A CLT FOR INFORMATION-THEORETIC STATISTICS OF NON-CENTERED GRAM RANDOM MATRICES

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Abstract. In this article, we study the fluctuations of the random variable:

$$\mathcal{I}_n(\rho) = \frac{1}{N}\log\det\left(\Sigma_n\Sigma_n^* + \rho I_N\right), \quad (\rho > 0)$$

where $\Sigma_n = n^{-1/2} D_n^{1/2} X_n \tilde{D}_n^{1/2} + A_n$, as the dimensions of the matrices go to infinity at the same pace. Matrices X_n and A_n are respectively random and deterministic $N \times n$ matrices; matrices D_n and $\tilde{D}_n$ are deterministic and diagonal, with respective dimensions $N \times N$ and $n \times n$; matrix $X_n = (X_{ij})$ has centered, independent and identically distributed entries with unit variance, either real or complex.

We prove that when centered and properly rescaled, the random variable $\mathcal{I}_n(\rho)$ satisfies a Central Limit Theorem and has a Gaussian limit. The variance of $\mathcal{I}_n(\rho)$ depends on the moment $\mathbb{E}X_{ij}^2$ of the variables X_{ij} and also on its fourth cumulant $\kappa = \mathbb{E}|X_{ij}|^4 - 2 - |\mathbb{E}X_{ij}^2|^2$.

The main motivation comes from the field of wireless communications, where $\mathcal{I}_n(\rho)$ represents the mutual information of a multiple antenna radio channel. This article closely follows the companion article "A CLT for Information-theoretic statistics of Gram random matrices with a given variance profile", *Ann. Appl. Probab. (2008)* by Hachem et al., however the study of the fluctuations associated to non-centered large random matrices raises specific issues, which are addressed here.

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1. Introduction

The model, the statistics, and the literature. Consider a $N \times n$ random matrix $\Sigma_n = (\xi_{ij}^n)$ which has the expression

$$\Sigma_n = \frac{1}{\sqrt{n}} D_n^{\frac{1}{2}} X_n \tilde{D}_n^{\frac{1}{2}} + A_n \ , \tag{1.1}$$

where $A_n = (a_{ij}^n)$ is a deterministic $N \times n$ matrix, D_n and $\tilde{D}_n$ are diagonal deterministic matrices with nonnegative entries, with respective dimensions $N \times N$ and $n \times n$; $X_n = (X_{ij})$ is a $N \times n$ matrix with the entries X_{ij}'s being centered, independent and identically distributed (i.i.d.) random variables with unit variance $\mathbb{E}|X_{ij}|^2 = 1$ and finite 16^{th} moment.

Consider the following linear statistics of the eigenvalues:

$$\mathcal{I}_n(\rho) = \frac{1}{N}\log\det\left(\Sigma_n\Sigma_n^* + \rho I_N\right) = \frac{1}{N}\sum_{i=1}^{N}\log(\lambda_i + \rho) \ ,$$

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where I_N is the $N \times N$ identity matrix, $\rho > 0$ is a given parameter and the λ_i's are the eigenvalues of matrix $\Sigma_n \Sigma_n^*$ (Σ_n^* stands for the Hermitian adjoint of Σ_n). This functional, known as the mutual information for multiple antenna radio channels, is fundamental in wireless communication as it characterizes the performance of a (coherent) communication over a wireless Multiple-Input Multiple-Output (MIMO) channel with gain matrix Σ_n. When Σ_n follows the model described by (1.1), the deterministic matrix A_n accounts for the so-called specular component, while D_n and $\tilde{D}_n$ account for the correlations in certain bases at the receiving and emitting sides, respectively.

Since the seminal work of Telatar [37], the study of the mutual information $\mathcal{I}_n(\rho)$ of a MIMO channel (and other performance indicators) in the regime where the dimensions of the gain matrix grow to infinity at the same pace has turned to be extremely fruitful. However, non-centered channel matrices have been comparatively less studied from this point of view, as their analysis is more difficult due to the presence of the deterministic matrix A_n. First order results can be found in Girko [15, 16]; Dozier and Silverstein [11, 12] established convergence results for the spectral measure; and the systematic study of the convergence of $\mathcal{I}_n(\rho)$ for a correlated Rician channel has been undertaken by Hachem et al. in [20, 13], etc. The fluctuations of $\mathcal{I}_n$ are important as well, for the computation of the outage probability of a MIMO channel for instance. With the help of the replica method, Taricco [35, 36] provided a closed-form expression for the asymptotic variance of $\mathcal{I}_n$ when the elements of X_n are Gaussian.

The purpose of this article is to establish a Central Limit Theorem (CLT) for $\mathcal{I}_n(\rho)$ in the following regime

$$n \to \infty \quad \text{and} \quad 0 < \liminf \frac{N}{n} \leq \limsup \frac{N}{n} < \infty \ ,$$

(simply denoted by $n \to \infty$ in the sequel) under mild assumptions for matrices X_n, A_n, D_n and $\tilde{D}_n$.

The contributions of this article are twofold. From a wireless communication perspective, the fluctuations of $\mathcal{I}_n$ are established, regardless of the Gaussianity of the entries and the CLT conjectured by Tarrico is fully proven. Also, this article concludes a series of studies devoted to Rician MIMO channels, initiated in [20] where a deterministic equivalent of the mutual information was provided, and continued in [13] where the computation of the ergodic capacity was addressed and an iterative algorithm proposed.

From a mathematical point of view, the study of the fluctuations of $\mathcal{I}_n$ is the first attempt (up to our knowledge) to establish a CLT for a linear statistics of the eigenvalues of a Gram non-centered matrix (so-called signal plus noise model in [11, 12]). It complements (but does not supersede) the CLT established in [21] for a centered Gram matrix with a given variance profile. The fact that matrix Σ_n is non-centered ($\mathbb{E}\,\Sigma_n = A_n$) raises specific issues, from a different nature than those addressed in close-by results [1, 4, 21], etc. These issues arise from the presence in the computations of bilinear forms $u_n^* Q_n(z)\, v_n$ where at least one of the vectors u_n or v_n is deterministic. Often, the deterministic vector is related to the columns of matrix A_n, and has to be dealt with in such a way that the assumption over the spectral norm of A_n is exploited.

Another important contribution of this paper is to establish the CLT regardless of specific assumptions on the real or complex nature of the underlying random variables. It is in particular *not* assumed that the random variables are Gaussian, neither that whenever the random variables X_{ij} are complex, their second moment $\mathbb{E}X_{ij}^2$ is zero; nor is assumed that the random variables are circular[1]. As we shall see, all these assumptions, if assumed, would have resulted in substantial simplifications. As a reward however, we obtain a variance expression which smoothly depends upon $\mathbb{E}X_{ij}^2$ whose value is 1 in the real case, and zero in the complex case where the real and imaginary parts are not correlated.

Interestingly, the mutual information $\mathcal{I}_n$ has a strong relationship with the Stieltjes transform $f_n(z) = \frac{1}{N}\mathrm{Trace}(\Sigma_n\Sigma_n^* - zI_N)^{-1}$ of the spectral measure of $\Sigma_n\Sigma_n^*$:

$$\mathcal{I}_n(\rho) = \log\rho + \int_\rho^\infty \left(\frac{1}{w} - f_n(-w)\right) \mathbf{d}w \ .$$

Accordingly, the study of the fluctuations of $\mathcal{I}_n$ is also an important step toward the study of general linear statistics of $\Sigma_n\Sigma_n^*$'s eigenvalues which can be expressed via the Stieltjes transform:

$$\frac{1}{N}\mathrm{Trace}\ h(\Sigma_n\Sigma_n) = \frac{1}{N}\sum_{i=1}^N h(\lambda_i) = -\frac{1}{2\mathbf{i}\pi}\oint_{\mathcal{C}} h(z) f_n(z)\,\mathbf{d}z\ ,$$

for some well-chosen contour $\mathcal{C}$ (see for instance [4]).

Fluctuations for particular linear statistics (and general classes of linear statistics) of large random matrices have been widely studied: CLTs for Wigner matrices can be traced back to Girko [14] (see also [17]). Results for this class of matrices have also been obtained by Khorunzhy et al. [27], Boutet de Monvel and Khorunzhy [7], Johansson [24], Sinai and Sochnikov [33], Soshnikov [34], Cabanal-Duvillard [8], Guionnet [18], Anderson and Zeitouni [1], Mingo and Speicher [29], Chatterjee [9], Lytova and Pastur [28], etc. The case of Gram matrices has been studied in Arharov [2], Jonsson [25], Bai and Silverstein [4], Hachem et al. [21], and also in [28, 29, 9]. Fluctuation results dedicated to wireless communication applications have been developed in the centered case ($A_n = 0$) by Debbah and Müller [10] and Tulino and Verdù [38] (based on Bai and Silverstein [4]), Hachem et al. [19] (for Gaussian entries) and [21]. Other fluctuation results either based on the replica method or on saddle-point analysis have been developed by Moustakas, Sengupta and coauthors [30, 31], and Tarrico [35, 36].

Presentation of the results. We first introduce the fundamental equations needed to express the deterministic approximation of the mutual information and the variance in the CLT.

Fundamental equations, deterministic equivalents. We collect here resuls from [20]. The following system of equations

$$\left\{\begin{array}{lcl} \delta_n(z) & = & \frac{1}{n}\mathrm{Tr}\, D_n\left(-z(I_N + \tilde{\delta}_n(z)D_n) + A_n(I_n + \delta_n(z)\tilde{D}_n)^{-1}A_n^*\right)^{-1} \\ \tilde{\delta}_n(z) & = & \frac{1}{n}\mathrm{Tr}\, \tilde{D}_n\left(-z(I_n + \delta_n(z)\tilde{D}_n) + A_n^*(I_N + \tilde{\delta}_n(z)D_n)^{-1}A_n\right)^{-1} \end{array}\right. ,\quad z\in\mathbb{C}-\mathbb{R}^+ \tag{1.2}$$

[1]A random variable $X \in \mathbb{C}$ is circular if the distribution of X is equal to the distribution of ρX for every $\rho\in\mathbb{C}$, $|\rho| = 1$. This assumption is very often relevant in wireless communication and has an important consequence; it implies that all the cross moments $\mathbb{E}|X|^k X^\ell$ ($\ell \geq 1$) are zero.

admits a unique solution $(\delta_n, \tilde{\delta}_n)$ in the class of Stieltjes transforms of nonnegative measures[2] with support in $\mathbb{R}^+$. Matrices $T_n(z)$ and $\tilde{T}_n(z)$ defined by

$$\begin{cases} T_n(z) &= \left(-z(I_N + \tilde{\delta}_n(z) D_n) + A_n (I_n + \delta_n \tilde{D}_n)^{-1} A_n^*\right)^{-1} \\ \tilde{T}_n(z) &= \left(-z(I_n + \delta_n(z) \tilde{D}_n) + A_n^* (I_N + \tilde{\delta}_n D_n)^{-1} A_n\right)^{-1} \end{cases} \tag{1.3}$$

are approximations of the resolvent $Q_n(z) = (\Sigma_n \Sigma_n^* - z I_N)^{-1}$ and the co-resolvent $\tilde{Q}_n(z) = (\Sigma_n^* \Sigma_n - z I_n)^{-1}$ in the sense that ($\xrightarrow{a.s.}$ stands for almost sure convergence):

$$\frac{1}{n} \mathrm{Tr}\, (Q_n(z) - T_n(z)) \xrightarrow[n\to\infty]{a.s.} 0, \quad z \in \mathbb{C} - \mathbb{R}^+$$

which readily gives a deterministic approximation of the Stieltjes transform $N^{-1} \mathrm{Tr}\, Q_n(z)$ of the spectral measure of $\Sigma_n \Sigma_n^*$ in terms of T_n (and similarly for $\tilde{Q}_n$ and $\tilde{T}_n$). Also proven in [22] is the convergence of bilinear forms

$$u_n^* (Q_n(z) - T_n(z)) v_n \xrightarrow[n\to\infty]{a.s.} 0, \quad z \in \mathbb{C} - \mathbb{R}^+ \tag{1.4}$$

where (u_n) and (v_n) are sequences of $N \times 1$ deterministic vectors with bounded Euclidean norms, which complements the picture of T_n approximating Q_n.

Matrices $T_n = (t_{ij}\ ; 1 \le i, j \le N)$ and $\tilde{T}_n = (\tilde{t}_{ij}\ ; 1 \le i, j \le n)$ will play a fundamental role in the sequel and enable us to express a deterministic equivalent to $\mathbb{E}\mathcal{I}_n(\rho)$. Define $V_n(\rho)$ by:

$$V_n(\rho) = \frac{1}{N} \log\det \left(\rho(I_N + \tilde{\delta}_n D_n) + A_n (I_n + \delta_n \tilde{D}_n)^{-1} A_n^*\right) + \frac{1}{N} \log\det(I_n + \delta_n \tilde{D}_n) - \frac{\rho n}{N} \delta_n \tilde{\delta}_n \ , \tag{1.5}$$

where δ_n and $\tilde{\delta}_n$ are evaluated at $z = -\rho$. Then the difference $\mathbb{E}\, \mathcal{I}_n(\rho) - V_n(\rho)$ goes to zero as $n \to \infty$.

In order to study the fluctuations $N(\mathcal{I}_n(\rho) - V_n(\rho))$ and to establish a CLT, we study separately the quantity $N(\mathcal{I}_n(\rho) - \mathbb{E}\mathcal{I}_n(\rho))$ from which the fluctuations arise and the quantity $N(\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho))$ which yields a bias.

The fluctuations. In every case where the fluctuations of the mutual information have been studied, the variance of $N\,(\mathcal{I}_n(\rho) - V_n(\rho))$ always proved to take a remarkably simple closed-form expression (see for instance [30, 36, 38] and in a more mathematical flavour [19, 21]). The same phenomenon again occurs for the matrix model Σ_n under consideration. Drop the subscripts N, n and let

$$\gamma = \frac{1}{n} \mathrm{Tr}\, DTDT\ ,\ \tilde{\gamma} = \frac{1}{n} \mathrm{Tr}\, \tilde{D}\tilde{T}\tilde{D}\tilde{T}\ ,\ \underline{\gamma} = \frac{1}{n} \mathrm{Tr}\, DTD\bar{T}\ ,\ \underline{\tilde{\gamma}} = \frac{1}{n} \mathrm{Tr}\, \tilde{D}\tilde{T}\tilde{D}\bar{\tilde{T}}\ , \tag{1.6}$$

where $\bar{M}$ stands for the (elementwise) conjugate of matrix M. Let

$$\vartheta = \mathbb{E}(X_{ij})^2 \quad \text{and} \quad \kappa = \mathbb{E}|X_{ij}|^4 - 2 - |\vartheta|^2\ .$$

[2]In fact, δ_n is the Stieltjes transform of a measure with total mass equal to $n^{-1}\mathrm{Tr} D_n$ while $\tilde{\delta}_n$ is the Stieltjes transform of a measure with total mass equal to $n^{-1}\mathrm{Tr}\tilde{D}_n$.

Let

$$\Theta_n = -\log\left(\left(1-\frac{1}{n}\mathrm{Tr}\, D^{\frac12}TA(I+\delta\tilde D)^{-1}\tilde D(I+\delta\tilde D)^{-1}A^*TD^{\frac12}\right)^2 - \rho^2\gamma\tilde\gamma\right)$$
$$-\log\left(\left|1-\vartheta\frac{1}{n}\mathrm{Tr}\, D^{\frac12}\bar T\bar A(I+\delta\tilde D)^{-1}\tilde D(I+\delta\tilde D)^{-1}A^*TD^{\frac12}\right|^2 - |\vartheta|^2\rho^2\underline{\gamma}\,\underline{\tilde\gamma}\right)$$
$$+\kappa\frac{\rho^2}{n^2}\sum_i d_i^2 t_{ii}^2\sum_j \tilde d_j^2\tilde t_{jj}^2\ ,$$

where $d_i = [D_n]_{ii}$, $\tilde d_j = [\tilde D_n]_{jj}$, and all the needed quantities are evaluated at $z=-\rho$. The CLT can then be expressed as:

$$\frac{N}{\sqrt{\Theta_n}}\left(\mathcal{I}_n - \mathbb{E}\mathcal{I}_n\right) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1)\ ,$$

where $\xrightarrow{\mathcal{D}}$ stands for convergence in distribution. Although complicated at first sight, variance Θ_n encompasses the case of standard real random variables ($\vartheta=1$), standard complex random variables ($\vartheta=0$) and all the intermediate cases $0<|\vartheta|<1$. Moreover, Θ_n often takes simpler forms if the variables are Gaussian, real, etc. (see for instance Remark 2.2).

The bias. When the entries of X_n are complex Gaussian with independent and identically distributed real and imaginary parts, $\kappa=\vartheta=0$, and it has already been proven in [13] that $\mathbb{E}\,\mathcal{I}_n(\rho) - V_n(\rho) = \mathcal{O}(n^{-2})$. When any of κ or ϑ is non zero, a bias term $\mathcal{B}_n(\rho)\neq 0$ appears in the sense that

$$N\left(\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho)\right) - \mathcal{B}_n(\rho) \xrightarrow[n\to\infty]{} 0\ .$$

We establish the existence of this bias and provide its expression in the case where $A=0$.

Outline of the article. In Section 2, we provide the main assumptions and state the main results of the paper: Definition of the variance Θ_n and asymptotic fluctuations of $N\left(\mathcal{I}_n(\rho)-\mathbb{E}\mathcal{I}_n(\rho)\right)$ (Theorem 2.2), asymptotic bias of $N\left(\mathbb{E}\mathcal{I}_n(\rho)-V_n(\rho)\right)$ (Proposition 2.3). Notations, important estimates and classical results are provided in Section 3. Sections 4, 5 and 6 are devoted to the proof of Theorem 2.2. In Section 4, the general framework of the proof is exposed; in Section 5, the central part of the CLT and of the identification of the variance are established; remaining proofs are provided in Section 6. Finally, proof of Proposition 2.3 (bias) is provided in Section 7.

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2. The Central Limit Theorem for $\mathcal{I}_n(\rho)$

2.1. **Notations, assumptions and first-order results.** Let $\mathbf{i}=\sqrt{-1}$. As usual, $\mathbb{R}^+ = \{x\in\mathbb{R}\ :\ x\geq 0\}$. Denote by $\xrightarrow{\mathcal{P}}$ the convergence in probability of random variables and by $\xrightarrow{\mathcal{D}}$ the convergence in distribution of probability measures. Denote by $\mathrm{diag}(a_i;\,1\leq i\leq k)$ the $k\times k$ diagonal matrix whose diagonal entries are the a_i's. Element (i,j) of matrix M will be either denoted m_{ij} or $[M]_{ij}$ depending on the notational context. If M is a $n\times n$ square matrix, $\mathrm{diag}(M)=\mathrm{diag}(m_{ii};1\leq i\leq n)$. Denote by M^T the matrix transpose of M, by M^* its Hermitian adjoint, by $\bar{M}$ the (elementwise) conjugate of matrix M, by $\mathrm{Tr}(M)$ its trace and $\det(M)$ its determinant (if M is square). When dealing with vectors, $\|\cdot\|$ will refer to the Euclidean norm. In the case of matrices, $\|\cdot\|$ will refer to the spectral norm. We shall denote by K a generic constant that does not depend on n and that might change from a line to another. If (u_n) is a sequence of real numbers, then $u_n = \mathcal{O}(v_n)$ stands for $|u_n| \le K|v_n|$ where constant K does not depend on n.

Recall that

$$\Sigma_n = \frac{1}{\sqrt{n}} D_n^{1/2} X_n \tilde{D}_n^{1/2} + A_n \ , \tag{2.1}$$

denote $D_n = \mathrm{diag}(d_i,\ 1 \le i \le N)$ and $\tilde{D}_n = \mathrm{diag}(\tilde{d}_j,\ 1 \le j \le n)$. When no confusion can occur, we shall often drop subscripts and superscripts n for readability. Recall also that the asymptotic regime of interest is:

$$n \to \infty \quad \text{and} \quad 0 \ < \ \liminf \frac{N}{n} \ \le \ \limsup \frac{N}{n} \ < \infty \ ,$$

and will be simply denoted by $n \to \infty$ in the sequel. We can assume without loss of generality that there exist nonnegative real numbers $\boldsymbol{\ell}^-$ and $\boldsymbol{\ell}^+$ such that:

$$0 \ < \ \boldsymbol{\ell}^- \ \le \ \frac{N}{n} \ \le \ \boldsymbol{\ell}^+ \ < \ \infty \quad \text{as} \quad n \to \infty \ . \tag{2.2}$$

Assumption A-1. *The random variables $(X_{ij}^n\ ;\ 1 \le i \le N,\ 1 \le j \le n\ ,\ n \ge 1)$ are complex, independent and identically distributed. They satisfy*

$$\mathbb{E}X_{11}^n = 0, \quad \mathbb{E}|X_{11}^n|^2 = 1 \quad \text{and} \quad \mathbb{E}|X_{11}^n|^{16} < \infty \ .$$

Remark 2.1. (Gaussian distributions) If X_{11} is a standard complex or real Gaussian random variable, then $\kappa = 0$. More precisely, in the complex case, $\mathrm{Re}(X_{11})$ and $\mathrm{Im}(X_{11})$ are independent real Gaussian random variables, then $\vartheta = \kappa = 0$; in the real case, then $\vartheta = 1$ while $\kappa = 0$.

Assumption A-2. *The family of deterministic $N \times n$ complex matrices $(A_n, n \ge 1)$ is bounded for the spectral norm:*

$$\boldsymbol{a}_{\max} = \sup_{n \ge 1} \|A_n\| < \infty \ .$$

Assumption A-3. *The families of real deterministic $N \times N$ and $n \times n$ matrices (D_n) and $(\tilde{D}_n)$ are diagonal with non-negative diagonal elements, and are bounded for the spectral norm as $n \to \infty$:*

$$\boldsymbol{d}_{\max} = \sup_{n \ge 1} \|D_n\| < \infty \quad \textit{and} \quad \tilde{\boldsymbol{d}}_{\max} = \sup_{n \ge 1} \|\tilde{D}_n\| < \infty \ .$$

Moreover,

$$\boldsymbol{d}_{\min} = \inf_n \frac{1}{n} \mathrm{Tr}\, D_n > 0 \quad \textit{and} \quad \tilde{\boldsymbol{d}}_{\min} = \inf_n \frac{1}{n} \mathrm{Tr}\, \tilde{D}_n > 0 \ .$$

Theorem 2.1 (First order results - [20, 13])**.** *Consider the $N \times n$ matrix Σ_n given by* (2.1) *and assume that* **A-1**, **A-2** *and* **A-3** *hold true. Then, the system* (1.2) *admits a unique solution $(\delta_n, \tilde{\delta}_n)$ in the class of Stieltjes transforms of nonnegative measures. Moreover,*

$$\frac{1}{n}\mathrm{Tr}\ (Q_n(z) - T_n(z)) \xrightarrow[n\to\infty]{a.s.} 0 \quad \textit{and} \quad \frac{1}{n}\mathrm{Tr}\ \left(\tilde{Q}_n(z) - \tilde{T}_n(z)\right) \xrightarrow[n\to\infty]{a.s.} 0 \quad \textit{for any } z \in \mathbb{C} - \mathbb{R}^+.$$

2.2. **The Central Limit Theorem.** In this section, we state the CLT then provide the asymptotic bias in some particular cases.

