Resource Allocation for Downlink Cellular OFDMA Systems: Part I—Optimal Allocation

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Abstract

In this pair of papers (Part I and Part II in this issue), we investigate the issue of power control and subcarrier assignment in a sectorized two-cell downlink OFDMA system impaired by multicell interference. As recommended for WiMAX, we assume that the first part of the available bandwidth is likely to be reused by different base stations (and is thus subject to multicell interference) and that the second part of the bandwidth is shared in an orthogonal way between the different base stations (and is thus protected from multicell interference).

Although the problem of multicell resource allocation is nonconvex in this scenario, we provide in Part I the general form of the global solution. In particular, the optimal resource allocation turns out to be "binary" in the sense that, except for at most one pivot-user in each cell, any user receives data either in the reused bandwidth or in the protected bandwidth, but not in both. The determination of the optimal resource allocation essentially reduces to the determination of the latter pivot-position.

Index Terms

OFDMA Networks, Multicell Resource Allocation, Distributed Resource Allocation.

I. Introduction

We consider the problem of resource allocation in the downlink of a sectorized two-cell OFDMA system with incomplete Channel State Information (CSI) at the Base Station (BS) side. In principle, performing resource allocation for cellular OFDMA systems requires to solve the problem of power and subcarrier allocation jointly in all the considered cells, taking into consideration the interaction between

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users of different cells via the multicell interference. Unfortunately, in most of the practical cases, this global optimization problem is not convex and does not have, therefore, simple closed-form solution. Practical alternative methods must thus to be proposed to perform the resource allocation. Most of the works in the literature on multicell resource allocation assumed perfect CSI on the transmitters side. In flat-fading scenarios with multi-user interference, a number of interesting alternative methods have been proposed in the literature. One of them is the *geometric programming* (GP) approach proposed in [1] for centralized power control scenarios. The author of this work showed that at high SNR, the GP technique turns the nonconvex constrained optimization problem of power control into a convex, thus tractable, optimization problem. Another efficient resource allocation technique was proposed in [2] for decentralized power control scenarios. This technique is based on a min-max formulation of the optimization problem, and is adapted to ad-hoc networks contexts. Unfortunately, the two above mentioned techniques are mainly intended for flat-fading scenarios, and are not directly suitable to general cellular OFDMA contexts. To the best of our knowledge, only few works investigate OFDMA multicell resource allocation. Authors of [3] addressed the optimization of the sum rate performance in a multicell network in order to perform power control and user scheduling. In this context, the authors proposed a decentralized algorithm that maximizes an upperbound on the network sum rate. Interestingly, this upperbound is proved to be tight in the asymptotic regime when the number of users per cell is allowed to grow to infinity. However, the proposed algorithm does not guaranty fairness among the different users. In [4], a centralized iterative allocation scheme allowing to adjust the the number of cells reusing each subcarrier was presented. The proposed algorithm does not suppose the so called “reuse partitioning” scheme but nonetheless it promotes allocating subcarriers with low reuse factors to users with bad channel conditions. It also provides an interference limitation procedure in order to reduce the number of users whose rate requirements is unsatisfied. Authors of [5] considered the problem of subcarrier assignment and power control that minimize the percentage of unsatisfied users under rate and power constraints. For that sake, a centralized algorithm based on reuse partitioning was proposed. In this algorithm, the reuse factor of the far users next to the cell borders is adapted according to the QoS requirements and the problem parameters. Other dynamic resource allocation schemes were proposed in [6]-[10]. The authors of [9] and [10] have particularly discussed the issue of frequency reuse planning. It is worth mentioning here that neither of the above cited works [4]-[10] provided analytical study of the performance of their respective proposed schemes. The issue of power control in distributed cooperative OFDMA networks was addressed in [11]. However, the proposed solution assumes that subcarrier allocation is performed independently from the power control. The solution is thus suboptimal for the problem of resource allocation for OFDMA networks, and a general solution for both power control and frequency resource allocation remains to be provided.

In contrast to previous works where perfect CSI was assumed, authors of [12] assumed the knowledge of only the statistics of users' channels and proposed an iterative algorithm for resource allocation in the multicell context. In this algorithm a frequency (or subcarrier) reuse factor equal to one was chosen, which means that each cell is supposed to use all available subcarriers. This assumption relatively simplifies solving the problem of multicell OFDMA resource allocation. A similar iterative multicell allocation algorithm was proposed in [13] and its convergence to the optimal solution of the multicell resource allocation problem was proved based on the framework developed in [14].

In this paper, our aim is to characterize the resource allocation strategy (power control and subcarrier assignment scheme) allowing to satisfy all users' rate requirements while spending the least power at the transmitters' side. Similarly to [12], we investigate the case where the transmitter CSI is limited to some channel statistics. However, contrary to [12] which assumes a frequency reuse factor equal to one, our model assumes that a certain part of the available bandwidth is shared orthogonally between the adjacent base stations (and is thus "protected" from multicell interference) while the remaining part is reused by different base stations (and is thus subject to multicell interference). Note that this so-called *fractional frequency reuse* is recommended in a number of standards *e.g.* in [15] for IEEE 802.16 (WiMax) [16]. We also assume that each user is likely to modulate in each of these two parts of the bandwidth. Thus, we stress the fact that *i)* no user is forced to modulate in a single frequency band, *ii)* we do not assume *a priori* a geographical separation of users modulating in the two different bands. On the opposite, we shall *demonstrate* that such a geographical separation is actually optimal w.r.t. our resource allocation problem. In this context, we provide an algorithm that permits to compute the optimal resource allocation.

The paper is organized as follows. In Section II we present the system model. In Section III we consider the problem of resource allocation in a single cell assuming that the interference generated by the other cells of the network is fixed. The problem consists in minimizing the transmit power of the considered cell assuming a fixed level of interference such that the rate requirements of users of this cell are satisfied and such that the interference produced by the cell itself is less than a certain value. Although resource allocation for users of the network requires in general solving a multicell optimization problem, the single cell problem of Section III turns out to be a useful tool to solve the more complicated multicell problem. Theorem 1 gives the solution to this single cell optimization problem. Except for at most one "pivot" user in the considered cell, any user receives data either in the interference bandwidth or in the protected bandwidth, but not in both. In Section IV we introduce the joint multicell resource allocation problem. This problem is equivalent to jointly determining the resource allocation parameters of users belonging to different interfering cells, such that all users' rate requirements are satisfied and such that the total transmit power is minimized. Theorem 2 characterizes the solution to this optimization problem as function of a small number of unknown parameters. The solution turns out to have in each cell the same binary form as the solution to the single cell problem. Although this geographical separation is frequently used in practice, no existing works prove the optimality of such a scheme to our knowledge. Subsection IV-C provides a method to calculate the optimal resource allocation. Finally, Section V is devoted to the numerical results.

II. System Model

A. OFDMA Signal Model

We consider a downlink OFDMA sectorized cellular network. In order to simplify the presentation of our results, the network is supposed to be one-dimensional (linear) as in a number of existing studies [12], [17], [18], [19], [20]. The motivation behind our choice of the one-dimensional network is that such a simple model can provide a good understanding on the problem while still grasping the main aspects of a real-world cellular system. It provides also some interesting guidelines that help to implement practical cellular systems. Generalization to 2D-networks is however possible (though much more involved) and is addressed in a separate work [21]. We consider the case of sectorized networks *i.e.*, users belonging to different sectors of the same cell are spatially orthogonal [22]. In this case, it is reasonable to assume that a given user is only subject to interference from the nearest interfering base station. Thus, we focus on two interfering sectors of two adjacent cells, say Cell A and Cell B, as illustrated by Figure 1. Denote by D the radius of each cell which is assumed to be identical for all cells without restriction. We denote by K^A and K^B the number of users in Cell A and B respectively. We denote by $K = K^A + K^B$ the total number of users in both cells. Each base station provides information to all its users following a OFDMA scheme. The total number of available subcarriers is denoted by N. For a given user $k \in 1, 2, \ldots, K^A$ in Cell A, we denote by $\mathcal{N}_k$ the set of indices corresponding to the subcarriers modulated by k. $\mathcal{N}_k$ is a subset of $\{0, 1, \ldots, N-1\}$. By definition of OFDMA, two distinct users k, k' belonging to Cell A are such that $\mathcal{N}_k \cap \mathcal{N}_{k'} = \emptyset$. For each user $k \in \{1, \ldots, K^A\}$ of Cell A, the signal received by k at the nth subcarrier ($n \in \mathcal{N}_k$) and at the mth OFDM block is given by

