}^{x_0} < \infty$.

Furthermore, if such a set $\mathcal{D}$ exists, the convergence

$$\frac{1}{T} \int_0^T \varphi(x_t^{x_0})\, dt \xrightarrow[T \to \infty]{\text{a.s.}} \int_{\mathbb{R}_+^n} \varphi(y) G(dy) \tag{6}$$

holds true for each real function φ integrable with respect to G.

Proof of Proposition 2. The infinitesimal generator $\mathcal{A}$ of the process (x_t) applied to a function $\varphi \in \mathcal{C}_c^2(\mathbb{R}^n; \mathbb{R})$ is

$$\mathcal{A}\varphi(x) = (\nabla \varphi(x) \cdot (x(1 + (\Sigma - I)x) + \phi)) + T \operatorname{tr} \nabla^2 \varphi(x) \operatorname{diag}(x).$$

Let $\varepsilon > 0$ be a small enough number to be fixed later, and put $V(x) = (1 \cdot (x - \log(x + \varepsilon)))$ for $x \in \mathbb{R}_+^n$. A formal application of $\mathcal{A}$ to V shows that

$$\begin{aligned}
\mathcal{A}V(x) &= \left(\left(1 - \frac{1}{x + \varepsilon} \right) \cdot (x + x((\Sigma - I)x) + \phi) \right) + T \left(1 \cdot \frac{x}{(x + \varepsilon)^2} \right) \\
&= (1 \cdot x) - (x \cdot (I - \Sigma)x) + \phi n - \left(1 \cdot \frac{x}{x + \varepsilon} \right) + \left(x \cdot (I - \Sigma) \frac{x}{x + \varepsilon} \right) - \phi \left(1 \cdot \frac{1}{x + \varepsilon} \right) + T \left(1 \cdot \frac{x}{(x + \varepsilon)^2} \right).
\end{aligned}$$

Writing $(1 \cdot x) + \left(x \cdot (I - \Sigma) \frac{x}{x+\varepsilon} \right) \leqslant C\,(1 \cdot x)$ for some $C > 0$, we obtain that

$$\begin{aligned}
\mathcal{A}V(x) &\leqslant -(x \cdot (I - \Sigma)x) - (\phi - T) \left(1 \cdot \frac{1}{x + \varepsilon} \right) + C\,(1 \cdot x) + \phi n \\
&\leqslant -\lambda_+^{\min} \|x\|^2 - (\phi - T) \left(1 \cdot \frac{1}{x + \varepsilon} \right) + C\,(1 \cdot x) + \phi n \\
&\leqslant -0.5 \lambda_+^{\min} \|x\|^2 - (\phi - T) \left(1 \cdot \frac{1}{x + \varepsilon} \right) + C'
\end{aligned}$$

for some $C' > 0$ which does not depend on ε. For $x = [x_i]_{i \in [n]} \in \mathbb{R}_+^n$, write $x_{\min} = \min_i x_i$. Fix ε as

$$\varepsilon = \frac{1}{2} \left(1 \wedge \frac{\phi - T}{C' + 1} \right),$$

and define the open domain $\mathcal{D} \subset \mathbb{R}_+^n$ as

$$\mathcal{D} = \left\{ x \in \mathbb{R}_+^n \ : \ \|x\|^2 < \frac{2(1+C')}{\lambda_+^{\min}} \text{ and } x_{\min} > \frac{\phi - T}{C'+1} - \varepsilon \right\}$$

(up to enlarging C' if necessary, we can consider that $\mathcal{D} \neq \varnothing$). We observe that $V > 0$ on $\mathbb{R}_+^n$ and that $\mathcal{D}$ satisfies the statement of Proposition 3. Furthermore, one can check that for our choice of $\mathcal{D}$, it holds that $\mathcal{A}V(x) \leqslant -1$ for each $x \in \mathbb{R}_+^n \backslash \mathcal{D}$.

Let x_0 be a constant vector in $\mathbb{R}_+^n \backslash \mathcal{D}$. For $a > 0$, define the stopping time $\eta_a^{x_0} = \inf\{t \geqslant 0 \ : \ \|x_t^{x_0}\| \geqslant a\}$. By Itô's formula, we have by the derivation we just made that

$$\mathbb{E}V(x^{x_0}_{t\wedge\tau_{\mathcal{D}}^{x_0}\wedge\eta_a^{x_0}}) = V(x_0) + \mathbb{E}\int_0^{t\wedge\tau_{\mathcal{D}}^{x_0}\wedge\eta_a^{x_0}} \mathcal{A}V(x_u^{x_0})du \leqslant V(x_0) - \mathbb{E}[t\wedge\tau_{\mathcal{D}}^{x_0}\wedge\eta_a^{x_0}].$$

Since $V > 0$ on $\mathbb{R}_+^n$, we obtain that $\mathbb{E}[t\wedge\tau_{\mathcal{D}}^{x_0}\wedge\eta_a^{x_0}] \leqslant V(x_0)$. Taking $a \to \infty$ then $t \to \infty$, we get that $\mathbb{E}\tau_{\mathcal{D}}^{x_0} < V(x_0)$, which implies that $\sup_{x_0\in\mathcal{K}} \mathbb{E}\tau_{\mathcal{D}}^{x_0} < \infty$ for every compact set $\mathcal{K} \subset \mathbb{R}_+^n \backslash \mathcal{D}$, and the condition provided on the statement of Proposition 3 is satisfied. Thus, the Markov process defined by (1) has an unique invariant distribution G, and the convergence (6) holds true.

It remains to show that the invariant distribution G takes the form provided in the statement of Proposition 2. To that end, it is enough to show that for G taking this form, it holds that

$$\forall \varphi \in \mathcal{C}_c^2(\mathbb{R}_+^n; \mathbb{R}), \qquad \int_{\mathbb{R}_+^n} \mathcal{A}\varphi(y)\ G(dy) = 0. \tag{7}$$

By an Integration by Parts, we have

$$\begin{aligned}\int_{\mathbb{R}_+^n} \mathcal{A}\varphi(y) e^{\mathscr{H}(y)/T}\, dy &= \int_{\mathbb{R}_+^n} \left\{ (\nabla\varphi(y)\cdot(y(1+\Sigma y - y)+\phi)) + T\operatorname{tr}\nabla^2\varphi(y)\operatorname{diag}(y) \right\} e^{\mathscr{H}(y)/T}\, dy \\ &= \int \varphi(y) \left\{ \nabla\cdot\left((y(-1-\Sigma y + y) - \phi) e^{\mathscr{H}(y)/T} \right) + T\sum_i \frac{\partial^2}{\partial y_i^2}\left(y_i e^{\mathscr{H}(y)/T}\right) \right\} dy.\end{aligned}$$

We therefore need to show that the term between $\{\cdot\}$ in the integrand above is zero for each $y \in (0,\infty)^n$. It is enough to show that

$$\forall i \in [n], \quad (y_i(-1-[\Sigma y]_i + y_i) - \phi)\, e^{\mathscr{H}(y)/T} = -T\frac{\partial}{\partial y_i}\left(y_i e^{\mathscr{H}(y)/T}\right).$$

This is directly obtained by developing the right hand side of this expression with the help of (5). □

In all the remainder of this paper, we shall assume that $T < \phi$.

3. Realizability bound for deformed GOE interaction matrix

From now on, we assume that our interaction matrix Σ_n is a large random $n \times n$ matrix described by Equation (3). The purpose of this section is to characterize the behavior of $\lambda_+^{\max}(\Sigma_n)$ as $n \to \infty$ for this model.

This problem was essentially solved by Montanari and Richard in the different context of the so-called non-negative Principal Component Analysis [19]. Here, we extend their analysis to the case where α can be negative, and we correct a small error in their proof.

In all the remainder of this paper, we shall often drop the index n from objects like Σ_n or W_n for readability.

Proposition 4. Define the real functions d, f, and g on $\mathbb{R}$ as

$$d(x) = \mathbb{E}\,(Z+x)_+^2, \quad f(x) = \frac{\mathbb{E}\,(Z+x)_+}{\sqrt{d(x)}}, \quad \text{and} \quad g(x) = \frac{\mathbb{E}Z\,(Z+x)_+}{\sqrt{d(x)}},$$

where $Z \sim \mathcal{N}(0,1)$. Then, it holds that

$$\lambda_+^{\max}(\Sigma_n) \xrightarrow[n\to\infty]{\text{a.s.}} \boldsymbol{\lambda}_+(\alpha,\kappa),$$

where

$$\boldsymbol{\lambda}_+(\alpha,\kappa) = \alpha f(c)^2 + 2\kappa g(c),$$

and $c \in \mathbb{R}$ is the unique solution of the equation

$$c = \frac{\alpha}{\kappa} f(c).$$

Remark 1. We observe that if $\boldsymbol{\lambda}_+(\alpha,\kappa) < 1$, then on the probability one set where $\lambda_+^{\max}(\Sigma_n) \to \boldsymbol{\lambda}_+(\alpha,\kappa)$, the function $\exp(\mathscr{H}(x)/T)$ is integrable on $\mathbb{R}_+^n$ for all large n, and the distribution G is well-defined. Alternatively, if $\boldsymbol{\lambda}_+(\alpha,\kappa) > 1$, then on the probability one set where $\lambda_+^{\max}(\Sigma_n) \to \boldsymbol{\lambda}_+(\alpha,\kappa)$, for all large enough n, the function $\exp(\mathscr{H}(x)/T)$ is not integrable on $\mathbb{R}_+^n$. The "realizability frontier" $\{(\alpha,\kappa) : \boldsymbol{\lambda}_+(\alpha,\kappa) = 1\}$ which is plotted on Figure 1 coincides with the curve provided in [9].

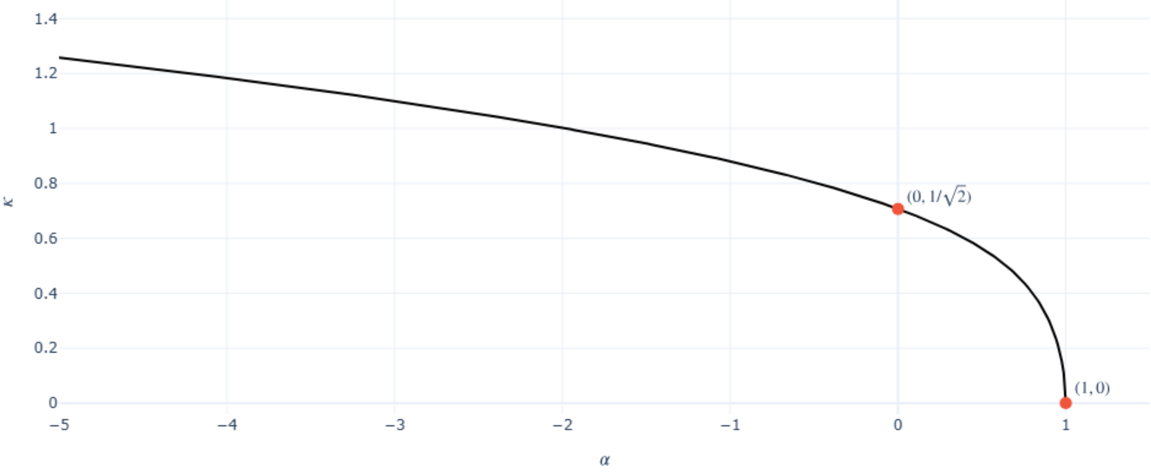

FIGURE 1. The curve $\boldsymbol{\lambda}_+(\alpha, \kappa) = 1$

Two particular cases deserve some attention:

Lemma 5. $\boldsymbol{\lambda}_+(0, \kappa) = \kappa\sqrt{2}$ and $\lim_{\kappa\to 0} \boldsymbol{\lambda}_+(\alpha, \kappa) = 0 \vee \alpha$.

This lemma explains the presence of the points $(\alpha, \kappa) = (0, 1/\sqrt{2})$ and $(\alpha, \kappa) = (1, 0)$ on the curve of Figure 1.

Modifications of the approach of [19] to prove Proposition 4. Noticing that

$$\lambda_+^{\max}(\Sigma_n) = \kappa\lambda_+^{\max}\left(\frac{1}{\sqrt{n}}W_n + \frac{\alpha}{\kappa}\frac{11^\top}{n}\right) \quad \text{and} \quad \boldsymbol{\lambda}_+(\alpha, \kappa) = \kappa\boldsymbol{\lambda}_+(\alpha/\kappa, 1),$$

it is enough to establish Proposition 4 for $\kappa = 1$ and α arbitrary.

We first focus on the well-definiteness of $\boldsymbol{\lambda}_+(\alpha, 1)$ as provided in the statement of the proposition, and provide a maximality property related with this quantity. To this end, we adapt the results of [19, Lemmata 18, 20, 22] to the cases where α can be negative.

Lemma 6. For each $\alpha \in \mathbb{R}$, the function $r_\alpha : \mathbb{R} \to \mathbb{R}$ defined as

$$r_\alpha(x) = \alpha f(x)^2 + 2g(x)$$

has an unique maximizer c, which is defined as the unique solution of the fixed point equation

$$x = \alpha f(x).$$

Proof. We first need some properties of the functions f and g, which can be obtained by a small adaptation of [19, Lemma 18]. These read: The function f is strictly positive and differentiable. It satisfies $f'(x) > 0$ for all $x \in \mathbb{R}$, $\lim_{x\to-\infty} f(x) = 0$, $f(0) = 1/\sqrt{\pi}$, and $\lim_{x\to+\infty} f(x) = 1$. The function g is a strictly positive differentiable function that satisfies $g'(x) > 0$ for all $x < 0$, $g'(0) = 0$, and $g'(x) < 0$ for all $x > 0$. Also, $\lim_{|x|\to\infty} g(x) = 0$.

We first observe that if $\alpha = 0$, then trivially, $c = 0$ and it is the unique maximizer of r_0 by the aforementioned properties of g. Assume $\alpha \neq 0$.

Let us define the real function function q on $\mathbb{R}\backslash\{0\}$ as $q(x) = f(x)/x$. On this domain of definition, it holds that

$$q'(x) = (xf'(x) - f(x))/x^2.$$

Therefore, it is clear that q is strictly decreasing on $(-\infty, 0)$. We also know from the proof of [19, Lemma 20] that q is also strictly decreasing on $(0, \infty)$. Finally, it is obvious from what precedes that $\lim_{|x|\to\infty} q(x) = 0$, $\lim_{x\to 0^-} q(x) = -\infty$, and $\lim_{x\to 0^+} q(x) = +\infty$. It results that the equation $x = \alpha f(x)$ has an unique solution $c \in \mathbb{R}$ for each $\alpha \in \mathbb{R}\backslash\{0\}$.

It remains to show that c is the unique maximum of r_α. A small calculation shows that $(\sqrt{d(x)})' = f(x)$ and that $\sqrt{d(x)} = xf(x) + g(x)$. Equating the derivatives at both sides of this last equation shows that $g'(x) = -xf'(x)$. We can therefore write

$$r'_\alpha(x) = 2\alpha f(x)f'(x) + 2g'(x) = 2xf'(x)\left(\alpha\frac{f(x)}{x} - 1\right) = 2xf'(x)\left(\alpha q(x) - 1\right).$$

Let us focus here on the case where $\alpha < 0$. The case $\alpha > 0$ is similar. From the previous results on the graph of q, we obtain that $c < 0$ and that

$$\begin{cases} x(\alpha q(x) - 1) > 0 & \text{if } x < c \\ x(\alpha q(x) - 1) = 0 & \text{if } x = c \\ x(\alpha q(x) - 1) < 0 & \text{if } x > c. \end{cases}$$

Recalling that $f' > 0$, we obtain that c is the unique maximizer of r_α. □

It remains to show that $\lambda_+^{\max}(\Sigma_n) \xrightarrow{\text{as}} \boldsymbol{\lambda}_+(\alpha, 1) = r_\alpha(c)$ to finish the proof of Proposition 4.

The first step consists in showing that $\limsup_n \lambda_+^{\max}(\Sigma_n) \leqslant r_\alpha(c)$ w.p.1. Introducing the set $\mathbb{S}_+^{n-1}(s)$ defined for $s \in [0, 1]$ as

$$\mathbb{S}_+^{n-1}(s) = \left\{ u \in \mathbb{S}_+^{n-1}, \left(u \cdot 1/\sqrt{n}\right) = s \right\},$$

we can characterize $\lambda_+^{\max}(\Sigma)$ by the identity

$$\begin{aligned} \lambda_+^{\max}(\Sigma) &= \max_{s \in [0,1]} \left(\alpha s^2 + \max_{u \in \mathbb{S}_+^{n-1}(s)} (u \cdot W u) \right) \\ &\triangleq \max_{s \in [0,1]} M(s). \end{aligned}$$

By a Gaussian concentration argument [19, Appendix B], it is enough to show that

$$\max_{s \in [0,1]} \limsup_n \mathbb{E} M(s) \leqslant r_\alpha(c) \tag{8}$$

to obtain that $\limsup_n \lambda_+^{\max}(\Sigma) \leqslant r_\alpha(c)$ w.p.1. This bound can be obtained by a Sudakov-Fernique argument for Gaussian processes [32]. Consider the Gaussian process $u \mapsto \xi_u = 2\,(u \cdot \boldsymbol{Z})$ on $\mathbb{S}_+^{n-1}(s)$ with $\boldsymbol{Z} \sim \mathcal{N}(0, n^{-1} I_n)$. We can easily show that $n^{-1}\mathbb{E}\left((u \cdot W u) - (v \cdot W v)\right)^2 \leqslant \mathbb{E}(\xi_u - \xi_v)^2$. By the Sudakov-Fernique inequality, we then have

$$\begin{aligned} \mathbb{E} M(s) &\leqslant \alpha s^2 + 2\mathbb{E} \max_{u \in \mathbb{S}_+^{n-1}(s)} (u \cdot \boldsymbol{Z}) \\ &\leqslant \alpha s^2 - 2\tilde{c}s + 2\mathbb{E} \max_{u \in \mathbb{S}_+^{n-1}} \left(u \cdot (\boldsymbol{Z} + \tilde{c}1/\sqrt{n})\right) \\ &= \alpha s^2 - 2\tilde{c}s + 2\mathbb{E}\|(\boldsymbol{Z} + \tilde{c}1/\sqrt{n})_+\| \\ &\leqslant \alpha s^2 - 2\tilde{c}s + 2\left(\mathbb{E}\|(\boldsymbol{Z} + \tilde{c}1/\sqrt{n})_+\|^2\right)^{1/2}, \end{aligned}$$

where $\tilde{c} \in \mathbb{R}$ is arbitrary. Taking n to infinity, we get that

$$\limsup_n \mathbb{E} M(s) \leqslant \alpha s^2 - 2\tilde{c}s + 2\sqrt{d(\tilde{c})}.$$

Let us now take $\tilde{c}$ as the unique solution of the equation $f(x) = s$. Using the identity $\sqrt{d(x)} = x f(x) + g(x)$, we then obtain

$$\limsup_n \mathbb{E} M(s) \leqslant \alpha f(\tilde{c})^2 - 2\tilde{c} f(\tilde{c}) + 2\sqrt{d(\tilde{c})} = r_\alpha(\tilde{c}) \leqslant r_\alpha(c)$$

by Lemma 6, and the bound (8) is established.