Theorem 2.2 (The CLT). *Consider the $N \times n$ matrix Σ_n given by* (2.1) *and assume that* **A-1**, **A-2** *and* **A-3** *hold true. Recall the definitions of δ and $\tilde{\delta}$ given by* (1.2)*, T and $\tilde{T}$ given by* (1.3)*, γ, $\tilde{\gamma}$, $\underline{\gamma}$ and $\underline{\tilde{\gamma}}$ given by* (1.6)*. Let $\rho > 0$. All the considered quantities are evaluated at $z = -\rho$. Define Δ_n and $\underline{\Delta}_n$ as*

$$\Delta_n = \left(1 - \frac{1}{n} \operatorname{Tr} D^{\frac{1}{2}} T A (I + \delta \tilde{D})^{-2} \tilde{D} A^* T D^{\frac{1}{2}}\right)^2 - \rho^2 \gamma \tilde{\gamma}$$

and

$$\underline{\Delta}_n = \left| 1 - \vartheta \frac{1}{n} \operatorname{Tr} D^{\frac{1}{2}} \bar{T} \bar{A} (I + \delta \tilde{D})^{-2} \tilde{D} A^* T D^{\frac{1}{2}} \right|^2 - |\vartheta|^2 \rho^2 \underline{\gamma}\, \underline{\tilde{\gamma}}.$$

Then the real numbers

$$\Theta_n = -\log \Delta_n - \log \underline{\Delta}_n + \kappa \frac{\rho^2}{n^2} \sum_{i=1}^N d_i^2 t_{ii}^2 \sum_{j=1}^n \tilde{d}_j^2 \tilde{t}_{jj}^2 \tag{2.3}$$

are well-defined and satisfy:

$$0 \quad < \quad \liminf_n \Theta_n \quad \leq \quad \limsup_n \Theta_n \quad < \quad \infty \tag{2.4}$$

as $n \to \infty$. Let

$$\mathcal{I}_n(\rho) = \frac{1}{N} \log \det \left(\Sigma_n \Sigma_n^* + \rho I_N \right) \ ,$$

then the following convergence holds true:

$$\frac{N}{\sqrt{\Theta_n}} \left(\mathcal{I}_n(\rho) - \mathbb{E} \mathcal{I}_n(\rho) \right) \xrightarrow[n \to \infty]{\mathcal{D}} \mathcal{N}(0,1) \ .$$

Remark 2.2. (Simpler forms for the variance) We consider here special cases where the variance Θ_n takes a simpler form.

(1) The standard complex Gaussian case. Assume that the X_{ij}'s are standard complex Gaussian random variables, i.e. that both the real and imaginary parts of X_{ij} are independent real Gaussian random variables, each with variance 1/2. In this case, $\vartheta = \kappa = 0$ and Θ_n is equal to $-\log \Delta_n$, and we in particular recover the variance formula given in [36].
(2) The standard real case. Assume that the X_{ij}'s are standard real random variables, assume also that A has real entries. Then Δ_n and $\underline{\Delta}_n$ are equal.
(3) The 'signal plus noise' model. In this case, $D_n = I_N$ and $\tilde{D}_n = I_n$, which already yields simplifications in the variance expression. In the case where $\vartheta = 0$, the variance is:

$$\Theta_n = -\log\left(\left(1 - n^{-1}(1+\delta)^{-2} \operatorname{Tr} T A A^* T\right)^2 - \rho^2 \gamma \tilde{\gamma} \right) + \frac{\kappa \rho^2}{n^2} \sum_i d_i^2 t_{ii}^2 \sum_j \tilde{d}_j^2 \tilde{t}_{jj}^2 \ .$$

As one may easily check, the first term of the variance only depends upon the spectrum of AA^*. The second term however also depends on the eigenvectors of AA^* (see for instance [26]).

A full study of the asymptotic bias turns out to be extremely involved and would have substantially increased the volume of this paper. In the following proposition, we restrict our study to two important particular cases:

Proposition 2.3 (The bias - particular cases). *Assume that the setting of Theorem 2.2 holds true.*

(i) *If the random variables* $(X_{ij}^n; i, j, n)$ *are complex with* $\mathrm{Re}(X_{ij}^n)$ *and* $\mathrm{Im}(X_{ij}^n)$ *independent, both with distribution* $\mathcal{N}(0, 1/2)$*, then:*

$$N\left(\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho)\right) = \mathcal{O}\left(\frac{1}{N}\right) .$$

(ii) *If* $A_n = 0$*, let the quantities* γ *and* $\tilde{\gamma}$ *be evaluated at* $z = -\rho$ *and consider*

$$\mathcal{B}_n = -\frac{\kappa}{2}\rho^2\gamma\tilde{\gamma} + \frac{1}{2}\log(1 - |\vartheta|^2\rho^2\gamma\tilde{\gamma}). \tag{2.5}$$

Then

$$N\left(\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho)\right) - \mathcal{B}_n \xrightarrow[n\to\infty]{} 0.$$

Remark 2.3. Observe that $T(z) = [-z(I + \tilde{\delta}(z)D)]^{-1}$ and $\tilde{T}(z) = [-z(I + \delta(z)\tilde{D})]^{-1}$ when $A = 0$. It is interesting to notice that $\mathcal{B}_n$ coincides in that case with $-0.5 \times$ the sum of the two last terms at the right hand side (r.h.s.) of (2.3).

Proof of Proposition 2.3 is deferred to Section 7.

3. Notations and classical results

3.1. **Further notations.** We denote by Y the $N \times n$ matrix $n^{-1/2}D^{1/2}X\tilde{D}^{1/2}$; by (η_j), (a_j) and (y_j) the columns of matrices Σ, A and Y. Denote by Σ_j, A_j, and Y_j the matrices Σ, A, and Y where column j has been removed. The associated resolvent is $Q_j(z) = (\Sigma_j\Sigma_j^* - zI_N)^{-1}$. We shall often write Q, Q_j, T for $Q(z)$, $Q_j(z)$, $T(z)$, etc. We denote by $\tilde{D}_j$ matrix $\tilde{D}$ where row and column j have been removed. We also denote by $A_{1:j}$ and $\Sigma_{1:j}$ the $N \times j$ matrices $A_{1:j} = [a_1, \cdots, a_j]$ and $\Sigma_{1:j} = [\eta_1, \cdots, \eta_j]$. Denote by $\mathbb{E}_j$ the conditional expectation with respect to the σ-field $\mathcal{F}_j$ generated by the vectors $(y_\ell, 1 \leq \ell \leq j)$. By convention, $\mathbb{E}_0 = \mathbb{E}$.

We introduce here intermediate quantities of constant use in the rest of the paper. For $1 \leq j \leq n$, let:

$$\begin{aligned}
\tilde{b}_j(z) &= \frac{-1}{z\left(1 + a_j^* Q_j(z) a_j + \frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j(z)\right)} , \\
e_j(z) &= \eta_j^* Q_j(z)\eta_j - \left(\frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j(z) + a_j^* Q_j(z) a_j\right) \\
&= y_j^* Q_j(z) y_j - \frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j(z) + a_j^* Q_j(z) y_j + y_j^* Q_j(z) a_j.
\end{aligned} \tag{3.1}$$

3.2. **Important identities.** Recall the following classical identity for the inverse of a perturbed matrix (see [23, Section 0.7.4]):

$$(A + XRY)^{-1} = A^{-1} - A^{-1}X\left(R^{-1} + YA^{-1}X\right)^{-1}YA^{-1} . \tag{3.2}$$

Identities involving the resolvents. The following identity expresses the diagonal elements $\tilde{q}_{jj}(z) = [\tilde{Q}(z)]_{jj}$ of the co-resolvent; the two following ones are obtained from (3.2).

$$\tilde{q}_{jj}(z) = \frac{-1}{z(1+\eta_j^* Q_j(z)\eta_j)} \ , \tag{3.3}$$

$$\begin{aligned} Q(z) &= Q_j(z) - \frac{Q_j(z)\eta_j\eta_j^* Q_j(z)}{1+\eta_j^* Q_j(z)\eta_j} \\ &= Q_j(z) + z\tilde{q}_{jj}(z)Q_j(z)\eta_j\eta_j^* Q_j(z) \end{aligned} \tag{3.4}$$

$$Q_j(z) = Q(z) + \frac{Q(z)\eta_j\eta_j^* Q(z)}{1-\eta_j^* Q(z)\eta_j} \tag{3.5}$$

$$1+\eta_j^* Q_j(z)\eta_j = \frac{1}{1-\eta_j^* Q(z)\eta_j} \tag{3.6}$$

Notice that

$$\tilde{q}_{jj}(z) = \tilde{b}_j(z) + z\tilde{q}_{jj}(z)\tilde{b}_j(z)e_j(z) \ . \tag{3.7}$$

and that $0 < \tilde{b}_j(-\rho), \tilde{q}_{jj}(-\rho) < \rho^{-1}$. These facts will be repeatedly used in the remainder. A useful consequence of (3.4) is:

$$\eta_j^* Q(z) = \frac{\eta_j^* Q_j(z)}{1+\eta_j^* Q_j(z)\eta_j} = -z\tilde{q}_{jj}(z)\eta_j^* Q_j(z) \ . \tag{3.8}$$

Identities involving the deterministic equivalents T and $\tilde{T}$. Define the $N \times N$ matrix $\mathcal{T}_j(z)$ as

$$\mathcal{T}_j(z) = \Big(-z(I_N + \tilde{\delta}(z)D) + A_j(I_{n-1} + \delta(z)\tilde{D}_j)^{-1}A_j^*\Big)^{-1} \ , \tag{3.9}$$

where δ and $\tilde{\delta}$ are defined in (1.2). Notice that matrix $\mathcal{T}_j$ is not obtained in general by solving the analogue of system (1.3) where A is replaced with A_j and when $\tilde{D}$ is truncated accordingly. This matrix naturally pops up when expressing the diagonal elements $\tilde{t}_{jj}$ of $\tilde{T}$. Indeed, we obtain (see Appendix A.1):

$$\tilde{t}_{jj}(z) = \frac{-1}{z\Big(1 + a_j^* \mathcal{T}_j(z) a_j + \tilde{d}_j \delta(z)\Big)} \ . \tag{3.10}$$

Let b be a given $N \times 1$ vector. The following identity is also shown in Appendix A.1:

$$-z\tilde{t}_{\ell\ell}(z) a_\ell^* \mathcal{T}_\ell(z) b = \frac{a_\ell^* T(z) b}{1+\tilde{d}_\ell \delta(z)}. \tag{3.11}$$

Thanks to (3.2), we also have

$$\tilde{T}(z) = -z^{-1}(I+\delta(z)\tilde{D})^{-1} + z^{-1}(I+\delta(z)\tilde{D})^{-1}A^*T(z)A(I+\delta(z)\tilde{D})^{-1}. \tag{3.12}$$

3.3. **Important estimates.** We gather in this section matrix estimates which will be of constant use in the sequel. In all the remainder, z will belong to the open negative real axis, and will be fixed to $z = -\rho$ until Section 7.

Let A and B be two square matrices. Then

$$|\mathrm{Tr}(AB)| \le \sqrt{\mathrm{Tr}(AA^*)}\sqrt{\mathrm{Tr}(BB^*)} \tag{3.13}$$

When B is Hermitian non negative, then a consequence of Von Neumann's trace theorem is

$$|\operatorname{Tr}(AB)| \quad \leq \quad \|A\| \operatorname{Tr} B\ . \tag{3.14}$$

The following lemma gives an estimate for a rank-one perturbation of the resolvent ([21, Lemma 6.3] and [32, Lemma 2.6]):

Lemma 3.1. *The resolvents Q and the perturbed resolvent Q_j satisfy for $z = -\rho$:*

$$|\mathrm{Tr}\ A\,(Q - Q_j)| \leq \frac{\|A\|}{\rho}$$

for any $N \times N$ matrix A.

The following results describe the asymptotic behaviour of quadratic forms based on the resolvent.

Lemma 3.2 (Bai and Silverstein, Lemma 2.7 in [3])**.** *Let $\boldsymbol{x} = (x_1, \cdots, x_n)$ be a $n \times 1$ vector where the x_i are centered i.i.d. complex random variables with unit variance. Let M be a $n \times n$ deterministic complex matrix. Then for any $p \geq 2$, there exists a constant K_p for which*

$$\mathbb{E}|\boldsymbol{x}^* M \boldsymbol{x} - \operatorname{Tr} M|^p \leq K_p \left(\left(\mathbb{E}|x_1|^4 \operatorname{Tr} MM^* \right)^{p/2} + \mathbb{E}|x_1|^{2p} \operatorname{Tr}(MM^*)^{p/2} \right)$$

Remark 3.1. There are some important consequences of the previous lemma. Let (M_n) be a sequence of $n \times n$ deterministic matrices with bounded spectral norm and $(\boldsymbol{x}_n)$ be a sequence of random vectors as in the statement of Lemma 3.2. Then for any $p \in [2; 8]$,

$$\max\left(\mathbb{E}\left| \frac{\boldsymbol{x}_n^* M_n \boldsymbol{x}_n}{n} - \frac{\operatorname{Tr} M_n}{n} \right|^p, \mathbb{E}|e_j|^p \right) \leq \frac{K}{n^{p/2}} \tag{3.15}$$

where e_j is given by (3.1) (the estimate $\mathbb{E}|e_j|^p = \mathcal{O}(n^{-p/2})$ is proven in Appendix A.2).

Remark 3.2. By replacing $\mathbb{E}|x_1|^4$ with $\max_i \mathbb{E}|x_i|^4$ and $\mathbb{E}|x_1|^{2p}$ with $\max_i \mathbb{E}|x_i|^{2p}$, Lemma 3.2 can be extended to the case where elements of vector $\boldsymbol{x}$ are independent but not necessarily identically distributed [5, Lemma B.26]. Accordingly, the results of this paper remain true when the X_{ij} are independent but not necessarily identically distributed, provided $\mathbb{E}|X_{ij}|^2 = 1$, $\mathbb{E}X_{ij}^2 = \vartheta$, and $\mathbb{E}|X_{ij}|^4 - 2 - |\vartheta|^2 = \kappa$ for all i, j, and $\sup_n \max_{i,j} \mathbb{E}|X_{ij}|^{16} < \infty$.

The following theorem is proven is Appendix A.3:

Theorem 3.3. *Assume that the setting of Theorem 2.2 holds true. Let (u_n) and (v_n) be two sequences of deterministic complex $N \times 1$ vectors bounded in the Euclidean norm:*

$$\sup_{n \geq 1} \max\left(\|u_n\|, \|v_n\|\right) < \infty,$$

and let (U_n) be a sequence of deterministic $N \times N$ matrices with bounded spectral norms:

$$\sup_{n \geq 1} \|U_n\| < \infty.$$

Then,

(1) *There exists a constant K for which*

$$\sum_{j=1}^{n} \mathbb{E}|u_n^* Q_j a_j|^2 \ \leq \ K.$$

(2) *The following holds true:*

$$\left|\frac{1}{n}\operatorname{Tr} U(T-\mathbb{E}Q)\right| \le \frac{K}{n}\ .$$

(3) *For every $p\in[1,2]$, there exists a constant K_p such that:*

$$\mathbb{E}\left|u_n^*(Q-T)v_n\right|^{2p} \le \frac{K_p}{n^p}\ .$$

(4) *For every $p\in[1,2]$, there exists a constant K_p such that:*

$$\mathbb{E}\left|u_n^*(Q_j-\mathcal{T}_j)v_n\right|^{2p} \le \frac{K_p}{n^p}\ .$$

(5) *There exists a constant K such that*

$$\mathbb{E}\left|\operatorname{Tr} U(Q-\mathbb{E}Q)\right|^2 < K.$$

The following results stem from Lemma 3.2 and Theorem 3.3 and will be of constant use in the sequel. Recalling (3.7) and (3.15) along with the bounds on $\tilde{q}_{jj}$ and $\tilde{b}_j$, we have for any $p\in[2,8]$

$$\mathbb{E}\left|\tilde{q}_{jj}-\tilde{b}_j\right|^p \le \frac{K}{n^{p/2}}. \tag{3.16}$$

Of course, the counterpart of Theorem 3.3 for the co-resolvent $\tilde{Q}$ and matrix $\tilde{T}$ holds true. In particular, taking the vectors u_n and v_n as the jth canonical vector of $\mathbb{C}^n$ yields the following estimate for any $p\in[2,4]$:

$$\mathbb{E}\left|\tilde{q}_{jj}-\tilde{t}_{jj}\right|^p \le \frac{K}{n^{p/2}}. \tag{3.17}$$

The following two lemmas, proven in Appendices A.4 and A.5, provide some important bounds:

Lemma 3.4. *Assume that the setting of Theorem 2.2 holds true. Then, the following quantities satisfy:*

$$\frac{\boldsymbol{d}_{\min}}{\rho+\boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max}+\boldsymbol{a}^2_{\max}} \le \delta_n \le \frac{\boldsymbol{\ell}^+\boldsymbol{d}_{\max}}{\rho}\ ,\quad \frac{\tilde{\boldsymbol{d}}_{\min}}{\rho+\boldsymbol{\ell}^+\boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max}+\boldsymbol{a}^2_{\max}} \le \tilde{\delta}_n \le \frac{\tilde{\boldsymbol{d}}_{\max}}{\rho}\ ,$$

$$\frac{\boldsymbol{d}_{\min}}{(\rho+\boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max}+\boldsymbol{a}^2_{\max})^2} \le \frac{1}{n}\operatorname{Tr} DT^2 \le \frac{\boldsymbol{\ell}^+\boldsymbol{d}_{\max}}{\rho^2}\ ,\quad \frac{\tilde{\boldsymbol{d}}_{\min}}{(\rho+\boldsymbol{\ell}^+\boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max}+\boldsymbol{a}^2_{\max})^2} \le \frac{1}{n}\operatorname{Tr}\tilde{D}\tilde{T}^2 \le \frac{\tilde{\boldsymbol{d}}_{\max}}{\rho^2}\ ,$$

$$\frac{\boldsymbol{d}^2_{\min}}{\boldsymbol{\ell}^+(\rho+\boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max}+\boldsymbol{a}^2_{\max})^2} \le \gamma_n \le \frac{\boldsymbol{\ell}^+\boldsymbol{d}^2_{\max}}{\rho^2}\ ,\quad \frac{\tilde{\boldsymbol{d}}^2_{\min}}{\left(\rho+\boldsymbol{\ell}^+\boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max}+\boldsymbol{a}^2_{\max}\right)^2} \le \tilde{\gamma}_n \le \frac{\tilde{\boldsymbol{d}}^2_{\max}}{\rho^2}$$

$$\frac{\boldsymbol{d}^2_{\min}}{\boldsymbol{\ell}^+(\rho+\boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max}+\boldsymbol{a}^2_{\max})^2} \le \frac{1}{n}\sum_{i=1}^N d_i^2t_{ii}^2 \le \frac{\boldsymbol{\ell}^+\boldsymbol{d}^2_{\max}}{\rho^2},\quad \frac{\tilde{\boldsymbol{d}}^2_{\min}}{\left(\rho+\boldsymbol{\ell}^+\boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max}+\boldsymbol{a}^2_{\max}\right)^2} \le \frac{1}{n}\sum_{j=1}^n \tilde{d}_j^2\tilde{t}_{jj}^2 \le \frac{\tilde{\boldsymbol{d}}^2_{\max}}{\rho^2}\ .$$

Lemma 3.5. *Assume that the setting of Theorem 2.2 holds true. Then*

$$\sup_n \frac{1}{n}\operatorname{Tr} D^{1/2}TA(I+\delta\tilde{D})^{-2}\tilde{D}A^*TD^{\frac{1}{2}}\ <\ 1.$$

Moreover, the sequence (Δ_n) as defined in Theorem 2.2 satisfies

$$\Delta_n \ge \frac{\rho}{n\delta}\operatorname{Tr} DT^2\frac{\rho}{n\tilde{\delta}}\operatorname{Tr}\tilde{D}\tilde{T}^2$$

and

$$\liminf_n \Delta_n > 0\ .$$

3.4. **Other important results.** The main result we shall rely on to establish the Central Limit Theorem is the following CLT for martingales:

Theorem 3.6 (CLT for martingales, Th. 35.12 in [6])**.** *Let $\gamma_1^{(n)}, \gamma_2^{(n)}, \dots, \gamma_n^{(n)}$ be a martingale difference sequence with respect to the increasing filtration $\mathcal{F}_1^{(n)}, \dots, \mathcal{F}_n^{(n)}$. Assume that there exists a sequence of real positive numbers Υ_n^2 such that*

$$\frac{1}{\Upsilon_n^2} \sum_{j=1}^n \mathbb{E}_{j-1} {\gamma_j^{(n)}}^2 \xrightarrow[n\to\infty]{\mathcal{P}} 1 \ . \tag{3.18}$$

Assume further that the Lyapounov condition ([6, Section 27]*) holds true:*

$$\exists \delta > 0, \quad \frac{1}{\Upsilon_n^{2(1+\delta)}} \sum_{j=1}^n \mathbb{E} \left| \gamma_j^{(n)} \right|^{2+\delta} \xrightarrow[n\to\infty]{} 0 \ .$$

Then $\Upsilon_n^{-1} \sum_{j=1}^n \gamma_j^{(n)}$ converges in distribution to $\mathcal{N}(0,1)$.

Remark 3.3. Note that if moreover $\liminf_n \Upsilon_n^2 > 0$, it is sufficient to prove:

$$\sum_{j=1}^n \mathbb{E}_{j-1} {\gamma_j^{(n)}}^2 - \Upsilon_n^2 \xrightarrow[n\to\infty]{\mathcal{P}} 0 \ , \tag{3.19}$$

instead of (3.18).

We now state a covariance identity (the proof of which is straightforward and therefore omitted) for quadratic forms based on non-centered vectors. This identity explains to some extent the various terms obtained in the variance.

Let $\boldsymbol{x} = (x_1, \cdots, x_N)^T$ be a $N \times 1$ vector where the x_i are centered i.i.d. complex random variables with unit variance. Let $\boldsymbol{y} = N^{-1/2} D^{1/2} \boldsymbol{x}$ where D is a $N \times N$ diagonal nonnegative deterministic matrix. Let $M = (m_{ij})$ and $P = (p_{ij})$ be $N \times N$ deterministic complex matrices and let $\boldsymbol{u}$ be a $N \times 1$ deterministic vector.

If M is an $N \times N$ matrix, $\mathrm{vdiag}(M)$ stands for the $N \times 1$ vector $[M_{11}, \cdots, M_{NN}]^T$.