$$y_k(n,m) = H_k(n,m) s_k(n,m) + w_k(n,m), \tag{1}$$

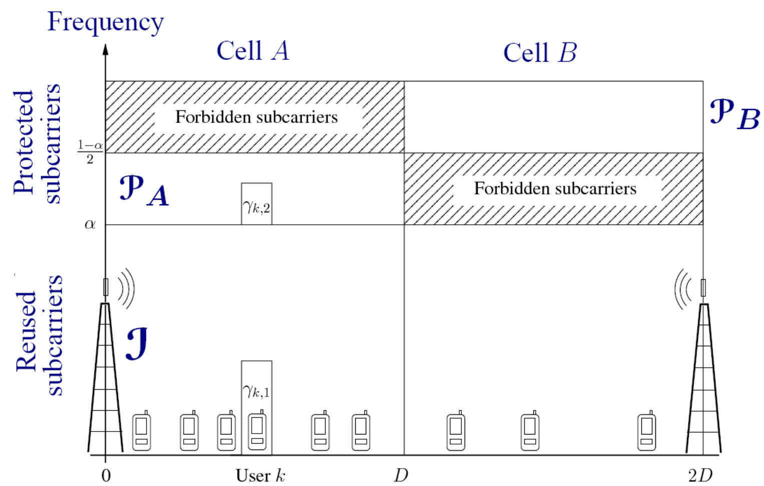

Figure 1. Two-Cell System model

where $s_k(n,m)$ represents the data symbol transmitted by Base Station A. Process $w_k(n,m)$ is an additive noise which encompasses the thermal noise and the possible multicell interference. Coefficient $H_k(n,m)$ is the frequency response of the channel at the subcarrier n and the OFDM block m. Random variables $H_k(n,m)$ are assumed to be Rayleigh distributed with variance

$$\rho_k = \mathbb{E}[|H_k(m,n)|^2] \; . \tag{2}$$

Note that the mean value ρ_k does not depend on the subcarrier index. This is satisfied for instance in the case of decorrelated channel taps in the time domain. For a given user k, $H_k(n,m)$ are identically distributed w.r.t. n,m, but are not supposed to be independent. Channel coefficients are supposed to be perfectly known at the receiver side, and unknown at the base station side. However, variances ρ_k are supposed to be known at the base station. This type of incomplete CSI is particularly adapted to fast fading scenarios. In such a context, sending feedback containing the instantaneous channel gain from users to the base station will result in a significant overhead.

As usual, we assume that ρ_k vanishes with the distance between Base Station A and user k, based on a given path loss model. In the sequel, it is convenient to assume (without restriction) that users $k = 1, 2, \ldots, K^A$ are numbered from the nearest to the base station to the farthest. Therefore, for all users k in Cell A,

$$\rho_1 > \rho_2 > \ldots > \rho_{K^A} . \tag{3}$$

B. Frequency Reuse

The frequency reuse scheme is illustrated by Figure 1. In practical cellular OFDMA systems, it is usually assumed that certain subcarriers $n \in \{0, \dots N-1\}$ used by Base Station A are reused by the adjacent Cell B. Denote by $\mathcal{I}$ this set of "$\mathcal{I}$nterfering" subcarriers, $\mathcal{I} \subset \{0, \dots, N-1\}$. If user k modulates such a subcarrier $n \in \mathcal{I}$, the additive noise $w_k(n,m)$ contains both thermal noise of variance σ^2 and interference. Therefore, the variance of $w_k(n,m)$ depends on k and is crucially related to the position of user k. We thus define

$$\forall n \in \mathcal{I}, \mathbb{E}[|w_k(n,k)|^2] = \sigma_k^2 \ .$$

Note that σ_k^2 is assumed to be a constant w.r.t. the subcarrier index n. This assumption is valid in OFDMA multicell systems using frequency hopping or random subcarrier assignment as in WiMax. If users $k = 1, 2 \dots K^A$ are numbered from the nearest to the base station to the farthest, it is reasonable to assume that

$$\sigma_1^2 < \sigma_2^2 < \dots < \sigma_{K^A}^2 \ , \tag{4}$$

meaning that the farthest users experience more multicell interference. The *reuse factor* α is defined as the ratio between the number of reused subcarriers and the total number of available subcarriers:

$$\alpha = \frac{\text{card}(\mathcal{I})}{N}$$

so that $\mathcal{I}$ contains αN subcarriers. The remaining $(1-\alpha)N$ subcarriers are shared by the two cells, A and B, in an orthogonal way. We assume that $\frac{1-\alpha}{2}N$ of these subcarriers are used by Base Station A only and are forbidden for B. Denote by $\mathcal{P}_A$ this set of "$\mathcal{P}$rotected" subcarriers. If user k modulates such a subcarrier $n \in \mathcal{P}_A$, the additive noise $w_k(n,m)$ contains only thermal noise. In other words, subcarrier n does not suffer from multicell interference. Then we simply write $\mathbb{E}[|w_k(n,m)|^2] = \sigma^2$, where σ^2 is the variance of the thermal noise only. Similarly, we denote by $\mathcal{P}_B$ the remaining $\frac{1-\alpha}{2}N$ subcarriers, such that each subcarrier $n \in \mathcal{P}_B$ is only used by Base Station B, and is not used by A. Finally, $\mathcal{I} \cup \mathcal{P}_A \cup \mathcal{P}_B = \{0, \dots, N-1\}$. Moreover, let $g_{k,1}$ (resp. $g_{k,2}$) be the channel Gain to Noise Ratio (GNR) in band $\mathcal{I}$ (resp. $\mathcal{P}_A$), namely $g_{k,1} = \rho_k/\sigma_k^2$ (resp. $g_{k,2} = \rho_k/\sigma^2$).

C. Resource Allocation Parameters

Of course, for a given user k of Cell A, the noise variance σ_k^2 depends on the particular resource allocation used in the adjacent Cell B. We assume that σ_k^2 is known at Base Station A, and that a given user may use subcarriers in both the "interference" bandwidth $\mathcal{I}$ and the "protected" bandwidth $\mathcal{P}_A$. We denote by $\gamma_{k,1}^A N$ (resp. $\gamma_{k,2}^A N$) the number of subcarriers modulated by user k in the set $\mathcal{I}$ (resp. $\mathcal{P}_A$). In other words,

$$\gamma_{k,1}^A = \text{card}(\mathcal{I} \cap \mathcal{N}_k)/N \qquad \gamma_{k,2}^A = \text{card}(\mathcal{P}_A \cap \mathcal{N}_k)/N.$$

Note that by definition of $\gamma_{k,1}^A$ and $\gamma_{k,2}^A$, $\sum_k \gamma_{k,1}^A \leq \alpha$ and $\sum_k \gamma_{k,2}^A \leq \frac{1-\alpha}{2}$, and that the superscript A (or B) is used to designate the cell in which user k is located. We assume in the sequel without restriction that the sharing factors $\{\gamma_{k,1}^A, \gamma_{k,2}^A\}_k$ are continuous real-valued variables and can take on any value in the interval $[0,1]$. Furthermore, we assume that a given user k of Cell A can modulate in both bands $\mathcal{I}$ and $\mathcal{P}_A$ using distinct powers in each band. For any modulated subcarrier $n \in \mathcal{N}_k$, we define $P_{k,1}^A = E[|s_k(n,m)|^2]$ if $n \in \mathcal{I}$, $P_{k,2}^A = E[|s_k(n,m)|^2]$ if $n \in \mathcal{P}_A$. Similarly, denote by $W_{k,i}^A = \gamma_{k,i}^A P_{k,i}^A$ the average power transmitted to user k in $\mathcal{I}$ if $i=1$ and in $\mathcal{P}_A$ if $i=2$. "Setting a resource allocation for Cell A" means setting a value for parameters $\{\gamma_{k,1}^A, \gamma_{k,2}^A, P_{k,1}^A, P_{k,2}^A\}_{k=1\dots K^A}$, or equivalently for parameters $\{\gamma_{k,1}^A, \gamma_{k,2}^A, W_{k,1}^A, W_{k,2}^A\}_{k=1\dots K^A}$.