We note here that an analogous argument is used in [19] to prove [19, Lemma 10]. However, in the context of Proposition 4 above, this argument requires the function $s \mapsto \alpha s^2 - 2\tilde{c}s$ to be concave, which is incorrect when $\alpha > 0$.

The second step towards showing Proposition 4 consists in showing that $\liminf_n \lambda_+^{\max}(\Sigma) \geqslant \boldsymbol{\lambda}_+(\alpha, 1)$ w.p.1. This is done in [19] with the help of an Approximate Message Passing algorithm. This argument can be applied to our situation practically without modification, completing the proof of Proposition 4.

Proof of Lemma 5. We have $\boldsymbol{\lambda}_+(0, \kappa) = 2\kappa g(0) = \kappa\sqrt{2}$ by a small calculation.

To establish the second result for $\alpha \neq 0$, let us focus on the equation $c = (\alpha/\kappa) f(c)$. By the properties of the function q provided above, we see that $c = c(\kappa)$ converges to ∞ as $\kappa \to 0$ when $\alpha > 0$, and to $-\infty$ when $\alpha < 0$. We also know that g is bounded. Therefore, $\lim_{\kappa \to 0} \boldsymbol{\lambda}_+(\alpha, \kappa) = \lim_{\kappa \to \infty} \alpha f(c(\kappa)) = 0 \vee \alpha$ by recalling the properties of f.

4. Asymptotics of the free energy

From now on, we assume that κ and α in our deformed GOE model are chosen in such a way that $\boldsymbol{\lambda}_+(\kappa, \alpha) < 1$, and we set $\varepsilon_\Sigma = (1 - \boldsymbol{\lambda}_+(\kappa, \alpha))/2$. Since Σ_n is now a large random matrix for which $\mathbb{P}\left[\lambda_+^{\max}(\Sigma_n) > 1\right] > 0$ for any fixed n, we need to replace our SDE (1) with the SDE

$$dx_t = x_t \left(1 + \left(\widetilde{\Sigma} - I\right) x_t\right) dt + \phi dt + \sqrt{2T x_t} dB_t$$

with

$$\widetilde{\Sigma} = \Sigma \mathbb{1}_{\lambda_+^{\max}(\Sigma) < 1 - \varepsilon_\Sigma}$$

to guarantee the well-definiteness of the solution. By a slight adaptation of Proposition 1 to the case where the interaction matrix is random, this SDE admits an unique strong solution. Furthermore, since $1 - \varepsilon_\Sigma = \boldsymbol{\lambda}_+(\kappa, \alpha) + \varepsilon_\Sigma$, it holds by Proposition 4 that with probability one, $\Sigma = \widetilde{\Sigma}$ for all n large enough. We use hereinafter the standard notation in spin glass theory $\beta = 1/T$, and recall our standing assumption $\phi\beta > 1$.

The invariant distribution G of the EDS (1), which existence is ensured by Proposition 2, becomes in our new setting the conditional distibution $\widetilde{G}(dx) = \widetilde{\mathscr{Z}}^{-1} \exp(\beta \widetilde{\mathscr{H}}(x)) dx$ given $\widetilde{\Sigma}$, with

$$\widetilde{\mathscr{H}}(x) = \frac{1}{2} x^\top \left(\widetilde{\Sigma} - I \right) x + (1 \cdot x) + (\phi - 1/\beta)(1 \cdot \log x),$$

and

$$\widetilde{\mathscr{Z}} = \int_{\mathbb{R}_+^n} \exp(\beta \widetilde{\mathscr{H}}(x)) \, dx < \infty.$$

In what follows, we shall use a more convenient expression for the measure $\exp(\beta \mathscr{H}(x)) \, dx$ on $\mathbb{R}_+^n$ by writing

$$\exp(\beta \widetilde{\mathscr{H}}(x)) \, dx = \exp \left(H_n(x) \mathbb{1}_{\lambda_+^{\max}(\Sigma_n) < 1 - \varepsilon_\Sigma} \right) \mu_\beta^{\otimes n}(dx)$$

where $H_n : \mathbb{R}_+^n \to \mathbb{R}$ is the new Hamiltonian

$$H_n(x) = \frac{\beta}{2} x^\top \Sigma_n x = \frac{\beta\kappa}{2\sqrt{n}} x^\top W_n x + \frac{\beta\alpha}{2} \frac{(x \cdot 1)^2}{n},$$

and $\mu_\beta^{\otimes n}(dx)$ is the n–fold product measure of the positive measure μ_β defined on $\mathbb{R}_+$ as

$$\mu_\beta(dx_1) = x_1^{\phi\beta - 1} \exp\left(-\beta x_1^2/2 + \beta x_1 \right) \, dx_1, \quad x_1 \geqslant 0.$$

Our purpose is to identify the asymptotic behavior of the free energy

$$\widetilde{F}_n = \frac{1}{n} \mathbb{E} \log \widetilde{\mathscr{Z}} = \frac{1}{n} \mathbb{E} \log \int_{\mathbb{R}_+^n} \exp \left(H_n(x) \mathbb{1}_{\lambda_+^{\max}(\Sigma_n) < 1 - \varepsilon_\Sigma} \right) \mu_\beta^{\otimes n}(dx).$$

We now construct the Parisi functional that will be used in our main theorem. To begin with, we set the notations for defining a so-called Parisi measure, a.k.a. the functional order parameter (f.o.p.). Let $D > 0$, let $K > 0$ be an integer, and let

$$\begin{gathered} 0 < \lambda_0 < \cdots < \lambda_{K-1} < 1, \quad \text{and} \\ 0 = b_0 < \cdots < b_{K-1} < b_K = D. \end{gathered} \tag{9}$$

To these reals we associate the Parisi measure ζ defined as

$$\zeta(\{b_k\}) = \lambda_k - \lambda_{k-1} \quad \text{with} \quad \lambda_{-1} = 0, \ \lambda_K = 1. \tag{10}$$

We shall denote the set of these finitely supported measures such that the maximum of the support is equal to D as $\mathrm{fop}_D \subset \mathcal{P}([0, D])$.

Let $a > 0$, $h \geqslant 0$ and $\gamma \in \mathbb{R}$, and let $(z_k)_{k \in [K]}$ be K independent standard Gaussians. Define the random variable

$$X_{K,a} = \log \int_0^a \exp \left(x\beta\kappa \sum_{k=1}^K z_k \sqrt{b_k - b_{k-1}} + \beta\alpha h x + \gamma x^2 \right) \mu_\beta(dx).$$

For $k = K - 1, \ldots, 0$, set

$$X_{k,a} = \frac{1}{\lambda_k} \log \mathbb{E}_{z_{k+1}} \exp\left(\lambda_k X_{k+1} \right),$$

where $\mathbb{E}_{z_{k+1}}$ is the expectation with respect to the distribution of z_{k+1}. With this, construction, $X_{0,a}$ is deterministic and depends on ζ, h, and γ. Denote this real number as $X_{0,a}(\zeta, h, \gamma)$. Our Parisi function will be

$$P_a(\zeta, h, \gamma) = X_{0,a}(\zeta, h, \gamma) - \frac{\beta^2\kappa^2}{4} \sum_{k=0}^{K-1} \lambda_k (b_{k+1}^2 - b_k^2).$$

We can now state our main result.

Theorem 7. Assume that $\boldsymbol{\lambda}_+(\kappa, \alpha) < 1$. Then, if $\alpha \leqslant 0$, it holds that

$$\widetilde{F}_n \xrightarrow[n\to\infty]{} \sup_{a \geqslant 0, D \geqslant 0} \ \inf_{\zeta \in \mathrm{fop}_D, h \geqslant 0, \gamma \in \mathbb{R}} \left(P_a(\zeta, h, \gamma) - \gamma D - \frac{\beta\alpha}{2} h^2 \right).$$

If $\alpha > 0$, then

$$\widetilde{F}_n \xrightarrow[n\to\infty]{} \sup_{a \geqslant 0, h \geqslant 0, D \geqslant 0} \ \inf_{\zeta \in \mathrm{fop}_D, \gamma \in \mathbb{R}} \left(P_a(\zeta, h, \gamma) - \gamma D - \frac{\beta\alpha}{2} h^2 \right).$$

To establish this result, we shall follow Panchenko's approach in [23]. The main particularities of our proof are related with the non-compactness of the support of μ_β and the presence of the factor α in the expression of the Hamiltonian H.

A comprehensive analysis of the limits stated by the previous theorem remains to be done. We conjecture that these limits are infinite when $\boldsymbol{\lambda}_+(\kappa, \alpha) > 1$. In the case where $\boldsymbol{\lambda}_+(\kappa, \alpha) < 1$, a challenge is to prove that each of the saddles that appear in the expressions of these limits is attained by an unique probability measure $\zeta_\star$ which is compactly supported. In the physics literature, the study of the structure of $\zeta_\star$ has attracted a great deal of interest. For instance, it is asserted that in the low temperature and low immigration rate regime $\beta \to \infty$ and $\phi \to 0$, the measure $\zeta_\star$ is reduced to a Dirac delta when $\kappa < 1/\sqrt{2}$ (see Figure 1), which corresponds to the so-called Replica Symmetry regime. For $\kappa > 1/\sqrt{2}$, the measure $\zeta_\star$ has an infinite support, corresponding to the so-called Full Replica Symmetry Breaking regime [5].

We now prove Theorem 7.

5. Proof of Theorem 7: some preparation

5.1. Ruelle probability cascades. It is well-known that a Parisi function is intimately connected with the so-called Ruelle Probability Cascades (RPC) which definition and properties are discussed at length in [22, Chapter 2]. We recall the main properties of these objects for the reader's convenience, and we provide an equivalent expression of our Parisi function $P_a(\zeta, h, \gamma)$ involving RPC's. Arranging the λ_k's in (9) in the vector $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_{K-1})$, the notation $\mathrm{RPC}_{\boldsymbol{\lambda}} \in \mathcal{P}(\mathcal{P}(\mathbb{N}^K))$ will denote the distribution of a RPC $(v_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{N}^K}$ with parameters the elements of the vector $\boldsymbol{\lambda}$. We shall repeatedly use the following result, obtained by a direct adaptation of [22, Theorem 2.9].

Proposition 8. Let $d > 0$ be an integer. Consider the vector $\boldsymbol{\lambda}$ above, K random vectors $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_K \overset{\text{iid}}{\sim} \mathcal{N}(0, I_d)$, and a real function $\mathcal{X}_K(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_K)$ satisfying $\mathbb{E} \exp \lambda_{K-1} \mathcal{X}_K < \infty$. For $k = K-1$ to 0, define recursively

$$\mathcal{X}_k = \frac{1}{\lambda_k} \log \mathbb{E}_{\boldsymbol{z}_{k+1}} e^{\lambda_k \mathcal{X}_{k+1}},$$

where $\mathbb{E}_{\boldsymbol{z}_{k+1}}$ is the expectation with respect to the distribution of $\boldsymbol{z}_{k+1}$. Consider a family of i.i.d. random vectors $(\boldsymbol{z}_{\boldsymbol{j}})_{\boldsymbol{j} \in \mathbb{N} \cup \mathbb{N}^2 \cup \cdots \cup \mathbb{N}^K}$ with distribution $\mathcal{N}(0, I_d)$. For $\boldsymbol{i} \in \mathbb{N}^k$, write $\boldsymbol{i} = (i_1, i_2, \ldots, i_k)$. Then, it holds that

$$\mathcal{X}_0 = \mathbb{E} \log \sum_{\boldsymbol{i} \in \mathbb{N}^K} v_{\boldsymbol{i}} \exp \mathcal{X}_K(\boldsymbol{z}_{i_1}, \boldsymbol{z}_{i_1 i_2}, \ldots, \boldsymbol{z}_{i_1, i_2, \ldots, i_K}),$$

where $(v_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{N}^K} \sim \mathrm{RPC}_{\boldsymbol{\lambda}}$ is independent of the family $(\boldsymbol{z}_{\boldsymbol{j}})$.

We will also need the following application of this result. Write $\boldsymbol{z}_k = [z_{k,l}]_{l=1}^d$, and assume that $\mathcal{X}_K$ is of the form $\mathcal{X}_K(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_K) = \sum_{l=1}^d \mathcal{U}_K(z_{1,l}, \ldots, z_{K,l})$ for some real function $\mathcal{U}_K$. Performing the induction $\mathcal{X}_K \to \mathcal{X}_{K-1} \to \cdots \to \mathcal{X}_0$ as in the statement of the previous proposition, we can check that $\mathcal{X}_0 = d\mathcal{U}_0$, where $\mathcal{U}_0$ is also obtained from $\mathcal{U}_K$ by the same kind of induction.

We now provide an expression of our Parisi function P_a in terms of a RPC with the help of the former proposition. As is usual in the SK literature, we introduce a function ξ capturing the covariance function of the Gaussian process $x \mapsto \beta\kappa x^\top W x/(2\sqrt{n})$. Namely, we write

$$\frac{\beta^2\kappa^2}{4n} \mathbb{E}(x^1)^\top W x^1 (x^2)^\top W x^2 = n\xi(R_{12}) \quad \text{with} \quad R_{ij} = \frac{(x^i \cdot x^j)}{n} \quad \text{and} \quad \xi(x) = \frac{\beta^2\kappa^2}{2} x^2.$$

We also find it more readable and more compliant with the SK literature to set $\theta(x) = x\xi'(x) - \xi(x)$, even though in our case, this function trivially reads $\theta(x) = \xi(x)$.

With the help of Proposition 8, we can provide an expression of the term $X_{0,a}$ in the expression of P_a in terms of a RPC. With our new notations, the function $X_{K,a}$ is re-written

$$X_{K,a} = \log \int_0^a \exp\left(x \sum_{k=1}^K z_k \sqrt{\xi'(b_k) - \xi'(b_{k-1})} + \beta\alpha h x + \gamma x^2 \right) \mu_\beta(dx).$$

Set $d = 1$, and let $\mathcal{X}_K = X_{K,a}$ in Proposition 8 with $\boldsymbol{z}_k = z_k$. Consider a family of independent standard Gaussians $(z_{\boldsymbol{j}})_{\boldsymbol{j} \in \mathbb{N} \cup \mathbb{N}^2 \cup \cdots \cup \mathbb{N}^K}$, and define the family of random variables $(q_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{N}^K}$ as follows. For $\boldsymbol{i} = (i_1, i_2, \ldots, i_K) \in \mathbb{N}^K$, set

$$q_{\boldsymbol{i}} = \beta\kappa \sum_{k=1}^K z_{i_1, i_2, \ldots, i_k} \sqrt{b_k - b_{k-1}} = \sum_{k=1}^K z_{i_1, i_2, \ldots, i_k} \sqrt{\xi'(b_k) - \xi'(b_{k-1})}. \tag{11}$$

Then, by Proposition 8 above, we have

$$X_{0,a}(\zeta, h, \gamma) = \mathbb{E} \log \sum_{\boldsymbol{i} \in \mathbb{N}^K} v_{\boldsymbol{i}} \int_0^a e^{x q_{\boldsymbol{i}} + \beta\alpha h x + \gamma x^2} \mu_\beta(dx).$$

We now consider the term $-(\beta^2\kappa^2/4)\sum_{k=0}^{K-1}\lambda_k(b_{k+1}^2-b_k^2)$ in the expression of P_a. First, it will be convenient to write

$$\frac{\beta^2\kappa^2}{4}\sum_{k=0}^{K-1}\lambda_k(b_{k+1}^2-b_k^2)=\frac{1}{2}\sum_{k=0}^{K-1}\lambda_k(\theta(b_{k+1})-\theta(b_k))$$
$$=-\frac{1}{2}\sum_{k=0}^{K}(\lambda_k-\lambda_{k-1})\theta(b_k)+\frac{1}{2}\theta(b_K)=-\frac{1}{2}\int\theta(b)\zeta(db)+\frac{1}{2}\theta(D).$$

Second, if we set $d=1$ and

$$\mathcal{X}_K(\boldsymbol{z}_1,\dots,\boldsymbol{z}_K)=\frac{\beta\kappa}{\sqrt{2}}\left(\boldsymbol{z}_1\sqrt{b_1^2-b_0^2}+\cdots+\boldsymbol{z}_K\sqrt{b_K^2-b_{K-1}^2}\right)=\sum_{k=1}^{K}\boldsymbol{z}_k\sqrt{\theta(b_k)-\theta(b_{k-1})}$$

in Proposition 8, then, using the expression of the moment generating function of a Gaussian, we easily obtain that

$$\mathcal{X}_0=\frac{1}{2}\sum_{k=0}^{K-1}\lambda_k(\theta(b_{k+1})-\theta(b_k))=-\frac{1}{2}\int\theta(b)\zeta(db)+\frac{1}{2}\theta(D).$$

similarly to the family $(q_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{N}^K}$, define the family of random variables $(y_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{N}^K}$ as

$$y_{\boldsymbol{i}}=\frac{\beta\kappa}{\sqrt{2}}\sum_{k=1}^{K}z_{i_1,i_2,\dots,i_k}\sqrt{b_k^2-b_{k-1}^2}=\sum_{k=1}^{K}z_{i_1,i_2,\dots,i_k}\sqrt{\theta(b_k)-\theta(b_{k-1})}. \tag{12}$$

Then, by Proposition 8, we also have

$$\mathcal{X}_0=\mathbb{E}\log\sum_{\boldsymbol{i}\in\mathbb{N}^K}v_{\boldsymbol{i}}e^{y_{\boldsymbol{i}}}.$$

In short, we can write

$$P_a(\zeta,h,\gamma)=X_{0,a}(\zeta,h,\gamma)+\frac{1}{2}\int\theta(b)\zeta(db)-\frac{1}{2}\theta(D)$$
$$=\mathbb{E}\log\sum_{\boldsymbol{i}\in\mathbb{N}^K}v_{\boldsymbol{i}}\int_0^a e^{xq_{\boldsymbol{i}}+\beta\alpha hx+\gamma x^2}\mu_\beta(dx)-\mathbb{E}\log\sum_{\boldsymbol{i}\in\mathbb{N}^K}v_{\boldsymbol{i}}e^{y_{\boldsymbol{i}}}$$

for $\zeta\in\mathrm{fop}_D$.

5.2. **Useful bounds.** We shall prove Theorem 7 by computing in turn an upper bound on $\limsup\widetilde{F}_n$ and a lower bound on $\liminf\widetilde{F}_n$. To this end, we shall rely on the two following lemmata thanks to which $\widetilde{F}_n$ will be replaced with more easily manageable free energies.