Denote by $\Upsilon(M)$ the random variable:

$$\Upsilon(M) = (\boldsymbol{y} + \boldsymbol{u})^* M (\boldsymbol{y} + \boldsymbol{u}) \ .$$

Then $\mathbb{E}\Upsilon(M) = \frac{1}{N} \mathrm{Tr}\, DM + \boldsymbol{u}^* M \boldsymbol{u}$ and the covariance between $\Upsilon(M)$ and $\Upsilon(P)$ is:

$$\begin{aligned}
&\mathbb{E}\left[\left(\Upsilon(M) - \mathbb{E}\Upsilon(M) \right) \left(\Upsilon(P) - \mathbb{E}\Upsilon(P) \right) \right] \\
&= \frac{1}{N^2} \mathrm{Tr}(MDPD) + \frac{1}{N} \left(\boldsymbol{u}^* MDP \boldsymbol{u} + \boldsymbol{u}^* PDM \boldsymbol{u} \right) \\
&\quad + \frac{|\mathbb{E}[x_1^2]|^2}{N^2} \mathrm{Tr}(MDP^T D) + \frac{\mathbb{E}[x_1^2]}{N} \boldsymbol{u}^* PDM^T \bar{\boldsymbol{u}} + \frac{\mathbb{E}[\bar{x}_1^2]}{N} \boldsymbol{u}^T M^T DP \boldsymbol{u} \\
&\quad + \frac{\mathbb{E}[|x_1|^2 x_1]}{N^{3/2}} \left(\boldsymbol{u}^* PD^{3/2} \mathrm{vdiag}(M) + \boldsymbol{u}^* MD^{3/2} \mathrm{vdiag}(P) \right) \\
&\quad + \frac{\mathbb{E}[|x_1|^2 \bar{x}_1]}{N^{3/2}} \left(\mathrm{vdiag}(P)^T D^{3/2} M \boldsymbol{u} + \mathrm{vdiag}(M)^T D^{3/2} P \boldsymbol{u} \right) \\
&\quad + \frac{\kappa}{N^2} \sum_{i=1}^N d_{ii}^2 m_{ii} p_{ii} \ ,
\end{aligned} \tag{3.20}$$

where $\kappa = \mathbb{E}|x_1|^4 - 2 - |\mathbb{E}x_1^2|^2$.

Remark 3.4. Identity (3.20) is the cornerstone for the proof of the CLT; it is the counterpart of Identity (1.15) in [4]. The complexity of Identity (3.20) with respect to [4, Identity (1.15)] lies in 8 extra terms and stems from two elements:

(1) The fact that matrix Σ is non-centered.
(2) The fact that the random variables X_{ij}'s are either real and complex with no particular assumption on their second moment (in particular, $\mathbb{E}X_{ij}^2$ can be non zero in the complex case).

It is this identity which induces to a large extent all the computations in the present article.

4. Proof of Theorem 2.2 (part I)

Decomposition of $\mathcal{I}_n - \mathbb{E}\mathcal{I}_n$, Cumulant and cross-moments terms in the variance

4.1. Decomposition of $\mathcal{I}_n - \mathbb{E}\mathcal{I}_n$ as a sum of martingale differences.

Denote by

$$\Gamma_j = \frac{\eta_j^* Q_j \eta_j - \left(\frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j + a_j^* Q_j a_j\right)}{1 + \frac{\tilde{d}_j}{n}\operatorname{Tr} DQ_j + a_j^* Q_j a_j} \ .$$

With this notation at hand, the decomposition of $\mathcal{I}_n - \mathbb{E}\mathcal{I}_n$ as

$$\mathcal{I}_n - \mathbb{E}\mathcal{I}_n = \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) \log(1 + \Gamma_j) \tag{4.1}$$

follows verbatim from [21, Section 6.2]. Moreover, it is a matter of bookkeeping to establish the following (cf. [21, Section 6.4]):

$$\sum_{j=1}^n \mathbb{E}_{j-1}\left((\mathbb{E}_j - \mathbb{E}_{j-1})\log(1+\Gamma_j)\right)^2 - \sum_{j=1}^n \mathbb{E}_{j-1}(\mathbb{E}_j \Gamma_j)^2 \xrightarrow[n\to\infty]{\mathcal{P}} 0 \ . \tag{4.2}$$

Hence, the details are omitted. In view of Theorem 3.6, Eq. (3.19), (4.1) and (4.2), the CLT will be established if one proves the following 3 results:

(1) (Lyapounov condition)

$$\exists \delta > 0, \quad \sum_{j=1}^n \mathbb{E}\left|\mathbb{E}_j \Gamma_j\right|^{2+\delta} \xrightarrow[n\to\infty]{} 0 \ ,$$

(2) (Martingale increments and variance)

$$\sum_{j=1}^n \mathbb{E}_{j-1}(\mathbb{E}_j \Gamma_j)^2 - \Theta_n \xrightarrow[n\to\infty]{\mathcal{P}} 0 \ .$$

(3) (estimates over the variance)

$$0 < \liminf_n \Theta_n \le \limsup_n \Theta_n < \infty$$

It is straightforward (and hence omitted) to verify Lyapounov condition. The convergence toward the variance is the cornerstone of the proof of the CLT: The rest of this section together with much of Section 5 are devoted to establish it. The estimates over the variance Θ_n, also central to apply Theorem 3.6, are established in Section 6.2.

Notice that $\mathbb{E}_{j-1}(\mathbb{E}_j\Gamma_j)^2 = \mathbb{E}_{j-1}(\mathbb{E}_j\rho\tilde{b}_j e_j)^2$. We prove hereafter that

$$\sum_{j=1}^{n}\mathbb{E}_{j-1}(\mathbb{E}_j\rho\tilde{b}_j e_j)^2 - \sum_{j=1}^{n}\rho^2\tilde{t}_{jj}^2\mathbb{E}_{j-1}(\mathbb{E}_j e_j)^2 \xrightarrow[n\to\infty]{\mathcal{P}} 0\ . \tag{4.3}$$

The inequality $\mathbb{E}|\tilde{b}_j - \tilde{t}_{jj}|^2 \le 2\mathbb{E}|\tilde{b}_j - \tilde{q}_{jj}|^2 + 2\mathbb{E}|\tilde{q}_{jj} - \tilde{t}_{jj}|^2$ in conjunction with Estimates (3.16) and (3.17) yield $\mathbb{E}|\tilde{b}_j - \tilde{t}_{jj}|^2 = \mathcal{O}(n^{-1})$. Moreover,

$$\begin{aligned}
&\mathbb{E}\left|\mathbb{E}_{j-1}(\mathbb{E}_j\rho\tilde{b}_j e_j)^2 - \mathbb{E}_{j-1}(\rho\tilde{t}_{jj}\mathbb{E}_j e_j)^2\right| \le \mathbb{E}\left|\left(\mathbb{E}_j\rho\tilde{b}_j e_j\right)^2 - \left(\mathbb{E}_j\rho\tilde{t}_{jj}e_j\right)^2\right| \\
&\quad = \mathbb{E}\left|\left(\mathbb{E}_j(\rho\tilde{b}_j - \rho\tilde{t}_{jj})e_j\right)\left(\mathbb{E}_j(\rho\tilde{b}_j + \rho\tilde{t}_{jj})e_j\right)\right| \le \mathbb{E}\left|(\rho\tilde{b}_j - \rho\tilde{t}_{jj})e_j\left(\mathbb{E}_j(\rho\tilde{b}_j + \rho\tilde{t}_{jj})e_j\right)\right| \\
&\qquad\qquad \le Kn^{-3/2}
\end{aligned}$$

using Cauchy-Schwarz inequality and (3.15). This implies (4.3). Let $\varsigma = \mathbb{E}(|X_{11}^2|X_{11})$. Using Identity (3.20), we develop the quantity $\mathbb{E}_{j-1}(\mathbb{E}_j e_j)^2$:

$$\begin{aligned}
&\sum_{j=1}^{n}\rho^2\tilde{t}_{jj}^2\mathbb{E}_{j-1}(\mathbb{E}_j e_j)^2 \\
&= \frac{\kappa}{n^2}\sum_{j=1}^{n}\rho^2\tilde{d}_j^2\tilde{t}_{jj}^2\sum_{i=1}^{N}d_i^2[\mathbb{E}_j Q_j]_{ii}^2 \\
&\quad + \frac{4}{n}\sum_{j=1}^{n}\rho^2\tilde{d}_j^{3/2}\tilde{t}_{jj}^2\,\mathrm{Re}\left(\varsigma\frac{a_j^*(\mathbb{E}_j Q_j)D^{3/2}\mathrm{vdiag}(\mathbb{E}_j Q_j)}{\sqrt{n}}\right) \\
&\quad + \frac{1}{n}\sum_{j=1}^{n}\rho^2\tilde{t}_{jj}^2\left(\frac{\tilde{d}_j^2}{n}\mathrm{Tr}(\mathbb{E}_j Q_j)D(\mathbb{E}_j Q_j)D + 2\tilde{d}_j a_j^*(\mathbb{E}_j Q_j)D(\mathbb{E}_j Q_j)a_j\right) \\
&\quad + \frac{1}{n}\sum_{j=1}^{n}\rho^2\tilde{t}_{jj}^2\left(|\vartheta|^2\frac{\tilde{d}_j^2}{n}\mathrm{Tr}(\mathbb{E}_j Q_j)D(\mathbb{E}_j\bar{Q}_j)D + 2\,\mathrm{Re}\left(\vartheta\tilde{d}_j a_j^*(\mathbb{E}_j Q_j)D(\mathbb{E}_j\bar{Q}_j)\bar{a}_j\right)\right) \\
&\triangleq \sum_{j=1}^{n}\chi_{1j} + \sum_{j=1}^{n}\chi_{2j} + \sum_{j=1}^{n}\chi_{3j} + \sum_{j=1}^{n}\chi_{4j}\ .
\end{aligned}$$

4.2. **Key lemmas for the identification of the variance.** The remainder of the proof of Theorem 2.2 is devoted to find deterministic equivalents for the terms $\sum_{j=1}^n \chi_{\ell j}$ for $\ell = 1,2,3,4$.

Lemma 4.1. *Assume that the setting of Theorem 2.2 holds true, then:*

$$\sum_{j=1}^{n}\chi_{1j} - \frac{\kappa\rho^2}{n^2}\sum_{i=1}^{N}\sum_{j=1}^{n}d_i^2 t_{ii}^2\tilde{d}_j^2\tilde{t}_{jj}^2 \xrightarrow[n\to\infty]{\mathcal{P}} 0\ .$$

Proof. Write

$$\frac{1}{n}\sum_{i=1}^{N} d_i^2[\mathbb{E}_j Q_j]_{ii}^2 - \frac{1}{n}\sum_{i=1}^{N} d_i^2[\mathbb{E}_j Q_j]_{ii} t_{ii} = \frac{1}{n}\sum_{i=1}^{N} d_i^2[\mathbb{E}_j Q_j]_{ii}\mathbb{E}_j([Q_j]_{ii}-[Q]_{ii}) + \frac{1}{n}\sum_{i=1}^{N} d_i^2[\mathbb{E}_j Q_j]_{ii}([\mathbb{E}_j Q]_{ii}-t_{ii}) = \varepsilon_{1,j}+\varepsilon_{2,j}.$$

The term $|\varepsilon_{1,j}| = n^{-1}|\mathbb{E}_j[\operatorname{Tr} D^2 \operatorname{diag}(\mathbb{E}_j Q_j)(Q_j - Q)]|$ is of order $\mathcal{O}(n^{-1})$ thanks to Lemma 3.1. Moreover, $\mathbb{E}|\varepsilon_{2,j}| = \mathcal{O}(n^{-1/2})$ by the analogue of (3.17) for the diagonal elements of the resolvent. Hence,

$$\sum_{j=1}^{n}\chi_{1j} - \frac{\kappa\rho^2}{n}\sum_{j=1}^{n}\tilde{d}_j^2\tilde{t}_{jj}^2\left(\frac{1}{n}\sum_{i=1}^{N} d_i^2[\mathbb{E}_j Q_j]_{ii}t_{ii}\right)\xrightarrow[n\to\infty]{\mathcal{P}} 0\ .$$

Iterating the same arguments, we can replace the remaining term $\mathbb{E}_j[Q_j]_{ii}$ by t_{ii} to obtain the desired result. ☐

Lemma 4.2. *Assume that the setting of Theorem 2.2 holds true. Then:*

$$\sum_{j=1}^{n}\chi_{2j}\xrightarrow[n\to\infty]{\mathcal{P}} 0\ .$$

Proof. We have

$$\begin{aligned}
\mathbb{E}\left|\sum_{j=1}^{n}\chi_{2j}\right| &\le \frac{K}{n}\sum_{j=1}^{n}\mathbb{E}\left|a_j^*(\mathbb{E}_j Q_j)D^{3/2}\frac{\operatorname{vdiag}(Q_j)}{\sqrt{n}}\right| \\
&\le \frac{K}{n}\sum_{j=1}^{n}\mathbb{E}\left|a_j^* Q_j D^{3/2}\frac{\operatorname{vdiag}(T)}{\sqrt{n}}\right| + \frac{K}{n}\sum_{j=1}^{n}\mathbb{E}\left|a_j^*(\mathbb{E}_j Q_j)D^{3/2}\frac{\operatorname{vdiag}(Q-T)}{\sqrt{n}}\right| \\
&\quad + \frac{K}{n}\sum_{j=1}^{n}\mathbb{E}\left|a_j^*(\mathbb{E}_j Q_j)D^{3/2}\frac{\operatorname{vdiag}(Q_j-Q)}{\sqrt{n}}\right|\ .
\end{aligned}\tag{4.4}$$

The first term satisfies

$$\sum_{j=1}^{n}\mathbb{E}\left|a_j^* Q_j D^{3/2}\frac{\operatorname{vdiag}(T)}{\sqrt{n}}\right| \le \sqrt{n}\left(\sum_{j=1}^{n}\mathbb{E}\left|a_j^* Q_j D^{3/2}\frac{\operatorname{vdiag}(T)}{\sqrt{n}}\right|^2\right)^{1/2}.$$

As $\|n^{-1/2}D^{3/2}\operatorname{vdiag}(T)\| = (n^{-1}\sum_{i=1}^{N} d_i^3 t_{ii}^2)^{1/2} \le K$, Theorem 3.3-(1) can be applied, and the first term at the r.h.s. of (4.4) is of order $n^{-1/2}$. We now deal with the second term at the r.h.s.

$$\mathbb{E}\left|a_j^*(\mathbb{E}_j Q_j)D^{3/2}\frac{\operatorname{vdiag}(Q-T)}{\sqrt{n}}\right| \le K\,\mathbb{E}\left\|\frac{\operatorname{vdiag}(Q-T)}{\sqrt{n}}\right\| \le \left(\frac{K}{n}\sum_{i=1}^{N}\mathbb{E}|q_{ii}-t_{ii}|^2\right)^{1/2} \le \frac{K}{\sqrt{n}}$$

by (3.17). We now consider the third term. Since $\|a_j^*(\mathbb{E}_j Q_j)D^{3/2}\|$ is uniformly bounded,

$$\frac{1}{n}\sum_{j=1}^n \mathbb{E}\left| a_j^*(\mathbb{E}_j Q_j)D^{3/2}\frac{\mathrm{vdiag}(Q_j-Q)}{\sqrt{n}}\right| \\ = \frac{1}{n^{3/2}}\sum_{j=1}^n \mathbb{E}\left|\mathrm{Tr}\left(\mathrm{diag}(a_j^*(\mathbb{E}_j Q_j)D^{3/2})(Q_j-Q)\right)\right| \le \frac{K}{\sqrt{n}}$$

by Lemma 3.1. □

Lemma 4.3. *Assume that the setting of Theorem 2.2 holds true, then:*

$$\sum_{j=1}^n \chi_{3j} + \log\left(\left(1-\frac{1}{n}\mathrm{Tr}\, D^{\frac12}TA(I+\delta\tilde{D})^{-2}\tilde{D}A^*TD^{\frac12}\right)^2 - \rho^2\gamma\tilde{\gamma}\right) \xrightarrow[n\to\infty]{\mathcal{P}} 0.$$

Lemma 4.4. *Assume that the setting of Theorem 2.2 holds true, then:*

$$\sum_{j=1}^n \chi_{4j} + \log\left(\left|1-\vartheta\frac{1}{n}\mathrm{Tr}\, D^{\frac12}\bar{T}\bar{A}(I+\delta\tilde{D})^{-2}\tilde{D}A^*TD^{\frac12}\right|^2 - |\vartheta|^2\rho^2\underline{\gamma}\,\underline{\tilde{\gamma}}\right) \xrightarrow[n\to\infty]{\mathcal{P}} 0\ .$$

The core of the paper is devoted to the proof of Lemma 4.3. This proof is provided in Section 5. The proof of Lemma 4.4 follows the same canvas with minor differences. Elements of this proof are given in Section 6.

5. Proof of Theorem 2.2 (part II)

This section is devoted to the proof of Lemma 4.3. We begin with the following lemma which implies that $\sum_{j=1}^n \chi_{3j}$ can be replaced by its expectation.

Lemma 5.1. *For any $N\times 1$ vector a with bounded Euclidean norm, we have,*

$$\max_j \mathrm{var}(a^*(\mathbb{E}_j Q)D(\mathbb{E}_j Q)a) \ = \ \mathcal{O}(n^{-1}) \quad \textit{and} \quad \max_j \mathrm{var}\left(\mathrm{Tr}\,(\mathbb{E}_j Q)D(\mathbb{E}_j Q)D\right) \ = \ \mathcal{O}(1).$$

Proof of Lemma 5.1 is postponed to Appendix B.1. Observe that:

$$\begin{aligned}\sum_{i=1}^n \chi_{3j} &= \frac{1}{n}\sum_{j=1}^n \rho^2\tilde{t}_{jj}^2\left(\frac{\tilde{d}_j^2}{n}\mathrm{Tr}(\mathbb{E}_j Q_j)D(\mathbb{E}_j Q_j)D + 2\tilde{d}_j a_j^*(\mathbb{E}_j Q_j)D(\mathbb{E}_j Q_j)a_j\right)\\ &= \frac{1}{n}\sum_{j=1}^n \rho^2\tilde{t}_{jj}^2\left(\frac{\tilde{d}_j^2}{n}\mathrm{Tr}(\mathbb{E}_j Q)D(\mathbb{E}_j Q)D + 2\tilde{d}_j a_j^*(\mathbb{E}_j Q_j)D(\mathbb{E}_j Q_j)a_j\right) + \mathcal{O}(n^{-1})\ ,\end{aligned}$$

due to Lemma 3.1. Consider the following notations:

$$\begin{aligned}\psi_j &= \frac{1}{n}\mathrm{Tr}\,\mathbb{E}\left[(\mathbb{E}_j Q)D(\mathbb{E}_j Q)D\right] = \frac{1}{n}\mathrm{Tr}\,\mathbb{E}\left[(\mathbb{E}_j Q)DQD\right]\ ,\\ \zeta_{kj} &= \mathbb{E}\left[a_k^*(\mathbb{E}_j Q)D(\mathbb{E}_j Q)a_k\right] = \mathbb{E}\left[a_k^*(\mathbb{E}_j Q)DQa_k\right]\ ,\\ \theta_{kj} &= \mathbb{E}\left[a_k^*(\mathbb{E}_j Q_k)D(\mathbb{E}_j Q_k)a_k\right] = \mathbb{E}\left[a_k^*(\mathbb{E}_j Q_k)DQ_k a_k\right]\ ,\\ \varphi_j &= \frac{1}{n}\sum_{k=1}^j \rho^2\tilde{d}_k\tilde{t}_{kk}^2\theta_{kj}\ .\end{aligned}$$

Thanks to Lemma 5.1, we only need to show that

$$\frac{1}{n}\sum_{j=1}^{n}\left(\rho^2\tilde{t}_{jj}^2\tilde{d}_j^2\psi_j+2\rho^2\tilde{d}_j\tilde{t}_{jj}^2\theta_{jj}\right)\quad+\quad\log\Delta_n\quad\xrightarrow[n\to\infty]{}0\ . \tag{5.1}$$

There are structural links between the various quantities ψ_j, ζ_{kj}, θ_{kj} and φ_j. The idea behind the proof is to establish the equations between these quantities. Solving these equations will yield explicit expressions which will enable to identify $\frac{1}{n}\sum_{j=1}^{n}\left(\rho^2\tilde{t}_{jj}^2\tilde{d}_j^2\psi_j+2\rho^2\tilde{d}_j\tilde{t}_{jj}^2\theta_{jj}\right)$ as the deterministic quantity $-\log\Delta_n$ up to a vanishing error term.

Proof of (5.1) is broken down into four steps. In the first step, we establish an equation between ζ_{kj}, ψ_j and φ_j (up to $\mathcal{O}(n^{-1/2})$): Eq. (5.7). In the second step, we establish an equation between ψ_j and φ_j: Eq. (5.11). In the third step, we establish an equation between ζ_{kj}, ψ_j and θ_{kj}: Eq. (5.12). Gathering these results, we obtain a 2×2 linear system (5.15) whose solutions are ψ_j and φ_j. In the fourth step, we solve this system and finally establish (5.1).