D. Multicell Interference Model

We define now more clearly the way interference levels $\sigma_1^2, \dots, \sigma_{K^A}^2$ depend on the adjacent Base Station B. In OFDMA system models which assume Frequency Hopping like Flash-OFDM system ([22] chapter 4, page 179-180, [23]), it is straightforward to show that for a given user k of Cell A, interference power σ_k^2 does not depend on the particular resource allocation in Cell B but only on i) the position of user k and ii) the average power $Q_1^B = \sum_{k=1}^{K^B} W_{k,1}^B$ transmitted by Base Station B in the interference bandwidth $\mathcal{I}$. More precisely,

$$\sigma_k^2 = \mathbb{E}\left[|\tilde{H}_k(n,m)|^2\right] Q_1^B + \sigma^2 \tag{5}$$

where $\tilde{H}_k(n,m)$ represents the channel between Base Station B and user k of Cell A at frequency n and OFDM block m. In particular, $\mathbb{E}\left[|\tilde{H}_k(n,m)|^2\right]$ only depends on the position of user k and on the path-loss exponent.

III. Single Cell Resource Allocation

Before tackling the problem of joint optimal resource allocation in the two considered cells, it is useful to consider first the simpler single cell problem. The single cell formulation focuses on resource allocation in one cell, and assumes that the resource allocation parameters of users in the other cell are fixed.

A. Single Cell Optimization Problem

Assume that each user k has a rate requirement of R_k nats/s/Hz. Our aim is to optimize the resource allocation for Cell A which i) allows to satisfy all target rates R_k of all users, and ii) minimizes the power used by Base Station A in order to achieve these rates. Considering a fast fading context (i.e. channel coefficients $H_k(n,m)$ vary w.r.t. m all along the code word), we assume as usual that successful transmission at rate R_k is possible provided that $R_k < C_k$, where C_k denotes the ergodic capacity associated with user k. Unfortunately, the exact expression of the ergodic capacity is difficult to obtain in our context due to the fact that the noise-plus-interference process $(w_k(n,m))_{n,m}$ is not a Gaussian process in general. Nonetheless, if we endow the input symbols $s_k(n,m)$ with Gaussian distribution, the mutual information between $s_k(n,m)$ and the received signal $y_k(n,m)$ in equation (1) is minimum when the interference-plus-noise $w_k(n,m)$ is Gaussian distributed. Therefore, the approximation of the multicell interference as a Gaussian random variable is widely used in the literature on OFDMA (see for instance [12], [24], [25]) as it provides a lower bound on the mutual information. For a given user k in Cell A, the ergodic capacity in the whole band is equal to the sum of the ergodic capacities corresponding to both bands $\mathcal{I}$ and $\mathcal{P}_A$. For instance, the part of the capacity corresponding to the protected band $\mathcal{P}_A$ is equal to $\gamma_{k,2}^A \mathbb{E}\left[\log\left(1 + P_{k,2}^A \frac{|H_k(n,m)|^2}{\sigma^2}\right)\right] = \gamma_{k,2}^A \mathbb{E}\left[\log\left(1 + \frac{W_{k,2}^A}{\gamma_{k,2}^A}\frac{|H_k(n,m)|^2}{\sigma^2}\right)\right]$, where factor $\gamma_{k,2}^A$ traduces the fact that the capacity increases with the number of subcarriers which are modulated by user k. In the latter expression, the expectation is calculated with respect to random variable $\frac{|H_k(m,n)|^2}{\sigma^2}$. Now, $\frac{H_k(m,n)|^2}{\sigma^2}$ has the same distribution as $\frac{\rho_k}{\sigma^2}Z = g_{k,2}Z$, where Z is a standard Chi-Square distributed random variable with two degrees of freedom. Finally, the ergodic capacity in the whole bandwidth is equal to

$$C_k(\gamma_{k,1}^A, \gamma_{k,2}^A, W_{k,1}^A, W_{k,2}^A) = \gamma_{k,1}^A \mathbb{E}\left[\log\left(1 + g_{k,1}\frac{W_{k,1}^A}{\gamma_{k,1}^A}Z\right)\right] + \gamma_{k,2}^A \mathbb{E}\left[\log\left(1 + g_{k,2}\frac{W_{k,2}^A}{\gamma_{k,2}^A}Z\right)\right] \quad (6)$$

where Z represents a standard Chi-Square distributed random variable with two degrees of freedom. The quantity Q^A defined by

$$Q^A = \sum_{k=1}^{K^A}(W_{k,1}^A + W_{k,2}^A) \quad (7)$$

denotes the average power spent by Base Station A during one OFDM block. The optimal resource allocation problem for Cell A consists in characterizing $\{\gamma_{k,1}^A, \gamma_{k,2}^A, W_{k,1}^A, W_{k,2}^A\}_{k=1\ldots K^A}$ allowing to satisfy all rate requirements of all users ($R_k < C_k$) so that the power Q^A to be spent is minimum. Furthermore, as we are targeting a multicell interference scenario, it is also legitimate to limit the interference which is *produced* by Base Station A. Therefore, we introduce the following “low nuisance constraint”: The power $Q_1^A = \sum_k W_{k,1}^A$ which is transmitted by Base Station A in the interference band $\mathcal{I}$ should not exceed a certain *nuisance level* $\mathcal{Q}$, which is assumed to be a predefined constant imposed by the system's requirements. The introduction of this constraint will be later revealed useful in Section IV when studying the solution to the joint multicell resource allocation problem. The single cell optimization problem can be formulated as follows.

Problem 1. *Minimize* Q^A *w.r.t.* $\{\gamma_{k,1}^A, \gamma_{k,2}^A, W_{k,1}^A, W_{k,2}^A\}_{k=1\ldots K^A}$ *under the following constraints.*

$$\mathbf{C1}: \forall k, R_k \le C_k \qquad \mathbf{C4}: \gamma_{k,1}^A \ge 0, \gamma_{k,2}^A \ge 0$$

$$\mathbf{C2}: \sum_{k=1}^{K^A} \gamma_{k,1}^A = \alpha \qquad \mathbf{C5}: W_{k,1}^A \ge 0, W_{k,2}^A \ge 0.$$

$$\mathbf{C3}: \sum_{k=1}^{K^A} \gamma_{k,2}^A = \frac{1-\alpha}{2} \qquad \mathbf{C6}: \sum_{k=1}^{K^A} W_{k,1}^A \le \mathcal{Q}\,.$$

Here, $\mathbf{C1}$ is the rate constraint, $\mathbf{C2}$-$\mathbf{C3}$ are the bandwidth constraints, $\mathbf{C4}$-$\mathbf{C5}$ are the positivity constraints. Note that $\mathbf{C6}$ is the low nuisance constraint imposed only on the power transmitted in the non protected band $\mathcal{I}$. The particular case where the maximum admissible nuisance level is set to $\mathcal{Q} = +\infty$ would correspond to a "selfish" resource allocation: Base Station A may transmit as much power as needed in the interference band $\mathcal{I}$ without caring about the nuisance which it produces on the adjacent cell. Note that in Problem 1 no power constraint is imposed on the total power Q^A transmitted by the base station in the two bands. Note also that the constraint set (the set of all feasible points) associated with Problem 1 is not empty as it contains at least the following trivial solution. This trivial solution consists in assigning zero power $W_{k,1}^A = 0$ on the subcarriers of the non protected band $\mathcal{I}$ (so that constraint $\mathbf{C6}$ will be satisfied), and in performing resource allocation only in the protected band $\mathcal{P}_A$. The main reason for expressing the resource allocation problems in terms of parameters $\gamma_{k,i}^A, W_{k,i}^A$ $(i = 1, 2)$ instead of $\gamma_{k,i}^A, P_{k,i}^A$ is that the ergodic capacity $C_k = C_k(\gamma_{k,1}^A, W_{k,1}^A, \gamma_{k,2}^A, W_{k,2}^A)$ is a concave function of $\gamma_{k,i}^A, W_{k,i}^A$. As a consequence, the constraint set is convex and Problem 1 is a convex optimization problem in $\{\gamma_{k,1}^A, \gamma_{k,2}^A, W_{k,1}^A, W_{k,2}^A\}_k$. Obviously, finding the optimal parameter set $\{\gamma_{k,1}^A, \gamma_{k,2}^A, W_{k,1}^A, W_{k,2}^A\}_k$ is equivalent to finding the optimal $\{\gamma_{k,1}^A, \gamma_{k,2}^A, P_{k,1}^A, P_{k,2}^A\}_k$ thanks to the simple relation $W_{k,i}^A = \gamma_{k,i}^A P_{k,i}^A$, $i = 1, 2$.