Given $A>0$, let $B_+^n(\sqrt{n}A)=\{x\in\mathbb{R}_+^n\ :\ \|x\|\leqslant\sqrt{n}A\}$ be the closed Euclidean $\sqrt{n}A$–ball of $\mathbb{R}_+^n$. Let F_n^A be the free energy defined as

$$F_n^A=\frac{1}{n}\mathbb{E}\log\int_{B_+^n(\sqrt{n}A)}e^{\beta\mathscr{H}(x))}dx=\frac{1}{n}\mathbb{E}\log\int_{B_+^n(\sqrt{n}A)}e^{H_n(x)}\mu_\beta^{\otimes n}(dx),$$

where the indicator $\mathbb{1}_{\lambda_+^{\max}(\Sigma)<1-\varepsilon_\Sigma}$ is absent, but where the integration is performed on $B_+^n(\sqrt{n}A)$. The free energy F_n^A is much more easily manageable that $\widetilde{F}_n$ because $H_n(x)$ is a Gaussian process, contrary to $H_n(x)\mathbb{1}_{\lambda_+^{\max}(\Sigma)<1-\varepsilon_\Sigma}$. The following lemma, proven in Appendix B, will let us focus our upper bound analysis on F_n^A:

Lemma 9. There exists $A>0$ large enough such that $\limsup_n\widetilde{F}_n\leqslant\limsup_n F_n^A$.

For $a>0$, define now the free energy

$$F_{a,n}=\frac{1}{n}\mathbb{E}\log\int_{[0,a]^n}e^{\beta\mathscr{H}(x))}dx=\frac{1}{n}\mathbb{E}\log\int_{[0,a]^n}e^{H_n(x)}\mu_\beta^{\otimes n}(dx).$$

The following lemma is proven is Appendix C.

Lemma 10. $\liminf_n\widetilde{F}_n\geqslant\sup_{a>0}\liminf_n F_{a,n}$.

6. Proof of Theorem 7: upper bound for $\alpha \leqslant 0$

Our purpose here is to establish the following proposition.

Proposition 11. Assume that $\alpha \leqslant 0$. Then,

$$\limsup_n \widetilde{F}_n \leqslant \sup_{a>0, D>0} \inf_{\zeta \in \mathrm{fop}_D, h \geqslant 0, \gamma \in \mathbb{R}} \left(P_a(\zeta, h, \gamma) - \gamma D - \frac{\beta\alpha}{2} h^2 \right).$$

Thanks to Lemma 9, we focus our analysis on F_n^A. Our technique will be based on the so-called Guerra's interpolation. Let $A \in (0, \infty]$ and $a \in (0, \infty]$. Let $\zeta \in \mathrm{fop}_D$ be defined as in (10) with the parameters in (9). Let $h \geqslant 0$ and $\gamma \in \mathbb{R}$, and let $(\boldsymbol{z}_k)_{k \in [K]}$ be K independent $\mathcal{N}(0, I_n)$ random vectors. Define the random variable

$$\boldsymbol{X}_{K,a}^A(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_K) = \log \int_{B_+(A\sqrt{n}) \cap [0,a]^n} \exp\left(\left(x \cdot \sum_{k=1}^K \sqrt{\xi'(b_k) - \xi'(b_{k-1})} \boldsymbol{z}_k \right) + \beta\alpha h \left(1 \cdot x\right) + \gamma \|x\|^2 \right) \mu_\beta^{\otimes n}(dx).$$

Applying Proposition 8 with $d = n$ and $\mathcal{X}_K(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_K) = \boldsymbol{X}_{K,a}^A(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_K)$, the result $\mathcal{X}_0$ of the recursion that we denote as $\mathcal{X}_0 = \boldsymbol{X}_{0,a}^A(\zeta, h, \gamma)$ can be written as

$$\boldsymbol{X}_{0,a}^A(\zeta, h, \gamma) = \mathbb{E} \log \sum_{\boldsymbol{i} \in \mathbb{N}^K} v_{\boldsymbol{i}} \int_{B_+(A\sqrt{n}) \cap [0,a]^n} \exp\left((x \cdot \boldsymbol{q}_{\boldsymbol{i}}) + \beta\alpha h \left(1 \cdot x\right) + \gamma \|x\|^2 \right) \mu_\beta^{\otimes n}(dx) \tag{13}$$

where the process $(\boldsymbol{q}_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{N}^K}$ with $\boldsymbol{q}_{\boldsymbol{i}} = [\boldsymbol{q}_{\boldsymbol{i},l}]_{l=1}^N$ is such that the n processes $(\boldsymbol{q}_{\boldsymbol{i},l})_{\boldsymbol{i} \in \mathbb{N}^K}$ for $l \in [n]$ are n independent copies of the process $(q_{\boldsymbol{i}})$ in (11), and the processes $(v_{\boldsymbol{i}})$ and $(\boldsymbol{q}_{\boldsymbol{i}})$ are independent.

With this, define

$$\boldsymbol{P}_a^A(\zeta, h, \gamma) = \frac{1}{n} \boldsymbol{X}_{0,a}^A(\zeta, h, \gamma) + \frac{1}{2} \int \theta d\zeta - \frac{\theta(D)}{2}.$$

We shall establish the following lemmata.

Lemma 12. $\limsup_n \left\{ F_n^A - \sup_{D>0} \inf_{\zeta \in \mathrm{fop}_D, h \in \mathbb{R}_+, \gamma \in \mathbb{R}} \left(\boldsymbol{P}_\infty^A(\zeta, h, \gamma) - \gamma D - \frac{\beta\alpha}{2} h^2 \right) \right\} \leqslant 0.$

Lemma 13. For each $D > 0$, it holds that

$$\lim_{a \to \infty} \inf_{\zeta \in \mathrm{fop}_D, h \in \mathbb{R}_+, \gamma \in \mathbb{R}} \left(\boldsymbol{P}_a^A(\zeta, h, \gamma) - \gamma D - \frac{\beta\alpha}{2} h^2 \right) = \inf_{\zeta \in \mathrm{fop}_D, h \in \mathbb{R}_+, \gamma \in \mathbb{R}} \left(\boldsymbol{P}_\infty^A(\zeta, h, \gamma) - \gamma D - \frac{\beta\alpha}{2} h^2 \right).$$

With the help of this lemma, we obtain that

$$\limsup_n F_n^A \leqslant \sup_{a>0, D>0} \inf_{\zeta \in \mathrm{fop}_D, h \in \mathbb{R}_+, \gamma \in \mathbb{R}} \left(\boldsymbol{P}_a^A(\zeta, h, \gamma) - \gamma D - \frac{\beta\alpha}{2} h^2 \right).$$

Notice that $\boldsymbol{P}_a^A(\zeta, h, \gamma)$ is increasing in A and converges as $A \to \infty$ to $\boldsymbol{P}_a^\infty(\zeta, h, \gamma) = P_a(\zeta, h, \gamma)$, since $\boldsymbol{X}_{K,a}^\infty$ is a sum as in the explanation that follows Proposition 8. We therefore get

$$\limsup_n F_n^A \leqslant \sup_{a>0, D>0} \inf_{\zeta \in \mathrm{fop}_D, h \in \mathbb{R}_+, \gamma \in \mathbb{R}} \left(P_a(\zeta, h, \gamma) - \gamma D - \frac{\beta\alpha}{2} h^2 \right). \tag{14}$$

Using Lemma 9, we obtain Proposition 11.

Proof of Lemma 12. Let $\zeta \in \mathrm{fop}_D$ be defined as in (10) with the parameters in (9). The basic object in the proof of this lemma is the interpolated Hamiltonian $V_t : \mathbb{R}_+^n \times \mathbb{N}^K \to \mathbb{R}$ defined for $t \in [0, 1]$ as

$$V_t(x, \boldsymbol{i}) = \sqrt{t} \frac{\beta\kappa}{2\sqrt{n}} x^\top W x + t \frac{\beta\alpha}{2} \frac{(x \cdot 1)^2}{n} + \sqrt{1-t}\,(x \cdot \boldsymbol{q}_{\boldsymbol{i}}) + \sqrt{t}\,(1 \cdot \boldsymbol{y}_{\boldsymbol{i}}) + (1-t)\beta\alpha h \left(1 \cdot x\right) \tag{15}$$

along with a RPC $(v_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{N}^K} \sim \mathrm{RPC}_{\boldsymbol{\lambda}}$. Here, the $\mathbb{R}^n$–valued random vectors $\boldsymbol{q}_{\boldsymbol{i}} = \left[q_{\boldsymbol{i},l}\right]_{l=1}^n$ and $\boldsymbol{y}_{\boldsymbol{i}} = \left[y_{\boldsymbol{i},l}\right]_{l=1}^n$ are constructed as follows: the n random processes $(q_{\boldsymbol{i},l})_{\boldsymbol{i} \in \mathbb{N}^K}$ for $l \in [n]$ are n independent copies of the the process $(q_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{N}^K}$ in (11), the n random processes $(y_{\boldsymbol{i},l})_{\boldsymbol{i} \in \mathbb{N}^K}$ are n independent copies of the the process $(y_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{N}^K}$ in (12), and $(\boldsymbol{q}_{\boldsymbol{i}})$, $(\boldsymbol{y}_{\boldsymbol{i}})$, $(v_{\boldsymbol{i}})$ and the matrix W are independent.

The standard way of establishing Lemma 12 with the help of the function $V_t(x, \boldsymbol{i})$ and the RPC $(v_{\boldsymbol{i}})$ is to set

$$\varphi(t) = \frac{1}{n} \mathbb{E} \log \sum_{\boldsymbol{i} \in \mathbb{N}^K} v_{\boldsymbol{i}} \int_{B_+^n(\sqrt{n}A)} \exp V_t(x, \boldsymbol{i})\ \mu_\beta^{\otimes n}(dx),$$

to relate $\varphi(0)$ with the Parisi function and $\varphi(1)$ with the free energy F^A, and to control $\partial_t \varphi(t)$ with the help of the well-known Gaussian Integration by Parts (IP) formula detailed in, *e.g.*, [22, Lemma 1.1] or [30, §1.3]. In the classical contexts of the SK model or the so-called spherical model [28], it holds by construction that the overlap R_{11} is equal to 1 since the replicas live on the sphere with radius $\sqrt{n}$. In this case, controlling $\partial_t \varphi(t)$ with the help of the Gaussian IP formula is a simple matter of book keeping. This property of R_{11} is however not satisfied in our model, and this creates a difficulty which can be circumvented by constaining the replicas to lie on thin spherical shells. This idea was implemented in [21], building on the proof developed in [29]. The paper [23] that we follow here also exploits this idea.

For $D \in (0, A^2)$ and $\varepsilon > 0$ small, denote as $\Delta_\varepsilon(D) \subset \mathbb{R}_+^n$ the set

$$\Delta_\varepsilon(D) = \left\{x \in \mathbb{R}_+^n \ : \ D - \varepsilon < \|x\|^2/n < D + \varepsilon\right\}$$

Associate with this set the free energy

$$F_n^{\Delta_\varepsilon(D)} = \frac{1}{n}\mathbb{E}\log\int_{\Delta_\varepsilon(D)} \exp\left(H_n(x)\right) \, \mu_\beta^{\otimes n}(dx).$$

Consider the Gibbs probability measure on the space $\Delta_\varepsilon(D) \times \mathbb{N}^K$ defined as

$$\Gamma_t^{\Delta_\varepsilon(D)}(dx, \boldsymbol{i}) \ \sim \ v_{\boldsymbol{i}} \exp V_t(x, \boldsymbol{i}) \ \mu_\beta^{\otimes n}(dx),$$

and let $\langle\cdot\rangle_t$ be the expectation w.r.t. $(\Gamma_t^{\Delta_\varepsilon(D)})^{\otimes\infty}$. Define the function

$$\varphi^{\Delta_\varepsilon(D)}(t) = \frac{1}{n}\mathbb{E}\log\sum_{\boldsymbol{i}\in\mathbb{N}^K} v_{\boldsymbol{i}} \int_{\Delta_\varepsilon(D)} \exp V_t(x, \boldsymbol{i}) \ \mu_\beta^{\otimes n}(dx).$$

We have

$$\begin{aligned}
\varphi^{\Delta_\varepsilon(D)}(1) &= F_n^{\Delta_\varepsilon(D)} + \frac{1}{n}\mathbb{E}\log\sum_{\boldsymbol{i}\in\mathbb{N}^K} v_{\boldsymbol{i}} \exp((1\cdot\boldsymbol{y}_{\boldsymbol{i}})) = F_n^{\Delta_\varepsilon(D)} + \frac{1}{2}\sum_{k=0}^{K-1}\lambda_k(\theta(b_{k+1}) - \theta(b_k)) \\
&= F_n^{\Delta_\varepsilon(D)} - \frac{1}{2}\int\theta d\zeta + \frac{\theta(D)}{2}
\end{aligned}$$

by the development that follows Proposition 8. We also have

$$\varphi^{\Delta_\varepsilon(D)}(0) = \frac{1}{n}\mathbb{E}\log\sum_{\boldsymbol{i}\in\mathbb{N}^K} v_{\boldsymbol{i}} \int_{\Delta_\varepsilon(D)} \exp\left((x\cdot\boldsymbol{q}_{\boldsymbol{i}}) + \beta\alpha h\,(1\cdot x)\right)\mu_\beta^{\otimes n}(dx)$$

On the set $\Delta_\varepsilon(D)$, it holds that

$$\forall\gamma\in\mathbb{R}, \quad \gamma(nD - \|x\|^2) \leqslant n|\gamma|\varepsilon,$$

therefore, for an arbitrary $\gamma \in \mathbb{R}$, it holds that

$$\begin{aligned}
\varphi^{\Delta_\varepsilon(D)}(0) &\leqslant \frac{1}{n}\mathbb{E}\log\sum_{\boldsymbol{i}\in\mathbb{N}^K} v_{\boldsymbol{i}} \int_{\Delta_\varepsilon(D)} \exp\left((x\cdot\boldsymbol{q}_{\boldsymbol{i}}) + \beta\alpha h\,(1\cdot x) + n|\gamma|\varepsilon - \gamma nD + \gamma\|x\|^2\right)\mu_\beta^{\otimes n}(dx) \\
&\leqslant -\gamma D + |\gamma|\varepsilon + \frac{1}{n}\mathbb{E}\log\sum_{\boldsymbol{i}\in\mathbb{N}^K} v_{\boldsymbol{i}} \int_{B_+(A\sqrt{n})} \exp\left((x\cdot\boldsymbol{q}_{\boldsymbol{i}}) + \beta\alpha h\,(1\cdot x) + \gamma\|x\|^2\right)\mu_\beta^{\otimes n}(dx) \\
&= -\gamma D + \frac{1}{n}\boldsymbol{X}_{0,\infty}^A(\zeta, h, \gamma) + |\gamma|\varepsilon.
\end{aligned}$$

To establish Guerra's bound, we compute the derivative $\partial_t\varphi^{\Delta_\varepsilon(D)}(t)$ by applying the Gaussian IP formula. We first observe that

$$\begin{aligned}
\partial_t\varphi^{\Delta_\varepsilon(D)}(t) &= \frac{1}{n}\mathbb{E}\left\langle\partial_t V_t(x, \boldsymbol{i})\right\rangle_t \\
&= \frac{1}{n}\mathbb{E}\left\langle\frac{1}{2\sqrt{t}}\frac{\beta\kappa}{2\sqrt{n}}x^\top Wx - \frac{1}{2\sqrt{1-t}}\,(x\cdot\boldsymbol{q}_{\boldsymbol{i}}) + \frac{1}{2\sqrt{t}}\,(1\cdot\boldsymbol{y}_{\boldsymbol{i}})\right\rangle_t + \frac{\beta\alpha}{2}\mathbb{E}\left\langle\frac{(x\cdot 1)^2}{n^2} - 2h\frac{(1\cdot x)}{n}\right\rangle_t.
\end{aligned}$$

We apply the Gaussian IP formula to compute the first term at the right hand side of the last display. To that end, we need to compute

$$\begin{aligned}
U((x, \boldsymbol{i}), (y, \boldsymbol{j})) &= \mathbb{E}\Bigg[\left(\frac{1}{2\sqrt{t}}\frac{\beta\kappa}{2\sqrt{n}}x^\top Wx - \frac{1}{2\sqrt{1-t}}\,(x\cdot\boldsymbol{q}_{\boldsymbol{i}}) + \frac{1}{2\sqrt{t}}\,(1\cdot\boldsymbol{y}_{\boldsymbol{i}})\right)\times \\
&\qquad\qquad \left(\sqrt{t}\frac{\beta\kappa}{2\sqrt{n}}y^\top Wy + \sqrt{1-t}\,(y\cdot\boldsymbol{q}_{\boldsymbol{j}}) + \sqrt{t}\,(1\cdot\boldsymbol{y}_{\boldsymbol{j}})\right)\Bigg] \\
&= \frac{n}{2}\left(\xi\left((x\cdot y)/n\right) - \left((x\cdot y)/n\right)\xi'(b_{\boldsymbol{i}\wedge\boldsymbol{j}}) + \theta(b_{\boldsymbol{i}\wedge\boldsymbol{j}})\right),
\end{aligned}$$

where, writing $\boldsymbol{i} = (i_1, \ldots i_K)$ and $\boldsymbol{j} = (j_1, \ldots, j_K)$, we set $\boldsymbol{i}\wedge\boldsymbol{j} = \max\{l : (i_1, \ldots, i_l) = (j_1, \ldots, j_l)\}$. Remembering that $b_K = D$ since our Parisi measure belongs to fop_D, we then have

$$\begin{aligned}
\partial_t\varphi^{\Delta_\varepsilon(D)}(t) &= \frac{1}{2}\mathbb{E}\left\langle\xi(R_{11}) - R_{11}\xi'(D) + \theta(D)\right\rangle_t - \frac{1}{2}\mathbb{E}\left\langle\xi(R_{12}) - R_{12}\xi'(b_{\boldsymbol{i}\wedge\boldsymbol{j}}) + \theta(b_{\boldsymbol{i}\wedge\boldsymbol{j}})\right\rangle_t \\
&\quad + \frac{\beta\alpha}{2}\mathbb{E}\left\langle\left((x\cdot 1)/n - h\right)^2\right\rangle_t - \frac{\beta\alpha}{2}h^2
\end{aligned}$$

For our function ξ, it holds that $\xi(x)-x\xi'(y)+\xi(y) = 0.5\beta^2\kappa^2(x-y)^2$. Therefore, remembering that $|R_{11}-D| < \varepsilon$ and that $\alpha \leqslant 0$, we obtain that

$$\begin{aligned}(16)\qquad \partial_t\varphi^{\Delta_\varepsilon(D)}(t) &= \frac{\beta^2\kappa^2}{4}\mathbb{E}\left\langle(R_{11}-D)^2\right\rangle_t - \frac{\beta^2\kappa^2}{4}\mathbb{E}\left\langle(R_{12}-b_{\boldsymbol{i}\wedge\boldsymbol{j}})^2\right\rangle_t + \frac{\beta\alpha}{2}\mathbb{E}\left\langle\left((x\cdot 1)/n-h\right)^2\right\rangle_t - \frac{\beta\alpha}{2}h^2 \\ &\leqslant \frac{\beta^2\kappa^2}{4}\varepsilon^2 - \frac{\beta\alpha}{2}h^2.\end{aligned}$$

We therefore have that $\varphi^{\Delta_\varepsilon(D)}(1) \leqslant \varphi^{\Delta_\varepsilon(D)}(0) + \frac{\beta^2\kappa^2}{4}\varepsilon^2 - \frac{\beta\alpha}{2}h^2$, which implies that

$$\begin{aligned}F_n^{\Delta_\varepsilon(D)} &\leqslant -\gamma D + \frac{1}{n}\boldsymbol{X}^A_{0,\infty}(\zeta,h,\gamma) - \frac{\beta\alpha}{2}h^2 + \frac{1}{2}\int\theta d\zeta - \frac{\theta(D)}{2} + \frac{\beta^2\kappa^2}{4}\varepsilon^2 + |\gamma|\varepsilon \\ &= \boldsymbol{P}^A_\infty(\zeta,h,\gamma) - \gamma D - \frac{\beta\alpha}{2}h^2 + \frac{\beta^2\kappa^2}{4}\varepsilon^2 + |\gamma|\varepsilon\end{aligned}$$

for arbitrary $\zeta \in \mathrm{fop}_D$, $h \geqslant 0$ and γ. By the argument of [23, Lemma 3], this implies Lemma 12. $\square$

We now turn to the proof of Lemma 13. Define the distance $\boldsymbol{d}$ on $\mathcal{P}([0,D])$ as

$$\boldsymbol{d}(\zeta,\nu) = \int_0^D |\zeta([0,t]) - \nu([0,t])|\, dt.$$

The following result will be needed.