5.1. Step 1: Expression of $\zeta_{kj}=\mathbb{E}[a_k^*(\mathbb{E}_jQ)DQa_k]$.

Writing

$$Q=T+T(T^{-1}-Q^{-1})Q=T+T\left(\rho\tilde{\delta}D+A(I+\delta\tilde{D})^{-1}A^*-\Sigma\Sigma^*\right)Q\ , \tag{5.2}$$

we have:

$$\begin{aligned}\zeta_{kj}&=\mathbb{E}\left[a_k^*\mathbb{E}_j\left[T+T\left(\rho\tilde{\delta}D+A(I+\delta\tilde{D})^{-1}A^*-\Sigma\Sigma^*\right)Q\right]DQa_k\right]\ ,\\&=\mathbb{E}[a_k^*TDQa_k]+\rho\tilde{\delta}\,\mathbb{E}[a_k^*TD(\mathbb{E}_jQ)DQa_k]\\&\quad+\mathbb{E}[a_k^*TA(I+\delta\tilde{D})^{-1}A^*(\mathbb{E}_jQ)DQa_k]-\mathbb{E}[a_k^*T(\mathbb{E}_j\Sigma\Sigma^*Q)DQa_k]\ , &(5.3)\\&\stackrel{\triangle}{=}a_k^*TDTa_k+\rho\tilde{\delta}\,\mathbb{E}[a_k^*TD(\mathbb{E}_jQ)DQa_k]+X+Z+\varepsilon\ , &(5.4)\end{aligned}$$

where X and Z are the last two terms at the r.h.s. of (5.3) and where $|\varepsilon|=\mathcal{O}(n^{-1/2})$ by Theorem 3.3-(3). Beginning with X, we have

$$\begin{aligned}X&=\sum_{\ell=1}^{n}\frac{\mathbb{E}[a_k^*Ta_\ell a_\ell^*(\mathbb{E}_jQ)DQa_k]}{1+\delta\tilde{d}_\ell}\\&=\sum_{\ell=1}^{n}\frac{\mathbb{E}[a_k^*Ta_\ell a_\ell^*(\mathbb{E}_jQ_\ell)DQa_k]}{1+\delta\tilde{d}_\ell}-\sum_{\ell=1}^{n}\frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell}a_k^*Ta_\ell a_\ell^*(\mathbb{E}_jQ_\ell\eta_\ell\eta_\ell^*Q_\ell)DQa_k]}{1+\delta\tilde{d}_\ell}+\varepsilon_1\ ,\\&=\sum_{\ell=1}^{n}\frac{\mathbb{E}[a_k^*Ta_\ell a_\ell^*(\mathbb{E}_jQ_\ell)DQa_k]}{1+\delta\tilde{d}_\ell}-\sum_{\ell=1}^{n}\frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell}a_k^*Ta_\ell a_\ell^*(\mathbb{E}_jQ_\ell a_\ell\eta_\ell^*Q_\ell)DQa_k]}{1+\delta\tilde{d}_\ell}+\varepsilon_1+\varepsilon_2\ ,\\&=\sum_{\ell=1}^{n}\frac{\mathbb{E}[a_k^*Ta_\ell a_\ell^*(\mathbb{E}_jQ_\ell)DQa_k]}{1+\delta\tilde{d}_\ell}-\sum_{\ell=1}^{n}\frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell}a_k^*Ta_\ell a_\ell^*\mathcal{T}_\ell a_\ell(\mathbb{E}_j\eta_\ell^*Q_\ell)DQa_k]}{1+\delta\tilde{d}_\ell}+\varepsilon_1+\varepsilon_2+\varepsilon_3\ ,\\&\stackrel{\triangle}{=}X_1+X_2+\varepsilon_1+\varepsilon_2+\varepsilon_3\ ,\end{aligned}$$

where

$$\begin{aligned}
\varepsilon_1 &= -\sum_{\ell=1}^n \frac{\mathbb{E}[a_k^* T a_\ell (\mathbb{E}_j\,(\rho\tilde{q}_{\ell\ell} - \rho\tilde{t}_{\ell\ell}) a_\ell^* Q_\ell \eta_\ell \eta_\ell^* Q_\ell) D Q a_k]}{1+\delta\tilde{d}_\ell}\ , \qquad (5.5)\\
\varepsilon_2 &= -\sum_{\ell=1}^n \frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell} a_k^* T a_\ell a_\ell^* (\mathbb{E}_j Q_\ell y_\ell \eta_\ell^* Q_\ell) D Q a_k]}{1+\delta\tilde{d}_\ell}\ ,\\
\varepsilon_3 &= -\sum_{\ell=1}^n \frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell} a_k^* T a_\ell (\mathbb{E}_j a_\ell (Q_\ell - \mathcal{T}_\ell) a_\ell\, \eta_\ell^* Q_\ell) D Q a_k]}{1+\delta\tilde{d}_\ell}\ .
\end{aligned}$$

Using (3.3) and (3.8), ε_1 can be written as:

$$\varepsilon_1 = \mathbb{E}\left[\mathbb{E}_j \left(a_k^* T A \ \mathrm{diag}(\xi_\ell)\,(I+\delta\tilde{D})^{-1}\Sigma^* Q\right) D Q a_k\right]\ ,$$

where $\xi_\ell = \rho(\tilde{q}_{\ell\ell} - \tilde{t}_{\ell\ell})(1+\eta_\ell^* Q_\ell \eta_\ell) a_\ell^* Q_\ell \eta_\ell$. Recalling that $\|\Sigma^* Q\|$ is bounded, we obtain $|\varepsilon_1| \le K\mathbb{E}\|a_k^* T A\,\mathrm{diag}(\xi_\ell)\| \le K(\sum_{\ell=1}^n |[a_k^* T A]_\ell|^2 \mathbb{E}\xi_\ell^2)^{1/2} \le K/\sqrt{n}$ by (3.17) and the boundedness of $\mathbb{E}|X_{11}|^{16}$ (Assumption **A1**). We show similarly that ε_2 and ε_3 (with the help of Theorem 3.3-(4)) are of order $\mathcal{O}(n^{-1/2})$. We now develop X_2 as:

$$\begin{aligned}
X_2 &= -\sum_{\ell=1}^n \frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell}\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, a_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k]}{1+\delta\tilde{d}_\ell} - \sum_{\ell=1}^j \frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell}\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, y_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k]}{1+\delta\tilde{d}_\ell}\ ,\\
&\triangleq U_1 + U_2\ .
\end{aligned}$$

The term U_2 can be expressed as:

$$\begin{aligned}
U_2 &= \sum_{\ell=1}^j \frac{\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}\tilde{q}_{\ell\ell}\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell\, \eta_\ell^* Q_\ell a_k]}{1+\delta\tilde{d}_\ell}\ ,\\
&= \sum_{\ell=1}^j \frac{\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell\, \eta_\ell^* Q_\ell a_k]}{1+\delta\tilde{d}_\ell} + \mathcal{O}(n^{-1/2})\ .
\end{aligned}$$

Write $\eta_\ell\eta_\ell^* = a_\ell a_\ell^* + a_\ell y_\ell^* + y_\ell y_\ell^* + y_\ell a_\ell^*$. The term in $a_\ell a_\ell^*$ is zero. Turning to the term in $a_\ell y_\ell^*$, we have $\mathbb{E}|y_\ell^*(\mathbb{E}_j Q_\ell) D Q_\ell a_\ell\, y_\ell^* Q_\ell a_k| = \mathcal{O}(n^{-1})$, hence

$$\sum_{\ell=1}^j \frac{\left|\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell a_\ell\, y_\ell^* Q_\ell a_k]\right|}{1+\delta\tilde{d}_\ell} \ \le\ \frac{K}{n}\sum_{\ell=1}^j |a_k^* T a_\ell| \ \le\ \frac{K}{\sqrt{n}}.$$

Moreover,

$$\begin{aligned}
&\sum_{\ell=1}^j \frac{\left|\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell\, y_\ell^* Q_\ell a_k]\right|}{1+\delta\tilde{d}_\ell}\\
&= \sum_{\ell=1}^j \frac{\left|\mathbb{E}\left[\rho^2 \tilde{t}_{\ell\ell}^2\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, \left(y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell - \tilde{d}_\ell n^{-1}\,\mathrm{Tr}\, D(\mathbb{E}_j Q_\ell) D Q_\ell\right) y_\ell^* Q_\ell a_k\right]\right|}{1+\delta\tilde{d}_\ell}\\
&\le \frac{K}{n}\sum_{\ell=1}^j |a_k^* T a_\ell| \quad = \quad \mathcal{O}(n^{-1/2})\ .
\end{aligned}$$

The term in $y_\ell a_\ell^*$ is written as

$$\sum_{\ell=1}^{j} \frac{\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell\, a_\ell^* Q_\ell a_k]}{1+\delta \tilde{d}_\ell} = \psi_j \sum_{\ell=1}^{j} \frac{\mathbb{E}[\rho^2 \tilde{d}_\ell \tilde{t}_{\ell\ell}^2\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, a_\ell^* Q_\ell a_k]}{1+\delta \tilde{d}_\ell} + \varepsilon$$

where $\varepsilon = \mathcal{O}(n^{-1})$ by Lemmas 5.1 and 3.1. The remaining term in the r.h.s. can be handled by the following lemma which is proven in appendix B.2:

Lemma 5.2. *Let $(u) = (u_n)_{n\in\mathbb{N}}$ be a sequence of vectors with bounded Euclidean norms. Let $(\alpha_\ell)_{1\le\ell\le n} = (\alpha_{\ell,n})_{1\le\ell\le n}$ be an array of bounded real numbers. Then:*

$$\sum_{\ell=1}^{j} \alpha_\ell\, u^* T a_\ell\, \mathbb{E}\left[a_\ell Q_\ell u\right] = \sum_{\ell=1}^{j} \frac{\alpha_\ell\, u^* T a_\ell\, a_\ell^* T u}{\rho \tilde{t}_{\ell\ell}(1+\tilde{d}_\ell \delta)} + \mathcal{O}(n^{-1/2}) \ .$$

Applying this lemma with $u = a_k$ and $\alpha_\ell = \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell (1+\tilde{d}_\ell)^{-1} a_\ell \mathcal{T}_\ell a_\ell$, we obtain

$$U_2 = \psi_j \sum_{\ell=1}^{j} \frac{\rho \tilde{d}_\ell \tilde{t}_{\ell\ell}\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, a_\ell^* T a_k}{(1+\delta \tilde{d}_\ell)^2} + \mathcal{O}(n^{-1/2}).$$

Gathering these results, and using the identity $(1-\rho \tilde{t}_{\ell\ell} a_\ell^* \mathcal{T}_\ell a_\ell) = \rho \tilde{t}_{\ell\ell}(1+\tilde{d}_\ell \delta)$ (see (3.10)), we obtain

$$X = \sum_{\ell=1}^{n} \rho \tilde{t}_{\ell\ell}\, a_k^* T a_\ell\, \mathbb{E}[a_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k] + \psi_j \sum_{\ell=1}^{j} \frac{\rho \tilde{d}_\ell \tilde{t}_{\ell\ell}\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, a_\ell^* T a_k}{(1+\tilde{d}_\ell \delta)^2} + \mathcal{O}(n^{-1/2}). \quad (5.6)$$

We now turn to the term Z in (5.4).

$$Z = -\sum_{\ell=1}^{n} \mathbb{E}[a_k^* T (\mathbb{E}_j \rho \tilde{q}_{\ell\ell} \eta_\ell \eta_\ell^* Q_\ell) D Q a_k] = -\sum_{\ell=1}^{n} \rho \tilde{t}_{\ell\ell} \mathbb{E}[a_k^* T (\mathbb{E}_j \eta_\ell \eta_\ell^* Q_\ell) D Q a_k] + \varepsilon \ ,$$

where

$$\varepsilon = \sum_{\ell=1}^{n} \mathbb{E}[a_k^* T \left(\mathbb{E}_j\, \rho(\tilde{q}_{\ell\ell} - \tilde{t}_{\ell\ell}) \eta_\ell \eta_\ell^* Q_\ell\right) D Q a_k]$$

satisfies $\varepsilon = \mathcal{O}(n^{-1/2})$ (same arguments as for ε_1 in (5.5)). Writing $\eta_\ell \eta_\ell^* = a_\ell a_\ell^* + y_\ell y_\ell^* + a_\ell y_\ell^* + y_\ell a_\ell^*$, we obtain:

$$\begin{aligned}
Z = & -\sum_{\ell=1}^{n} \rho \tilde{t}_{\ell\ell}\, a_k^* T a_\ell\, \mathbb{E}[a_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k] \\
& - \left(\sum_{\ell=1}^{j} \rho \tilde{t}_{\ell\ell}\, \mathbb{E}[a_k^* T y_\ell\, y_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k] + \frac{1}{n} \sum_{\ell=j+1}^{n} \rho \tilde{t}_{\ell\ell} \tilde{d}_\ell\, \mathbb{E}[a_k^* T D (\mathbb{E}_j Q_\ell) D Q a_k] \right) \\
& - \sum_{\ell=1}^{j} \rho \tilde{t}_{\ell\ell}\, a_k^* T a_\ell\, \mathbb{E}[y_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k] - \sum_{\ell=1}^{j} \rho \tilde{t}_{\ell\ell}\, \mathbb{E}[a_k^* T y_\ell\, a_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k] + \mathcal{O}(n^{-1/2}) \\
& \stackrel{\triangle}{=} Z_1 + Z_2 + Z_3 + Z_4 + \mathcal{O}(n^{-1/2}).
\end{aligned}$$

The term Z_1 cancels with the first term in the decomposition of X (first term at the r.h.s. of (5.6)). The term Z_2 can be written as:

$$
\begin{aligned}
Z_2 &= -\left(\sum_{\ell=1}^{j}\rho\tilde{t}_{\ell\ell}\,\mathbb{E}[a_k^* T y_\ell\, y_\ell^*(\mathbb{E}_j Q_\ell)DQ_\ell a_k] + \frac{1}{n}\sum_{\ell=j+1}^{n}\rho\tilde{t}_{\ell\ell}\tilde{d}_\ell\,\mathbb{E}[a_k^* TD(\mathbb{E}_j Q_\ell)DQa_k]\right) \\
&\quad + \sum_{\ell=1}^{j}\rho^2\tilde{t}_{\ell\ell}^2\,\mathbb{E}[a_k^* T y_\ell\, y_\ell^*(\mathbb{E}_j Q_\ell)DQ_\ell \eta_\ell\,\eta_\ell^* Q_\ell a_k] + \varepsilon \\
&\stackrel{\triangle}{=} W_1 + W_2 + \varepsilon\ ,
\end{aligned}
$$

where ε follows from the substitution of $\rho\tilde{q}_{\ell\ell}$ with $\rho\tilde{t}_{\ell\ell}$ and satisfies $\varepsilon = \mathcal{O}(n^{-1/2})$ as in (5.5). Consider first W_1:

$$
W_1 = -\left(\frac{1}{n}\sum_{\ell=1}^{j}\rho\tilde{t}_{\ell\ell}\tilde{d}_\ell\mathbb{E}[a_k^* TD(\mathbb{E}_j Q_\ell)DQ_\ell a_k] + \frac{1}{n}\sum_{\ell=j+1}^{n}\rho\tilde{t}_{\ell\ell}\tilde{d}_\ell\mathbb{E}[a_k^* TD(\mathbb{E}_j Q_\ell)DQa_k]\right).
$$

Write:

$$
(\mathbb{E}_j Q_\ell)DQ_\ell - (\mathbb{E}_j Q)DQ = (\mathbb{E}_j Q_\ell)D(Q_\ell - Q) + (\mathbb{E}_j Q_\ell - \mathbb{E}_j Q)DQ\ .
$$

Using (3.5) and (3.6),

$$
\begin{aligned}
\frac{1}{n}\sum_{\ell=1}^{j}\rho\tilde{t}_{\ell\ell}\tilde{d}_\ell\left|\mathbb{E}[a_k^* TD(\mathbb{E}_j Q_\ell)D(Q_\ell - Q)a_k]\right| &\le \frac{K}{n}\sum_{\ell=1}^{j}\left(\mathbb{E}|(1+\eta_\ell^* Q_\ell \eta_\ell)|^2\right)^{1/2}\left(\mathbb{E}|\eta_\ell^* Q a_k|^2\right)^{1/2} \\
&\le \frac{K}{\sqrt{n}}(\mathbb{E}a_k^* Q\Sigma_{1:j}\Sigma_{1:j}^* Q a_k)^{1/2} = \mathcal{O}(n^{-1/2})\ ,
\end{aligned}
$$

and the same arguments apply to the term $(\mathbb{E}_j Q_\ell - \mathbb{E}_j Q)DQ$. Hence,

$$
W_1 = -\rho\tilde{\delta}\,\mathbb{E}[a_k^* TD(\mathbb{E}_j Q)DQa_k] + \mathcal{O}(n^{-1/2}).
$$

Turning to W_2, we have:

$$
\begin{aligned}
&\sum_{\ell=1}^{j}\rho^2\tilde{t}_{\ell\ell}^2\,\mathbb{E}[a_k^* T y_\ell\, y_\ell^*(\mathbb{E}_j Q_\ell)DQ_\ell y_\ell\, a_\ell^* Q_\ell a_k] \\
&\qquad = \sum_{\ell=1}^{j}\rho^2\tilde{t}_{\ell\ell}^2\,\mathbb{E}\Big[a_k^* T y_\ell\left(y_\ell^*(\mathbb{E}_j Q_\ell)DQ_\ell y_\ell - \frac{\tilde{d}_\ell}{n}\operatorname{Tr} D(\mathbb{E}_j Q_\ell)DQ_\ell\right)a_\ell^* Q_\ell a_k\Big]
\end{aligned}
$$

whose modulus is of order $\mathcal{O}(n^{-1/2})$. The term

$$
\sum_{\ell=1}^{j}\rho^2\tilde{t}_{\ell\ell}^2\mathbb{E}[a_k^* T y_\ell y_\ell^*(\mathbb{E}_j Q_\ell)DQ_\ell a_\ell y_\ell^* Q_\ell a_k]
$$

can be handled similarly.

The term

$$
\sum_{\ell=1}^{j}\rho^2\tilde{t}_{\ell\ell}^2\mathbb{E}[a_k^* T y_\ell y_\ell^*(\mathbb{E}_j Q_\ell)DQ_\ell a_\ell a_\ell^* Q_\ell a_k] = \frac{1}{n}\sum_{\ell=1}^{j}\rho^2\tilde{t}_{\ell\ell}^2\tilde{d}_\ell\mathbb{E}[a_k^* TD(\mathbb{E}_j Q_\ell)DQ_\ell a_\ell a_\ell^* Q_\ell a_k]
$$

is bounded by $Kn^{-1/2}$. Finally,

$$\begin{aligned} W_2 &= \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \mathbb{E}[y_\ell^* Q_\ell a_k a_k^* T y_\ell \, y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell] + \mathcal{O}(n^{-1/2}) \ , \\ &\overset{(a)}{=} \psi_j \, a_k^* T D T a_k \, \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 + \mathcal{O}(n^{-1/2}) \ , \end{aligned}$$

where (a) follows by standard arguments as those already developed.

The term Z_3 satisfies

$$Z_3 = \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 a_k^* T a_\ell \mathbb{E}[y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell \, \eta_\ell^* Q_\ell a_k] + \mathcal{O}(n^{-1/2}).$$

Writing $\eta_\ell \eta_\ell^* = y_\ell y_\ell^* + a_\ell y_\ell^* + a_\ell a_\ell^* + y_\ell a_\ell^*$ and relying arguments as those already developed, one can check that the only non-negligible contribution stems from the term containing $y_\ell a_\ell^*$. Hence,

$$\begin{aligned} Z_3 &= \psi_j \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell a_k^* T a_\ell \mathbb{E}[a_\ell^* Q_\ell a_k] + \mathcal{O}(n^{-1/2}) \ , \\ &= \psi_j \sum_{\ell=1}^{j} \frac{\rho \tilde{t}_{\ell\ell} \tilde{d}_\ell a_k^* T a_\ell a_\ell^* T a_k}{1 + \tilde{d}_\ell \delta} + \mathcal{O}(n^{-1/2}) \ , \end{aligned}$$

by Lemma (5.2). Similarly,

$$\begin{aligned} Z_4 &= \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \, \mathbb{E}[a_k^* T y_\ell \, a_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell \, \eta_\ell^* Q_\ell a_k] + \mathcal{O}(n^{-1/2}) \ , \\ &= a_k^* T D T a_k \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{d}_\ell \tilde{t}_{\ell\ell}^2 \mathbb{E}[a_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell a_\ell] + \mathcal{O}(n^{-1/2}) \ , \\ &= a_k^* T D T a_k \varphi_j + \mathcal{O}(n^{-1/2}) \ . \end{aligned}$$

Gathering these results, we obtain

$$\begin{aligned} Z = &-\sum_{\ell=1}^{n} \rho \tilde{t}_{\ell\ell} \, a_k^* T a_\ell \, \mathbb{E}[a_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k] - \rho \tilde{\delta} \, \mathbb{E}[a_k^* T D (\mathbb{E}_j Q) D Q a_k] \\ &+ \psi_j \, a_k^* T D T a_k \, \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 + \psi_j \sum_{\ell=1}^{j} \frac{\rho \tilde{t}_{\ell\ell} \tilde{d}_\ell a_k^* T a_\ell a_\ell^* T a_k}{1 + \tilde{d}_\ell \delta} + a_k^* T D T a_k \, \varphi_j + \mathcal{O}(n^{-1/2}). \end{aligned}$$

Plugging this and Eq. (5.6) into (5.4), and noticing that $\rho \tilde{t}_{\ell\ell} (a_\ell \mathcal{T}_\ell a_\ell (1 + \tilde{d}_\ell \delta)^{-1} + 1) = (1 + \tilde{d}_\ell \delta)^{-1}$, we obtain:

$$\zeta_{kj} = a_k^* T D T a_k + \psi_j \left(\sum_{\ell=1}^{j} \frac{a_k^* T a_\ell \, \tilde{d}_\ell \, a_\ell^* T a_k}{(1 + \tilde{d}_\ell \delta)^2} \ + \ a_k^* T D T a_k \, \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 \right) + a_k^* T D T a_k \, \varphi_j + \mathcal{O}(n^{-1/2}) \ . \quad (5.7)$$