B. Optimal Single Cell Resource Allocation

In order to solve convex Problem 1, we use the Lagrange Karush-Kuhn-Tucker (KKT) conditions. Define the following function on $\mathbb{R}_+$

$$f(x) = \frac{\mathbb{E}\left[\log(1 + xZ)\right]}{\mathbb{E}\left[\frac{Z}{1+xZ}\right]} - x\,, \tag{8}$$

where Z represents a standard Chi-Square distributed random variable with two degrees of freedom. It can be shown that function $f(x)$ is increasing from 0 to ∞ on $\mathbb{R}_+$. The following theorem provides the general form of any global solution to Problem 1. Its proof is provided in Appendix A.

Theorem 1. *Any global solution* $\{\gamma_{k,1}^A, \gamma_{k,2}^A, W_{k,1}^A, W_{k,2}^A\}_{k=1\dots K^A}$ *to Problem 1 is as follows. There exists an integer* $L \in \{1, \dots, K^A\}$ *and three nonnegative numbers* β_1, β_2 *and* ξ *such that*

1) *For each* $k < L$,

$$\begin{array}{l|l} P_{k,1}^A = g_{k,1}^{-1} f^{-1}\left(\dfrac{g_{k,1}}{1+\xi}\beta_1\right) & P_{k,2}^A = 0 \\ \gamma_{k,1}^A = \dfrac{R_k}{\mathbb{E}\left[\log\left(1+g_{k,1}P_{k,1}^A Z\right)\right]} & \gamma_{k,2}^A = 0 \end{array} \tag{9}$$

2) *For each* $k > L$,

$$\begin{array}{l|l} P_{k,1}^A = 0 & P_{k,2}^A = g_{k,2}^{-1} f^{-1}(g_{k,2}\beta_2) \\ \gamma_{k,1}^A = 0 & \gamma_{k,2}^A = \dfrac{R_k}{\mathbb{E}\left[\log\left(1+g_{k,2}P_{k,2}^A Z\right)\right]} \end{array} \tag{10}$$

3) *For* $k = L$

$$\begin{array}{l|l} P_{k,1}^A = g_{k,1}^{-1} f^{-1}\left(\dfrac{g_{k,1}}{1+\xi}\beta_1\right) & P_{k,2}^A = g_{k,2}^{-1} f^{-1}(g_{k,2}\beta_2) \\ \gamma_{k,1}^A = \alpha - \displaystyle\sum_{l=1}^{k-1}\gamma_{l,1}^A & \gamma_{k,2}^A = \dfrac{1-\alpha}{2} - \displaystyle\sum_{l=k+1}^{K^A}\gamma_{l,2}^A, \end{array} \tag{11}$$

where β_1, β_2 *and* ξ *are the Lagrange multipliers associated with constraints* $\mathbf{C2}$, $\mathbf{C3}$ *and* $\mathbf{C6}$ *respectively. Determination of* L, β_1, β_2 *and* ξ *is provided by Proposition 1.*

Comments on Theorem 1:

a) Theorem 1 states that the optimal resource allocation scheme is "binary": except for at most one user ($k = L$), any user receives data either in the interference bandwidth $\mathcal{I}$ or in the protected bandwidth $\mathcal{P}_A$, but not in both. Intuitively, it seems clear that users who are the farthest from the base station should mainly receive data in the protected bandwidth $\mathcal{P}_A$, as they are subject to an significant multicell interference and hence need to be protected. Now, a closer look at our result shows that the farthest users should only receive in the protected bandwidth $\mathcal{P}_A$. On the other hand, nearest users should only receive in the interference bandwidth $\mathcal{I}$.

b) Nonzero resource allocation parameters $\gamma_{k,1}^A, P_{k,1}^A$ (for $k \le L$) and $\gamma_{k,2}^A, P_{k,2}^A$ (for $k \ge L$) are expressed as functions of three parameters β_1, β_2, ξ. It can be easily seen from Appendix A that β_1, β_2, ξ are the Lagrange multipliers associated with constraints $\mathbf{C2}$, $\mathbf{C3}$ and $\mathbf{C6}$ respectively. It is quite intuitive that, when the admissible nuisance level is large (take for instance $\mathcal{Q} = +\infty$), constraint **C6** holds with strict inequality. Thus, $\xi = 0$ from complementary slackness condition. In the general case, the values of parameters β_1, β_2, ξ can be obtained from KKT conditions. The determination of β_1, β_2, ξ and the pivot-user L is given by Proposition 1.

c) As expected, the optimal resource allocation depends on the resource allocation in Cell B via parameters $\sigma_1^2, \ldots, \sigma_{K^A}^2$. Joint optimization of the resource allocation in both cells, A and B, is investigated in Section IV.

While Theorem 1 provides the form of any global solution to the single cell problem, the following proposition proves that the global solution to this problem is unique and provides a practical method to compute it. Its proof is provided in Appendix B. Before proceeding, define for each $x \geq 0$,

$$F(x) = \mathbb{E}\left[\frac{Z}{1 + f^{-1}(x)Z}\right] \tag{12}$$

$$C(x) = \mathbb{E}[\log(1 + f^{-1}(x)Z)] \,. \tag{13}$$

Proposition 1. *The global solution to the single cell Problem 1 is unique and is given by equations (9)-(10)-(11), where parameters L, β_1, β_2 and ξ are unique and determined as follows. For each l, define a_l^A and b_l^A as the unique positive numbers such that:*

$$\sum_{k=1}^{l} \frac{R_k}{C\left(g_{k,1} a_l^A\right)} = \alpha \quad \textit{and} \quad \sum_{k=l+1}^{K^A} \frac{R_k}{C(g_{k,2} b_l^A)} = \frac{1-\alpha}{2},$$

with $a_0^A = b_{K^A}^A = 0$ by convention. Consider the following system of equations.

$$\left[\alpha - \sum_{k<L} \frac{R_k}{C\left(\frac{g_{k,1}}{1+\xi}\beta_1\right)}\right] \frac{C\left(\frac{g_{L,1}}{1+\xi}\beta_1\right)}{R_L} + \left[\frac{1-\alpha}{2} - \sum_{k>L} \frac{R_k}{C(g_{k,2}\beta_2)}\right] \frac{C(g_{L,2}\beta_2)}{R_L} = 1 \tag{14}$$

$$\frac{g_{L,1}}{1+\xi} F\left(\frac{g_{L,1}}{1+\xi}\beta_1\right) = g_{L,2} F(g_{L,2}\beta_2) \tag{15}$$

$$L = \min\left\{ l = 1 \ldots K^A / \frac{g_{l,1}}{1+\xi} F\left(g_{l,1} a_l^A\right) \leq g_{l,2} F\left(g_{l,2} b_l^A\right) \right\} \tag{16}$$

$$\sum_{k \leq L} \gamma_{k,1}^A P_{k,1}^A = \mathcal{Q} \,. \tag{17}$$

The following procedure permits the determination of parameters L, β_1, β_2 and ξ.

1) *Assuming $\xi = 0$, evaluate L by (16) and (β_1, β_2) as the unique solution to the system of equations (14)-(15) satisfying $\left(\frac{\beta_1}{1+\xi}, \beta_2\right) \in [a_{L-1}^A, a_L^A] \times [b_L^A, b_{L-1}^A]$. Then evaluate $Q_1^A = \sum_k \gamma_{k,1}^A P_{k,1}^A$.*
2) *Stop if $Q_1^A \leq \mathcal{Q}$ (constraint* **C6** *is met) otherwise continue.*
3) *Evaluate (L, β_1, β_2, ξ) as the unique solution to the system of equations (14)-(15)-(16)-(17).*

IV. Joint Multicell Resource Allocation

A. Optimization Problem

Our aim now is to jointly optimize the resource allocation for the two cells which i) allows to satisfy all target rates R_k of all users, and ii) minimizes the power used by the two base stations in order to achieve these rates. The ergodic capacity associated with user k in Cell A is given by equation (6), where coefficient $g_{k,1}$ in that equation coincides with

$$g_{k,1}(Q_1^B) = \frac{\rho_k}{\mathbb{E}\left[|\tilde{H}_k(n,m)|^2\right] Q_1^B + \sigma^2},$$

where $\tilde{H}_k(n,m)$ represents the channel between Base Station B and user k of Cell A at frequency n and OFDM block m. Coefficient $g_{k,1}(Q_1^B)$ represents the signal to interference plus noise ratio in the interference band $\mathcal{I}$. Here, $g_{k,1}(Q_1^B)$ not only depends on the position of user k in Cell A, but also on the power $Q_1^B = \sum_{k=1}^{K^B} W_{k,1}^B$ transmitted by the adjacent Base Station B in band $\mathcal{I}$. We now solve the following multicell resource allocation problem.