Lemma 14. Given $\zeta,\tilde{\zeta} \in \mathrm{fop}_D$, it holds that

$$\frac{1}{n}\left|\boldsymbol{X}^A_{0,a}(\zeta,h,\gamma) - \boldsymbol{X}^A_{0,a}(\tilde{\zeta},h,\gamma)\right| \leqslant \beta^2\kappa^2A^2\boldsymbol{d}(\zeta,\tilde{\zeta}).$$

The analogue of this lemma in the context of the SK model is a well known result that dates back to Guerra [12], [13, Theorem 1]. One can also consult [11, Lemma 6.2 and Proposition 6.3] for the proof technique. The adaptations of these results to our context are minor; we only outline the main steps of the proof in Appendix D.

Proof of Lemma 13. To prove this proposition, our first step is to show that $\boldsymbol{X}^A_{0,a}(\zeta,h,\gamma)$ is a convex function. To this end, we start by characterizing $\boldsymbol{X}^A_{0,a}$ through a multi-dimensional Parisi PDE (Partial Differential Equation). Let $\zeta \in \mathrm{fop}_D$. Defining the function $\boldsymbol{g}^A_a : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$\boldsymbol{g}^A_a(v,h,\gamma) = \log\int_{B_+(A\sqrt{n})\cap[0,a]^n} \exp\left((x\cdot v) + \beta\alpha h\,(1\cdot x) + \gamma\|x\|^2\right)\mu_\beta^{\otimes n}(dx),$$

our PDE reads

$$\begin{aligned}&\partial_t\boldsymbol{\Phi}(t,v) + \frac{\xi''(t)}{2}\left(\Delta\boldsymbol{\Phi}(t,v) + \zeta([0,t])\|\nabla\boldsymbol{\Phi}(t,v)\|^2\right) = 0, \quad (t,v) \in (0,D)\times\mathbb{R}^n, \\ &\boldsymbol{\Phi}(D,v) = \boldsymbol{g}^A_a(v,h,\gamma)\end{aligned}$$

where $\nabla\boldsymbol{\Phi}(t,v)$ and $\Delta\boldsymbol{\Phi}(t,v)$ are respectively the gradient of $\boldsymbol{\Phi}(t,v)$ and the Laplacian of $\boldsymbol{\Phi}(t,v)$ with respect to v. Getting back to the recursive construction starting from $\boldsymbol{X}^A_{K,a}$ and leading to $\boldsymbol{X}^A_{0,a}$, and using the so-called Cole-Hopf transformation, we know that

$$\boldsymbol{X}^A_{0,a} = \boldsymbol{\Phi}(0,0).$$

see, *e.g.*, [11, Chapter 6] for a more detailed treatment of this PDE.

In [7] and [15], the PDE solution is given a variational form, which can be verified without difficulty in our case. Define as $\mathcal{A}$ the class of $\mathbb{R}^n$–valued bounded random processes which are progressively measurable on $[0,D]$ with respect to a multi-dimensional $\mathbb{R}^n$–valued Brownian Motion B_t on $[0,D]$. Then, it holds that

$$(17)\qquad \boldsymbol{X}^A_{0,a}(\zeta,h,\gamma) = \sup_{f\in\mathcal{A}}\mathbb{E}\left[-\frac{1}{2}\int_0^1\xi''(t)\zeta([0,t])\|f_t\|^2dt + \boldsymbol{g}^A_a(Z^f_D,h,\gamma)\right],$$

where the $\mathbb{R}^n$–valued random process $(Z^f_t)_{t\in[0,D]}$ solves the SDE

$$dZ^f_t = \xi''(t)\zeta([0,t])f_tdt + \sqrt{\xi''(t)}dB_t.$$

Let us quickly check that $\boldsymbol{g}^A_a(v,h,\gamma)$ is convex on $\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$. Write $u = (v,h,\gamma) \in \mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$. For a given u, let $\langle\cdot\rangle$ be the expectation operator for the probability measure G supported by $B_+(A\sqrt{n})\cap[0,a]^n$, and that satisfies $G(dx) \sim \exp\left((x\cdot v)+\beta\alpha h\,(1\cdot x)+\gamma\|x\|^2\right)\mu_\beta^{\otimes n}(dx)$, and let $r(x) = [x^\top, \beta\alpha\,(1\cdot x), \|x\|^2]^\top \in \mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$. Then, we see that $\nabla\boldsymbol{g}^A_a(u) = \langle r(x)\rangle$ and $\nabla^2\boldsymbol{g}^A_a(u) = \left\langle r(x)r(x)^\top\right\rangle - \langle r(x)\rangle\langle r(x)\rangle^\top \geqslant 0$, which shows that $\boldsymbol{g}^A_a$ is convex. With this at hand, the convexity of $\boldsymbol{X}^A_{0,a}(\zeta,h,\gamma)$ is obtained through a straightforward adaptation of the beginning of the proof of [15, Theorem 20], relying on the variational representation (17) of $\boldsymbol{X}^A_{0,a}(\zeta,h,\gamma)$.

From Lemma 14 in conjunction with the Dominated Convergence Theorem, we obtain that the function $(\zeta,h,\gamma) \mapsto \boldsymbol{X}^A_{0,a}(\zeta,h,\gamma)$ can be continuously extended from $\mathrm{fop}_D\times\mathbb{R}_+\times\mathbb{R}$ to $\mathcal{P}([0,D])\times\mathbb{R}_+\times\mathbb{R}$. Furthermore, by the previous result, this extension is convex on $\mathcal{P}([0,D]) \times \mathbb{R}_+ \times \mathbb{R}$. We still denote as $\boldsymbol{X}^A_{0,a}$ this extension. Similarly, we still denote as $\boldsymbol{P}^A_a$ the extension of $\boldsymbol{P}^A_a$ to $\mathcal{P}([0,D]) \times \mathbb{R}_+ \times \mathbb{R}$, and we write $\boldsymbol{Q}^A_a(\zeta, h, \gamma) = \boldsymbol{P}^A_a(\zeta, h, \gamma) - \beta\alpha h^2/2$. The function $\boldsymbol{Q}^A_a$ is convex on $\mathcal{P}([0,D]) \times \mathbb{R}_+ \times \mathbb{R}$ since the term $\int \theta d\zeta$ is linear and the term $-\beta\alpha h^2/2$ is convex for $\alpha \leqslant 0$. It is moreover continuous.

Given $\zeta \in \mathcal{P}([0,D])$, define the sequence $\boldsymbol{m}^\zeta = \left(\int_0^D x^k \zeta(dx)/k!\right)_{k\geqslant 1}$ which belongs to $\ell^2(\mathbb{N})$. Define the function $S : \mathcal{P}([0,D]) \times \mathbb{R}_+ \times \mathbb{R} \to \ell^2(\mathbb{N})$ as

$$S((\zeta, h, \gamma)) = \left(h, \gamma, \boldsymbol{m}^\zeta\right)$$

where the right hand side is meant to be the $\ell^2(\mathbb{N})$ sequence obtained by preceding the sequence $\boldsymbol{m}^\zeta$ with (h,γ). The function S is an injection from $\mathcal{P}([0,D]) \times \mathbb{R}_+ \times \mathbb{R}$ to the set $\mathcal{D} = S(\mathcal{P}([0,D]) \times \mathbb{R}_+ \times \mathbb{R})$ since each element of $\mathcal{P}([0,D])$ is determined by its moments. Define the function $\mathfrak{Q}^A_a$ as

$$\begin{array}{cccl} \mathfrak{Q}^A_a & : & \ell^2(\mathbb{N}) & \to \mathbb{R} \cup \{\infty\} \\ & & f & \mapsto \begin{cases} \boldsymbol{Q}^A_a(S^{-1}(f)) & \text{if } f \in \mathcal{D} \\ \infty & \text{otherwise.} \end{cases} \end{array}$$

This function is proper. Moreover, it is convex since its domain $\mathcal{D}$ is convex, $S^{-1}(uf_1 + (1-u)f_2) = uS^{-1}(f_1) + (1-u)S^{-1}(f_2)$ for each $u \in [0,1]$, and $\boldsymbol{Q}^A_a$ is convex.

The convergence in $\ell^2(\mathbb{N})$ of the elements of the set $\mathcal{D}_{\mathcal{P}} = \{\boldsymbol{m}^\zeta \ : \ \zeta \in \mathcal{P}([0,D])\}$ is equivalent to the finite dimensional convergence. Furthermore, since $\mathcal{P}([0,D])$ is a compact space and since the narrow convergence in this space is equivalent to the convergence of the moments, the set $\mathcal{D}_{\mathcal{P}}$ is a compact. This implies that $\mathcal{D}$ is a closed subset of $\ell^2(\mathbb{N})$. Given $c \in \mathbb{R}$, let $\mathrm{lev}_{\leqslant c}\, \mathfrak{Q}^A_a$ be the c–level set of $\mathfrak{Q}^A_a$, assumed non-empty, and let (f_k) be a sequence in $\mathrm{lev}_{\leqslant c}\, \mathfrak{Q}^A_a$ that converges to $f \in \ell^2(\mathbb{N})$. Since $\mathcal{D}$ is closed, $f \in \mathcal{D}$. By the continuity of $\boldsymbol{Q}^A_a$, $f \in \mathrm{lev}_{\leqslant c}\, \mathfrak{Q}^A_a$. Thus, $\mathfrak{Q}^A_a$ is a lower semicontinuous function on $\ell^2(\mathbb{N})$ for each $a \in (0,\infty]$.

Write an element $f \in \ell^2(\mathbb{N})$ as $f = (h, \gamma, (m_k)_{k\geqslant 1})$. By the continuity of $\boldsymbol{Q}^A_a$, we have

$$\begin{aligned} - \inf_{\zeta \in \mathrm{fop}_D, h \in \mathbb{R}_+, \gamma \in \mathbb{R}} \boldsymbol{Q}^A_a(\zeta, h, \gamma) - \gamma D &= \sup_{\zeta \in \mathcal{P}([0,D]), h \in \mathbb{R}_+, \gamma \in \mathbb{R}} \gamma D - \boldsymbol{Q}^A_a(\zeta, h, \gamma) \\ &= \sup_{(h,\gamma,(m_k)) \in \ell^2(\mathbb{N})} \gamma D - \mathfrak{Q}^A_a((h, \gamma, (m_k))) \\ &= (\mathfrak{Q}^A_a)^*((0, D, (0, 0, \ldots))), \end{aligned}$$

where $(\mathfrak{Q}^A_a)^*$ is the Fenchel-Legendre transform of $\mathfrak{Q}^A_a$. The family of proper, convex, and lower semicontinuous functions $(\mathfrak{Q}^A_a)_{a\in(0,\infty]}$ is increasing with a and converges in the pointwise sense to $\mathfrak{Q}^A_\infty$. Therefore, this convergence takes place in the so-called Mosco (or epigraphic) sense, see [6] for a detailed account on this convergence. In this situation, it is well-known that $(\mathfrak{Q}^A_a)^* \to_a (\mathfrak{Q}^A_\infty)^*$ in the Mosco sense [20], or equivalently, since $(\mathfrak{Q}^A_a)^*$ is decreasing in a, in a pointwise sense. Consequently,

$$\begin{aligned} \lim_{a\to\infty} \inf_{\zeta \in \mathrm{fop}_D, h \in \mathbb{R}_+, \gamma \in \mathbb{R}} \boldsymbol{Q}^A_a(\zeta, h, \gamma) - \gamma D &= -(\mathfrak{Q}^A_\infty)^*((0, D, (0, 0, \ldots))) \\ &= \inf_{\zeta \in \mathcal{P}([0,D]), h \in \mathbb{R}_+, \gamma \in \mathbb{R}} \boldsymbol{Q}^A_\infty(\zeta, h, \gamma) - \gamma D \\ &= \inf_{\zeta \in \mathrm{fop}_D, h \in \mathbb{R}_+, \gamma \in \mathbb{R}} \boldsymbol{P}^A_\infty(\zeta, h, \gamma) - \gamma D - \beta\alpha h^2/2, \end{aligned}$$

and Lemma 13 is proven. □

7. Proof of Theorem 7: lower bound

The purpose of this section is to establish the following proposition:

Proposition 15. It holds that

$$\liminf_n \widetilde{F}_n \geqslant \sup_{D>0, a>0} \inf_{\zeta \in \mathrm{fop}_D, h \geqslant 0, \gamma \in \mathbb{R}} \left(P_a(\zeta, h, \gamma) - \gamma D - \frac{\beta\alpha}{2} h^2 \right).$$

To prove this proposition, we shall lower bound $\liminf_n F_{a,n}$ for $a > 0$ and use Lemma 10. For $D \in (0, a^2)$ and $\varepsilon > 0$ small, we still define the set $\Delta^\varepsilon_n(D) \subset \mathbb{R}^n_+$ as above, and we circumvent the domain of integration for $F_{a,n}$, working with $F^{\Delta^\varepsilon_n(D)}_{a,n}$ defined as

$$F^{\Delta^\varepsilon_n(D)}_{a,n} = \frac{1}{n} \mathbb{E} \log \mathcal{Z}^{\Delta^\varepsilon_n(D)}_{a,n} \quad \text{with} \quad \mathcal{Z}^{\Delta^\varepsilon_n(D)}_{a,n} = \int_{\Delta^\varepsilon_n(D) \cap [0,a]^n} e^{H_n(x)} \mu_\beta^{\otimes n}(dx).$$

The proof relies on the so-called Aizenman-Sims-Starr (ASS) scheme, which goes as follows in our context. The equations that we will write right away will just serve to set the stage; we shall need to modify them afterwards.

Fixing an integer $m > 0$, we partition a vector $v \in \mathbb{R}_+^{n+m}$ as $v = (u, x)$ with $u \in \mathbb{R}_+^m$ and $x \in \mathbb{R}_+^n$. Since

$$\Delta_{n+m}^\varepsilon(D) \supset \left\{ v = (u,x) \in \mathbb{R}_+^n \,:\, u \in \Delta_m^\varepsilon(D), x \in \Delta_n^\varepsilon(D) \right\},$$

we can write

$$\begin{aligned}
\liminf_n F_{a,n}^{\Delta_n^\varepsilon(D)} &\geqslant \liminf_n \frac{1}{m}\left(\mathbb{E}\log \mathcal{Z}_{a,n+m}^{\Delta_{n+m}^\varepsilon(D)} - \mathbb{E}\log \mathcal{Z}_{a,n}^{\Delta_n^\varepsilon(D)} \right) \\
&\geqslant \liminf_n \frac{1}{m}\Bigg(\mathbb{E}\log \int_{\Delta_m^\varepsilon(D)\cap[0,a]^m} \mu_\beta^{\otimes m}(du) \int_{\Delta_n^\varepsilon(D)\cap[0,a]^n} \mu_\beta^{\otimes n}(dx)\, e^{H_{n+m}(u,x)} \\
&\qquad\qquad - \mathbb{E}\log \int_{\Delta_n^\varepsilon(D)\cap[0,a]^n} e^{H_n(x)} \mu_\beta^{\otimes n}(dx) \Bigg) \\
&\triangleq \liminf_n \chi_{m,n}.
\end{aligned} \tag{18}$$

Adapting a standard derivation to our case, see [22, §1.3], with the small particularity that we need now to consider the term $(\beta\alpha/2)\,(v\cdot 1)^2/(n+m)$ in the expression of $H_{n+m}(v)$, we obtain

$$\begin{aligned}
\chi_{m,n} &= \boldsymbol{\chi}_{m,n} + o_n(1), \quad \text{with} \\
\boldsymbol{\chi}_{m,n} &= \frac{1}{m}\mathbb{E}\log \left\langle \int_{\Delta_m^\varepsilon(D)\cap[0,a]^m} \exp\left((u\cdot Q(x)) + \beta\alpha\,(u\cdot 1_m)\, \frac{(x\cdot 1_n)}{n} \right) \mu_\beta^{\otimes m}(du) \right\rangle' \\
&\quad - \frac{1}{m}\mathbb{E}\log \left\langle \exp\left(\sqrt{m} Y(x) + m\frac{\beta\alpha}{2}\frac{(x\cdot 1_n)^2}{n^2} \right) \right\rangle',
\end{aligned} \tag{19}$$

with the following notations: the expectation $\langle\cdot\rangle'$ is taken with respect to $(G_n')^{\otimes\infty}$, where $G_n'(dx) \in \mathcal{P}(\Delta_n^\varepsilon(D) \cap [0,a]^n)$ is the random probability measure defined as

$$G_n'(dx) \sim \exp\left(H_n'(x)\right) \mu_\beta^{\otimes n}(dx),$$

with $H_n'(x)$ being the Hamiltonian

$$H_n'(x) = \frac{\beta\kappa}{2\sqrt{n+m}} x^\top W_n x + \frac{\beta\alpha}{2}\frac{(x\cdot 1_n)^2}{n+m},$$

and $Q : \mathbb{R}_+^n \to \mathbb{R}^m$ and $Y : \mathbb{R}_+^n \to \mathbb{R}$ are two Gaussian centered processes, independent of W_n, and which probability distributions are defined through the matrix and scalar covariances

$$\begin{aligned}
\mathbb{E}Q(x^1)Q(x^2)^\top &= \beta^2\kappa^2 R_{12} I_m = \xi'(R_{12}) I_m, \text{ and} \\
\mathbb{E}Y(x^1)Y(x^2) &= \frac{\beta^2\kappa^2}{2} R_{12}^2 = \theta(R_{12}),
\end{aligned}$$

with $R_{12} = \left(x^1\cdot x^2\right)/n$.