5.2. **Step 2: Expression of $\psi_j = n^{-1}\operatorname{Tr}\mathbb{E}[(\mathbb{E}_j Q)DQD]$.** Using Identity (5.2), we obtain:

$$
\begin{aligned}
\psi_j &= \frac{1}{n}\operatorname{Tr}\mathbb{E}[TDQD] + \frac{\rho\tilde{\delta}}{n}\operatorname{Tr}\mathbb{E}[TD(\mathbb{E}_j Q)DQD] \\
&\quad + \frac{1}{n}\operatorname{Tr}\mathbb{E}[TA(I+\delta\tilde{D})^{-1}A^*(\mathbb{E}_j Q)DQD] - \frac{1}{n}\operatorname{Tr}\mathbb{E}[T(\mathbb{E}_j \Sigma\Sigma^* Q)DQD]\ , \qquad (5.8) \\
&= \frac{1}{n}\operatorname{Tr} DTDT + \frac{\rho\tilde{\delta}}{n}\operatorname{Tr}\mathbb{E}[TD(\mathbb{E}_j Q)DQD] + X + Z + \varepsilon\ , \qquad (5.9)
\end{aligned}
$$

where X and Z are the last two terms of the r.h.s. of (5.8), and where $\varepsilon = \mathcal{O}(n^{-1})$ by Theorem 3.3-(2). Due to the presence of the multiplying factor n^{-1}, the treatment of X and Z is simpler here than the treatment of their analogues for ζ_{kj}. We skip hereafter the details related to the bounds over the ε's. The term X satisfies

$$
\begin{aligned}
X &= \frac{1}{n}\sum_{\ell=1}^{n}\frac{\mathbb{E}[a_\ell^*(\mathbb{E}_j Q)DQDTa_\ell]}{1+\delta\tilde{d}_\ell}\ , \\
&= \frac{1}{n}\sum_{\ell=1}^{n}\frac{\mathbb{E}[a_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell]}{1+\delta\tilde{d}_\ell} - \frac{1}{n}\sum_{\ell=1}^{n}\frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell}\,(\mathbb{E}_j a_\ell^* Q_\ell \eta_\ell\, \eta_\ell^* Q_\ell)DQDTa_\ell]}{1+\delta\tilde{d}_\ell} + \varepsilon\ , \\
&= \frac{1}{n}\sum_{\ell=1}^{n}\frac{\mathbb{E}[a_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell]}{1+\delta\tilde{d}_\ell} - \frac{1}{n}\sum_{\ell=1}^{n}\frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell}\, a_\ell^*\mathcal{T}_\ell a_\ell\, a_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell]}{1+\delta\tilde{d}_\ell} \\
&\quad - \frac{1}{n}\sum_{\ell=1}^{j}\frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell}\, a_\ell^*\mathcal{T}_\ell a_\ell\, y_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell]}{1+\delta\tilde{d}_\ell} + \varepsilon'\ ,
\end{aligned}
$$

where $\max(|\varepsilon|, |\varepsilon'|) = \mathcal{O}(n^{-1/2})$. As $1 - \rho\tilde{t}_{\ell\ell} a_\ell^* \mathcal{T}_\ell a_\ell = \rho\tilde{t}_{\ell\ell}(1+\tilde{d}_\ell\delta)$,

$$
\begin{aligned}
X &= \frac{1}{n}\sum_{\ell=1}^{n}\mathbb{E}[\rho\tilde{t}_{\ell\ell}\, a_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell] - \frac{1}{n}\sum_{\ell=1}^{j}\frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell}\, a_\ell^*\mathcal{T}_\ell a_\ell\, y_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell]}{1+\delta\tilde{d}_\ell} + \mathcal{O}(n^{-1/2})\ , \\
&= \frac{1}{n}\sum_{\ell=1}^{n}\mathbb{E}[\rho\tilde{t}_{\ell\ell}\, a_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell] + \frac{1}{n}\sum_{\ell=1}^{j}\frac{\mathbb{E}[\rho^2\tilde{t}_{\ell\ell}^2\, a_\ell^*\mathcal{T}_\ell a_\ell\, y_\ell^*(\mathbb{E}_j Q_\ell)DQ_\ell\eta_\ell\, \eta_\ell^* Q_\ell DTa_\ell]}{1+\delta\tilde{d}_\ell} + \mathcal{O}(n^{-1/2})\ , \\
&= \frac{1}{n}\sum_{\ell=1}^{n}\mathbb{E}[\rho\tilde{t}_{\ell\ell}\, a_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell] + \frac{\psi_j}{n}\sum_{\ell=1}^{j}\frac{\tilde{d}_\ell\, a_\ell^* Ta_\ell\, a_\ell^* TDTa_\ell}{(1+\delta\tilde{d}_\ell)^3} + \mathcal{O}(n^{-1/2})\ , \qquad (5.10)
\end{aligned}
$$

where (3.11) is used to obtain the last equation. The term Z can be expressed as:

$$\begin{aligned}
Z &= -\frac{1}{n}\sum_{\ell=1}^{n} \operatorname{Tr}\mathbb{E}[\rho\tilde{t}_{\ell\ell}T(\mathbb{E}_j\eta_\ell\eta_\ell^* Q_\ell)DQD] + \mathcal{O}(n^{-1/2}) \ , \\
&= -\frac{1}{n}\sum_{\ell=1}^{n} \mathbb{E}[\rho\tilde{t}_{\ell\ell}a_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell] \\
&\quad - \left(\frac{1}{n}\sum_{\ell=1}^{j}\mathbb{E}[\rho\tilde{t}_{\ell\ell}y_\ell^*(\mathbb{E}_j Q_\ell)DQDTy_\ell] + \frac{1}{n}\sum_{\ell=j+1}^{n}\frac{1}{n}\operatorname{Tr}\mathbb{E}[\rho\tilde{d}_\ell\tilde{t}_{\ell\ell}TD(\mathbb{E}_j Q_\ell)DQD]\right) \\
&\quad - \frac{1}{n}\sum_{\ell=1}^{j}\mathbb{E}[\rho\tilde{t}_{\ell\ell}y_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell] - \frac{1}{n}\sum_{\ell=1}^{j}\mathbb{E}[\rho\tilde{t}_{\ell\ell}a_\ell^*(\mathbb{E}_j Q_\ell)DQDTy_\ell] + \mathcal{O}(n^{-1/2}) \ , \\
&\stackrel{\triangle}{=} Z_1 + Z_2 + Z_3 + Z_4 + \mathcal{O}(n^{-1/2}).
\end{aligned}$$

The term Z_1 cancels with the first term in the r.h.s. of X's decomposition (5.10). The terms Z_2, Z_3 and Z_4 satisfy:

$$\begin{aligned}
Z_2 &= -\frac{\rho\tilde{\delta}}{n}\operatorname{Tr}\mathbb{E}[TD(\mathbb{E}_j Q)DQD] + \frac{1}{n}\sum_{\ell=1}^{j}\mathbb{E}[\rho^2\tilde{t}_{\ell\ell}^2 y_\ell^*(\mathbb{E}_j Q_\ell)DQ_\ell\eta_\ell\,\eta_\ell^* Q_\ell DTy_\ell] + \mathcal{O}(n^{-1/2}) \\
&= -\frac{\rho\tilde{\delta}}{n}\operatorname{Tr}\mathbb{E}[TD(\mathbb{E}_j Q)DQD] + \psi_j\frac{1}{n}\operatorname{Tr}DTDT\frac{1}{n}\sum_{\ell=1}^{j}\rho^2\tilde{d}_\ell^2\tilde{t}_{\ell\ell}^2 + \mathcal{O}(n^{-1/2}) \ , \\
Z_3 &= \frac{1}{n}\sum_{\ell=1}^{j}\mathbb{E}[\rho^2\tilde{t}_{\ell\ell}^2 y_\ell^*(\mathbb{E}_j Q_\ell)DQ_\ell\eta_\ell\,\eta_\ell^* Q_\ell DTa_\ell] + \mathcal{O}(n^{-1/2}) \\
&= \psi_j\frac{1}{n}\sum_{\ell=1}^{j}\rho\tilde{d}_\ell\tilde{t}_{\ell\ell}\,\frac{a_\ell^*TDTa_\ell}{1+\tilde{d}_\ell\delta} + \mathcal{O}(n^{-1/2}) \quad \text{(see Th.3.3-(4) and (3.11))}, \\
Z_4 &= \frac{1}{n}\sum_{\ell=1}^{j}\mathbb{E}[\rho^2\tilde{t}_{\ell\ell}^2 a_\ell^*(\mathbb{E}_j Q_\ell)DQ_\ell\eta_\ell\,\eta_\ell^* Q_\ell DTy_\ell] + \mathcal{O}(n^{-1/2}) \\
&= \frac{1}{n}\operatorname{Tr}DTDT\frac{1}{n}\sum_{\ell=1}^{j}\rho^2\tilde{d}_\ell\tilde{t}_{\ell\ell}^2\mathbb{E}[a_\ell^*(\mathbb{E}_j Q_\ell)DQ_\ell a_\ell] + \mathcal{O}(n^{-1/2}).
\end{aligned}$$

Plugging these terms in (5.9), we obtain:

$$\begin{aligned}
\psi_j &= \gamma + \psi_j\left(\frac{1}{n}\sum_{\ell=1}^{j}a_\ell^*TDTa_\ell\left(\frac{\tilde{d}_\ell\,a_\ell^*Ta_\ell}{(1+\delta\tilde{d}_\ell)^3} + \frac{\rho\tilde{d}_\ell\tilde{t}_{\ell\ell}}{1+\tilde{d}_\ell\delta}\right) + \frac{\gamma}{n}\sum_{\ell=1}^{j}\rho^2\tilde{d}_\ell^2\tilde{t}_{\ell\ell}^2\right) + \gamma\varphi_j + \mathcal{O}(n^{-1/2}) \ , \\
&= \gamma + \psi_j\left(\frac{1}{n}\sum_{\ell=1}^{j}\frac{\tilde{d}_\ell\,a_\ell^*TDTa_\ell}{(1+\delta\tilde{d}_\ell)^2} + \frac{\gamma}{n}\sum_{\ell=1}^{j}\rho^2\tilde{d}_\ell^2\tilde{t}_{\ell\ell}^2\right) + \gamma\varphi_j + \mathcal{O}(n^{-1/2}) \ , \qquad (5.11)
\end{aligned}$$

using (3.10) and (3.11).

5.3. **Step 3: Relation between ζ_{kj} and θ_{kj} for $k \le j$.** The term ζ_{kj} can be written as

$$\begin{aligned}
\zeta_{kj} &= \mathbb{E}[a_k^* \mathbb{E}_j (Q_k - \rho \tilde{q}_{kk} Q_k \eta_k \eta_k^* Q_k) D (Q_k - \rho \tilde{q}_{kk} Q_k \eta_k \eta_k^* Q_k) a_k] \\
&= \theta_{kj} - \rho \tilde{t}_{kk} \mathbb{E}[a_k^* \mathbb{E}_j (Q_k \eta_k \eta_k^* Q_k) D Q_k a_k] - \rho \tilde{t}_{kk}\ \mathbb{E}[a_k^* \mathbb{E}_j (Q_k) D Q_k \eta_k\ \eta_k^* Q_k a_k] \\
&\quad + \rho^2 \tilde{t}_{kk}^2\ \mathbb{E}[a_k^* \mathbb{E}_j (Q_k \eta_k \eta_k^* Q_k) D Q_k \eta_k\ \eta_k^* Q_k a_k] + \mathcal{O}(n^{-1/2}) \\
&\stackrel{\triangle}{=} \theta_{kj} + X_1 + X_2 + X_3 + \mathcal{O}(n^{-1/2})\ .
\end{aligned}$$

Using similar arguments as those developed previously, we get:

$$\begin{aligned}
X_1 &= -\rho \tilde{t}_{kk} a_k^* \mathcal{T}_k a_k\, \mathbb{E}[a_k^* \mathbb{E}_j (Q_k) D Q_k a_k] + \mathcal{O}(n^{-1/2}) \ = \ -\rho \tilde{t}_{kk} a_k^* \mathcal{T}_k a_k\, \theta_{kj} + \mathcal{O}(n^{-1/2}), \\
X_2 &= -\rho \tilde{t}_{kk} a_k^* \mathcal{T}_k a_k\ \theta_{kj} + \mathcal{O}(n^{-1/2})\ .
\end{aligned}$$

As $k \le j$,

$$\begin{aligned}
X_3 &= \rho^2 \tilde{t}_{kk}^2\ (a_k^* \mathcal{T}_k a_k)^2\ \mathbb{E}[\eta_k^* \mathbb{E}_j (Q_k) D Q_k \eta_k] + \mathcal{O}(n^{-1/2})\ , \\
&= \rho^2 \tilde{t}_{kk}^2\ (a_k^* \mathcal{T}_k a_k)^2\ \left(\theta_{kj} + \tilde{d}_k \psi_j \right) + \mathcal{O}(n^{-1/2})\ .
\end{aligned}$$

Using (3.10) and (3.11), we finally obtain:

$$\zeta_{kj} = \rho^2 \tilde{t}_{kk}^2 (1 + \tilde{d}_k \delta)^2\, \theta_{kj} + \tilde{d}_k \left(\frac{a_k^* T a_k}{1 + \tilde{d}_k \delta} \right)^2 \psi_j + \mathcal{O}(n^{-1/2})\ . \tag{5.12}$$

5.4. **Step 4: A system of perturbed linear equations in (ψ_j, φ_j). Proof of (5.1).** Combining (5.12) with (5.7), we obtain

$$\begin{aligned}
\rho^2 \tilde{d}_k \tilde{t}_{kk}^2 \theta_{kj} &= \tilde{d}_k \frac{a_k^* T D T a_k}{(1 + \tilde{d}_k \delta)^2} \\
&+ \left(\sum_{\ell=1}^{j} \frac{\tilde{d}_k\, a_k^* T a_\ell\, \tilde{d}_\ell\, a_\ell^* T a_k}{(1 + \tilde{d}_k \delta)^2 (1 + \tilde{d}_\ell \delta)^2} + \frac{\tilde{d}_k\, a_k^* T D T a_k}{(1 + \tilde{d}_k \delta)^2} \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 - \tilde{d}_k^2 \frac{(a_k^* T a_k)^2}{(1 + \tilde{d}_k \delta)^4} \right) \psi_j \\
&\qquad + \frac{\tilde{d}_k\, a_k^* T D T a_k}{(1 + \tilde{d}_k \delta)^2} \varphi_j + \mathcal{O}(n^{-1/2})
\end{aligned} \tag{5.13}$$

which implies that $\varphi_j = \frac{1}{n} \sum_{k=1}^{j} \rho^2 \tilde{d}_k \tilde{t}_{kk}^2 \theta_{kj}$ satisfies

$$(1 - F_j) \varphi_j - (G_j + F_j M_j)\, \psi_j = F_j + \mathcal{O}(n^{-1/2})$$

where

$$\begin{aligned}
F_j &= \frac{1}{n} \sum_{k=1}^{j} \frac{a_k^* T D T a_k\, \tilde{d}_k}{(1 + \tilde{d}_k \delta)^2}\ , \\
M_j &= \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2\ , \\
G_j &= \frac{1}{n} \sum_{k=1}^{j} \sum_{\substack{\ell=1 \\ \ell \ne k}}^{j} \frac{\tilde{d}_k \tilde{d}_\ell\, |a_k^* T a_\ell|^2}{(1 + \tilde{d}_k \delta)^2 (1 + \tilde{d}_\ell \delta)^2}\ .
\end{aligned} \tag{5.14}$$

With these new notations, equation (5.11) is rewritten

$$-\gamma \varphi_j + (1 - F_j - \gamma M_j) \psi_j = \gamma + \mathcal{O}(n^{-1/2}),$$

and we end up with a system of two perturbed linear equations in (φ_j, ψ_j):

$$\left\{ \begin{array}{lcl} (1-F_j)\varphi_j - (G_j + F_j M_j)\,\psi_j & = & F_j + \mathcal{O}(n^{-1/2}) \\ -\gamma\varphi_j + (1-F_j-\gamma M_j)\psi_j & = & \gamma + \mathcal{O}(n^{-1/2}) \end{array} \right. . \tag{5.15}$$

The determinant of this system is $\boldsymbol{\Delta}_j = (1-F_j)^2 - \gamma M_j - \gamma G_j$. The following lemma establishes the link between the $\boldsymbol{\Delta}_j$'s and Δ_n as defined in Theorem 2.2.

Lemma 5.3. *Recall the definition of* Δ_n *:*

$$\Delta_n = \left(1 - \frac{1}{n}\operatorname{Tr} D^{\frac12} T A (I+\delta\tilde{D})^{-2}\tilde{D}A^* T D^{\frac12}\right)^2 - \rho^2\gamma\tilde{\gamma}\ .$$

The determinants $\boldsymbol{\Delta}_j$ *decrease as* j *goes from* 1 *to* n*; moreover,* $\boldsymbol{\Delta}_n$ *coincides with* Δ_n*.*

Proof of Lemma 5.3 is postponed to Appendix B.3.

Solving this system of equations and using the lemma in conjunction with the fact $\liminf \Delta_n > 0$, established in Lemma 3.5, we obtain:

$$\begin{bmatrix}\varphi_j \\ \psi_j\end{bmatrix} = \frac{1}{\boldsymbol{\Delta}_j}\begin{bmatrix} F_j(1-F_j) + \gamma G_j \\ \gamma \end{bmatrix} + \boldsymbol{\varepsilon}_j\ ,$$

where $\|\boldsymbol{\varepsilon}_j\| = \mathcal{O}(n^{-1/2})$. Replacing into (5.13), we obtain

$$\begin{aligned} \frac{2\rho^2\tilde{d}_j\tilde{t}_{jj}^2\theta_{jj}}{n} &= 2(F_j - F_{j-1}) + (G_j - G_{j-1} + 2M_j(F_j - F_{j-1}))\psi_j + 2(F_j - F_{j-1})\varphi_j + \mathcal{O}(n^{-3/2}) \\ &= 2(F_j - F_{j-1}) + \frac{\gamma(G_j - G_{j-1}) + 2\gamma M_j(F_j - F_{j-1})}{\boldsymbol{\Delta}_j} \\ &\qquad + \frac{2(F_j - F_{j-1})(F_j(1-F_j) + \gamma G_j)}{\boldsymbol{\Delta}_j} + \mathcal{O}(n^{-3/2}) \end{aligned}$$

which leads to

$$\begin{aligned} &\frac{1}{n}\sum_{j=1}^n \left(\rho^2\tilde{t}_{jj}^2\tilde{d}_j^2\psi_j + 2\rho^2\tilde{d}_j\tilde{t}_{jj}^2\theta_{jj}\right) \\ &\qquad = \sum_{j=1}^n \frac{2(F_j - F_{j-1})(1-F_j) + \gamma(M_j - M_{j-1}) + \gamma(G_j - G_{j-1})}{\boldsymbol{\Delta}_j} + \mathcal{O}(n^{-1/2}). \end{aligned}$$

On the other hand, $\boldsymbol{\Delta}_{j-1} - \boldsymbol{\Delta}_j = 2(F_j - F_{j-1})(1-F_j) + \gamma(M_j - M_{j-1}) + \gamma(G_j - G_{j-1}) + \mathcal{O}(n^{-2})$, hence, due to Lemma 5.3 and to $\liminf \Delta_n > 0$,

$$\begin{aligned} \frac{1}{n}\sum_{j=1}^n \left(\rho^2\tilde{t}_{jj}^2\tilde{d}_j^2\psi_j + 2\rho^2\tilde{d}_j\tilde{t}_{jj}^2\theta_{jj}\right) &= \sum_{j=1}^n \frac{\boldsymbol{\Delta}_{j-1} - \boldsymbol{\Delta}_j}{\boldsymbol{\Delta}_j} + \mathcal{O}(n^{-1/2}) \\ &= \sum_{j=1}^n \log\left(1 + \frac{\boldsymbol{\Delta}_{j-1} - \boldsymbol{\Delta}_j}{\boldsymbol{\Delta}_j}\right) + \mathcal{O}(n^{-1/2}) \\ &= \sum_{j=1}^n \log\frac{\boldsymbol{\Delta}_{j-1}}{\boldsymbol{\Delta}_j} + \mathcal{O}(n^{-1/2}) = -\log(\Delta_n) + \mathcal{O}(n^{-1/2}) \end{aligned}$$

which proves (5.1). Lemma 4.3 is proven.

6. Proof of Theorem 2.2 (part III)

In this section, we complete the proof of Theorem 2.2. Proof of Lemma 4.4 is very close to the proof of Lemma 4.3; we therefore only provide its main landmarks. We finally establish the main estimates over (Θ_n).

6.1. **Elements of proof for Lemma 4.4.**

Proof of Lemma 4.4 relies on the following counterpart of Lemma 3.2:

Lemma 6.1. *Assume that the setting of Lemma 3.2 holds true; and let $\mathbb{E}x^2 = \vartheta$. Then for any $p \geq 2$,*

$$\mathbb{E}|\boldsymbol{x}^T M \boldsymbol{x} - \vartheta \operatorname{Tr} M|^p \leq K_p \left(\left(\mathbb{E}|x_1|^4 \operatorname{Tr} MM^* \right)^{p/2} + \mathbb{E}|x_1|^{2p} \operatorname{Tr}(MM^*)^{p/2} \right) .$$

Proof. The result is obtained upon noticing that

$$\boldsymbol{x}^T M \boldsymbol{x} = \frac{1}{4} \sum_{k=0}^{3} \mathbf{i}^k \left(\mathbf{i}^k \bar{\boldsymbol{x}} + \boldsymbol{x} \right)^* M \left(\mathbf{i}^k \bar{\boldsymbol{x}} + \boldsymbol{x} \right)$$

and using Lemma 3.2. □

Here are the main steps of the proof. Introducing the notations

$$\begin{aligned}
\underline{\psi}_j &= \frac{1}{n} \operatorname{Tr} \mathbb{E} \left[(\mathbb{E}_j Q) D \bar{Q} D \right] , \\
\underline{\zeta}_{kj} &= \mathbb{E} \left[a_k^* (\mathbb{E}_j Q) D \bar{Q} \bar{a}_k \right] , \\
\underline{\theta}_{kj} &= \mathbb{E} \left[a_k^* (\mathbb{E}_j Q_k) D \bar{Q}_k \bar{a}_k \right] , \\
\underline{\varphi}_j &= \frac{1}{n} \sum_{k=1}^{j} \rho^2 \tilde{d}_k \tilde{t}_{kk}^2 \underline{\theta}_{kj} ,
\end{aligned}$$

and adapting Lemma 5.1, we only need to prove that:

$$\frac{1}{n} \sum_{j=1}^{n} \left(\rho^2 \tilde{t}_{jj}^2 \tilde{d}_j^2 |\vartheta|^2 \underline{\psi}_j + 2\rho^2 \tilde{d}_j \tilde{t}_{jj}^2 \operatorname{Re} \left(\vartheta \underline{\theta}_{jj} \right) \right) \quad + \quad \log \underline{\boldsymbol{\Delta}}_n \quad \xrightarrow[n \to \infty]{} 0 .$$

Similar derivations as those performed in Steps 1-3 in Section 5 yield the perturbed system:

$$\left\{ \begin{array}{rcl}
(1 - \vartheta \underline{F}_j) \underline{\varphi}_j - (\bar{\vartheta} \underline{G}_j + |\vartheta|^2 \underline{F}_j M_j) \underline{\psi}_j &=& \underline{F}_j + \mathcal{O}(n^{-1/2}) \\
-\vartheta \underline{\gamma} \underline{\varphi}_j + (1 - \bar{\vartheta} \bar{\underline{F}}_j - \underline{\gamma} |\vartheta|^2 M_j) \underline{\psi}_j &=& \underline{\gamma} + \mathcal{O}(n^{-1/2})
\end{array} \right. ,$$

where

$$\underline{F}_j = \frac{1}{n} \sum_{k=1}^{j} \frac{a_k^* T D \bar{T} \bar{a}_k \, \tilde{d}_k}{(1 + \tilde{d}_k \delta)^2} \in \mathbb{C}, \qquad \underline{G}_j = \frac{1}{n} \sum_{k=1}^{j} \sum_{\substack{\ell=1 \\ \ell \neq k}}^{j} \frac{\tilde{d}_k \tilde{d}_\ell \left(a_k^* T a_\ell \right)^2}{(1 + \tilde{d}_k \delta)^2 (1 + \tilde{d}_\ell \delta)^2} \in \mathbb{R} ,$$

$$M_j = \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 .$$

The determinant of this system is:

$$\underline{\boldsymbol{\Delta}}_j = \left| 1 - \vartheta \underline{F}_j \right|^2 - |\vartheta|^2 \underline{\gamma} \left(M_j + \underline{G}_j \right) .$$

By (3.13), $0 \le \underline{\gamma} \le \gamma$; furthermore, $|\vartheta| \le 1$, $|\underline{F}_j| \le F_j$, and $|\underline{G}_j| \le G_j$. As a result, $\underline{\boldsymbol{\Delta}}_j \ge \boldsymbol{\Delta}_j$. Hence, by Lemma 5.3, the perturbation remains of order $\mathcal{O}(n^{-1/2})$ after solving the system. Performing the same derivations as in Step 4 in Section 5, it can be established that $\underline{\boldsymbol{\Delta}}_n = \underline{\Delta}_n$. We finally end up with:

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\left(\rho^2\tilde{t}_{jj}^2\tilde{d}_j^2|\vartheta|^2\underline{\psi}_j + 2\rho^2\tilde{d}_j\tilde{t}_{jj}^2\,\mathrm{Re}\left(\vartheta\underline{\theta}_{jj}\right)\right) &= \sum_{j=1}^{n}\frac{\underline{\boldsymbol{\Delta}}_{j-1}-\underline{\boldsymbol{\Delta}}_j}{\underline{\boldsymbol{\Delta}}_j} + \mathcal{O}(n^{-1/2})\ ,\\ &= -\log(\underline{\Delta}_n) + \mathcal{O}(n^{-1/2})\ ,\end{aligned}$$

which is the desired result.