Problem 2. *Minimize the total power spent by both base stations* $Q = \sum_{c=A,B} \sum_{k=1}^{K^c} (W_{k,1}^c + W_{k,2}^c)$ *with respect to* $\{\gamma_{k,1}^c, \gamma_{k,2}^c, W_{k,1}^c, W_{k,2}^c\}_{\substack{c=A,B \\ k=1\ldots K^c}}$ *under the following constraints.*

$$\mathbf{C1}: \forall k, R_k \le C_k \qquad \mathbf{C4}: \gamma_{k,1}^c \ge 0, \gamma_{k,2}^c \ge 0$$

$$\mathbf{C2}: \sum_{k=1}^{K^c} \gamma_{k,1}^c = \alpha \qquad \mathbf{C5}: W_{k,1}^c \ge 0, W_{k,2}^c \ge 0.$$

$$\mathbf{C3}: \sum_{k=1}^{K^c} \gamma_{k,2}^c = \frac{1-\alpha}{2}$$

It can be easily seen that the above optimization problem is feasible as soon as $\alpha < 1$. Indeed, a naive but nevertheless feasible point can be easily constructed by forcing each user to modulate in the protected band only (force $\gamma_{k,1}^c = 0$ for each user). Cells thus become orthogonal, and all users rate requirements R_k can be satisfied provided that enough power is transmitted in the protected band. Unfortunately, the ergodic capacity C_k of user k is not a convex function with respect to the optimization variables. This is due to the fact that the gain-to-noise ratio $g_{k,1}(Q_1^B)$ is a function of the resource allocation parameters of users belonging to the interfering cell. Therefore, Problem 2 is nonconvex, and cannot be solved by classical convex optimization methods. Nonetheless, we manage to characterize its solution. In fact, we prove that the solution has the same simple binary form of the single cell optimal solution.

B. Optimal Resource Allocation

For each cell $c \in \{A, B\}$, denote by $\bar{c}$ the adjacent cell ($\overline{A} = B$ and $\overline{B} = A$). The following result is proved in Appendix C.

Theorem 2.

(A) *Any global solution to Problem 2 has the following form. For each Cell c, there exists an integer $L^c \in \{1, \ldots, K^c\}$, and there exist four positive numbers β_1^c, β_2^c, ξ^c, $Q_1^{\bar{c}}$ such that*

1) *For each $k < L^c$,*

$$\left.\begin{array}{l} P_{k,1}^c = g_{k,1}(Q_1^{\bar{c}})^{-1} f^{-1}\left(\dfrac{g_{k,1}(Q_1^{\bar{c}})}{1+\xi^c}\beta_1^c\right) \\ \gamma_{k,1}^c = \dfrac{R_k}{\mathbb{E}\left[\log\left(1+g_{k,1}(Q_1^{\bar{c}})P_{k,1}^c Z\right)\right]} \end{array}\right| \begin{array}{l} P_{k,2}^c = 0 \\ \gamma_{k,2}^c = 0 \end{array} \tag{18}$$

2) *For each $k > L^c$,*

$$\begin{array}{l} P_{k,1}^c = 0 \\ \gamma_{k,1}^c = 0 \end{array}\left| \begin{array}{l} P_{k,2}^c = g_{k,2}^{-1} f^{-1}(g_{k,2}\beta_2^c) \\ \gamma_{k,2}^c = \dfrac{R_k}{\mathbb{E}\left[\log\left(1+g_{k,2}P_{k,2}^c Z\right)\right]} \end{array}\right. \tag{19}$$

3) *For $k = L^c$*

$$\left.\begin{array}{l} P_{k,1}^c = g_{k,1}(Q_1^{\bar{c}})^{-1} f^{-1}\left(\dfrac{g_{k,1}(Q_1^{\bar{c}})}{1+\xi^c}\beta_1\right) \\ \gamma_{k,1}^c = \alpha - \displaystyle\sum_{l=1}^{k-1}\gamma_{l,1}^c \end{array}\right| \begin{array}{l} P_{k,2}^c = g_{k,2}^{-1} f^{-1}(g_{k,2}\beta_2^c) \\ \gamma_{k,2}^c = \dfrac{1-\alpha}{2} - \displaystyle\sum_{l=k+1}^{K^c}\gamma_{l,2}^c. \end{array} \tag{20}$$

(B) *For each $c = A, B$, the system $\mathcal{S}^c(Q_1^A, Q_1^B)$ formed by the following four equations is satisfied.*

$$L^c = \min\left\{ l = 1 \ldots K^c / \frac{g_{l,1}(Q_1^{\bar{c}})}{1+\xi^c} F\left(\frac{g_{l,1}(Q_1^{\bar{c}})}{1+\xi^c} a_l\right) \leq g_{l,2} F\left(g_{l,2} b_l\right)\right\} \tag{21}$$

$$\frac{g_{L^c,1}(Q_1^{\bar{c}})}{1+\xi^c} F\left(\frac{g_{L^c,1}(Q_1^{\bar{c}})}{1+\xi^c}\beta_1^c\right) = g_{L^c,2} F(g_{L^c,2}\beta_2^c) \tag{22}$$

$$\gamma_{L^c,1}^c C\left(\frac{g_{L^c,1}(Q_1^{\bar{c}})}{1+\xi^c}\beta_1^c\right) + \gamma_{L^c,2}^c C(g_{L,2}\beta_2^c) = R_{L^c} \tag{23}$$

$$\sum_{k}^{L^c} \gamma_{k,1}^c P_{k,1}^c = Q_1^c\,, \tag{24}$$

where the values of $\gamma_{k,1}^c$ and $P_{k,1}^c$ in (24) are the functions of $(\beta_1^c, \beta_2^c, \xi^c)$ defined by equation (18).

(C) *Furthermore, for each $c = A, B$ and for any arbitrary values $\tilde{Q}_1^A$ and $\tilde{Q}_1^B$, the system of equations $\mathcal{S}^c(\tilde{Q}_1^A, \tilde{Q}_1^B)$ admits at most one solution $(L^c, \beta_1^c, \beta_2^c, \xi^c)$.*

Comments on Theorem 2:

a) The joint multicell resource allocation problem required initially the determination of $4K$ parameters (where K is the total number of users). Theorem 2 allows to reduce the search to only two parameters, namely Q_1^A and Q_1^B. Once the value of these parameters is fixed, the resource allocation parameters for each user can be obtained from the above results. As a consequence, the only remaining task is to determine the value of (Q_1^A, Q_1^B). This task is addressed in Subsection IV-C.

b) We observe that the system $\mathcal{S}^c(Q_1^A, Q_1^B)$ is very similar to the system obtained in the single cell case at equations (14), (15), (16) and (17). In fact, as stated by the proof later, the optimal resource allocation in the multicell case can be interpreted as the solution to a certain single-cell problem.

c) As a consequence of the above remark, Theorem 2 states that the optimal multicell resource allocation scheme has the same "binary" form as in the single cell case. Even if optimal resource allocation is achieved *jointly* for both interfering cells, there still exists a pivot-user L^c in each Cell c which separates the users modulating respectively in bands $\mathcal{I}$ and $\mathcal{P}_c$.

d) It is worth noticing that this binary resource allocation strategy is already proposed in a number of recent standards. One of the contributions introduced by Theorem 2 is the proof that such a strategy in not only simple and intuitive, but is also optimal.