By adapting the proof of [22, Theorem 1.3], see also [23, Page 878], we obtain that for each fixed $m > 0$, $\boldsymbol{\chi}_{m,n}$ is a continuous functional of the distribution of the couple $\left((R_{i,j})_{i,j\geqslant 1}, \left(\left(x^k\cdot 1\right)/n\right)_{k\geqslant 1} \right)$ of the infinite array of overlaps $(R_{i,j})_{i,j\geqslant 1}$ and the infinite vector $(\left(x^k\cdot 1\right)/n)_{k\geqslant 1}$ under the distribution $\mathbb{E}(G_n')^{\otimes\infty}$. Here the continuity is with the respect to the topology of the narrow convergence of the finite dimensional distributions of $\left((R_{i,j})_{i,j\geqslant 1}, (\left(x^k\cdot 1\right)/n)_{k\geqslant 1} \right)$.

Leaving aside the vector $(\left(x^k\cdot 1\right)/n)_{k\geqslant 1}$ for a moment, the principle of the proof for the limit inferior of the free energy stands as follows. It is usually required that the distribution of the array of overlaps $(R_{i,j})_{i,j\geqslant 1}$ satisfies the celebrated Ghirlanda-Guerra (GG) identities in the large–n limit, so that in this limit, these overlaps can be seen as issued from replicas sampled from a Gibbs measure on a Hilbert space described in terms of a Ruelle probability cascade. Applying one of the important ideas in spin glass theory, the GG identities can be obtained by properly perturbing the Hamiltonian of the measure G_n' without much affecting the free energy [31, Chapter 12], [22, Chapter 3]. In our context, this should be complemented with another idea, dating back to [23]: since our replicas live in a thickening $\Delta_n^\varepsilon(D)$ of a sphere, and not exactly on this sphere, a transformation of these replicas is needed before applying the perturbation on the Hamiltonian in order to obtain the GG identities for large n. It will be the array of overlaps of the transformed replicas, denoted as $(\widetilde{R}_{ij})$ below, that will satisfy the GG identities in the large–n limit.

In our specific model, we also need to manage the vector of empirical means $(\left(x^k\cdot 1\right)/n)_{k\geqslant 1}$, requiring these empirical means to concentrate in the large–n limit. To that end, we shall add a supplementary perturbation to the Hamiltonian of G_n'. This is the main specificity of our proof as regards the lower bound on $F_{a,n}$.

To implement these ideas, we now resume our argument from the beginning by perturbing our Hamiltonian $H_n(x)$. Keeping our $\varepsilon > 0$, let $D_\varepsilon = D\mathbb{1}_{D\geqslant\sqrt{\varepsilon}}$. For $x \in \Delta_n^\varepsilon(D)$, define the function $x \mapsto \tilde{x}$ as

$$\tilde{x} = \begin{cases} \sqrt{\dfrac{D}{\|x\|^2/n}}\, x & \text{if } D \geqslant \sqrt{\varepsilon} \\ 0 & \text{if not.} \end{cases}, \tag{20}$$

in such a way that $\tilde{x}$ lives on the sphere of radius $\sqrt{D_\varepsilon}$. For a given $n > 0$, let $(g_{n,j})_{j\geqslant 1}$ is a sequence of scalar independent centered Gaussian processes on $\mathbb{R}_+^n$ such that $\mathbb{E} g_{n,j}(x^1) g_{n,j}(x^2) = \left(\left(x^1 \cdot x^2\right)/n\right)^j$, see [22, §3.2] for the construction of such processes. Writing

$$\eta_n(x) = \frac{(x \cdot 1)}{n},$$

our perturbed version H_n^{pert} of the Hamiltonian H_n will take the form

$$H_n^{\text{pert}}(x) = H_n(x) + n^\varrho \sum_{j\geqslant 1} (2a)^{-j} w_j g_{n,j}(\tilde{x}) + n^\delta s \eta_n(x),$$

where $\varrho, \delta > 0$, the elements of the sequence $(w_j)_{j\geqslant 1}$ take their values in the interval $[0, 3]$, and $s \in [0, 3]$. Let

$$G_{a,n}^{\text{pert},\Delta_n^\varepsilon(D)}(dx) = \frac{e^{H_n^{\text{pert}}(x)}}{\mathcal{Z}_{a,n}^{\text{pert},\Delta_n^\varepsilon(D)}} \mu_\beta^{\otimes n}(dx) \in \mathcal{P}(\Delta_n^\varepsilon(D) \cap [0,a]^n), \quad \text{with}$$

$$\mathcal{Z}_{a,n}^{\text{pert},\Delta_n^\varepsilon(D)} = \int_{\Delta_n^\varepsilon(D)\cap[0,a]^n} e^{H_n^{\text{pert}}(x)} \mu_\beta^{\otimes n}(dx)$$

be the Gibbs measure constructed from this Hamiltonian, and let $\langle\cdot\rangle$ be the mean with respect to $(G_{a,n}^{\text{pert},\Delta_n^\varepsilon(D)})^{\otimes\infty}$. We also define the overlaps $\widetilde{R}_{ij} = \left(\tilde{x}^i \cdot \tilde{x}^j\right)/n$, where x^i and x^j are two replicas under $(G_{a,n}^{\text{pert},\Delta_n^\varepsilon(D)})^{\otimes\infty}$, and where $\tilde{x}^i$ and $\tilde{x}^j$ are the transformations of x^i and x^j by (20) respectively.

Lemma 16. Assume that $\varrho \in (1/4, 1/2)$ and $\delta \in (1/2, 1)$. Then, the free energy $F_{a,n}^{\text{pert},\Delta_n^\varepsilon(D)} = n^{-1}\mathbb{E}\log\mathcal{Z}_{a,n}^{\text{pert},\Delta_n^\varepsilon(D)}$ satisfies

$$F_{a,n}^{\Delta_n^\varepsilon(D)} - F_{a,n}^{\text{pert},\Delta_n^\varepsilon(D)} \xrightarrow[n\to\infty]{} 0. \tag{21}$$

Assume now that $(s, w_1, w_2, \ldots)$ is a sequence of i.i.d. random variables distributed uniformly on the interval $[1, 2]$ and independent of W_n, and denote as $\mathbb{E}_{s,w}$ the expectation with respect to this sequence. For each integers $k \geqslant 2$ and $p \geqslant 1$ and each bounded measurable function $f = f((\widetilde{R}_{ij})_{1\leqslant i,j\leqslant k})$ of the overlaps $(\widetilde{R}_{ij})_{1\leqslant i,j\leqslant k}$, it holds that

$$\mathbb{E}_{s,w}\left|\mathbb{E}\left\langle f\widetilde{R}_{1,k+2}^p\right\rangle - \frac{1}{k}\mathbb{E}\langle f\rangle\mathbb{E}\left\langle\widetilde{R}_{12}^p\right\rangle - \frac{1}{k}\sum_{i=2}^k \mathbb{E}\left\langle f\widetilde{R}_{1,i}^p\right\rangle\right| \xrightarrow[n\to\infty]{} 0.$$

Finally, writing $\eta = (x \cdot 1)/n$ with x being distributed as $G_{a,n}^{\text{pert},\Delta_n^\varepsilon(D)}$, it holds that

$$\mathbb{E}_{s,w}\mathbb{E}\langle|\eta - \mathbb{E}\langle\eta\rangle|\rangle \xrightarrow[n\to\infty]{} 0.$$

Proof. Writing $g(\tilde{x}) = \sum_{j\geqslant 1}(2a)^{-j} w_j g_{n,j}(\tilde{x})$, we have by Jensen's inequality

$$\mathbb{E}\log\frac{\int\exp(H + n^\varrho g)d\mu_\beta^{\otimes n}}{\int\exp(H)d\mu_\beta^{\otimes n}} \geqslant \mathbb{E}\frac{\int n^\varrho g\exp(H)d\mu_\beta^{\otimes n}}{\int\exp(H)d\mu_\beta^{\otimes n}} = 0,$$

therefore,

$$F_{a,n}^{\Delta_n^\varepsilon(D)} = \frac{1}{n}\mathbb{E}\log\mathcal{Z}_{a,n}^{\Delta_n^\varepsilon(D)} \leqslant \frac{1}{n}\mathbb{E}\log\int\exp(H + n^\varrho g)d\mu_\beta^{\otimes n} \leqslant \frac{1}{n}\mathbb{E}\log\int\exp(H + n^\varrho g + n^\delta s\eta)d\mu_\beta^{\otimes n} = F_{a,n}^{\text{pert},\Delta_n^\varepsilon(D)}.$$

By Jensen's inequality involving this time the expectation with respect to the law of the process g, we also have

$$\begin{aligned} F_{a,n}^{\text{pert},\Delta_n^\varepsilon(D)} &\leqslant 3an^{\delta-1} + \frac{1}{n}\mathbb{E}\log\int\exp(H + n^\varrho g)d\mu_\beta^{\otimes n} \leqslant 3an^{\delta-1} + \frac{1}{n}\mathbb{E}\log\int\exp(H)\exp(n^{2\varrho}\mathbb{E}g^2/2)d\mu_\beta^{\otimes n} \\ &\leqslant 3an^{\delta-1} + 1.5n^{2\varrho-1} + F_{a,n}^{\Delta_n^\varepsilon(D)}, \end{aligned}$$

hence the convergence (21).

The second convergence result is obtained by a straightforward adaptation of the proof of [22, Theorem 3.2] towards dealing with the overlaps $\widetilde{R}_{ij}$. Here, the replacement of x with $\tilde{x}$ in the expression of $g(\tilde{x})$ plays an important role.

To establish the last convergence, we actually follow the same canvas as for the proof of [22, Theorem 3.2]. Recall the presence of the term $n^\delta s\eta_n$ in the expression of $H_n^{\rm pert}$ above, and define the function $s \mapsto \varphi(s) = \log \mathcal{Z}_{a,n}^{{\rm pert},\Delta_n^\varepsilon(D)}$. It is clear that $\varphi'(s) = n^\delta \langle \eta \rangle$ and $\varphi''(s) = n^{2\delta}\left(\langle \eta^2\rangle - \langle \eta\rangle^2\right) \geqslant 0$. With this at hand, we have

$$n^{2\delta}\int_1^2 \mathbb{E}\left\langle (\eta - \langle\eta\rangle)^2\right\rangle ds = \mathbb{E}\int_0^1 \varphi''(s)ds = \mathbb{E}\varphi'(2) - \mathbb{E}\varphi'(1) \leqslant \mathbb{E}\varphi'(2) \leqslant n^\delta a.$$

It therefore holds that

$$\int_1^2 \mathbb{E}\left\langle |\eta - \langle\eta\rangle|\right\rangle ds \leqslant \frac{\sqrt{a}}{n^{\delta/2}}.$$

We now bound

$$\mathbb{E}|\langle\eta\rangle - \mathbb{E}\langle\eta\rangle| = \frac{\mathbb{E}|\varphi'(s) - \mathbb{E}\varphi'(s)|}{n^\delta}.$$

Here, we shall need the qantitative version of the so-called Griffith lemma, given by [22, Lemma 3.2]. Namely, if $f, \boldsymbol{f} : \mathbb{R} \to \mathbb{R}$ are two convex differentiable functions, then, for any $\epsilon > 0$, it holds that

$$|f'(s) - \boldsymbol{f}'(s)| \leqslant \boldsymbol{f}'(s+\epsilon) - \boldsymbol{f}'(s-\epsilon) + \frac{|f(s+\epsilon) - \boldsymbol{f}(s+\epsilon)| + |f(s-\epsilon) - \boldsymbol{f}(s-\epsilon)| + |f(s) - \boldsymbol{f}(s)|}{\epsilon}.$$

We shall use this result with $f = \varphi$ and $\boldsymbol{f} = \mathbb{E}\varphi$. Observing that $\mathbb{E}(H(x) + n^\varrho g(\tilde{x}))^2 \leqslant nR_{11} + 3n^{2\varrho}$, we obtain by Gaussian concentration [22, Theorem 1.2] that

$$\sup\{\mathbb{E}|\varphi(v) - \mathbb{E}\varphi(v)| \ : \ s, w_1, w_2, \ldots \in [0,3]\} \leqslant C\sqrt{n}.$$

Therefore, since φ is an^δ–Lipschitz, we have

$$\begin{aligned}
\int_1^2 \mathbb{E}|\langle\eta\rangle - \mathbb{E}\langle\eta\rangle| ds &= n^{-\delta}\int_1^2 \mathbb{E}|\varphi'(s) - \mathbb{E}\varphi'(s)|ds \\
&\leqslant n^{-\delta}\int_1^2 \mathbb{E}\varphi'(s+\epsilon)ds - \int_1^2 \mathbb{E}\varphi'(s-\epsilon)ds + C\frac{\sqrt{n}}{\epsilon n^\delta} \\
&= \frac{\mathbb{E}\varphi(2+\epsilon) - \mathbb{E}\varphi(2-\epsilon) - (\mathbb{E}\varphi(1+\epsilon) - \mathbb{E}\varphi(1-\epsilon))}{n^\delta} + C\frac{\sqrt{n}}{\epsilon n^\delta} \\
&\leqslant 2a\epsilon + C\frac{\sqrt{n}}{\epsilon n^\delta}.
\end{aligned}$$

Taking $\epsilon = n^{1/4-\delta/2}$, we obtain our last convergence. ☐

We now follow the approach of [23] by pointing out the specificity of our model related with the presence of the empirical means of the replicas. Take $\varrho \in (1/4, 1/2)$ and $\delta \in (1/2, 1)$. Then, the convergence (21) holds true by the previous lemma. Recall the expression of $H_n'(x)$, let $G_{a,n}^{{\rm pert}',\Delta_n^\varepsilon(D)}$ be the Gibbs measure constructed from the Hamiltonian

$$H_n^{{\rm pert}'}(x) = H_n'(x) + n^\varrho \sum_{j\geqslant 1}(2a)^{-j}w_j g_{n,j}(\tilde{x}) + n^\delta s\eta_n(x),$$

and denote as $\langle\cdot\rangle'$ the expectation with respect to $(G_{a,n}^{{\rm pert}',\Delta_n^\varepsilon(D)})^{\otimes\infty}$. Define the overlaps $\widetilde{R}_{ij} = \left(\tilde{x}^i \cdot \tilde{x}^j\right)/n$, where x^i and x^j are two replicas under $(G_{a,n}^{{\rm pert}',\Delta_n^\varepsilon(D)})^{\otimes\infty}$, and where $\tilde{x}^i$ and $\tilde{x}^j$ are the transformations of x^i and x^j by (20) respectively. For each integers $k \geqslant 2$ and $p \geqslant 1$ and each bounded measurable function $f = f((\widetilde{R}_{ij})_{1\leqslant i,j\leqslant k})$, write

$$E(f,k,p) = \mathbb{E}\left\langle f\widetilde{R}_{1,k+2}^p\right\rangle - \frac{1}{k}\mathbb{E}\langle f\rangle\mathbb{E}\left\langle \widetilde{R}_{12}^p\right\rangle - \frac{1}{k}\sum_{i=2}^k \mathbb{E}\left\langle f\widetilde{R}_{1,i}^p\right\rangle.$$

Then, by a slight modification of the proof of [22, Lemma 3.3] based on Lemma 16 above, we can show that for each n, there exists a deterministic sequence $(s^n, w_1^n, w_2^n, \ldots)$ entering the construction of $H_n^{{\rm pert}'}$, and such that

$$\begin{aligned}
&E(f,k,p) \xrightarrow[n\to\infty]{} 0 \quad \text{for each } k \geqslant 2, p \geqslant 1, \text{ and monomial } f = f((\widetilde{R}_{ij})_{1\leqslant i,j\leqslant k}), \text{ and} \\
&\mathbb{E}\langle|\eta - \mathbb{E}\langle\eta\rangle|\rangle \xrightarrow[n\to\infty]{} 0.
\end{aligned} \tag{22}$$

Furthermore, similarly to what we obtained in Equations (18) and (19) above through the ASS scheme, it holds that

$$\liminf_n F_{a,n}^{{\rm pert},\Delta_n^\varepsilon(D)} \geqslant \liminf_n \boldsymbol{\chi}_{m,n}^{\rm pert}, \tag{23}$$

where $\boldsymbol{\chi}_{m,n}^{\rm pert}$ has the same expression as $\boldsymbol{\chi}_{m,n}$ in (19) with the difference that $\langle\cdot\rangle'$ is now the expectation with respect to $(G_{a,n}^{{\rm pert}',\Delta_n^\varepsilon(D)})^{\otimes\infty}$. In the remainder, the sequence $(s^n, w_1^n, w_2^n, \ldots)$ is chosen for each n in such a way that all these properties are satisfied.

As a next step, we need to replace the processes $Q(x)$ and $Y(x)$ in the expression of $\chi_{m,n}^{\rm pert}$ with $Q(\tilde{x})$ and $Y(\tilde{x})$ respectively. Writing

$$\tilde{\chi}_{m,n}^{\rm pert} = \frac{1}{m}\mathbb{E}\log\left\langle \int_{\Delta_m^\varepsilon(D)\cap[0,a]^m} \exp\left((u\cdot Q(\tilde{x})) + \beta\alpha\,(u\cdot 1_m)\ \frac{(x\cdot 1_n)}{n}\right)\mu_\beta^{\otimes m}(du)\right\rangle' - \frac{1}{m}\mathbb{E}\log\left\langle \exp\left(\sqrt{m}Y(\tilde{x}) + m\frac{\beta\alpha}{2}\frac{(x\cdot 1_n)^2}{n^2}\right)\right\rangle',$$

it is true that

$$\left|\tilde{\chi}_{m,n}^{\rm pert} - \chi_{m,n}^{\rm pert}\right| \leqslant C\varepsilon^{1/4}, \tag{24}$$

see [23, §6], which allows us to focus the lower bound analysis on $\tilde{\chi}_{m,n}^{\rm pert}$. This quantity is a continuous functional of the distribution of the couple $\left((\widetilde{R}_{i,j})_{i,j\geqslant 1}, \left(\left(x^k\cdot 1\right)/n\right)_{k\geqslant 1}\right)$, under the distribution $\mathbb{E}(G_{a,n}^{{\rm pert}',\Delta_n^\varepsilon(D)})^{\otimes\infty}$, in the topology of the narrow convergence of the finite dimensional distributions of $\left((\widetilde{R}_{i,j})_{i,j\geqslant 1}, \left(\left(x^k\cdot 1\right)/n\right)_{k\geqslant 1}\right)$.