6.2. **Estimates over Θ_n.** In order to conclude the proof of Theorem 2.2, it remains to prove that $0 < \liminf_n \Theta_n \le \limsup_n \Theta_n < \infty$.

Consider first the upper bound. By Lemma 3.5, $\sup_n(-\log\Delta_n) < \infty$. As $\underline{\Delta}_n \ge \Delta_n$, $\log\underline{\Delta}_n$ is defined and $\sup_n(-\log\underline{\Delta}_n) < \infty$. By Lemma 3.4, the cumulant term in the expression of Θ_n is bounded, hence $\limsup_n \Theta_n < \infty$.

We now prove that $\liminf \Theta_n > 0$. To this end, write:

$$\begin{aligned}\Theta_n &= \sum_{j=1}^{n}\left(\frac{\boldsymbol{\Delta}_{j-1}-\boldsymbol{\Delta}_j}{\boldsymbol{\Delta}_j} + \frac{\underline{\boldsymbol{\Delta}}_{j-1}-\underline{\boldsymbol{\Delta}}_j}{\underline{\boldsymbol{\Delta}}_j}\right) + \kappa\,\frac{\rho^2}{n^2}\sum_{i=1}^{N}d_i^2t_{ii}^2\sum_{j=1}^{n}\tilde{d}_j^2\tilde{t}_{jj}^2 + \mathcal{O}(n^{-1/2})\ ,\\ &= \sum_{j=1}^{n}\left(\frac{\gamma(G_j-G_{j-1})}{\boldsymbol{\Delta}_j} + \frac{|\vartheta|^2\underline{\gamma}(\underline{G}_j-\underline{G}_{j-1})}{\underline{\boldsymbol{\Delta}}_j}\right)\\ &\quad+\sum_{j=1}^{n}\left(\frac{2(F_j-F_{j-1})(1-F_j)}{\boldsymbol{\Delta}_j} + \frac{2\,\mathrm{Re}\left(\vartheta(\underline{F}_j-\underline{F}_{j-1})(1-\bar{\vartheta}\bar{\underline{F}}_j)\right)}{\underline{\boldsymbol{\Delta}}_j}\right)\\ &\quad+\frac{\rho^2}{n}\sum_{j=1}^{n}\tilde{d}_j^2\tilde{t}_{jj}^2\left(\frac{\gamma}{\boldsymbol{\Delta}_j} + |\vartheta|^2\frac{\underline{\gamma}}{\underline{\boldsymbol{\Delta}}_j} + \frac{\kappa}{n}\sum_{i=1}^{N}d_i^2t_{ii}^2\right) + \mathcal{O}(n^{-1/2})\ ,\\ &\triangleq Z_{1,n} + Z_{2,n} + Z_{3,n} + \mathcal{O}(n^{-1/2})\ .\end{aligned}$$

We prove in the sequel that $Z_{1,n} \ge 0$, $Z_{2,n} \ge 0$, and that $\liminf_n Z_{3,n} > 0$. It has already been noticed that $\underline{\boldsymbol{\Delta}}_j \ge \boldsymbol{\Delta}_j$; moreover, it can be proven by direct computation that $|\underline{G}_j - \underline{G}_{j-1}| \le G_j - G_{j-1}$, hence $Z_{1,n} \ge 0$. As

$$(1-F_j) - \frac{\gamma(M_j+G_j)}{1-F_j} \le \left|1-\vartheta\underline{F}_j\right| - \frac{|\vartheta|^2\underline{\gamma}\left(M_j+\underline{G}_j\right)}{\left|1-\vartheta\underline{F}_j\right|},$$

this implies that $\underline{\boldsymbol{\Delta}}_j^{-1}|1-\vartheta\underline{F}_j| \le \boldsymbol{\Delta}_j^{-1}(1-F_j)$. Noticing in addition that $|\underline{F}_j - \underline{F}_{j-1}| \le F_j - F_{j-1}$, we get $Z_{2,n} \ge 0$. The cumulant $\kappa = \mathbb{E}|X_{11}|^4 - 2 - |\vartheta|^2$ satisfies $\kappa \ge -1 - |\vartheta|^2$, hence

$$
\begin{aligned}
Z_{3,n} \geq \frac{\rho^2}{n^2} \sum_{j=1}^{n} \tilde{d}_j^2 \tilde{t}_{jj}^2 \left(\left(\frac{1}{\mathbf{\Delta}_j} - 1 \right) + |\vartheta|^2 \left(\frac{1}{\underline{\mathbf{\Delta}}_j} - 1 \right) \right) \sum_{i=1}^{N} d_i^2 t_{ii}^2 \\
+ \frac{\rho^2}{n} \sum_{j=1}^{n} \tilde{d}_j^2 \tilde{t}_{jj}^2 \Big(\frac{1}{n \mathbf{\Delta}_j} \sum_{\substack{k,\ell=1 \\ k \neq \ell}}^{N} d_k^2 \, |t_{kl}|^2 + \frac{1}{n \underline{\mathbf{\Delta}}_j} \sum_{\substack{k,\ell=1 \\ k \neq \ell}}^{N} d_k^2 \, (t_{kl})^2 \Big) \ , \\
\geq \frac{\rho^2}{n^2} \sum_{j=1}^{n} \tilde{d}_j^2 \tilde{t}_{jj}^2 \left(\left(\frac{1}{\mathbf{\Delta}_j} - 1 \right) + |\vartheta|^2 \left(\frac{1}{\underline{\mathbf{\Delta}}_j} - 1 \right) \right) \sum_{i=1}^{N} d_i^2 t_{ii}^2 = \frac{\rho^2}{n^2} \sum_{j=1}^{n} \tilde{d}_j^2 \tilde{t}_{jj}^2 p_j \sum_{i=1}^{N} d_i^2 t_{ii}^2 \ .
\end{aligned}
$$

As the term p_j is linear in $|\vartheta|^2 \in [0,1]$, $p_j \geq \min\big(\mathbf{\Delta}_j^{-1}(1-\mathbf{\Delta}_j), \mathbf{\Delta}_j^{-1} + \underline{\mathbf{\Delta}}_j^{-1} - 2\big)$. We have

$$
\begin{aligned}
\mathbf{\Delta}_j \underline{\mathbf{\Delta}}_j \quad &\leq \quad \big((1-F_j)^2 - \gamma(M_j+G_j)\big)\big((1+F_j)^2 + \gamma(M_j+G_j)\big) \ , \\
&= \quad (1-F_j^2)^2 - \gamma^2 (M_j+G_j)^2 - 4\gamma F_j (M_j+G_j) \quad \leq \quad 1 \ .
\end{aligned}
$$

Hence $\mathbf{\Delta}_j^{-1} + \underline{\mathbf{\Delta}}_j^{-1} - 2 \geq \mathbf{\Delta}_j^{-1} + \mathbf{\Delta}_j - 2 = \mathbf{\Delta}_j^{-1}(1-\mathbf{\Delta}_j)^2$. As $1 - \mathbf{\Delta}_j \geq \gamma M_j$, we get $p_j \geq \gamma^2 M_j^2$, which implies that

$$
\begin{aligned}
Z_{3,n} &\geq \frac{\rho^2 \gamma^2}{n^2} \sum_{j=1}^{n} \tilde{d}_j^2 \tilde{t}_{jj}^2 M_j^2 \sum_{i=1}^{N} d_i^2 t_{ii}^2 = \gamma^2 \Big(\sum_{j=1}^{n} M_j^2 (M_j - M_{j-1}) \Big) \frac{1}{n} \sum_{i=1}^{N} d_i^2 t_{ii}^2 \\
&= \frac{\gamma^2}{3} \Big(\sum_{j=1}^{n} M_j^3 - M_{j-1}^3 \Big) \frac{1}{n} \sum_{i=1}^{N} d_i^2 t_{ii}^2 + \mathcal{O}(n^{-1}) = \frac{\gamma^2 M_n^3}{3} \frac{1}{n} \sum_{i=1}^{N} d_i^2 t_{ii}^2 + \mathcal{O}(n^{-1}) \ ,
\end{aligned}
$$

whose liminf is positive by Lemma 3.4.

The estimates over the variance are therefore established. This completes the proof of Theorem 2.2.

7. Proposition 2.3 (bias): Main steps of the proof

Proof of Proposition 2.3-(i) can be found in [13, Theorem 2]. Let us prove (ii). The same arguments as in the companion article [21] allow to write the bias term as:

$$
N\left(\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho)\right) = \int_\rho^\infty \operatorname{Tr}\left(T(-\omega) - \mathbb{E}Q(-\omega)\right) \mathbf{d}\omega,
$$

Recall that in the centered case where $A = 0$, $T(-\omega)$ and $\tilde{T}(-\omega)$ take the simple form $T(-\omega) = [\omega(I_N + \tilde{\delta}(-\omega)D)]^{-1}$ and $\tilde{T}(-\omega) = [\omega(I_N + \delta(-\omega)\tilde{D})]^{-1}$, which implies that $\underline{\gamma} = \gamma$ and $\underline{\tilde{\gamma}} = \tilde{\gamma}$. We introduce the following intermediate quantities:

$$
\begin{aligned}
\alpha(-\omega) &= \frac{1}{n} \operatorname{Tr} D \mathbb{E} Q(-\omega), & \tilde{\alpha}(-\omega) &= \frac{1}{n} \operatorname{Tr} \tilde{D} \mathbb{E} \tilde{Q}(-\omega), \\
C(-\omega) &= \Big(\omega (I_N + \tilde{\alpha}(-\omega) D) \Big)^{-1}, & \tilde{C}(-\omega) &= \Big(\omega (I_n + \alpha(-\omega) \tilde{D}) \Big)^{-1}.
\end{aligned}
$$

From Theorem 3.3, $n^{-1} \operatorname{Tr} U(C - T) \to 0$ and $n^{-1} \operatorname{Tr} \tilde{U}(\tilde{C} - \tilde{T}) \to 0$ for any sequences of deterministic matrices U and $\tilde{U}$ with bounded spectral norms.

The proof consists of two steps:

7.1. **Step 1.** Given $\omega > 0$, let

$$\beta_n(\omega) = \left(\kappa + \frac{|\vartheta|^2}{1-\omega^2|\vartheta|^2\gamma\tilde{\gamma}}\right) R_n(\omega)$$

where

$$R_n(\omega) = \frac{\frac{\omega^3}{n}\operatorname{Tr} DT^2}{1-\omega^2\gamma\tilde{\gamma}}\left(\frac{\gamma}{n}\operatorname{Tr}(\tilde{D}\tilde{T})^3 - \frac{\omega\tilde{\gamma}^2}{n}\operatorname{Tr}(DT)^3\right) - \frac{\omega^2\tilde{\gamma}}{n}\operatorname{Tr} D^2T^3.$$

The purpose of this step is to show that $\int_\rho^\infty |\beta_n(\omega)|\,\mathbf{d}(\omega) < \infty$ and that

$$N(\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho)) - \int_\rho^\infty \beta_n(\omega)\,\mathbf{d}(\omega) \xrightarrow[n\to\infty]{} 0\ . \tag{7.1}$$

By inspecting the expression of $\beta_n(\omega)$, by using Lemmas 3.4 and 3.5 and by recalling that $1-\omega^2\gamma\tilde{\gamma} = \Delta_n$ taken at $z=-\omega$, we obtain after a small derivation that $|\beta_n(\omega)| \le K/\omega^3$ on $[\rho,\infty)$ where K does not depend on n nor on ω. This proves the integrability of $|\beta_n(\omega)|$. By taking up the poof of [21, Inequality (7.10)] with minor modifications, we also show that

$$|\operatorname{Tr}(T(-\omega) - \mathbb{E}Q(-\omega))| \le \frac{K}{\omega^2}$$

hence, showing

$$\operatorname{Tr}(T(-\omega) - \mathbb{E}Q(-\omega)) - \beta(\omega) \xrightarrow[n\to\infty]{} 0 \tag{7.2}$$

and applying the Dominated Convergence Theorem leads to (7.1). In order to show (7.2), we start by writing for $z=-\omega$

$$\operatorname{Tr}(T-\mathbb{E}Q) = \operatorname{Tr}(T-C) + \operatorname{Tr}(C-\mathbb{E}Q)\,.$$

Using the decomposition $\operatorname{Tr}(T-C) = \operatorname{Tr} C(C^{-1}-T^{-1})T = \omega\left(\tilde{\alpha}-\tilde{\delta}\right)\operatorname{Tr} DCT$, we obtain

$$\operatorname{Tr}(T-\mathbb{E}Q) = \omega n\left(\tilde{\alpha}-\tilde{\delta}\right)\frac{1}{n}\operatorname{Tr} DCT + \operatorname{Tr}(C-\mathbb{E}Q)\ . \tag{7.3}$$

On the other hand, writing $n(\alpha-\delta) = \operatorname{Tr} D(\mathbb{E}Q - C) + \operatorname{Tr} D(C-T) = \operatorname{Tr} D(\mathbb{E}Q-C) - \omega(\tilde{\alpha}-\tilde{\delta})\operatorname{Tr} D^2CT$ and similarly for $n(\tilde{\alpha}-\tilde{\delta})$, we obtain the system

$$\begin{bmatrix} 1 & \omega n^{-1}\operatorname{Tr} D^2CT \\ \omega n^{-1}\operatorname{Tr}\tilde{D}^2\tilde{C}\tilde{T} & 1\end{bmatrix}\begin{bmatrix} n(\alpha-\delta) \\ n(\tilde{\alpha}-\tilde{\delta})\end{bmatrix} = \begin{bmatrix}\operatorname{Tr} D(\mathbb{E}Q-C) \\ \operatorname{Tr}\tilde{D}(\mathbb{E}\tilde{Q}-\tilde{C})\end{bmatrix}. \tag{7.4}$$

Consequently, in order to show (7.2), we need to look for approximations of $\operatorname{Tr} U(\mathbb{E}Q-C)$ and $\operatorname{Tr}\tilde{U}(\mathbb{E}\tilde{Q}-\tilde{C})$ for deterministic matrices U and $\tilde{U}$ with bounded spectral norms:

Lemma 7.1. *Assume that the setting of Proposition 2.3 holds true. Fix $z=-\omega<0$ and let $(U_n)_n$ (resp. $(\tilde{U}_n)_n$) be a sequence of $N\times N$ (resp. $n\times n$) diagonal deterministic matrices such that* $\sup_n \max(\|U_n\|,\|\tilde{U}_n\|) < \infty$. *Then,*

$$\operatorname{Tr} U_n(C-\mathbb{E}Q) + \kappa\frac{\omega^2\tilde{\gamma}}{n}\operatorname{Tr} UD^2T^3 + |\vartheta|^2\frac{\omega^2\tilde{\gamma}}{1-\omega^2|\vartheta|^2\gamma\tilde{\gamma}}\frac{1}{n}\operatorname{Tr} UD^2T^3 \xrightarrow[n\to\infty]{} 0, \tag{7.5}$$

$$\operatorname{Tr}\tilde{U}_n(\tilde{C}-\mathbb{E}\tilde{Q}) + \kappa\frac{\omega^2\gamma}{n}\operatorname{Tr}\tilde{U}\tilde{D}^2\tilde{T}^3 + |\vartheta|^2\frac{\omega^2\gamma}{1-\omega^2|\vartheta|^2\gamma\tilde{\gamma}}\frac{1}{n}\operatorname{Tr}\tilde{U}\tilde{D}^2\tilde{T}^3 \xrightarrow[n\to\infty]{} 0. \tag{7.6}$$

Recalling that $n^{-1}\operatorname{Tr} U(C-T)\to 0$ and taking $U=D^2T$, we have $n^{-1}\operatorname{Tr} D^2CT-\gamma\to 0$. Similarly, $n^{-1}\operatorname{Tr}\tilde D^2\tilde C\tilde T-\tilde\gamma\to 0$. Solving system (7.4) and using this lemma with $U=D$ and $\tilde U=\tilde D$, we obtain

$$\begin{aligned} n(\tilde\alpha-\tilde\delta) &= \frac{1}{1-\omega^2\gamma\tilde\gamma}\Big(-\omega\tilde\gamma\operatorname{Tr} D(\mathbb{E}Q-C)+\operatorname{Tr}\tilde D(\mathbb{E}\tilde Q-\tilde C)\Big)+\varepsilon \\ &= \frac{\kappa}{1-\omega^2\gamma\tilde\gamma}\left(\frac{\omega^2\gamma}{n}\operatorname{Tr}(\tilde D\tilde T)^3-\frac{\omega^3\tilde\gamma^2}{n}\operatorname{Tr}(DT)^3\right) \\ &\quad+\frac{|\vartheta|^2}{(1-\omega^2\gamma\tilde\gamma)(1-\omega^2|\vartheta|^2\gamma\tilde\gamma)}\left(\frac{\omega^2\gamma}{n}\operatorname{Tr}(\tilde D\tilde T)^3-\frac{\omega^3\tilde\gamma^2}{n}\operatorname{Tr}(DT)^3\right)+\varepsilon' \end{aligned}$$

where $\varepsilon,\varepsilon'\to 0$. Using Lemma 7.1 again in (7.3) with $U=I$, we obtain (7.2).

The remainder of this paragraph is devoted to the proof of Lemma 7.1.

Recall the following notations:

$$\tilde b_j(-\omega)=\frac{1}{\omega\left(1+\frac{\tilde d_j}{n}\operatorname{Tr} DQ_j(-\omega)\right)}\quad\text{and}\quad e_j(-\omega)=\eta_j^*Q_j(-\omega)\eta_j-\frac{\tilde d_j}{n}\operatorname{Tr} DQ_j(-\omega)\ .$$

Starting with

$$\operatorname{Tr} U\,(\mathbb{E}Q-C)=\operatorname{Tr}\mathbb{E}[UC(C^{-1}-Q^{-1})Q]=-\operatorname{Tr}\mathbb{E}[UC\Sigma\Sigma^*Q]+\omega\tilde\alpha\operatorname{Tr}\mathbb{E}[UCDQ]$$

and using (3.8) and (3.7), we obtain $\operatorname{Tr} U\,(\mathbb{E}Q-C)=Z_1+Z_2+Z_3$ where

$$\begin{aligned} Z_1 &= \sum_{j=1}^n \mathbb{E}\left[\omega^2\tilde b_j^2\,e_j\,\eta_j^*Q_jUC\eta_j\right], \\ Z_2 &= -\sum_{j=1}^n \mathbb{E}\left[\omega^3\tilde q_{jj}\tilde b_j^2\,e_j^2\,\eta_j^*Q_jUC\eta_j\right], \\ Z_3 &= \omega\,\tilde\alpha\,\mathbb{E}\operatorname{Tr} UCDQ-\frac{\omega}{n}\sum_{j=1}^n\mathbb{E}\,\tilde b_j\tilde d_j\operatorname{Tr} DQ_jUC\ . \end{aligned}$$

In the remainder, we omit the study of the negligible terms to focus on the deterministic equivalent formulas; in this spirit, we shall denote by ε a negligible term whose value might change from line to line.

The term Z_1 is

$$Z_1=\sum_{j=1}^n\mathbb{E}\left[\omega^2\tilde b_j^2\,e_j\,\left(\eta_j^*Q_jUC\eta_j-\tilde d_j\frac{\operatorname{Tr} DQ_jUC}{n}\right)\right].$$

Using Identity (3.20) with $M=Q_j$, $P=Q_jUC$ and $\boldsymbol{u}=0$, we obtain:

$$\begin{aligned} Z_1=\frac1n\sum_{j=1}^n\mathbb{E}\Bigg[\omega^2\tilde b_j^2\Bigg(&\frac{\tilde d_j^2}{n}\operatorname{Tr} Q_jDQ_jUCD+\frac{|\vartheta|^2}{n}\tilde d_j^2\operatorname{Tr} Q_jDCU\bar Q_jD \\ &+\frac{\kappa}{n}\sum_{i=1}^N\tilde d_j^2d_i^2[Q_j]_{ii}[Q_jUC]_{ii}\Bigg)\Bigg]+\varepsilon. \end{aligned}$$

It is not difficult to check that

$$Z_2 = -\sum_{j=1}^n \mathbb{E}\left[\omega^3 \tilde{b}_j^3\, \tilde{d}_j \frac{\operatorname{Tr} DQ_jUC}{n}\, e_j^2\right] + \varepsilon.$$

Turning to Z_3, we have

$$\begin{aligned}
Z_3 &= \frac{\omega}{n}\sum_{j=1}^n \mathbb{E}\left[\tilde{d}_j \mathbb{E}\tilde{q}_{jj} \operatorname{Tr} UCDQ - \tilde{d}_j \tilde{b}_j \operatorname{Tr} UCDQ_j\right] \\
&= \frac{\omega}{n}\sum_{j=1}^n \tilde{d}_j \mathbb{E}\left[\mathbb{E}\tilde{q}_{jj}\left(\operatorname{Tr} UCDQ_j - \omega\tilde{q}_{jj} \operatorname{Tr} UCDQ_j\eta_j\eta_j^* Q_j\right) - \tilde{b}_j \operatorname{Tr} UCDQ_j\right] \\
&= \frac{\omega}{n}\sum_{j=1}^n \tilde{d}_j \mathbb{E}\left[(\mathbb{E}\tilde{q}_{jj} - \tilde{b}_j) \operatorname{Tr} UCDQ_j\right] - \frac{\omega^2}{n}\sum_{j=1}^n \tilde{d}_j \mathbb{E}\tilde{q}_{jj} \mathbb{E}\left[\tilde{b}_j \eta_j^* Q_j UCDQ_j\eta_j\right] + \varepsilon.
\end{aligned}$$

Replacing $\tilde{b}_j$ by $\tilde{q}_{jj} + \omega\tilde{b}_j^2 e_j - \omega^2 \tilde{q}_{jj}\tilde{b}_j^2 e_j^2$, we have

$$\mathbb{E}\left[(\mathbb{E}\tilde{q}_{jj} - \tilde{b}_j) \operatorname{Tr} UCDQ_j\right] = \mathbb{E}\left[(\mathbb{E}\tilde{q}_{jj} - \tilde{q}_{jj}) \operatorname{Tr} UCDQ_j\right] + \mathbb{E}\left[\omega^2 \tilde{q}_{jj}\tilde{b}_j^2 e_j^2\, \operatorname{Tr} UCDQ_j\right].$$