C. Optimal Distributed Algorithm

Once the relevant values of Q_1^A and Q_1^B have been determined, each base station can easily compute the optimal resource allocation based on Theorem 2. As a consequence, the only remaining task is to determine the value of (Q_1^A, Q_1^B). To that end, we propose to perform an exhaustive search on (Q_1^A, Q_1^B). *i)* For each point $(\tilde{Q}_1^A, \tilde{Q}_1^B)$ on a certain 2D-grid (whose determination will be discussed later on), each base station $c = A, B$ solves the system $\mathcal{S}^c(\tilde{Q}_1^A, \tilde{Q}_1^B)$ introduced by Theorem 2. Solving $\mathcal{S}^c(\tilde{Q}_1^A, \tilde{Q}_1^B)$ for arbitrary values $(\tilde{Q}_1^A, \tilde{Q}_1^B)$ can be easily achieved by base station c thanks to a simple *single-cell* procedure. Focus for instance on Cell A.

- Base station A solves the single cell resource allocation Problem 1 assuming that the interference level coincides with $\tilde{Q}_1^B$, and that the nuisance constraint $\mathcal{Q}$ is set to $\mathcal{Q} = \tilde{Q}_1^A$. The (unique) solution is provided by Theorem 1 and Proposition 1.
- If the resulting power $\sum_k \gamma_{k,1}^A P_{k,1}^A$ transmitted in the interference band $\mathcal{P}_A$ is equal to the nuisance constraint $\tilde{Q}_1^A$ (*i.e.* constraint **C6** holds with equality), then the resulting value of $(L^A, \beta_1^A, \beta_2^A, \xi^A)$ coincides with the unique solution to system $\mathcal{S}^A(\tilde{Q}_1^A, \tilde{Q}_1^B)$. This claim is the immediate consequence of Proposition 1.

- If the power $\sum_k \gamma_{k,1}^A P_{k,1}^A$ is less than $\tilde{Q}_1^A$ (*i.e.* constraint **C6** holds with strict inequality), then $(L^A, \beta_1^A, \beta_2^A, \xi^A)$ is clearly not a solution to system $\mathcal{S}^A(\tilde{Q}_1^A, \tilde{Q}_1^B)$, as equality (24) does not hold. In this case, it can easily be seen that $\mathcal{S}^A(\tilde{Q}_1^A, \tilde{Q}_1^B)$ has no solution. The point $(\tilde{Q}_1^A, \tilde{Q}_1^B)$ cannot correspond to a global solution as stated by Theorem 2 and is thus eliminated.

ii) Base station A evaluates the power

$$Q_T^A(\tilde{Q}_1^A, \tilde{Q}_1^B) = \sum_k \gamma_{k,1}^A P_{k,1}^A + \gamma_{k,2}^A P_{k,2}^A$$

that would be transmitted if the interference level and the nuisance constraint were respectively equal to $\tilde{Q}_1^B$ and $\tilde{Q}_1^A$. This value is then communicated to Base Station B. Base station B proceed in a similar way.

iii) The final value of (Q_1^A, Q_1^B) is defined as the argument of the minimum power transmitted by the network:

$$(Q_1^A, Q_1^B) = \arg \min_{(\tilde{Q}_1^A, \tilde{Q}_1^B)} Q_T^A(\tilde{Q}_1^A, \tilde{Q}_1^B) + Q_T^B(\tilde{Q}_1^A, \tilde{Q}_1^B) \,.$$

Note that the optimal resource allocation algorithm as described above does not require the intervention of a central controlling unit supposed to have access to the two base stations and to users' information (position and data rate). We only assume that both base stations can communicate via a special link dedicated to this task. The algorithm is thus distributed. This special link will be only used to exchange a limited number of messages. Indeed, the only values that need to be exchanged between the two base stations are $Q_T^A(\tilde{Q}_1^A, \tilde{Q}_1^B)$ and $Q_T^B(\tilde{Q}_1^A, \tilde{Q}_1^B)$ corresponding to the couples $(\tilde{Q}_1^A, \tilde{Q}_1^B)$ for which the two systems of equations $\mathcal{S}^A(\tilde{Q}_1^A, \tilde{Q}_1^B)$ and $\mathcal{S}^B(\tilde{Q}_1^A, \tilde{Q}_1^B)$ have a solution.

Determination of the search domain in (Q_1^A, Q_1^B).

In order to limit the complexity of the proposed approach, the search for (Q_1^A, Q_1^B) should be restricted to a certain compact domain, say

$$Q_1^c \in [0, \mathcal{Q}_{max}]$$

for each c. For instance, a possible value for $\mathcal{Q}_{max}$ can be defined as the total power that would be spent by the two base stations if one would use the naive and suboptimal resource allocation which consists in only transmitting in the protected bands $\mathcal{P}_A$ and $\mathcal{P}_B$. Clearly, the latter value of $\mathcal{Q}_{max}$ is a constant w.r.t. Q_1^A and Q_1^B and can be computed beforehand. A second way to restrict the search domain is to make use of a simple suboptimal multicell resource allocation algorithm prior to the use of our algorithm (see for instance the suboptimal algorithm defined in Part II of this work). In this case, it is possible to restrict the search for (Q_1^A, Q_1^B) to a well-chosen neighborhood of the couple $(Q_1^A, Q_1^B)_{\text{subopt}}$ provided by the suboptimal solution.

Complexity Analysis.

In order to get an idea about the cost of applying the optimal allocation, we provide in the following a computational complexity analysis of this algorithm as function of the number of users K in the system. In other words, we study how the number of operations involved in the algorithm increases when the number of users grows. For that sake, recall that the system of equations $\mathcal{S}^c(Q_1^A, Q_1^B)$ must be solved for each possible value of (Q_1^A, Q_1^B) inside a 2D-grid contained in a compact interval. Denote by M the number of couples (Q_1^A, Q_1^B) in the 2D-grid. The overall computational complexity of the algorithm can be obtained by multiplying the cost of solving $\mathcal{S}^c(Q_1^A, Q_1^B)$ for each point of the 2D-grid by M the number of points in the grid.

For each point (Q_1^A, Q_1^B) of the 2D-grid, solving $\mathcal{S}^c(Q_1^A, Q_1^B)$ consists in the determination of L^c, β_1^c, β_2^c, ξ^c such that equations (21)-(24) are satisfied. Equation (21) in particular permits the determination of L^c independently of β_1^c and β_2^c by solving $L^c = \min\left\{ l = 1 \dots K^c / \ \frac{g_{l,1}}{1+\xi^c} F\left(g_{l,1} a_l^c\right) \leq g_{l,2} F\left(g_{l,2} b_l^c\right) \right\}$, provided that the value of ξ^c is fixed. Note that solving the latter equation requires that parameters a_l^c, b_l^c should be computed first. It can be shown that the number of operations required to compute a_l^c, b_l^c is of order $O(K^c)$. Furthermore, we argued in Section III that the determination of L^c can be done by dichotomy, computing a_l^A and b_l^A only for a limited number, for instance $\log_2 K^c$, of values of l. The overall complexity of finding L^c for a fixed ξ^c is therefore of the order of $O(K^c \log_2 K^c)$. The system of equation $\mathcal{S}^c(Q_1^A, Q_1^B)$ can now be reduced to a system of three equations (22), (23), (24) in variables $\beta_1^c, \beta_2^c, \xi^c$. This system of non linear equations can be solved using Newton-like iterative methods. For a fixed value of L^c, one can verify from (22), (23), (24) and by referring to [26] that each iteration of Newton method requires a computational complexity of order $O(K^c)$. We conclude that the computational complexity associated with each iteration of Newton method is dominated by the cost of computing L^c, which is of order $O(K^c \log_2 K^c)$. We can now compute the overall computational complexity of solving $\mathcal{S}^c(Q_1^A, Q_1^B)$ by multiplying the cost associated with solving each iteration of Newton method by the number of iterations needed till convergence. Denote by N_i this number of iterations. The overall computational complexity of solving $\mathcal{S}^c(Q_1^A, Q_1^B)$ is therefore proportional to $O(N_i K^c \log_2 K^c)$. The overall cost of the algorithm for each Base station c can be obtained by multiplying the complexity of solving $\mathcal{S}^c(Q_1^A, Q_1^B)$ by M, the number of couples (Q_1^A, Q_1^B) in the search grid. The overall computational complexity of the optimal allocation is therefore of the order of $O(MN_i K^A \log_2 K^A) + O(MN_i K^B \log_2 K^B)$, which is itself of order $O(MN_i K \log_2 K)$ in the particular case $K^A \sim K^B \sim K/2$. Of course, the value of M and N_i

should be chosen such that the required accuracy of the final solution *i.e.*, its distance from the optimal solution, is achieved.