Let $\zeta \in {\rm fop}_D$ be defined as in (9) and (10), and let $(v_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{N}^K} \sim {\rm RPC}_{\boldsymbol{\lambda}}$. Consider the two Gaussian processes $(q_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{N}^K}$ and $(y_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{N}^K}$ independent of $(v_{\boldsymbol{i}})$ and such that

$$\mathbb{E}q_{\boldsymbol{i}^1}q_{\boldsymbol{i}^2} = \xi'(b_{\boldsymbol{i}^1\wedge\boldsymbol{i}^2}) \quad\text{and}\quad \mathbb{E}y_{\boldsymbol{i}^1}y_{\boldsymbol{i}^2} = \theta(b_{\boldsymbol{i}^1\wedge\boldsymbol{i}^2}).$$

Given an integer $m > 0$, let $(\boldsymbol{q}_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{N}^K}$ be a $\mathbb{R}^m$–valued Gaussian process made of m independent copies of $(q_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{N}^K}$. Define the functions

$$f_m^1(\Delta_m^\varepsilon(D), a, \zeta, h) = \frac{1}{m}\mathbb{E}\log\sum_{\boldsymbol{i}\in\mathbb{N}^K} v_{\boldsymbol{i}}\int_{\Delta_m^\varepsilon(D)\cap[0,a]^m}\exp\left((u\cdot\boldsymbol{q}_{\boldsymbol{i}}) + \beta\alpha h\,(u\cdot 1_m)\right)\mu_\beta^{\otimes m}(du), \quad\text{and}$$

$$f^2(\zeta) = \frac{1}{m}\mathbb{E}\log\sum_{\boldsymbol{i}\in\mathbb{N}^K} v_{\boldsymbol{i}}\exp\left(\sqrt{m}y_{\boldsymbol{i}}\right) \stackrel{f^2}{=} \frac{1}{2}\sum_{k=0}^{K-1}\lambda_k(\theta(b_{k+1}) - \theta(b_k)),$$

where the identity $\stackrel{f^2}{=}$ is obtained by Proposition 8. We also define

$$\Phi_a(\zeta, h, \gamma) = \mathbb{E}\log\sum_{\boldsymbol{i}\in\mathbb{N}^K} v_{\boldsymbol{i}}\int_0^a \exp\left(uq_{\boldsymbol{i}} + \beta\alpha h u + \gamma u^2\right)\mu_\beta(du).$$

We have the two following lemmas:

Lemma 17. $\liminf_m f_m^1(\Delta_m^\varepsilon(D), a, \zeta, h) \geqslant \inf_\gamma\left(\Phi_a(\zeta, h, \gamma) - \gamma D\right)$.

This lemma is proven by a straightforward adaptation of the proof of [23, Lemma 6].

Lemma 18. It holds that

$$\left|f_m^1(\Delta_m^\varepsilon(D), a, \zeta, h) - f_m^1(\Delta_m^\varepsilon(D), a, \tilde{\zeta}, \tilde{h})\right| \leqslant C\left(\boldsymbol{d}(\zeta,\tilde{\zeta}) + |h-\tilde{h}|\right).$$

for each measures $\zeta, \tilde{\zeta} \in {\rm fop}_D$ and reals $h, \tilde{h} \geqslant 0$, where $C > 0$ is independent of m. Moreover,

$$\left|f^2(\zeta) - f^2(\tilde{\zeta})\right| \leqslant C\boldsymbol{d}(\zeta,\tilde{\zeta}).$$

Sketch of proof. The second bound is well-known. The proof for the first bound is a slight modification of the proof of [24, Lemma 7] that we succinctly explain. We can assume without generality loss that the two measures ζ and $\tilde{\zeta}$ take the forms that follow (28) in the proof of Lemma 14 below. Let $(\boldsymbol{q}_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{N}^K}$ be the $\mathbb{R}^m$–valued Gaussian process constructed from ζ as in the definition of $f_m^1(\Delta_m^\varepsilon(D), a, \zeta, h)$ above, and let $(\tilde{\boldsymbol{q}}_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{N}^K}$ a $\mathbb{R}^m$–valued Gaussian process independent of $(\boldsymbol{q}_{\boldsymbol{i}})$ and $(v_{\boldsymbol{i}})$, and constructed from $\tilde{\zeta}$ similarly to $(\boldsymbol{q}_{\boldsymbol{i}})$. For $t \in [0,1]$, define the process $(\boldsymbol{q}_{\boldsymbol{i}}(t))_{\boldsymbol{i}\in\mathbb{N}^K}$ as $\boldsymbol{q}_{\boldsymbol{i}}(t) = \sqrt{t}\boldsymbol{q}_{\boldsymbol{i}} + \sqrt{1-t}\tilde{\boldsymbol{q}}_{\boldsymbol{i}}$. Also let $h(t) = th + (1-t)\tilde{h}$, and let $G_t(\boldsymbol{i}, dx) \in \mathcal{P}(\mathbb{N}^K \times (\Delta_m^\varepsilon(D)\cap[0,a]^m))$ be the Gibbs measure $G_t(\boldsymbol{i}, du) \sim v_{\boldsymbol{i}}\exp\left((u\cdot\boldsymbol{q}_{\boldsymbol{i}}(t)) + \beta\alpha h(t)\,(u\cdot 1_m)\right)\mu_\beta^{\otimes m}(du)$ with mean $\langle\cdot\rangle_t$. Define

$$\varphi(t) = \frac{1}{m}\mathbb{E}\log\sum_{\boldsymbol{i}\in\mathbb{N}^K} v_{\boldsymbol{i}}\int_{\Delta_m^\varepsilon(D)\cap[0,a]^m}\exp\left((u\cdot\boldsymbol{q}_{\boldsymbol{i}}(t)) + \beta\alpha h(t)\,(u\cdot 1_m)\right)\mu_\beta^{\otimes m}(du),$$

in such a way that $f_m^1(\Delta_m^\varepsilon(D), \zeta, h) = \varphi(1)$ and $f_m^1(\Delta_m^\varepsilon(D), \tilde{\zeta}, \tilde{h}) = \varphi(0)$. We have

$$\varphi'(t) = \frac{1}{2m}\mathbb{E}\left\langle\frac{(u\cdot\boldsymbol{q}_{\boldsymbol{i}})}{\sqrt{t}} - \frac{(u\cdot\tilde{\boldsymbol{q}}_{\boldsymbol{i}})}{\sqrt{1-t}}\right\rangle + \frac{\beta\alpha}{m}(h-\tilde{h})\mathbb{E}\left\langle(u\cdot 1_m)\right\rangle.$$

The first term is treated by the IP formula as in the proof of [24, Lemma 7], and leads to the $\boldsymbol{d}(\zeta,\tilde{\zeta})$ term in the statement. The second term is bounded by $a\beta|\alpha||h-\tilde{h}|$. $\square$

We can now finish the proof of Proposition 15. Recalling the convergence (21) and the bounds (23) and (24), we obtain that

$$\liminf_n F_{a,n}^{\Delta_n^\varepsilon(D)} = \liminf_n F_{a,n}^{\mathrm{pert},\Delta_n^\varepsilon(D)} \geqslant \liminf_m \liminf_n \tilde{\boldsymbol{\chi}}_{m,n}^{\mathrm{pert}} - C\varepsilon^{1/4}.$$

We now apply the well-known theory detailed in, *e.g.*, [22, §3.6], as regards the treatment of the $\liminf_n$ at the right hand side of this inequality. For a given m, consider a sub-sequence of (n) converging to infinity, along which $\tilde{\boldsymbol{\chi}}_{m,n}^{\mathrm{pert}}$ converges to its limit inferior in n, and the couple $\left((\widetilde{R}_{i,j})_{i,j\geqslant 1}, (\left(x^k\cdot 1\right)/n)_{k\geqslant 1}\right)$ converges in distribution under $\mathbb{E}(G_{a,n}^{\mathrm{pert}',\Delta_n^\varepsilon(D)})^{\otimes\infty}$. By the convergences (22), the limit distribution of the array $(\widetilde{R}_{i,j})_{i,j\geqslant 1}$ satisfies the GG identities, and the replicas $\left(x^k\cdot 1\right)/n$ converge in probability towards a deterministic number $h^m \in [0,\sqrt{D}]$. Along this sub-sequence that we re-denote as (n), we have

$$\begin{aligned}\lim_n \tilde{\boldsymbol{\chi}}_{m,n}^{\mathrm{pert}} &= \lim_n \frac{1}{m}\mathbb{E}\log\left\langle \int_{\Delta_m^\varepsilon(D)\cap[0,a]^m} \exp\left((u\cdot Q(\tilde{x})) + \beta\alpha h^m\left(u\cdot 1_m\right)\right)\mu_\beta^{\otimes m}(du)\right\rangle' \\ &\quad - \lim_n \frac{1}{m}\mathbb{E}\log\left\langle \exp\left(\sqrt{m}Y(\tilde{x})\right)\right\rangle' - (h^m)^2\frac{\beta\alpha}{2}.\end{aligned}$$

Furthermore, it is known that we can approximate the limit distribution of the overlap $\widetilde{R}_{12}$ with a measure $\zeta^m \in \mathrm{fop}_D$. By absorbing the approximation error into, *e.g.*, the small number ε, we obtain that

$$\liminf_n \tilde{\boldsymbol{\chi}}_{m,n}^{\mathrm{pert}} \geqslant f_m^1(\Delta_m^\varepsilon(D), a, \zeta^m, h^m) - f^2(\zeta^m) - \frac{\beta\alpha}{2}(h^m)^2 - \varepsilon.$$

Now, consider a sub-sequence of (m) converging to infinity along which h^m converges to a real number h^∞, ζ^m converges narrowly, and $f_m^1(\Delta_m^\varepsilon(D), a, \zeta^m, h^m) - f^2(\zeta^m) - \frac{\beta\alpha}{2}(h^m)^2$ converges to its limit inferior. By Lemma 18, there exists $\zeta^\infty \in \mathrm{fop}_D$ such that

$$\begin{aligned}\liminf_m f_m^1(\Delta_m^\varepsilon(D), a, \zeta^m, h^m) - f^2(\zeta^m) - \frac{\beta\alpha}{2}(h^m)^2 &\geqslant \liminf_m f_m^1(\Delta_m^\varepsilon(D), a, \zeta^\infty, h^\infty) - f^2(\zeta^\infty) - \frac{\beta\alpha}{2}(h^\infty)^2 - \varepsilon \\ &\geqslant \inf_\gamma \left(\Phi_a(\zeta^\infty, h^\infty, \gamma) - \gamma D\right) - f^2(\zeta^\infty) - \frac{\beta\alpha}{2}(h^\infty)^2 - \varepsilon \\ &= \inf_{\gamma\in\mathbb{R}} \left(P_a(\zeta^\infty, h^\infty, \gamma) - \gamma D\right) - \frac{\beta\alpha}{2}(h^\infty)^2 - \varepsilon,\end{aligned}$$

where the second inequality is due to Lemma 17. This leads to the bound

$$\liminf_n F_{a,n}^{\Delta_n^\varepsilon(D)} \geqslant \inf_{\zeta\in\mathrm{fop}_D, h\geqslant 0, \gamma\in\mathbb{R}} \left(P_a(\zeta, h, \gamma) - \gamma D - \frac{\beta\alpha}{2}h^2\right) - C\varepsilon^{1/4}.$$

We therefore have $\liminf F_{a,n} \geqslant \sup_D \liminf F_{a,n}^{\Delta_n^\varepsilon(D)} \geqslant \sup_D \inf_{\zeta\in\mathrm{fop}_D, h\geqslant 0,\gamma} \left(P_a(\zeta,h,\gamma) - \gamma D - \beta\alpha h^2/2\right) - C\varepsilon^{1/4}$. Since ε is arbitrary, we obtain that $\liminf F_{a,n} \geqslant \sup_D \inf_{\zeta\in\mathrm{fop}_D, h\geqslant 0,\gamma} \left(P_a(\zeta,h,\gamma) - \gamma D - \beta\alpha h^2/2\right)$, and Proposition 15 follows from Lemma 10.

Theorem 7 for $\alpha \leqslant 0$ results from Propositions 11 and 15.

8. Proof of Theorem 7 for $\alpha > 0$.

We still use the notation $\eta_n(x) = \left(x\cdot 1\right)/n$. The term $\beta\alpha n\eta_n(x)^2/2$ in the Hamiltonian $H_n(x)$ is reminiscent of the ferro-magnetic interaction that appears in, *e.g.*, the Curie-Weiss model. The proof principle for dealing with this term when $\alpha > 0$ is well known, and can be found in [10] in the SK case with ferro-magnetic interaction.

Lemma 19. Given a real number $h \in \mathbb{R}$, consider the Hamiltonian with external field $H_n^{\mathrm{EF}}(\cdot; h)$ defined on $\mathbb{R}_+^n$ as

$$H_n^{\mathrm{EF}}(x;h) = \frac{\beta\kappa}{2\sqrt{n}} x^\top W x + \beta\alpha h n\,\eta_n(x).$$

Given $a, A > 0$, define the free energies $F_n^{\mathrm{EF},A}(h)$ and $F_{a,n}^{\mathrm{EF}}(h)$ as

$$\begin{aligned}F_n^{\mathrm{EF},A}(h) &= \frac{1}{n}\mathbb{E}\log\int_{B_+^n(\sqrt{n}A)} \exp\left(H_n^{\mathrm{EF}}(x;h)\right)\mu_\beta^{\otimes n}(dx), \quad \text{and} \\ F_{a,n}^{\mathrm{EF}}(h) &= \frac{1}{n}\mathbb{E}\log\int_{[0,a]^n} \exp\left(H_n^{\mathrm{EF}}(x;h)\right)\mu_\beta^{\otimes n}(dx).\end{aligned}$$

Then,

$$\limsup_n F_n^{\mathrm{EF},A}(h) \leqslant \sup_{a,D>0} \inf_{\zeta\in\mathrm{fop}_D,\gamma\in\mathbb{R}} \left(P_a(\zeta,h,\gamma) - \gamma D\right), \quad \text{and} \tag{25}$$

$$\lim_n F_{a,n}^{\mathrm{EF}}(h) = \sup_{D>0} \inf_{\zeta\in\mathrm{fop}_D,\gamma\in\mathbb{R}} \left(P_a(\zeta,h,\gamma) - \gamma D\right). \tag{26}$$

Proof. We can write

$$F_n^{\mathrm{EF},A}(h) = \frac{1}{n}\mathbb{E}\log\int_{B_+^n(\sqrt{n}A)} \exp\left(\frac{\beta\kappa}{2\sqrt{n}}x^\top Wx\right)\nu_{\beta,h}^{\otimes n}(dx),$$

with $\nu_{\beta,h}$ being the positive measure on $\mathbb{R}_+$ defined as

$$\nu_{\beta,h}(dx_1) = x_1^{\phi\beta-1}\exp\left(-\beta x_1^2/2 + \beta(1+\alpha h)x_1\right)\,dx_1.$$

Therefore, we can apply to this free energy the development of Section 6 after replacing the measure μ_β with $\nu_{\beta,h}$, and considering that $\alpha = 0$ in the Hamiltonian. This leads to the bound (25) which is the analogue of (14). We stress that in (25), we take the infimum over $\zeta \in \mathrm{fop}_D$ and $\gamma \in \mathbb{R}$ only because there is no term in $\eta_n(x)^2$ in the Hamiltonian $H^{\mathrm{EF}}(\cdot;h)$.

To establish the convergence (26), write

$$F_{a,n}^{\mathrm{EF}}(h) = \frac{1}{n}\mathbb{E}\log\int_{[0,a]^n} \exp\left(\frac{\beta\kappa}{2\sqrt{n}}x^\top Wx\right)\nu_{\beta,h}^{\otimes n}(dx),$$

and use [23, Theorem 1]. □

We shall work with $\liminf \widetilde{F}_n$ then with $\limsup \widetilde{F}_n$.