Since

$$\tilde{q}_{jj} - \mathbb{E}\tilde{q}_{jj} = \frac{\mathbb{E}(\eta_j^* Q_j\eta_j) - \eta_j^* Q_j \eta_j}{\omega(1+\eta_j^* Q_j\eta_j)(1+\mathbb{E}(\eta_j^* Q_j\eta_j))} + \mathbb{E}\left[\frac{\eta_j^* Q_j\eta_j - \mathbb{E}(\eta_j^* Q_j\eta_j)}{\omega(1+\eta_j^* Q_j\eta_j)(1+\mathbb{E}(\eta_j^* Q_j\eta_j))}\right]$$

we have $\mathbb{E}(\tilde{q}_{jj} - \mathbb{E}\tilde{q}_{jj})^2 = \mathcal{O}(1/n)$. Hence

$$\mathbb{E}\left[(\mathbb{E}\tilde{q}_{jj} - \tilde{q}_{jj}) \operatorname{Tr} UCDQ_j\right] = \mathbb{E}\left[(\mathbb{E}\tilde{q}_{jj} - \tilde{q}_{jj})(\operatorname{Tr} UCDQ_j - \mathbb{E}\operatorname{Tr} UCDQ_j)\right] = \mathcal{O}(n^{-1/2})$$

by Theorem 3.3-(5). It follows that $\mathbb{E}\left[(\mathbb{E}\tilde{q}_{jj} - \tilde{b}_j) \operatorname{Tr} UCDQ_j\right] = \mathbb{E}\left[\omega^2\tilde{b}_j^3 e_j^2\, \operatorname{Tr} UCDQ_j\right] + \varepsilon$, hence

$$Z_2 + Z_3 = -\frac{1}{n}\sum_{j=1}^n \mathbb{E}\left[\omega^2 \tilde{b}_j^2 \tilde{d}_j^2 \frac{1}{n} \operatorname{Tr} Q_jDQ_jUCD\right] + \varepsilon.$$

Taking the sum $Z_1 + Z_2 + Z_3$, the terms that do not depend on ϑ nor on κ cancel out, and we are left with

$$\operatorname{Tr} U(\mathbb{E}Q - C) = |\vartheta|^2 \omega^2 \tilde{\gamma} \frac{1}{n} \mathbb{E}\left[\operatorname{Tr} QDCU\bar{Q}D\right] + \kappa\omega^2\tilde{\gamma}\frac{1}{n}\operatorname{Tr} UD^2T^3 + \varepsilon.$$

where we relied on the usual approximations for the diagonal entries of the resolvent (see Lemma 4.1) to obtain the term in κ. We now briefly characterize the asymptotic behavior of $n^{-1}\operatorname{Tr}\mathbb{E}\, QDCU\bar{Q}D$. Starting with $Q = T + \omega\tilde{\delta}TDQ - T\Sigma\Sigma^* Q$, we have

$$\frac{1}{n}\operatorname{Tr}\mathbb{E}QDCU\bar{Q}D = \frac{1}{n}\operatorname{Tr}\mathbb{E}TDCU\bar{Q}D + \frac{\omega\tilde{\delta}}{n}\operatorname{Tr}\mathbb{E}TDQDCU\bar{Q}D - \frac{1}{n}\operatorname{Tr}\mathbb{E}T\Sigma\Sigma^* QDCU\bar{Q}D,$$

and

$$
\begin{aligned}
-\frac{1}{n}\operatorname{Tr}\mathbb{E}T\Sigma\Sigma^* QDCU\bar{Q}D &= -\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\omega\tilde{q}_{jj}\eta_j^* Q_j DCU\bar{Q}DT\eta_j \\
&= -\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\omega\tilde{q}_{jj}\eta_j^* Q_j DCU\bar{Q}_j DT\eta_j \\
&\qquad\qquad + \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\omega^2\tilde{q}_{jj}^2\eta_j^* Q_j DCU\bar{Q}_j\bar{\eta}_j\,\eta_j^T DT\eta_j \\
&= -\frac{\omega\tilde{\delta}}{n}\operatorname{Tr}\mathbb{E}TDQDCU\bar{Q}D + |\vartheta|^2\omega^2\gamma\tilde{\gamma}\frac{\operatorname{Tr}\mathbb{E}QDCU\bar{Q}D}{n} + \varepsilon.
\end{aligned}
$$

We therefore get

$$
\frac{1}{n}\operatorname{Tr}\,\mathbb{E}\,QDCU\bar{Q}D = \frac{\frac{1}{n}\operatorname{Tr}\,UD^2T^3}{1-\omega^2|\vartheta|^2\gamma\tilde{\gamma}} + \varepsilon
$$

and Convergence (7.5) of lemma 7.1 is shown. Convergence (7.6) is proven similarly. Lemma 7.1 is proven, and Step 1 of the proof of Proposition 2.3-(ii) is established.

7.2. **Step 2.** The purpose of this step is to show that

$$
R(\omega) = \frac{1}{2}\frac{\mathbf{d}}{\mathbf{d}\omega}\left(\omega^2\gamma(-\omega)\tilde{\gamma}(-\omega)\right).
$$

Plugging into the expression of $\beta_n(\omega)$, it is straightforward to show that $\int_\rho^\infty \beta_n(\omega)\,\mathbf{d}(\omega)$ coincides with $\mathcal{B}_n$ given by (2.5).

Recall that

$$
R(\omega) = \frac{\omega^3\gamma}{1-\omega^2\gamma\tilde{\gamma}}\frac{\operatorname{Tr}DT^2}{n}\frac{\operatorname{Tr}(\tilde{D}\tilde{T})^3}{n} - \frac{\omega^4\tilde{\gamma}^2}{1-\omega^2\gamma\tilde{\gamma}}\frac{\operatorname{Tr}DT^2}{n}\frac{\operatorname{Tr}(DT)^3}{n} - \omega^2\tilde{\gamma}\frac{\operatorname{Tr}D^2T^3}{n} = R_1+R_2+R_3
$$

Our method is similar to [19, §V.B]. We start by showing that the derivatives of $\tilde{\gamma}(-\omega)$ and $\gamma(-\omega)$ that we denote respectively as $\tilde{\gamma}'$ and γ' are

$$
\begin{aligned}
\tilde{\gamma}' &= \frac{\mathbf{d}\tilde{\gamma}(-\omega)}{\mathbf{d}\omega} = -\frac{2}{\omega}\tilde{\gamma} + \frac{2\omega}{1-\omega^2\gamma\tilde{\gamma}}\frac{\operatorname{Tr}(\tilde{D}\tilde{T})^3}{n}\frac{\operatorname{Tr}DT^2}{n} \\
\gamma' &= \frac{\mathbf{d}\gamma(-\omega)}{\mathbf{d}\omega} = -\frac{2}{\omega}\gamma + \frac{2\omega}{1-\omega^2\gamma\tilde{\gamma}}\frac{\operatorname{Tr}(DT)^3}{n}\frac{\operatorname{Tr}\tilde{D}\tilde{T}^2}{n}.
\end{aligned}
\tag{7.7}
$$

We have

$$
\tilde{\gamma}' = \frac{1}{n}\sum_{j=1}^{n}\tilde{d}_j^2\frac{\mathbf{d}}{\mathbf{d}\omega}\left(\frac{1}{\omega^2(1+\tilde{d}_j\delta(-\omega))^2}\right) = -\frac{2}{\omega}\tilde{\gamma} - 2\omega\delta'\frac{\operatorname{Tr}(\tilde{D}\tilde{T})^3}{n} \tag{7.8}
$$

where we put $\delta' = \mathbf{d}\delta(-\omega)/\mathbf{d}\omega$. This derivative can be expressed as

$$
\delta' = \frac{1}{n}\sum_{i=1}^{N} d_i\frac{\mathbf{d}}{\mathbf{d}\omega}\left(\frac{1}{\omega(1+d_i\tilde{\delta}(-\omega))}\right) = -\frac{\delta}{\omega} - \omega\gamma\tilde{\delta}'
$$

where $\tilde{\delta}' = \mathbf{d}\tilde{\delta}(-\omega)/\mathbf{d}\omega$. Similarly, $\tilde{\delta}' = -\omega^{-1}\tilde{\delta} - \omega\tilde{\gamma}\delta'$. Combining the two equations, we obtain

$$
\delta' = \frac{\gamma\tilde{\delta} - \omega^{-1}\delta}{1-\omega^2\gamma\tilde{\gamma}}. \tag{7.9}
$$

Since $T = [\omega(I + \tilde{\delta} D)]^{-1}$ and $\tilde{T} = [\omega(I + \delta \tilde{D})]^{-1}$, we have

$$T = \omega^{-1} I_N - \tilde{\delta} D T, \quad \tilde{T} = \omega^{-1} I_n - \delta \tilde{D} \tilde{T}. \tag{7.10}$$

This leads to

$$\frac{1}{n} \operatorname{Tr} D T^2 = \frac{1}{n} \operatorname{Tr} D T((\omega^{-1} I - \tilde{\delta} D T) = \omega^{-1} \delta - \gamma \tilde{\delta}, \text{ and } \frac{1}{n} \operatorname{Tr} \tilde{D} \tilde{T}^2 = \omega^{-1} \tilde{\delta} - \tilde{\gamma} \delta. \tag{7.11}$$

Combining with (7.9) and (7.8), we obtain the first equation of (7.7), the second being obtained similarly. Using the first equation, the term R_1 can be expressed as

$$R_1 = \frac{1}{2} \left(\omega^2 \gamma \tilde{\gamma}' + 2 \omega \gamma \tilde{\gamma} \right).$$

Turning to R_2, we have

$$\begin{aligned}
R_2 &= -\frac{\omega^4 \tilde{\gamma}^2}{1 - \omega^2 \gamma \tilde{\gamma}} \frac{\operatorname{Tr}(DT)^3}{n} \frac{\operatorname{Tr} DT(\omega^{-1} I - \tilde{\delta} DT)}{n} \\
&= -\frac{\omega^3 \delta \tilde{\gamma}^2}{1 - \omega^2 \gamma \tilde{\gamma}} \frac{\operatorname{Tr}(DT)^3}{n} + \frac{\omega^4 \tilde{\delta} \gamma \tilde{\gamma}^2}{1 - \omega^2 \gamma \tilde{\gamma}} \frac{\operatorname{Tr}(DT)^3}{n} \\
&\overset{(a)}{=} \frac{\omega^3 \tilde{\gamma}}{1 - \omega^2 \gamma \tilde{\gamma}} \frac{\operatorname{Tr}(DT)^3}{n} \frac{\operatorname{Tr} \tilde{D} \tilde{T}^2}{n} - \omega^2 \tilde{\gamma} \tilde{\delta} \frac{\operatorname{Tr}(DT)^3}{n} \\
&\overset{(b)}{=} \frac{1}{2} \left(\omega^2 \gamma' \tilde{\gamma} + 2 \omega \gamma \tilde{\gamma} \right) - \omega^2 \tilde{\gamma} \tilde{\delta} \frac{\operatorname{Tr}(DT)^3}{n}
\end{aligned}$$

where (a) is due to $\tilde{\gamma} \delta = \omega^{-1} \tilde{\delta} - n^{-1} \operatorname{Tr} \tilde{D} \tilde{T}^2$, see (7.11), and (b) is due to (7.7). Considering R_3, we have by (7.10),

$$-\omega^2 \tilde{\gamma} \tilde{\delta} \frac{\operatorname{Tr}(DT)^3}{n} - \omega^2 \tilde{\gamma} \frac{\operatorname{Tr} D^2 T^3}{n} = -\omega^2 \tilde{\gamma} \frac{\operatorname{Tr} D^2 T^2 (T + \tilde{\delta} DT)}{n} = -\omega \gamma \tilde{\gamma}.$$

We therefore have $R(\omega) = 0.5 \left(\omega^2 \gamma' \tilde{\gamma} + \omega^2 \gamma \tilde{\gamma}' + 2 \omega \gamma \tilde{\gamma} \right) = 0.5 (\omega^2 \gamma \tilde{\gamma})'$. Proposition 2.3 is proven.

Appendix A. Proofs for Section 3

A.1. **Proofs of Eq.** (3.10) **and Eq.** (3.11)**.** Proof of Eq. (3.10) mainly relies on matrix identity (3.2) and on the following identity for the inverse of a partitioned matrix (see for instance [23, Section 0.7.3]):

$$\text{If} \quad A = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad \text{then} \quad \left(A^{-1} \right)_{11} = \left(a_{11} - A_{12} A_{22}^{-1} A_{21} \right)^{-1}. \tag{A.1}$$

To lighten the computations, let us introduce the following notations:

$$\mathcal{I} = \left(I_n + \tilde{\delta} D \right)^{-1}, \qquad \tilde{\mathcal{I}} = \left(I_{N-1} + \delta \tilde{D}_1 \right)^{-1}.$$

In order to express a diagonal element of $\tilde{T}$, say $\tilde{t}_{11}$ (without loss of generality), let us first write:

$$\tilde{T} = \begin{bmatrix} -z(1 + \delta \tilde{d}_1) + a_1^* \mathcal{I} a_1 & a_1^* \mathcal{I} A_1 \\ A_1^* \mathcal{I} a_1 & -z \tilde{\mathcal{I}}^{-1} + A_1^* \mathcal{I} A_1 \end{bmatrix}^{-1}.$$

Hence, according to (A.1):

$$\begin{aligned}
\frac{1}{\tilde{t}_{11}} &= -z(1+\delta\tilde{d}_1) + a_1^*\mathcal{I}a_1 - a_1^*\mathcal{I}A_1\left[-z\tilde{\mathcal{I}}^{-1} + A_1^*\mathcal{I}A_1\right]^{-1} A_1^*\mathcal{I}a_1 \\
&\overset{(a)}{=} -z(1+\delta\tilde{d}_1) + a_1^*\mathcal{I}a_1 - a_1^*\mathcal{I}A_1\left[-\frac{1}{z}\tilde{\mathcal{I}} + \frac{1}{z}\tilde{\mathcal{I}}A_1^*\mathcal{T}_1A_1\tilde{\mathcal{I}}\right] A_1^*\mathcal{I}a_1 \\
&\overset{(b)}{=} -z(1+\delta\tilde{d}_1) + a_1^*\mathcal{I}a_1 + \frac{1}{z}a_1^*\mathcal{I}\left(\mathcal{T}_1^{-1} + z\mathcal{I}^{-1}\right)\mathcal{I}a_1 \\
&\qquad -\frac{1}{z}a_1^*\mathcal{I}A_1\tilde{\mathcal{I}}A_1^*\left[I_N + z\mathcal{T}_1\mathcal{I}^{-1}\right]\mathcal{I}a_1 \\
&\overset{(b)}{=} -z(1+\delta\tilde{d}_1) + 2a_1^*\mathcal{I}a_1 + \frac{1}{z}a_1^*\mathcal{I}\mathcal{T}_1^{-1}\mathcal{I}a_1 \\
&\qquad -\frac{1}{z}a_1^*\mathcal{I}\left[\mathcal{T}_1^{-1} + z\mathcal{I}^{-1}\right]\left[I_N + z\mathcal{T}_1\mathcal{I}^{-1}\right]\mathcal{I}a_1 \\
&= -z(1+\delta\tilde{d}_1) - za_1^*\mathcal{T}_1a_1\ ,
\end{aligned}$$

where (a) follows from (3.2), (b) from equalities

$$\mathcal{T}_1A_1\tilde{\mathcal{I}}A_1^* = I_N + z\mathcal{T}_1\mathcal{I}^{-1} \quad \text{and} \quad A_1\tilde{\mathcal{I}}A_1^* = \mathcal{T}_1^{-1} + z\mathcal{I}^{-1}$$

which follow from the mere definition of $\mathcal{T}_1$. Finally, (3.10) is established.

Let us now turn to the proof of (3.11). Notice first that T can be expressed as

$$T = \left[-z(I_N + \tilde{\delta}D) + A_1(I_{n-1} + \delta\tilde{D}_1)^{-1}A_1^* + \frac{a_1a_1^*}{1+\delta\tilde{d}_1}\right]^{-1}\ .$$

Applying (3.2) readily yields:

$$T = \mathcal{T}_1 - \mathcal{T}_1a_1\left(1 + \delta\tilde{d}_1 + a_1^*\mathcal{T}_1a_1\right)a_1^*\mathcal{T}_1\ .$$

It remains to multiply by a_1^* (left), b (right) and to use (3.10) to establish (3.11).

A.2. **Proof of Inequality** (3.15)**.** We provide here some elements to establish that $\mathbb{E}|e_j|^p = \mathcal{O}(n^{-p/2})$. Recall the definition (3.1) of e_j and write:

$$\mathbb{E}|e_j|^p \le K\left\{\mathbb{E}\left|y_j^*Q_jy_j - \frac{\tilde{d}_j}{n}\mathrm{Tr}DQ_j\right|^p + \mathbb{E}\left|a_j^*Q_jy_j\right|^p + \mathbb{E}\left|y_j^*Q_ja_j\right|^p\right\}\ .$$

The first term of the r.h.s. can be directly estimated with the help of Lemma 3.2. The two remaining terms are similar and can be estimated in the following way:

$$\begin{aligned}
\mathbb{E}\left|a_j^*Q_jy_j\right|^p &= \mathbb{E}\left(y_j^*Q_j^*a_ja_j^*Q_jy_j\right)^{p/2} \\
&\le K\left(\mathbb{E}\left|y_j^*Q_j^*a_ja_j^*Q_jy_j - \frac{d_j^2}{n}\mathrm{Tr}Q_j^*a_ja_j^*Q_j\right|^{p/2} + \mathbb{E}\left|\frac{1}{n}\mathrm{Tr}Q_j^*a_ja_j^*Q_j\right|^{p/2}\right)\ .
\end{aligned}$$

The first term of the r.h.s. can be handled with the help of Lemma 3.2 (notice that $Q_j^*a_ja_j^*Q_j$ is of rank one and has a bounded spectral norm), and the second term is directly of the right order.

A.3. **Proof of Theorem 3.3.** Items (1)–(3) of Theorem 3.3 are shown in [22]. Let us show Theorem 3.3-(4). Denote by $(\delta_j, \tilde{\delta}_j)$ the solution of System (1.2) when A and $\tilde{D}$ are replaced with A_j and $\tilde{D}_j$ respectively. Let T_j and $\tilde{T}_j$ be the matrices associated to $(\delta_j, \tilde{\delta}_j)$ as in Eq. (1.3). Then $\mathbb{E}|u^*(Q_j - T_j)v|^{2p} \le K_p n^{-p}$ by Item (3), and we only need to show that $|u^*(\mathcal{T}_j - T_j)v| \le K/\sqrt{n}$. We have

$$\begin{aligned}|\delta - \delta_j| &= \frac{1}{n}\left|\operatorname{Tr} D(T - T_j)\right| \\ &\le \frac{1}{n}\left|\operatorname{Tr} \mathbb{E} D(T - Q)\right| + \frac{1}{n}\left|\operatorname{Tr} \mathbb{E} D(Q - Q_j)\right| + \frac{1}{n}\left|\operatorname{Tr} \mathbb{E} D(Q_j - T_j)\right| = \mathcal{O}(n^{-1})\end{aligned}$$

by Item (2) and Lemma 3.1. Moreover,

$$|\tilde{\delta} - \tilde{\delta}_j| \le \frac{1}{n}\left|\operatorname{Tr} \mathbb{E} \tilde{D}(\tilde{T} - \tilde{Q})\right| + \frac{1}{n}\left|\mathbb{E}(\operatorname{Tr} \tilde{D}\tilde{Q} - \operatorname{Tr} \tilde{D}_j \tilde{Q}_j)\right| + \frac{1}{n}\left|\operatorname{Tr} \mathbb{E} \tilde{D}_j(\tilde{Q}_j - \tilde{T}_j)\right|.$$

In order to deal with the middle term at the r.h.s., assume without generality loss that $j = 1$. Using the identity in [23, Section 0.7.3] for the inverse of a partitioned matrix, we obtain

$$\tilde{Q} = \begin{bmatrix} \tilde{q}_{11} & \times \\ \times & \tilde{Q}_1 + \tilde{q}_{11}\tilde{Q}_1 \Sigma_1^* \eta_1 \eta_1^* \Sigma_1 \tilde{Q}_1 \end{bmatrix}$$

hence $\mathbb{E}(\operatorname{Tr} \tilde{D}\tilde{Q} - \operatorname{Tr} \tilde{D}_1 \tilde{Q}_1) = \mathcal{O}(1)$, which shows that $|\tilde{\delta} - \tilde{\delta}_j| = \mathcal{O}(n^{-1})$. We now have

$$\begin{aligned}u^*(\mathcal{T}_j - T_j)v &= u^* \mathcal{T}_j \left(T_j^{-1} - \mathcal{T}_j^{-1}\right) T_j v \\ &= u^* \mathcal{T}_j \left(\rho(\tilde{\delta}_j - \tilde{\delta}) D + (\delta - \delta_j) A_j (I + \delta_j \tilde{D}_j)^{-1} \tilde{D}_j (I + \delta \tilde{D}_j)^{-1} A_j^*\right) T_j v = \mathcal{O}(n^{-1})\end{aligned}$$

which proves Item (4). In order to prove Item (5), we develop $\operatorname{Tr} U(Q - \mathbb{E}Q)$ as a sum of martingale differences:

$$\begin{aligned}\operatorname{Tr} U(Q - \mathbb{E}Q) &= \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) \operatorname{Tr} UQ \\ &= \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) \operatorname{Tr} U(Q - Q_j) = -\sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1})(\rho \tilde{q}_{jj} \eta_j^* Q_j U Q_j \eta_j)\end{aligned}$$

by (3.4), hence $\mathbb{E}\left|\operatorname{Tr} U(Q - \mathbb{E}Q)\right|^2 = \sum_{j=1}^n \mathbb{E}\left|(\mathbb{E}_j - \mathbb{E}_{j-1})(\rho \tilde{q}_{jj} \eta_j^* Q_j U Q_j \eta_j)\right|^2$. We now use (3.7). We have

$$\begin{aligned}&\mathbb{E}\left|(\mathbb{E}_j - \mathbb{E}_{j-1})(\rho \tilde{b}_j \eta_j^* Q_j U Q_j \eta_j)\right|^2 \\ &\qquad = \mathbb{E}\left|\mathbb{E}_j\left(\rho \tilde{b}_j \left(\eta_j^* Q_j U Q_j \eta_j - \tilde{d}_j \frac{\operatorname{Tr} D Q_j U Q_j}{n} - a_j^* Q_j U Q_j a_j\right)\right)\right|^2 = \mathcal{O}(n^{-1})\end{aligned}$$

by Lemma 3.2, and furthermore,

$$\mathbb{E}\left|(\mathbb{E}_j - \mathbb{E}_{j-1})(\rho^2 \tilde{q}_{jj} \tilde{b}_j e_j \eta_j^* Q_j U Q_j \eta_j)\right|^2 \le K\left(\mathbb{E} e_j^4 \ \mathbb{E}|\eta_j^* Q_j U Q_j \eta_j|^4\right)^{1/2} = \mathcal{O}(n^{-1})$$

by Lemma 3.2 and (3.15). This shows Th.3.3-(5).