Note from the above discussion that the determination of the pivot-user L^c in each cell for each value of (Q_1^A, Q_1^B) is one of the costliest operations in solving $\mathcal{S}^c(Q_1^A, Q_1^B)$ and that it dominates the overall complexity. This is why we propose in Part II of this work a simplified resource allocation algorithm which uses a predefined value for the pivot distance. The simplified algorithm turns out to have a computational complexity of order $O(K)$, as opposed to the computational complexity of the optimal algorithm which is of the order of $O(MN_iK\log_2 K)$.

V. SIMULATIONS

In our simulations, we considered a Free Space Loss model (FSL) characterized by a path loss exponent $s = 2$ as well as the so-called Okumura-Hata (O-H) model for open areas [27] with a path loss exponent $s = 3$. The carrier frequency is $f_0 = 2.4GHz$. At this frequency, path loss in dB is given by $\rho_{dB}(x) = 20\log_{10}(x) + 100.04$ in the case where $s = 2$, where x is the distance in kilometers between the BS and the user. In the case $s = 3$, $\rho_{dB}(x) = 30\log_{10}(x) + 97.52$. The signal bandwidth B is equal to 5 MHz and the thermal noise power spectral density is equal to $N_0 = -170$ dBm/Hz. Each cell has a radius $D = 500$m and contains the same number of randomly distributed users ($K^A = K^B$). The rate requirement of user k in bits/sec/Hz is designated by R_k. The distance separating each user from the base station is considered a random variable with a uniform distribution on the interval $[0, D]$. The joint resource allocation problem for Cells A and B (Problem 2) was solved for a large number of realizations of this random distribution of users and the values of the resulting transmit power were averaged. Computing the mean value of the total transmit power with respect to the random positions of users is intended to get results that do not depend on the particular position of each user in the cell but rather on global information about the geographic distribution of users in the cell. We give now more details on the way simulation were carried out.

Define $\mathbf{X}$ as the vector containing the positions of all the users in the system *i.e*, $\mathbf{X} = (x_1, x_2, \ldots, x_{K^c})_{c=A,B}$. Recall that $\forall k$, x_k is a random variable with a uniform distribution on $[0, D]$. For each realization of $\mathbf{X}$, denote by $Q_T(\mathbf{X}, \alpha)$ the minimal total transmit power that results from a global solution to the multicell resource allocation problem (Problem 2) *i.e.*, $Q_T(\mathbf{X}, \alpha) = \sum_{c=A,B}\left(\sum_{k=1}^{L^c} W_{k,1}^c + \sum_{k=L^c}^{K^c} W_{k,2}^c\right)$ where $(\gamma_{k,1}^c, W_{k,1}^c, \gamma_{k,2}^c, W_{k,2}^c)_{c\in\{A,B\},k=1,\ldots,K^c}$ is a global solution to Problem 2 described by Theorem 2. Define $r_t = \sum_{k=1}^{K^c} R_k B$ as the sum rate of the users of Cell c measured in bits/sec. We consider first the case where all the users have the same rate requirement $R_1 = R_2 = \ldots = R_{K^c}$. Figures 2 and 3

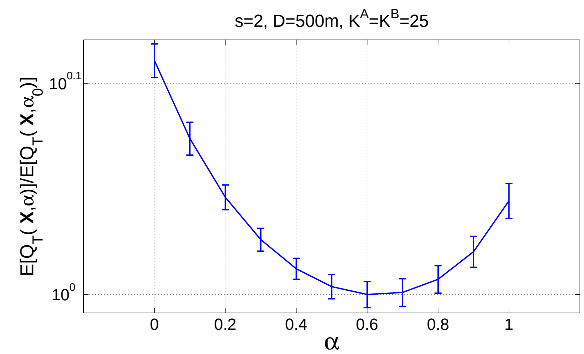

Figure 2. Power vs. α for $s = 2$, $D = 500$ m, $K^A = K^B = 25$, $r_t = 5$ Mbps

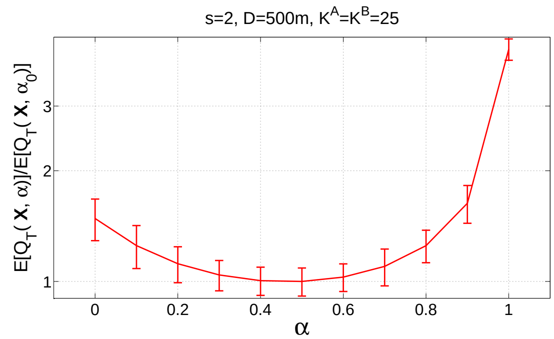

Figure 3. Power vs. α for $s = 2$, $D = 500$ m, $K^A = K^B = 25$, $r_t = 10$ Mbps

represent, for a sum rate requirement of $r_t = 5$ Mbps (Mega bits/sec) and $r_t = 10$ Mbps respectively and assuming $s = 2$, the mean value of $Q_T(\mathbf{X}, \alpha)$ normalized by its minimum value w.r.t α *i.e.*, the ratio $\mathbb{E}_{\mathbf{X}}[Q_T(\mathbf{X}, \alpha)]/\mathbb{E}_{\mathbf{X}}[Q_T(\mathbf{X}, \alpha_0)]$, where α_0 is the value of the reuse factor α that minimizes $\mathbb{E}[Q_T(\mathbf{X}, \alpha)]$. Figures 4 and 5 plot the same quantity for $r_t = 5$ Mbps and $r_t = 10$ Mbps respectively, but with the difference that it assumes $s = 3$. The error bars in the aforementioned four figures represent the variance of $Q_T(\mathbf{X}, \alpha)$ *i.e.*, $\mathbb{E}_{\mathbf{X}}[(Q_T(\mathbf{X}, \alpha) - \mathbb{E}_{\mathbf{X}}[Q_T(\mathbf{X}, \alpha)])^2]$.

For each value of $\mathbf{X}$ and of the reuse factor α, $Q_T(\mathbf{X}, \alpha)$ was computed using the optimal resource allocation algorithm of Section IV. Power gains are considerable compared to the extreme cases $\alpha = 0$ (the available bandwidth is shared in an orthogonal way between Cells A and B) and $\alpha = 1$ (all the available bandwidth is reused in the two cells). Note also that for $r_t = 10$ Mbps, α_0 the optimal value of the reuse factor that minimizes $Q_T(\mathbf{X}, \alpha)$ is smaller than the optimal value of the reuse factor for $r = 5$ Mbps. This result is expected, given that higher values of r_t will lead to higher transmit powers in order to satisfy users' rate requirements, and consequently to higher levels of interference. More users will need thus to be protected from the higher interference. For that purpose, a larger part of the available bandwidth must be reserved for the protected bands $\mathcal{P}_A$ and $\mathcal{P}_B$. We also remark that in the case where $s = 3$, the value of the reuse factor α_0 is larger than its value for $s = 2$. This is due to the fact that when the path loss exponent is larger, the interference produced by the adjacent base station will undergo more fading than in the case when the path loss exponent is smaller. As a result, less users need to be protected from interference in the case $s = 3$ compared to the case $s = 2$. (see table V which provides, in the two cases, the percentage of protected users to the total number of users for $r_t = 5$ and $r_t = 10$ Mbps, provided that the corresponding value of α_0 is used in each case).