8.1. **Lower bound.** Considering the free energy $F_{a,n}$ used in the statement of Lemma 10, and using that $\eta(x)^2 \geqslant 2h\eta(x) - h^2$ for an arbitrary $h \in \mathbb{R}$, we have

$$F_{a,n} = \frac{1}{n}\mathbb{E}\log\int_{[0,a]^n} \exp\left(\frac{\beta\kappa}{2\sqrt{n}}x^\top Wx + \frac{\beta\alpha n}{2}\eta(x)^2\right)\mu_\beta^{\otimes n}(dx) \geqslant F_{a,n}^{\mathrm{EF}}(h) - \frac{\beta\alpha}{2}h^2.$$

By the previous lemma,

$$\liminf F_{a,n} \geqslant \sup_{h,D>0}\ \inf_{\zeta\in\mathrm{fop}_D,\gamma\in\mathbb{R}}\left(P_a(\zeta,h,\gamma) - \gamma D - \frac{\beta\alpha}{2}h^2\right),$$

and we obtain by Lemma 10 that

$$\liminf \widetilde{F}_n \geqslant \sup_{a,h,D>0}\ \inf_{\zeta\in\mathrm{fop}_D,\gamma\in\mathbb{R}}\left(P_a(\zeta,h,\gamma) - \gamma D - \frac{\beta\alpha}{2}h^2\right).$$

8.2. **Upper bound.** Write the free energy F_n^A in the statement of Lemma 9 as $F_n^A = \mathbb{E}\mathcal{X}_n^A$ where $\mathcal{X}_n^A$ is the so-called free energy density. Given a measurable bounded set $\mathcal{S} \subset \mathbb{R}_+^n$, write

$$F_n^{\mathcal{S}} = \frac{1}{n}\mathbb{E}\log\int_{\mathcal{S}} e^{H_n(x)}\mu_\beta^{\otimes n}(dx) \quad\text{and}\quad \mathcal{X}_n^{\mathcal{S}} = \frac{1}{n}\log\int_{\mathcal{S}} e^{H_n(x)}\mu_\beta^{\otimes n}(dx)$$

(in such a way that $F_n^A = F_n^{B_+^n(\sqrt{n}A)}$ and $\mathcal{X}_n^A = \mathcal{X}_n^{B_+^n(\sqrt{n}A)}$). Fix a large number $N > 0$ independently of n. For an integer $i > 0$, consider the ℓ^1 ring $\mathcal{R}_i \subset \mathbb{R}_+^n$ defined as

$$\mathcal{R}_i = \{x \in \mathbb{R}_+^n \ :\ (i-1)/N < \eta_n(x) \leqslant i/N\}.$$

By Cauchy-Schwarz, each $x \in B_+^n(\sqrt{n}A)$ satisfies $\eta_n(x) \leqslant A$. Therefore, writing $M = \lceil NA\rceil$, we have that $\mathcal{R}_1 \cap B_+^n(\sqrt{n}A), \ldots, \mathcal{R}_M \cap B_+^n(\sqrt{n}A)$ is a partition of $B_+^n(\sqrt{n}A)$. Thus,

$$\mathcal{X}_n^A \leqslant \frac{\log M}{n} + \max_{i\in[M]} \mathcal{X}_n^{\mathcal{R}_i\cap B_+^n(\sqrt{n}A)}.$$

By Gaussian concentration, see, *e.g.*, [22, Theorem 1.2], it holds that

$$\forall i \in [M], \forall t > 0, \quad \mathbb{P}\left[\left|\mathcal{X}_n^{\mathcal{R}_i\cap B_+^n(\sqrt{n}A)} - F_n^{\mathcal{R}_i\cap B_+^n(\sqrt{n}A)}\right| \geqslant t\right] \leqslant 2\exp(-nt^2/(2\beta^2\kappa^2A^2)),$$

which implies that $\mathbb{E}\left(\mathcal{X}_n^{\mathcal{R}_i\cap B_+^n(\sqrt{n}A)} - F_n^{\mathcal{R}_i\cap B_+^n(\sqrt{n}A)}\right)^2 \leqslant 4\beta^2\kappa^2A^2/n$. Consequently,

$$F_n^A = \mathbb{E}\mathcal{X}_n^A \leqslant \frac{\log M}{n} + \mathbb{E}\max_{i\in[M]} \mathcal{X}^{\mathcal{R}_i\cap B_+^n(\sqrt{n}A)} \leqslant \frac{\log M}{n} + \frac{CM}{\sqrt{n}} + \max_{i\in[M]} F_n^{\mathcal{R}_i\cap B_+^n(\sqrt{n}A)}.$$

When $\eta \in ((i-1)/N, i/N]$, it holds that $\eta^2 \leqslant 2\eta i/N - (i/N)^2 + 1/N^2$. We can thus write

$$\begin{aligned} F_n^{\mathcal{R}_i\cap B_+^n(\sqrt{n}A)} &= \frac{1}{n}\mathbb{E}\log\int_{\mathcal{R}_i\cap B_+^n(\sqrt{n}A)} \exp\left(\frac{\beta\kappa}{2\sqrt{n}}x^\top Wx + \frac{\beta\alpha n}{2}\eta(x)^2\right)\mu_\beta^{\otimes n}(dx) \\ &\leqslant F_n^{\mathrm{EF},A}(i/N) - \frac{\beta\alpha}{2}\left(\frac{i}{N}\right)^2 + \frac{\beta\alpha}{2N^2}, \end{aligned}$$

and we obtain by the previous lemma that

$$\begin{aligned}\limsup_n F_n^A &\leqslant \max_{i\in[M]}\left(\limsup_n F_n^{\mathrm{EF}}(i/N)-\frac{\beta\alpha}{2}\left(\frac{i}{N}\right)^2\right)+\frac{\beta\alpha}{2N^2}\\ &\leqslant \sup_{a,h,D>0}\inf_{\zeta\in\mathrm{fop}_D,\gamma\in\mathbb{R}}\left(P_a(\zeta,h,\gamma)-\gamma D-\frac{\beta\alpha}{2}h^2\right)+\frac{\beta\alpha}{2N^2}.\end{aligned}$$

Recalling that N is arbitrarily large and using Lemma 9, we obtain that

$$\limsup_n \widetilde{F}_n \leqslant \sup_{a,h,D>0}\inf_{\zeta\in\mathrm{fop}_D,\gamma\in\mathbb{R}}\left(P_a(\zeta,h,\gamma)-\gamma D-\frac{\beta\alpha}{2}h^2\right),$$

and Theorem 7 is proven for $\alpha>0$.

Appendix A. Proof of Proposition 1

Using the martingale representation theorem, we know from, *e.g.*, [14, Chapter 4, Theorem 2.2 and 2.3 and Remark. 2.1] that for each probability measure $\mu\in\mathcal{P}(\mathbb{R}_+^n)$, there exists a weak solution to the SDE (1) such that $x_0\sim\mu$ and $x_0 \perp\!\!\!\perp B$. Defining the explosion time of the process (x_t) as

$$\tau_\infty=\inf\{t\geqslant 0\ :\ \|x_t\|=\infty\},$$

we now show that (2) (or, equivalently, $\lambda_+^{\min}>0$) is satisfied, then $\tau_\infty=\infty$ with probability one, which means that the solutions of (1) never explode.

To this end, it is enough to assume that x_0 is an arbitrary deterministic vector in $\mathbb{R}_+^n$. Writing $V(x)=(1\cdot x)$, any solution of the SDE (1) starting with x_0 satisfies

$$V(x_t)=V(x_0)+\int_0^t\left((x_u\cdot 1)+\phi n+(x_u\cdot(\Sigma-I)\,x_u)\right)du+\sqrt{2T}\int_0^t\left(\sqrt{x_u}\cdot dB_u\right).$$

Given $a>0$, define the stopping time

$$\tau_a=\inf\left\{t\geqslant 0\ :\ V(x_t)\geqslant a\right\}.$$

Observing that τ_∞ is a $\bar{\mathbb{R}}$–valued random variable given as $\tau_\infty=\lim_{a\to\infty}\tau_a$, our purpose is to show that $\mathbb{P}[\tau_\infty=\infty]=1$. The techniques for establishing this convergence are well-known [18]. In our case, we write

$$V(x_{t\wedge\tau_a})=V(x_0)+\int_0^{t\wedge\tau_a}\left((x_u\cdot 1)+\phi n+(x_u\cdot(\Sigma-I)\,x_u)\right)du+\sqrt{2T}\int_0^{t\wedge\tau_a}\left(\sqrt{x_u}\cdot dB_u\right),$$

thus,

$$\begin{aligned}\mathbb{E}V(x_{t\wedge\tau_a})&=V(x_0)+\mathbb{E}\int_0^{t\wedge\tau_a}\left((x_u\cdot 1)+\phi n-(x_u\cdot(I-\Sigma)\,x_u)\right)du\\ &\leqslant V(x_0)+\mathbb{E}\int_0^{t\wedge\tau_a}\left(C-\lambda_+^{\min}\|x_u\|^2/2\right)du\\ &\leqslant V(x_0)+C\mathbb{E}[t\wedge\tau_a].\end{aligned}$$

Assume there exists $T,\varepsilon>0$ such that $\mathbb{P}[\tau_\infty\leqslant T]\geqslant\varepsilon$, which implies that $\mathbb{P}[\tau_a\leqslant T]\geqslant\varepsilon$ for all a. Setting $t=T$, we get that

$$\mathbb{E}V(x_{T\wedge\tau_a})\leqslant V(x_0)+C\mathbb{E}(T\wedge\tau_a)\leqslant V(x_0)+CT.$$

On the event $\mathcal{E}_a=[\tau_a\leqslant T]$, we have $V(x_{\tau_a})=a$. Therefore,

$$V(x_0)+CT\geqslant\mathbb{E}V(x_{T\wedge\tau_a})\geqslant\mathbb{E}\mathbb{1}_{\mathcal{E}_a}V(x_{\tau_a})\geqslant\varepsilon a.$$

Making $a\to\infty$, we obtain the contradiction

$$V(x_0)+CT\geqslant\infty.$$

The last step of the proof is to establish the pathwise uniqueness of the solution of (1). Indeed, pathwise uniqueness implies the existence of a unique strong solution for (1) [14, Chapter 4, Theorem 1.1].

For notational simplicity, we rewrite Eq. (1) as $dx_t=f(x_t)dt+\sigma(x_t)dB_t$ with f being the vector function $f(x)=[f_i(x)]_{i\in[n]}=x(1+(\Sigma-I)x)+\phi$ and $\sigma(x)=\sqrt{2Tx}$. Let $x_t^1=[x_{i,t}^1]_{i\in[n]}$ and $x_t^2=[x_{i,t}^2]_{i\in[n]}$ be two solutions starting at the same point x_0 and defined with the same BM $B_t=[B_{i,t}]_{i\in[n]}$. Observing that the function σ satisfies the inequality $|\sigma(x)-\sigma(y)|\leqslant\rho(|x-y|)$ with $\rho(x)=\sqrt{2T}\sqrt{x}$ satisfying $\int_0^1\rho^{-2}(x)dx=\infty$, we can use the technique of the proof of [14, Chapter 4, Theorem 3.2] to construct a sequence of $\mathbb{R}\to\mathbb{R}_+$ functions $(\varphi_k)_{k\geqslant 1}$ such that $\varphi_k\in\mathcal{C}^2(\mathbb{R};\mathbb{R})$, $\varphi_k(x)\uparrow|x|$ as $k\to\infty$, $|\varphi_k'(x)|\leqslant 1$, and $0\leqslant\varphi_k''(x)\leqslant 2\rho^{-2}(x)/k$.

Let $\Delta_{i,t}=x_{i,t}^1-x_{i,t}^2$, consider the SDE

$$\begin{bmatrix}dx_t^1\\ dx_t^2\end{bmatrix}=\begin{bmatrix}f(x_t^1)\\ f(x_t^2)\end{bmatrix}dt+\begin{bmatrix}\operatorname{diag}\sigma(x_t^1)\\ \operatorname{diag}\sigma(x_t^2)\end{bmatrix}dB_t,$$

and, denoting as $\|\cdot\|_1$ the ℓ_1 norm in $\mathbb{R}^n$, define the stopping time

$$\eta_a = \inf\left\{t \geqslant 0 \ : \ \|x^1_t\|_1 \vee \|x^2_t\|_1 \geqslant a\right\}$$

for $a > 0$. Applying Itô's formula to the SDE above, we obtain

$$\begin{aligned}\sum_i \varphi_k(\Delta_{i,t\wedge\eta_a}) = \int_0^{t\wedge\eta_a} \sum_i \varphi_k'(\Delta_{i,u}) \left(f_i(x_u^1) - f_i(x_u^2)\right) du + \frac{1}{2}\int_0^{t\wedge\eta_a} \sum_i \varphi_k''(\Delta_{i,u}) \left(\sigma(x^1_{i,u}) - \sigma_i(x^2_u)\right)^2 du \\ + \int_0^{t\wedge\eta_a} \sum_i \varphi_k'(\Delta_{i,u}) \left(\sigma(x^1_{i,u}) - \sigma_i(x^2_u)\right) dB_{i,u}.\end{aligned}$$

Observing that the function $b : (\mathbb{R}^n_+, \|\cdot\|_1) \to (\mathbb{R}^n, \|\cdot\|_1)$ is Lipschitz on the ball $\{x \in \mathbb{R}^n_+ \ : \ \|x\|_1 \leqslant a\}$ with the Lipschitz constant C_a, we obtain

$$\begin{aligned}\sum_i \mathbb{E}\varphi_k(\Delta_{i,t\wedge\eta_a}) &= \mathbb{E}\int_0^{t\wedge\eta_a} \sum_i \varphi_k'(\Delta_{i,u}) \left(f_i(x_u^1) - f_i(x_u^2)\right) du + \frac{1}{2}\mathbb{E}\int_0^{t\wedge\eta_a} \sum_i \varphi_k''(\Delta_{i,u}) \left(\sigma(x^1_{i,u}) - \sigma_i(x^2_{i,u})\right)^2 du \\ &\leqslant C_a \mathbb{E}\int_0^{t\wedge\eta_a} \sum_i |\Delta_{i,u}|\, du + \frac{n}{k}\mathbb{E}\int_0^{t\wedge\eta_a} du \\ &\leqslant C_a \mathbb{E}\int_0^{t\wedge\eta_a} \sum_i |\Delta_{i,u}|\, du + \frac{nt}{k}.\end{aligned}$$

By taking $k \to \infty$, we then obtain by monotone convergence that

$$\sum_i \mathbb{E}\,|\Delta_{i,t\wedge\eta_a}| \leqslant C_a \mathbb{E}\int_0^{t\wedge\eta_a} \sum_i |\Delta_{i,u}|\, du \leqslant C_a \int_0^t \sum_i \mathbb{E}\,|\Delta_{i,u\wedge\eta_a}|\, du.$$

Using Grönwall's lemma, we obtain that $\sum_i \mathbb{E}|\Delta_{i,t\wedge\eta_a}| = 0$, thus, $\sum_i \mathbb{E}|\Delta_{i,t}| = 0$, which shows that x^1 and x^2 are indistinguishable by continuity.

Appendix B. Proof of Lemma 9

Define the event

$$\mathcal{E} = \left[\lambda_+^{\max}(\Sigma) < 1 - \varepsilon_\Sigma\right],$$

and recall that $\mathbb{1}_{\mathcal{E}^c} \to_n 0$ almost surely by Proposition 4. Given a number $a > 0$, define the sets $\mathfrak{C}_+(a) \subset \mathbb{R}^n_+$ and $\mathfrak{C}(a) \subset \mathbb{R}^n$ as

$$\mathfrak{C}_+(a) = \left\{x = [x_i] \in \mathbb{R}^n_+, \ \forall i \in [n], x_i \geqslant a\right\} \quad \text{and} \quad \mathfrak{C}(a) = \{x = [x_i] \in \mathbb{R}^n, \ \forall i \in [n], |x_i| \geqslant a\}.$$

Let $B^n(a)$ be the closed ball of $\mathbb{R}^n$ with radius a, and let $Z \sim \mathcal{N}(0, I_n)$.

Let $A > 0$. Whether we set $\boldsymbol{H}(x) = \widetilde{\mathscr{H}}(x)$ or $\boldsymbol{H}(x) = \mathscr{H}(x)$, obtain by inspecting the expressions of the Hamiltonians $\widetilde{\mathscr{H}}$ and $\mathscr{H}$ that

$$\begin{aligned}\int_{B_+(\sqrt{n}A)} e^{\beta \boldsymbol{H}(x)} dx &\geqslant \int_{B_+(\sqrt{n}A)\cap\mathfrak{C}_+(1)} e^{-\beta(\|\Sigma\|+1)\|x\|^2/2} dx = 2^{-n}\int_{B(\sqrt{n}A)\cap\mathfrak{C}(1)} e^{-\beta(\|\Sigma\|+1)\|x\|^2/2} dx \\ &= \left(\frac{\pi}{2\beta(\|\Sigma\|+1)}\right)^{n/2} \mathbb{P}_Z\left[Z \in B\left(\sqrt{n\beta(\|\Sigma\|+1)}A\right) \cap \mathfrak{C}\left(\sqrt{\beta(\|\Sigma\|+1)}\right)\right] \\ &\geqslant \left(\frac{\pi}{2\beta(\|\Sigma\|+1)}\right)^{n/2} \left(\mathbb{P}_Z\left[Z \in \mathfrak{C}\left(\sqrt{\beta(\|\Sigma\|+1)}\right)\right] - \mathbb{P}_Z\left[Z \in B\left(\sqrt{n\beta(\|\Sigma\|+1)}A\right)^{\mathrm{c}}\right]\right).\end{aligned}$$

Using, *e.g.*, [1, 7.1.13] to lower bound the Gaussian tail function, we obtain that there exists a constant C_β depending on β only such that $\mathbb{P}_Z\left[Z \in \mathfrak{C}\left(\sqrt{\beta(\|\Sigma\|+1)}\right)\right] \geqslant C_\beta^n \exp(-n\beta(\|\Sigma\|+1))$. By Gaussian concentration, we also have that $\mathbb{P}_Z\left[Z \in B\left(\sqrt{n\beta(\|\Sigma\|+1)}A\right)^{\mathrm{c}}\right] \leqslant \exp(-n\beta(\|\Sigma\|+1)A^2/2)$ for A large enough, which is negligible with respect to $C_\beta^n \exp(-n\beta(\|\Sigma\|+1))$ for large A. Putting things together, we obtain that

$$\int_{B_+(\sqrt{n}A)} e^{\beta \boldsymbol{H}(x)} dx \geqslant C^n (\|\Sigma\|+1)^{-n/2} \exp(-n\beta(\|\Sigma\|+1)) \tag{27}$$

for A large enough.

Now, setting $B_+(\sqrt{n}A)^{\mathrm{c}} = \mathbb{R}^n_+ \backslash B_+(\sqrt{n}A)$, we write

$$\begin{aligned}\widetilde{F} &= \frac{1}{n}\mathbb{E}\mathbb{1}_{\mathcal{E}} \log\left(\int_{B_+(\sqrt{n}A)} e^{\beta\widetilde{\mathscr{H}}} + \int_{B_+(\sqrt{n}A)^{\mathrm{c}}} e^{\beta\widetilde{\mathscr{H}}}\right) + \frac{1}{n}\mathbb{E}\mathbb{1}_{\mathcal{E}^c} \log\int_{\mathbb{R}^n_+} e^{\beta\widetilde{\mathscr{H}}} \\ &\triangleq \frac{1}{n}\mathbb{E}\mathbb{1}_{\mathcal{E}} \log\left(\mathcal{I}_{B_+(\sqrt{n}A)} + \mathcal{I}_{B_+(\sqrt{n}A)^{\mathrm{c}}}\right) + \frac{1}{n}\mathbb{E}\mathbb{1}_{\mathcal{E}^c} \log\int_{\mathbb{R}^n_+} e^{\beta\widetilde{\mathscr{H}}}.\end{aligned}$$

To manage the term with the indicator $\mathbb{1}_{\mathcal{E}^c}$, we observe that $\widetilde{\Sigma} = 0$ on the event $\mathcal{E}^c$. With this, the integral in this third term is deterministic and can be easily shown to satisfy $n^{-1} \log \int_{\mathbb{R}_+^n} \exp(\beta \widetilde{\mathscr{H}}) \leqslant C$. Therefore, this term is negligible because $\mathbb{P}[\mathcal{E}^c] \to 0$.