A.4. **Proof of Lemma 3.4.** The two first upper bounds are easy to obtain, given that δ_n and $\tilde{\delta}_n$ are Stieltjes transforms of nonnegative measures with respective total mass $n^{-1}\operatorname{Tr} D$ and $n^{-1}\operatorname{Tr}\tilde{D}$. Now $\operatorname{Tr} DT^2 \le \boldsymbol{d}_{\max}\operatorname{Tr} T^2$ by Inequality (3.14), which in turn is smaller than $N\boldsymbol{d}_{\max}\rho^{-2}$, hence the third upper bound, and the other upper bounds can be proven similarly. Let us now prove the first lower bound.

$$\begin{aligned}
\operatorname{Tr} D &= \operatorname{Tr}(T^{\frac12} D T^{\frac12} T^{-1}) \overset{(a)}{\le} \operatorname{Tr}(DT)\times \|T^{-1}\| \ , \\
&\le \operatorname{Tr}(DT)\times\Big(\rho(1+\tilde{\delta}_n \boldsymbol{d}_{\max}) + \boldsymbol{a}^2_{\max}\|(I+\delta_n\tilde{D})^{-1}\|\Big) \ , \\
&\overset{(b)}{\le} \operatorname{Tr}(DT)\times\Big(\rho+\boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max}+\boldsymbol{a}^2_{\max}\Big) \ ,
\end{aligned}$$

where (a) follows from (3.14) and (b) from the upper bound on $\tilde{\delta}_n$. This readily yields δ_n's lower bound and $\tilde{\delta}_n$'s lower bound which can be proven similarly. Writing $\operatorname{Tr} D \le (\operatorname{Tr} DT^2)\|T^{-1}\|^2$, we obtain the lower bounds on $n^{-1}\operatorname{Tr} DT^2$ and $n^{-1}\operatorname{Tr}\tilde{D}\tilde{T}^2$ similarly. The lower bound for γ_n follows from the same ideas:

$$\begin{aligned}
\left(\frac{1}{N}\operatorname{Tr} D\right)^2 &\le \frac1N \operatorname{Tr} D^2 = \frac1N\operatorname{Tr}(T^{\frac12}D^2T^{\frac12}T^{-1}) \ , \\
&\le \frac1N\operatorname{Tr}(T^{\frac12}D^2T^{\frac12})\times\|T^{-1}\| = \frac1N \operatorname{Tr}(T^{\frac12}DT^{\frac12}T^{-1}T^{\frac12}DT^{\frac12})\times\|T^{-1}\| \ , \\
&\le \frac1N\operatorname{Tr}(TDTD)\times\|T^{-1}\|^2 \ ,
\end{aligned}$$

and one readily obtains γ_n's lower bound (and similarly $\tilde{\gamma}_n$'s lower bound) using Assumption **A-3** and the upper estimate previously obtained for $\|T^{-1}\|$.

The two last series of inequalities related to $n^{-1}\sum_{i=1}^N d_i^2 t_{ii}^2$ and $n^{-1}\sum_{j=1}^n \tilde{d}_j^2\tilde{t}_{jj}^2$ can be proven with similar arguments (lower bounds are in fact easier to obtain as one can directly get lower bounds for t_{ii} and $\tilde{t}_{jj}$ - using (3.10) for instance).

A.5. **Proof of Lemma 3.5.** From (1.3), $TA(I+\delta\tilde{D})^{-1}A^* = I-\rho T(I+\tilde{\delta}D)$. Moreover, $(I+\delta\tilde{D})^{-1}\tilde{D} = \delta^{-1}I - \delta^{-1}(I+\delta\tilde{D})^{-1}$. Hence

$$\begin{aligned}
\frac1n\operatorname{Tr} D^{1/2}TA(I+\delta\tilde{D})^{-2}\tilde{D}A^*TD^{\frac12} &\le \frac{1}{n\delta}\operatorname{Tr} DTA(I+\delta\tilde{D})^{-1}A^*T \\
&= 1-\frac{\rho}{n\delta}\operatorname{Tr} DT^2 - \rho\frac{\tilde{\delta}}{\delta}\gamma \qquad \text{(A.2)}
\end{aligned}$$

which proves the first assertion with the help of the results of Lemma 3.4. Similarly,

$$\frac1n\operatorname{Tr}\tilde{D}^{1/2}\tilde{T}A^*(I+\tilde{\delta}D)^{-2}DA\tilde{T}\tilde{D}^{\frac12} \le 1-\frac{\rho}{n\tilde{\delta}}\operatorname{Tr}\tilde{D}\tilde{T}^2 - \rho\frac{\delta}{\tilde{\delta}}\tilde{\gamma} \qquad \text{(A.3)}$$

We now show that the left hand sides (l.h.s.) of (A.2) and (A.3) are equal. Using the well known matrix identity $(I+UV)^{-1}U = U(I+VU)^{-1}$,

$$\begin{aligned}
TA(I+\delta\tilde{D})^{-1} &= \rho^{-1}(I+\tilde{\delta}D)^{-1}\Big(I+\rho^{-1}A(I+\delta\tilde{D})^{-1}A^*(I+\tilde{\delta}D)^{-1}\Big)^{-1}A(I+\delta\tilde{D})^{-1} \\
&= \rho^{-1}(I+\tilde{\delta}D)^{-1}A(I+\delta\tilde{D})^{-1}\Big(I+\rho^{-1}A^*(I+\tilde{\delta}D)^{-1}A(I+\delta\tilde{D})^{-1}\Big)^{-1} \\
&= (I+\tilde{\delta}D)^{-1}A\tilde{T}.
\end{aligned}$$

Plugging this identity in the l.h.s. of (A.2), and identifying with the l.h.s. of (A.3), we obtain the result. As a consequence, we have

$$\left(1-\frac{1}{n}\operatorname{Tr} D^{1/2}TA(I+\delta\tilde{D})^{-2}\tilde{D}A^*TD^{\frac{1}{2}}\right)^2 \geq \left(\frac{\rho}{n\delta}\operatorname{Tr} DT^2+\rho\frac{\tilde{\delta}}{\delta}\gamma\right)\left(\frac{\rho}{n\tilde{\delta}}\operatorname{Tr}\tilde{D}\tilde{T}^2+\rho\frac{\delta}{\tilde{\delta}}\tilde{\gamma}\right)$$
$$\geq \rho^2\gamma\tilde{\gamma}+\frac{\rho}{n\delta}\operatorname{Tr} DT^2\frac{\rho}{n\tilde{\delta}}\operatorname{Tr}\tilde{D}\tilde{T}^2$$

which is the second assertion. By Lemma 3.4, this leads to $\liminf \Delta_n > 0$. Lemma 3.5 is proven.

Appendix B. Additional proofs for Section 5

B.1. **Proof of Lemma 5.1.** Let us show that $\max_j \operatorname{var}(a^*\mathbb{E}_jQD\mathbb{E}_jQa) = \mathcal{O}(n^{-1})$. We have

$$\begin{aligned}
a^*\mathbb{E}_jQD\mathbb{E}_jQa - \mathbb{E}a^*\mathbb{E}_jQD\mathbb{E}_jQa &= \sum_{i=1}^{j}(\mathbb{E}_i-\mathbb{E}_{i-1})(a^*\mathbb{E}_jQD\mathbb{E}_jQa)\\
&= \sum_{i=1}^{j}(\mathbb{E}_i-\mathbb{E}_{i-1})\left\|D^{1/2}\mathbb{E}_j(Q_i-\rho\tilde{q}_{ii}Q_i\eta_i\eta_i^*Q_i)a\right\|^2\\
&= \sum_{i=1}^{j}(\mathbb{E}_i-\mathbb{E}_{i-1})\Big[-2\rho\operatorname{Re}\left(a^*(\mathbb{E}_jQ_i)D(\mathbb{E}_j\tilde{q}_{ii}Q_i\eta_i\eta_i^*Q_i)a\right)\\
&\qquad\qquad + \|\mathbb{E}_j(\rho\tilde{q}_{ii}\,\eta_i^*Q_ia\,D^{1/2}Q_i\eta_i)\|^2\Big]\\
&\triangleq 2\operatorname{Re}(X)+Z\ , \qquad\qquad (\text{B.1})
\end{aligned}$$

and the variance of $a^*\mathbb{E}_jQD\mathbb{E}_jQa$ is the sum of the variances of these martingale increments. Consider the term X. Recalling that $\tilde{q}_{ii}=\tilde{b}_i-\rho\tilde{q}_{ii}\tilde{b}_ie_i$,

$$\begin{aligned}
X &= -\rho\sum_{i=1}^{j}(\mathbb{E}_i-\mathbb{E}_{i-1})(\tilde{b}_i\,\eta_i^*Q_iaa^*(\mathbb{E}_jQ_i)DQ_i\eta_i)+\rho^2\sum_{i=1}^{j}(\mathbb{E}_i-\mathbb{E}_{i-1})(\tilde{b}_i\tilde{q}_{ii}e_i\eta_i^*Q_iaa^*(\mathbb{E}_jQ_i)DQ_i\eta_i)\\
&\triangleq X_1+X_2\ .
\end{aligned}$$

Let $M_i = Q_iaa^*(\mathbb{E}_jQ_i)DQ_i$. The term X_1 satisfies

$$\mathbb{E}|X_1|^2=\rho^2\sum_{i=1}^{j}\mathbb{E}\left|\mathbb{E}_i\tilde{b}_i\left(y_i^*M_iy_i-\frac{\tilde{d}_i}{n}\operatorname{Tr} DM_i+y_i^*M_ia_i+a_i^*M_iy_i\right)\right|^2\ . \qquad (\text{B.2})$$

Since M_i is a rank one matrix, $\sum_{i=1}^{j}\mathbb{E}|y_i^*M_iy_i-\tilde{d}_i\operatorname{Tr} DM_i/n|^2\leq K/n$. Moreover,

$$\sum_{i=1}^{j}\mathbb{E}\left|y_i^*M_ia_i\right|^2=\frac{1}{n}\sum_{i=1}^{j}\tilde{d}_i\mathbb{E}\left(a^*Q_iDQ_ia\ \ |a^*(\mathbb{E}_jQ_i)DQ_ia_i|^2\right)\leq\frac{K}{n}\sum_{i=1}^{j}\mathbb{E}\left|a^*(\mathbb{E}_jQ_i)DQ_ia_i\right|^2.$$

The summand at the r.h.s. of the inequality satisfies:

$$\begin{aligned}
|a^*(\mathbb{E}_jQ_i)DQ_ia_i|^2 &\leq 4\Big(|a^*(\mathbb{E}_jQ)DQa_i|^2+|a^*(\mathbb{E}_j(Q_i-Q))D(Q_i-Q)a_i|^2+\\
&\qquad |a^*(\mathbb{E}_jQ)D(Q_i-Q)a_i|^2+|a^*(\mathbb{E}_j(Q_i-Q))DQa_i|^2\Big)\\
&\triangleq 4(W_{i,1}+W_{i,2}+W_{i,3}+W_{i,4}).
\end{aligned}$$

Recalling that $A_{1:j} = [a_1, \cdots, a_j]$, we have

$$\sum_{i=1}^{j} W_{i,1} = a^*(\mathbb{E}_j Q) D Q A_{1:j} A_{1:j}^* Q D (\mathbb{E}_j Q) a \le K \ .$$

Recalling (3.5) and (3.6), and writing $\xi_i = 1 + \eta_i^* Q_i \eta_i$, we have:

$$W_{i,2} \le \frac{a_i^* Q \eta_i \eta_i^* Q a_i}{1 - \eta_i^* Q \eta_i} \times a^* \mathbb{E}_j \left(\xi_i \, Q \eta_i \eta_i^* Q \right) \ D \frac{Q \eta_i \eta_i^* Q}{1 - \eta_i^* Q \eta_i} D \ \mathbb{E}_j \left(\xi_i \, Q \eta_i \eta_i^* Q \right) a \ .$$

As $\|(1 - \eta_i^* Q \eta_i)^{-1} Q \eta_i \eta_i^* Q\| = \|Q - Q_i\| \le K$ and $\|Q\Sigma\| \le K$, we have $\sum_{i=1}^{j} \mathbb{E} W_{i,2} \le \sum_{i=1}^{j} \mathbb{E} \left[|a^* Q \eta_i|^2 |\xi_i|^2 \right]$. Writing $\xi_i = (\rho \tilde{b}_i)^{-1} + e_i$, and noticing that $(\rho \tilde{b}_i)^{-1}$ is bounded, we obtain:

$$\sum_{i=1}^{j} \mathbb{E} W_{i,2} \le 2 \mathbb{E} a^* Q \Sigma_{1:j} \operatorname{diag}((\rho \tilde{b}_1)^{-1}, \ldots, (\rho \tilde{b}_j)^{-1}) \Sigma_{1:j}^* Q a \ + \ K \sum_{i=1}^{j} \mathbb{E} |e_i|^2 \le K.$$

The terms $W_{i,3}$ and $W_{i,4}$ can be handled by similar derivations.
We get that $\sum_{i=1}^{j} \mathbb{E} |y_i^* M_i a_i|^2 \le K/n$. The terms $a_i^* M_i y_i$ on the right hand side of (B.2) satisfy $\sum_{i=1}^{j} \mathbb{E} \left| a_i^* M_i y_i \right|^2 \le K n^{-1} \sum_{i=1}^{j} \mathbb{E} |a_i^* Q_i a|^2 \le K/n$, which proves that $\mathbb{E}|X_1|^2 \le K/n$.

We now consider X_2, which satisfies $\mathbb{E}|X_2|^2 \le 2\rho^4 \sum_{i=1}^{j} \mathbb{E} |\tilde{b}_i \tilde{q}_{ii} e_i \eta_i^* M_i \eta_i|^2$. We have

$$\begin{aligned} \sum_{i=1}^{j} \mathbb{E} \left| \tilde{b}_i \tilde{q}_{ii} e_i a_i^* M_i a_i \right|^2 \quad &\le \quad K \sum_{i=1}^{j} \mathbb{E} \mathbb{E}^{(i)} \left| e_i a_i^* M_i a_i \right|^2 \\ &\le \quad \frac{K}{n} \sum_{i=1}^{j} \mathbb{E} |a_i^* M_i a_i|^2 \le \frac{K}{n} \sum_{i=1}^{j} \mathbb{E} |a_i^* Q_i a|^2 \le \frac{K}{n} \ , \end{aligned}$$

where $\mathbb{E}^{(i)} = \mathbb{E}[\cdot | y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n]$. Moreover,

$$\sum_{i=1}^{j} \mathbb{E} \left| \tilde{b}_i \tilde{q}_{ii} e_i y_i^* M_i a_i \right|^2 \le K \sum_{i=1}^{j} (\mathbb{E}|e_i|^4)^{1/2} (\mathbb{E}|y_i^* M_i a_i|^4)^{1/2} \le \frac{K}{n}$$

and similarly for the terms in $a_i^* M_i y_i$ and in $y_i^* M_i y_i$. We get that $\mathbb{E}|X_2|^2 = \mathcal{O}(n^{-1})$. We now turn to the term Z of equation (B.1). To control the variance of Z, we only need to control the variances of the terms:

$$\begin{aligned} Z_1 &= \sum_{i=1}^{j} (\mathbb{E}_i - \mathbb{E}_{i-1}) (\mathbb{E}_j \tilde{q}_{ii} a^* Q_i y_i \eta_i^* Q_i) D (\mathbb{E}_j \tilde{q}_{ii} Q_i \eta_i a_i^* Q_i a), \\ Z_2 &= \sum_{i=1}^{j} (\mathbb{E}_i - \mathbb{E}_{i-1}) \|\mathbb{E}_j (\tilde{q}_{ii} y_i^* Q_i a \, D^{1/2} Q_i \eta_i)\|^2, \\ Z_3 &= \sum_{i=1}^{j} (\mathbb{E}_i - \mathbb{E}_{i-1}) \|\mathbb{E}_j (\tilde{q}_{ii} a_i^* Q_i a \, D^{1/2} Q_i \eta_i)\|^2. \end{aligned}$$

The first term satisfies

$$\begin{aligned}\mathbb{E}|Z_1|^2 &\le 2\sum_{i=1}^{j}\mathbb{E}\left|(\mathbb{E}_j\tilde{q}_{ii}a^*Q_iy_i\eta_i^*Q_i)D(\mathbb{E}_j\tilde{q}_{ii}Q_i\eta_ia_i^*Q_ia)\right|^2\\ &\le 2\sum_{i=1}^{j}\mathbb{E}\left|(\mathbb{E}_ja^*Q_iy_i\eta_i^*Q_i)DQ_i\eta_ia_i^*Q_ia\right|^2\\ &= 2\sum_{i=1}^{j}\mathbb{E}\left[|a_i^*Q_ia|^2\,\mathbb{E}^{(i)}\left|(\mathbb{E}_ja^*Q_iy_i\eta_i^*Q_i)DQ_i\eta_i\right|^2\right]\\ &\le \frac{K}{n}\sum_{i=1}^{j}\mathbb{E}|a_i^*Q_ia|^2 \quad = \quad \mathcal{O}(n^{-1})\ ,\end{aligned}$$

where the second inequality comes from $\mathbb{E}|\mathbb{E}_j(X)\mathbb{E}_j(Y)|^2 = \mathbb{E}|\mathbb{E}_j(X\mathbb{E}_j(Y))|^2 \le \mathbb{E}|X\mathbb{E}_j(Y)|^2$. The terms Z_2 and Z_3 can be handled similarly; details are omitted. The result is $\mathbb{E}|Z|^2 \le K/n$.

Hence, $\operatorname{var}\left(a^*(\mathbb{E}_jQ)^2a\right) = \mathcal{O}(n^{-1})$. The estimate $\operatorname{var}\left(\operatorname{Tr}(\mathbb{E}_jQ)D(\mathbb{E}_jQ)\right) = \mathcal{O}(1)$ can be established similarly.

B.2. **Proof of Lemma 5.2.** Recalling the expression (3.10) of $\tilde{t}_{\ell\ell}$, we notice that $(1-\rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell) = \rho\tilde{t}_{\ell\ell}(1+\tilde{d}_\ell\delta)$ is bounded below. It follows from Theorem 3.3-(3) that

$$\sum_{\ell=1}^{j}\frac{\alpha_\ell\,u^*Ta_\ell\,\mathbb{E}\,[a_\ell^*Qu]}{1-\rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell} = \sum_{\ell=1}^{j}\frac{\alpha_\ell\,u^*Ta_\ell\,a_\ell^*Tu}{\rho\tilde{t}_{\ell\ell}(1+\tilde{d}_\ell\delta)} + \mathcal{O}(n^{-1/2})$$

Moreover,

$$\begin{aligned}\sum_{\ell=1}^{j}\alpha_\ell\,u^*Ta_\ell\left(\frac{\mathbb{E}\,[a_\ell^*Qu]}{1-\rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell} - \mathbb{E}\,[a_\ell^*Q_\ell u]\right) &= \sum_{\ell=1}^{j}\alpha_\ell\,u^*Ta_\ell\mathbb{E}\left[a_\ell^*Q_\ell u\left(\frac{1-\rho\tilde{q}_{\ell\ell}a_\ell^*Q_\ell\eta_\ell}{1-\rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell}-1\right)\right]\\ &\quad - \sum_{\ell=1}^{j}\alpha_\ell\,u^*Ta_\ell\frac{\mathbb{E}[\rho\tilde{q}_{\ell\ell}a_\ell^*Q_\ell\eta_\ell\,y_\ell^*Q_\ell u]}{1-\rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell} = \varepsilon_1+\varepsilon_2\end{aligned}$$

We have $\varepsilon_1 = \sum_{\ell=1}^{j}\alpha_\ell\,u^*Ta_\ell\mathbb{E}\,[a_\ell^*Q_\ell u\,\xi_\ell]$ where $\mathbb{E}|\xi_\ell|^p \le Kn^{-p/2}$ for $p\ge 2$. It follows that $|\varepsilon_1| \le \left(\sum_{\ell=1}^{j}\alpha_\ell^2|u^*Ta_\ell|^2\mathbb{E}\xi_\ell^2\right)^{1/2}\left(\sum_{\ell=1}^{j}\mathbb{E}|a_\ell^*Q_\ell u|^2\right)^{1/2} \le K/\sqrt{n}$ by Theorem 3.3-(1). By writing

$$\varepsilon_2 = -\sum_{\ell=1}^{j}\alpha_\ell\,u^*Ta_\ell\frac{\mathbb{E}\,[(\rho\tilde{q}_{\ell\ell}a_\ell^*Q_\ell\eta_\ell - \mathbb{E}[\rho\tilde{q}_{\ell\ell}a_\ell^*Q_\ell\eta_\ell])\;y_\ell^*Q_\ell u]}{1-\rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell}$$

and proceeding similarly to ε_1, we obtain $|\varepsilon_2| \le K/\sqrt{n}$, which completes the proof of Lemma 5.2.

B.3. **Proof of Lemma 5.3.** The F_j increase to $F_n = n^{-1}\operatorname{Tr}D^{1/2}TA(1+\delta\tilde{D})^{-2}\tilde{D}A^*TD^{1/2} < 1$ by Lemma 3.5. As $\gamma>0$ and M_j and G_j are increasing, $\boldsymbol{\Delta}_j$ is decreasing. In order to show that $\boldsymbol{\Delta}_n = \Delta_n$, we only need to show that $M_n + G_n = \rho^2\tilde{\gamma}$. We have

$$G_n = \frac{1}{n}\operatorname{Tr}\tilde{D}(I+\delta\tilde{D})^{-2}A^*TA\tilde{D}(I+\delta\tilde{D})^{-2}A^*TA - \frac{1}{n}\sum_{k=1}^{n}\left(\frac{\tilde{d}_k\,a_k^*Ta_k}{(1+\delta\tilde{d}_k)^2}\right)^2$$

Recall from (3.10) and (3.11) that $(1+\delta\tilde{d}_k)^{-2}a_k^* T a_k = (1+\delta\tilde{d}_k)^{-1} - \rho\tilde{t}_{kk}$. Hence

$$\frac{1}{n}\sum_{k=1}^{n}\left(\frac{\tilde{d}_k\, a_k^* T a_k}{(1+\delta\tilde{d}_k)^2}\right)^2 = \frac{1}{n}\sum_{k=1}^{n}\rho^2\tilde{d}_k^2\tilde{t}_{kk}^2 - \frac{1}{n}\sum_{k=1}^{n}\frac{\rho\tilde{d}_k^2\tilde{t}_{kk}}{(1+\delta\tilde{d}_k)} + \frac{1}{n}\sum_{k=1}^{n}\frac{\tilde{d}_k^2\, a_k^* T a_k}{(1+\delta\tilde{d}_k)^3}$$

which results in

$$M_n + G_n = \frac{\rho}{n}\operatorname{Tr}\tilde{D}\tilde{T}\tilde{D}(I+\delta\tilde{D})^{-1} - \frac{1}{n}\operatorname{Tr}\tilde{D}^2(I+\delta\tilde{D})^{-3}A^*TA + \frac{1}{n}\operatorname{Tr}\tilde{D}(I+\delta\tilde{D})^{-2}A^*TA\tilde{D}(I+\delta\tilde{D})^{-2}A^*TA.$$

Now, one can check with the help of (3.12) that $\rho^2\tilde{\gamma} = \rho^2 n^{-1}\operatorname{Tr}\tilde{D}\tilde{T}\tilde{D}\tilde{T}$ is equal to the r.h.s. of this equation. Lemma 5.3 is proven.

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