We now manage the terms $\mathcal{I}_{B_+(\sqrt{n}A)}$ and $\mathcal{I}_{B_+(\sqrt{n}A)^c}$, essentially showing that the latter is negligible with respect to the former. On the event $\mathcal{E}$, it holds that $x^\top(\Sigma - I)x \leqslant -\varepsilon_\Sigma \|x\|^2$ on $\mathbb{R}_+^n$, thus, there exists a constant $c = c(\phi, \beta)$ such that on this event,

$$\mathcal{I}_{B_+(\sqrt{n}A)^c} \leqslant \int_{B_+(\sqrt{n}A)^c} e^{-\frac{\beta}{2}(\varepsilon_\Sigma \|x\|^2 - 2c(1\cdot x))} dx = e^{\frac{\beta c^2}{2\varepsilon_\Sigma} n} \int_{B_+(\sqrt{n}A)^c} e^{-\frac{\beta}{2}\left\|\sqrt{\varepsilon_\Sigma} x - \frac{c}{\sqrt{\varepsilon_\Sigma}} 1\right\|^2} dx.$$

By making the variable change $u = \sqrt{\varepsilon_\Sigma} x - \frac{c}{\sqrt{\varepsilon_\Sigma}} 1$ and noticing that $\|x\| \geqslant \sqrt{n}A \Rightarrow \|u\| \geqslant \sqrt{n\varepsilon_\Sigma} A - c\sqrt{n/\varepsilon_\Sigma}$, we obtain that

$$\mathcal{I}_{B_+(\sqrt{n}A)^c} \leqslant \varepsilon_\Sigma^{-n/2} e^{\frac{\beta c^2}{2\varepsilon_\Sigma} n} \int_{B(\sqrt{n}(\sqrt{\varepsilon_\Sigma} A - c/\sqrt{\varepsilon_\Sigma}))^c} e^{-\beta \|u\|^2/2} du$$

on $\mathcal{E}$ for A large enough, where $B(a)^c = \mathbb{R}^n \backslash B(a)$. By Gaussian concentration, we finally get that there

$$\mathcal{I}_{B_+(\sqrt{n}A)^c} \leqslant \exp(-nC(A^2 - 1))$$

on $\mathcal{E}$ for A large enough.

We now write

$$\frac{1}{n} \mathbb{E} \mathbb{1}_{\mathcal{E}} \log\left(\mathcal{I}_{B_+(\sqrt{n}A)} + \mathcal{I}_{B_+(\sqrt{n}A)^c}\right) = \frac{1}{n} \mathbb{E} \mathbb{1}_{\mathcal{E}} \log \mathcal{I}_{B_+(\sqrt{n}A)} + \frac{1}{n} \mathbb{E} \mathbb{1}_{\mathcal{E}} \log\left(1 + \mathcal{I}^{-1}_{B_+(\sqrt{n}A)} \mathcal{I}_{B_+(\sqrt{n}A)^c}\right).$$

Using Inequality (27), we have

$$\frac{1}{n} \mathbb{E} \mathbb{1}_{\mathcal{E}} \log\left(1 + \mathcal{I}^{-1}_{B_+(\sqrt{n}A)} \mathcal{I}_{B_+(\sqrt{n}A)^c}\right) \leqslant \frac{1}{n} \mathbb{E} \log\left(1 + e^{nC(\|\Sigma\| + 1 - A^2)}\right),$$

and since $\log(1 + e^x) \leqslant e^x \mathbb{1}_{x<0} + (x+1)\mathbb{1}_{x \geqslant 0}$, we can write

$$\frac{1}{n} \mathbb{E} \log\left(1 + e^{nC(\|\Sigma\| + 1 - A^2)}\right) \leqslant \frac{1}{n} + C\mathbb{E}(\|\Sigma\| + 1 - A^2) \mathbb{1}_{\|\Sigma\| \geqslant A^2 - 1}.$$

Using, *e.g.*, [32, Corollary 4.4.8], we know that $\mathbb{E}\|\Sigma\|^2 \leqslant 2\kappa^2 \mathbb{E}\|W\|^2 + 2\alpha^2 < C$. Thus, for A large enough, the right hand side converges to zero by the Cauchy-Schwarz inequality and the standard results on the behavior of $\|\Sigma\| = \|\kappa W + \alpha 1 1^\top / n\|$ for large n.

Getting back to the expression of $\widetilde{F}_n$ provided above, we then obtain that

$$\widetilde{F}_n = \frac{1}{n} \mathbb{E} \mathbb{1}_{\mathcal{E}} \log \mathcal{I}_{B_+(\sqrt{n}A)} + o_n(1).$$

Turning to F_n^A, we write

$$F_n^A = \frac{1}{n} \mathbb{E} \mathbb{1}_{\mathcal{E}} \log \int_{B_+(\sqrt{n}A)} e^{\beta \mathscr{H}} + \frac{1}{n} \mathbb{E} \mathbb{1}_{\mathcal{E}^c} \log \int_{B_+(\sqrt{n}A)} e^{\beta \mathscr{H}} = \frac{1}{n} \mathbb{E} \mathbb{1}_{\mathcal{E}} \log \mathcal{I}_{B_+(\sqrt{n}A)} + \frac{1}{n} \mathbb{E} \mathbb{1}_{\mathcal{E}^c} \log \int_{B_+(\sqrt{n}A)} e^{\beta \mathscr{H}}$$

since $\Sigma = \widetilde{\Sigma}$ on the event $\mathcal{E}$. By (27), using that $\log(1 + |a|) \leqslant |a|$, we obtain that the second term satisfies

$$\frac{1}{n} \mathbb{E} \mathbb{1}_{\mathcal{E}^c} \log \int_{B_+(\sqrt{n}A)} e^{\beta \mathscr{H}} \geqslant -C \mathbb{E} \mathbb{1}_{\mathcal{E}^c} (1 + \|\Sigma\|).$$

By the Cauchy-Schwarz inequality, the right hand side is negligible.

We therefore have that $F_n^A - \widetilde{F}_n \geqslant o_n(1)$ for A large enough, and Lemma 9 is proven.

Appendix C. Proof of Lemma 10

Still writing $\mathcal{E} = \left[\lambda_+^{\max}(\Sigma) < 1 - \varepsilon_\Sigma\right]$, we have

$$F_{a,n} = \frac{1}{n} \mathbb{E} \mathbb{1}_{\mathcal{E}} \log \int_{[0,a]^n} e^{\beta \mathscr{H}(x)} dx + \frac{1}{n} \mathbb{E} \mathbb{1}_{\mathcal{E}^c} \log \int_{[0,a]^n} e^{\beta \mathscr{H}(x)} dx \triangleq \chi_{1,n} + \chi_{2,n}.$$

We bound the term $\chi_{2,n}$ by writing

$$\begin{aligned}
\chi_{2,n} &\leqslant \frac{1}{n} \mathbb{E} \mathbb{1}_{\mathcal{E}^c} \log \int_{[0,a]^n} \exp\left(\beta \|\Sigma - I\| \|x\|^2/2 + \beta (1 \cdot x) + (\beta\phi - 1)(1 \cdot \log x)\right) dx \\
&= \mathbb{E} \mathbb{1}_{\mathcal{E}^c} \log \int_0^a \exp\left(\beta \|\Sigma - I\| x^2/2 + \beta x + (\beta\phi - 1) \log x\right) dx \\
&\leqslant \mathbb{E} \mathbb{1}_{\mathcal{E}^c} \log\left(a \exp\left(\beta \|\Sigma - I\| a^2/2 + \beta a + (\beta\phi - 1) \log a\right)\right) \\
&\leqslant \mathbb{E} \mathbb{1}_{\mathcal{E}^c} \left(\beta\phi \log a + \beta a^2 \|\Sigma - I\|/2 + \beta a\right)
\end{aligned}$$

which goes to zero as $n \to \infty$ by using [32, Corollary 4.4.8] and Cauchy-Schwarz. We also have

$$\widetilde{F}_n \geqslant \frac{1}{n}\mathbb{E}\mathbb{1}_{\mathcal{E}} \log \int_{[0,a]^n} e^{\beta \mathscr{H}} + \frac{1}{n}\mathbb{E}\mathbb{1}_{\mathcal{E}^c} \log \int_{\mathbb{R}_+^n} e^{\beta \widetilde{\mathscr{H}}}.$$

As in Appendix B, the second term at the right hand side is negligible. This proves Lemma 10.

Appendix D. Lemma 14: Sketch of proof

Let $\zeta \in \mathrm{fop}_D$ be a measure of the form

$$\zeta = \sum_{k=0}^{K} (\lambda_k - \lambda_{k-1})\, \delta_{b_k},$$

where

$$0 < \lambda_0 < \cdots < \lambda_{K-1} < 1, \quad 0 = b_0 < \cdots < b_{K-1} < b_K = D.$$

Consider the random Gibbs probability measure $\mathcal{G}$ on $(B_+(A\sqrt{n}) \cap [0,a]^n) \times \mathbb{N}^K$, whose density with respect to the product of the Lebesgue measure on $\mathbb{R}^n$ and the counting measure on $\mathbb{N}^K$ is proportional to

$$v_{\boldsymbol{i}} \, \exp\big((\boldsymbol{x} \cdot \boldsymbol{q}_{\boldsymbol{i}}) + \beta\alpha h\, (1 \cdot \boldsymbol{x}) + \gamma \|\boldsymbol{x}\|^2\big)\, \mu_\beta^{\otimes n}(d\boldsymbol{x}),$$

where $(v_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{N}^K} \sim \mathrm{RPC}_{\boldsymbol{\lambda}}$ and $\boldsymbol{q}_{\boldsymbol{i}} = (q_{i1}, \ldots, q_{in})^\top$ satisfies

$$q_{\boldsymbol{i}\ell} \overset{d}{=} \sum_{k=1}^{K} z^{(\ell)}_{i_1,\ldots,i_k} \sqrt{\xi'(b_k) - \xi'(b_{k-1})}, \quad z^{(\ell)}_{i_1,\ldots,i_k} \sim \mathcal{N}(0,1) \text{ independent.}$$

Let $\mathcal{Z}$ denote the associated partition function and recall that $\boldsymbol{X}^A_{0,a} = \mathbb{E}\log\mathcal{Z}$. For any measurable function $f(\boldsymbol{x}, \boldsymbol{i})$, we write

$$\langle f(\boldsymbol{x},\boldsymbol{i})\rangle = \mathbb{E}\left[\frac{1}{\mathcal{Z}} \sum_{\boldsymbol{i}\in\mathbb{N}^K} v_{\boldsymbol{i}} \int_{B_+(A\sqrt{n})\cap[0,a]^n} f(\boldsymbol{x},\boldsymbol{i})\, e^{(\boldsymbol{x}\cdot\boldsymbol{q}_{\boldsymbol{i}})+\beta\alpha h(1\cdot\boldsymbol{x})+\gamma\|\boldsymbol{x}\|^2}\, \mu_\beta^{\otimes n}(d\boldsymbol{x})\right].$$

Step 1. Derivative with respect to b_k. Following [11, Lemma 6.2], for any $k \in [K-1]$ we compute

$$\partial_{b_k} X^A_{0,a} = \mathbb{E}\langle \partial_{b_k} (\boldsymbol{x}\cdot\boldsymbol{q}_{\boldsymbol{i}})\rangle.$$

Since only the k-th and $(k+1)$-th terms depend on b_k, we obtain

$$\partial_{b_k} X^A_{0,a} = \frac{\xi''(b_k)}{2\sqrt{\xi'(b_k) - \xi'(b_{k-1})}}\, \mathbb{E}\langle(\boldsymbol{x}\cdot\boldsymbol{z}_{i_1,\ldots,i_k})\rangle - \frac{\xi''(b_k)}{2\sqrt{\xi'(b_{k+1}) - \xi'(b_k)}}\, \mathbb{E}\big\langle\big(\boldsymbol{x}\cdot\boldsymbol{z}_{i_1,\ldots,i_{k+1}}\big)\big\rangle,$$

where $\boldsymbol{z}_{i_1,\ldots,i_k} := (z^{(\ell)}_{i_1,\ldots,i_k})_{\ell\in[n]}$.

Now in order to perform Gaussian integration by parts we first define the Gaussian processes

$$\mathcal{X}(\boldsymbol{x},\boldsymbol{i}) := (\boldsymbol{x}\cdot\boldsymbol{z}_{i_1,\ldots,i_k}), \qquad \mathcal{Y}(\boldsymbol{x},\boldsymbol{i}) := (\boldsymbol{x}\cdot\boldsymbol{q}_{\boldsymbol{i}}) + \beta\alpha h\,(1\cdot\boldsymbol{x}) + \gamma\|\boldsymbol{x}\|^2.$$

For two replicas $(\boldsymbol{x}^1, \boldsymbol{i}^1)$ and $(\boldsymbol{x}^2, \boldsymbol{i}^2)$, the covariance between $\mathcal{X}$ and $\mathcal{Y}$ reads

$$\begin{aligned} C((\boldsymbol{x}^1,\boldsymbol{i}^1),(\boldsymbol{x}^2,\boldsymbol{i}^2)) &:= \mathbb{E}\big[\mathcal{X}(\boldsymbol{x}^1,\boldsymbol{i}^1)\mathcal{Y}(\boldsymbol{x}^2,\boldsymbol{i}^2)\big] \\ &= \mathbb{E}\Big[\Big(\boldsymbol{x}^1\cdot\boldsymbol{z}_{i^1_1,\ldots,i^1_k}\Big)\,\big(\boldsymbol{x}^2\cdot\boldsymbol{q}_{\boldsymbol{i}^2}\big)\Big] \\ &= \sqrt{\xi'(b_k) - \xi'(b_{k-1})}\, \sum_{\ell=1}^{n} x^1_\ell x^2_\ell \mathbb{E}\Big[z^{(\ell)}_{i^1_1,\ldots,i^1_k} z^{(\ell)}_{i^2_1,\ldots,i^2_k}\Big]. \end{aligned}$$

The last expectation equals 1 if $\boldsymbol{i}^1 \wedge \boldsymbol{i}^2 \geqslant k$, and 0 otherwise, hence

$$C((\boldsymbol{x}^1,\boldsymbol{i}^1),(\boldsymbol{x}^2,\boldsymbol{i}^2)) = \sqrt{\xi'(b_k) - \xi'(b_{k-1})}\, \big(\boldsymbol{x}^1\cdot\boldsymbol{x}^2\big)\, \mathbb{1}_{\{\boldsymbol{i}^1\wedge\boldsymbol{i}^2\geqslant k\}}.$$

By Gaussian integration by parts (see [22, Lemma 1.1]),

$$\mathbb{E}\langle(\boldsymbol{x}\cdot\boldsymbol{z}_{i_1,\ldots,i_k})\rangle = \sqrt{\xi'(b_k) - \xi'(b_{k-1})}\, \mathbb{E}\big\langle\|\boldsymbol{x}^1\|^2 - \big(\boldsymbol{x}^1\cdot\boldsymbol{x}^2\big)\, \mathbb{1}_{\{\boldsymbol{i}^1\wedge\boldsymbol{i}^2\geqslant k\}}\big\rangle.$$

Similarly,

$$\mathbb{E}\big\langle\big(\boldsymbol{x}\cdot\boldsymbol{z}_{i_1,\ldots,i_{k+1}}\big)\big\rangle = \sqrt{\xi'(b_{k+1}) - \xi'(b_k)}\, \mathbb{E}\big\langle\|\boldsymbol{x}^1\|^2 - \big(\boldsymbol{x}^1\cdot\boldsymbol{x}^2\big)\, \mathbb{1}_{\{\boldsymbol{i}^1\wedge\boldsymbol{i}^2\geqslant k+1\}}\big\rangle.$$

Substituting these into the expression for $\partial_{b_k} X^A_{0,a}$ gives

$$\partial_{b_k} X^A_{0,a} = \frac{\xi''(b_k)}{2}\mathbb{E}\big\langle\big(\boldsymbol{x}^1\cdot\boldsymbol{x}^2\big)\,\big(\mathbb{1}_{\{\boldsymbol{i}^1\wedge\boldsymbol{i}^2\geqslant k+1\}} - \mathbb{1}_{\{\boldsymbol{i}^1\wedge\boldsymbol{i}^2\geqslant k\}}\big)\big\rangle = \frac{\xi''(b_k)}{2}\mathbb{E}\big\langle\big(\boldsymbol{x}^1\cdot\boldsymbol{x}^2\big)\,\mathbb{1}_{\{\boldsymbol{i}^1\wedge\boldsymbol{i}^2= k\}}\big\rangle.$$

Step 2. As a next step and without loss of generality, we may assume that ζ and $\tilde{\zeta}$ admit the same coefficients (λ_k), i.e. it can be shown that without modifying neither $\boldsymbol{X}^A_{0,a}(\zeta,h,\gamma)$ nor $\boldsymbol{X}^A_{0,a}(\tilde{\zeta},h,\gamma)$, nor $\boldsymbol{d}(\zeta,\tilde{\zeta})$, one can assume that ζ and $\tilde{\zeta}$ take the following forms: there exists an integer $K>0$ and real numbers

$$\begin{aligned}
&0<\lambda_0<\cdots<\lambda_{K-1}<1,\\
&0=b_0\leqslant\cdots\leqslant b_{K-1}\leqslant b_K=D, \quad \text{and}\\
&0=\tilde{b}_0\leqslant\cdots\leqslant \tilde{b}_{K-1}\leqslant \tilde{b}_K=D,
\end{aligned} \tag{28}$$

such that $\zeta([0,t])=\lambda_k$ if $t\in[b_k,b_{k+1})$ and such that $\tilde{\zeta}([0,t])=\lambda_k$ if $t\in[\tilde{b}_k,\tilde{b}_{k+1})$. See [11, Proposition 6.3] for a justification. For $s\in[0,1]$, define $b_k(s)=sb_k+(1-s)\tilde{b}_k$ and set

$$\zeta_s := \sum_{k=0}^{K}(\lambda_k-\lambda_{k-1})\,\delta_{b_k(s)},$$

so that $\zeta_1=\zeta$ and $\zeta_0=\tilde{\zeta}$. Then

$$\boldsymbol{X}^A_{0,a}(\zeta,h,\gamma)-\boldsymbol{X}^A_{0,a}(\tilde{\zeta},h,\gamma)=\int_0^1\partial_s\boldsymbol{X}^A_{0,a}(\zeta_s,h,\gamma)\,ds=\sum_{k=1}^{K-1}(b_k-\tilde{b}_k)\int_0^1\partial_{b_k(s)}\boldsymbol{X}^A_{0,a}(\zeta_s,h,\gamma)\,ds.$$

Using $|\left(\boldsymbol{x}^1\cdot\boldsymbol{x}^2\right)|\leqslant nA^2$ and $\mathbb{E}\left\langle\mathbb{1}_{\{\boldsymbol{i}^1\wedge\boldsymbol{i}^2=k\}}\right\rangle=\lambda_k-\lambda_{k-1}$ from the general properties of RPC, we deduce

$$\left|\partial_{b_k}X^A_{0,a}\right|\leqslant nA^2\frac{|\xi''(b_k)|}{2}, \quad \text{and hence}$$

$$\left|\boldsymbol{X}^A_{0,a}(\zeta,h,\gamma)-\boldsymbol{X}^A_{0,a}(\tilde{\zeta},h,\gamma)\right|\leqslant nA^2\max_{t\in[0,D]}\frac{|\xi''(t)|}{2}\sum_{k=1}^{K-1}(\lambda_k-\lambda_{k-1})|b_k-\tilde{b}_k|.$$

Finally, observing that

$$\sum_{k=1}^{K}(\lambda_k-\lambda_{k-1})|b_k-\tilde{b}_k|=\int_0^D|\zeta(t)-\tilde{\zeta}(t)|\,dt,$$

we obtain the desired result.

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