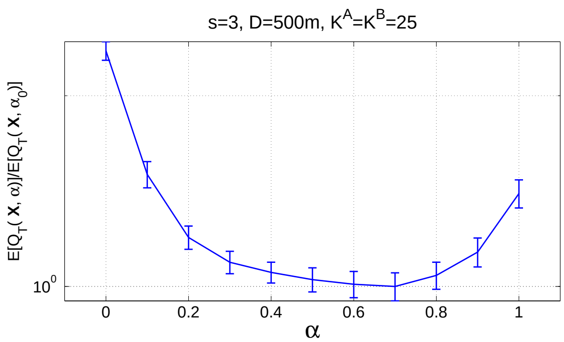

Figure 4. Power vs. α for $s = 3$, $D = 500$ m, $K^A = K^B = 25$, $r_t = 5$ Mbps

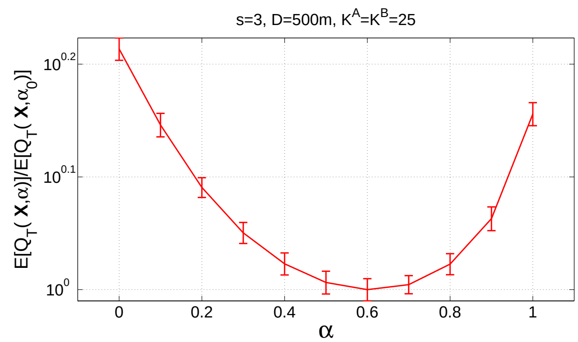

Figure 5. Power vs. α for $s = 3$, $D = 500$ m, $K^A = K^B = 25$, $r_t = 10$ Mbps

	$s = 2$	$s = 3$
$r_t = 5$ Mbps	19.8%	11.6%
$r_t = 10$ Mbps	30.0%	18.7%

Table I

PERCENTAGE OF THE PROTECTED USERS TO THE TOTAL NUMBER OF USERS

We now compare the performance of our proposed resource allocation with the distributed scheme proposed in [13]. The latter scheme assumes a reuse factor α equal to one (all the subcarriers can be reused in all the cells), in contrast to our scheme which uses an optimized value of α. Figure 6 plots the average total transmit power $\mathbb{E}[Q_T^{(K)}(\mathbf{X}, \alpha_0)]$ that results when our proposed scheme is applied compared to the power that results from applying the scheme of [13]. This comparison was carried out assuming $K^A = K^B = 25$, $s = 2$ and $r_t = 5$ Mbps. The gain obtained when the proposed scheme is applied is clear from the figure, and it increases with respect to the total rate r_t. We consider now the case when the rate requirement is not the same for all users. In particular, we assume that the rate requirement of each user is a random variable that can take on one of two values with the same probability. For example, consider the case $K^A = K^B = 25$ and assume that the rate requirement of each user can either be equal to 250 kbps (kilo bits/sec) with probability 0.5 or to 150 kbps with the same probability. This means that the mean rate per user is equal to 200 kbps and that the mean total rate per sector is equal to $r_t = 25 * 200$ kbps = 5 Mbps. Figure 7 represents, assuming $s = 2$, the mean value of $Q_T(\mathbf{X}, \alpha)$ normalized by its minimum value w.r.t α *i.e.*, the ratio $\mathbb{E}_{\mathbf{X}}[Q_T(\mathbf{X}, \alpha)]/\mathbb{E}_{\mathbf{X}}[Q_T(\mathbf{X}, \alpha_0)]$, where α_0 is the value of the reuse factor α that minimizes $\mathbb{E}[Q_T(\mathbf{X}, \alpha)]$. The error bars in the above figure represent

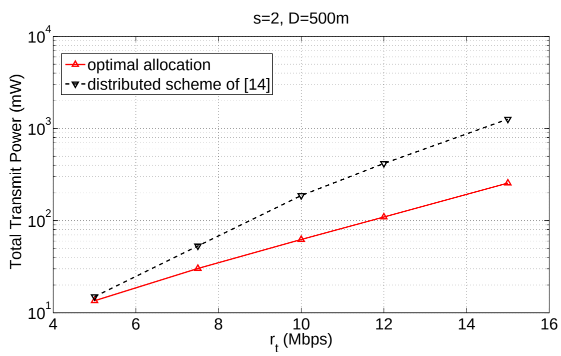

Figure 6. Comparison between the proposed optimal scheme and the distributed scheme of [13] for $K^A = K^B = 25$.

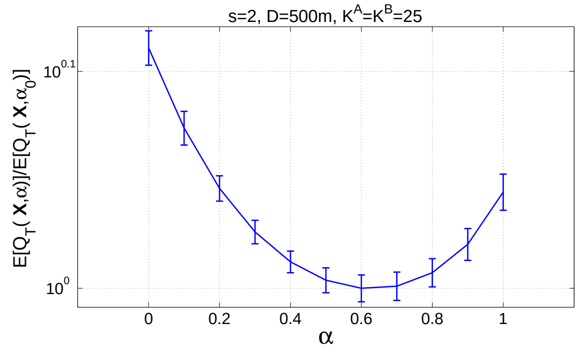

Figure 7. Power vs. α for $s = 2$, $D = 500$ m, $K^A = K^B = 25$ assuming random rate requirements.

the variance of $Q_T(\mathbf{X}, \alpha)$ *i.e.*, $\mathbb{E}_{\mathbf{X}}[(Q_T(\mathbf{X}, \alpha) - \mathbb{E}_{\mathbf{X}}[Q_T(\mathbf{X}, \alpha)])^2]$. By comparing Figures 2 and 7 we note that the normalized mean value $\mathbb{E}_{\mathbf{X}}[Q_T(\mathbf{X}, \alpha)]/\mathbb{E}_{\mathbf{X}}[Q_T(\mathbf{X}, \alpha_0)]$ is practically the same in the two figures. Only the variance $\mathbb{E}_{\mathbf{X}}[(Q_T(\mathbf{X}, \alpha) - \mathbb{E}_{\mathbf{X}}[Q_T(\mathbf{X}, \alpha)])^2]$ is slightly different (its value is slightly larger in the case of random rate requirements).

VI. Conclusions

In this paper, the resource allocation problem for a sectorized downlink OFDMA system has been studied in the context of a partial reuse factor $\alpha \in [0, 1]$. The general solution to the (nonconvex) optimization problem has been provided. It has been proved that the solution admits a simple form and that the initial tedious problem reduces to the identification of a restricted number of parameters. As a noticeable property, it has been proved that the optimal resource allocation policy is “binary”: there exists a pivot-distance to the base station such that users who are farther than this distance should only modulate protected subcarriers, while closest users should only modulate reused subcarriers.

Appendix A
Proof of Theorem 1

When the resource allocation parameters of users in Cell B are fixed, it is straightforward to show that the ergodic capacity $C_k = C_k(\gamma_{k,1}^A, \gamma_{k,2}^A, W_{k,1}^A, W_{k,2}^A)$ defined by (6) is a concave function of $\gamma_{k,1}^A$, $\gamma_{k,2}^A$, $W_{k,1}^A$, $W_{k,2}^A$ (and hence $-C_k(\gamma_{k,1}^A, \gamma_{k,2}^A, W_{k,1}^A, W_{k,2}^A)$ is convex). This is essentially due to the fact that $g_{k,1} = g_{k,1}(Q_1^B)$ can be treated as a constant and does not depend on the optimization parameters. Thus, the single cell resource allocation problem (Problem 1) is convex in $\{\gamma_{k,1}^A, \gamma_{k,2}^A, W_{k,1}^A, W_{k,2}^A\}_{k\in\{1,\ldots,K^A\}}$.

In the following, we derive the KKT conditions in order to obtain the general form of the solution and to prove the existence of L, β_1, β_2, ξ as stated by Theorem 1. In particular, we prove that any optimal resource allocation is binary *i.e.*, there exists a certain pivot-integer L such that $\gamma_{k,2}^A = 0$ for $k < L$ and $\gamma_{k,1}^A = 0$ for $k > L$. Furthermore, we prove that there exist three parameters β_1, β_2 and ξ such that equations (9), (10) and (11) hold. As explained above, β_1, β_2, ξ are the Lagrange multipliers associated with constraints $\mathbf{C2}$, $\mathbf{C3}$ and $\mathbf{C6}$ respectively.

KKT Conditions for Problem 1

In order to simplify the notations and since we are only interested in users of Cell A, we simply omit the superscript A in the sequel and define $Q = Q^A$, $\gamma_{k,1} = \gamma_{k,1}^A$, etc. Denote by $\mathbf{x}_A$ the vector of resource allocation parameters of users in Cell A *i.e,* $\mathbf{x}_A = [(\mathbf{W})^T, (\boldsymbol{\gamma})^T]^T$ where $\mathbf{W} = [W_{1,1}, W_{1,2}, \ldots, W_{K^A,1}, W_{K^A,2}]^T$ and $\boldsymbol{\gamma} = [\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{K^A,1}, \gamma_{K^A,2}]^T$. The associated Lagrangian is equal to:

$$\begin{aligned} \mathcal{L} = Q - \sum_k \lambda_k C_k + \beta_1 \left(\sum_k \gamma_{k,1} \right) + \beta_2 \left(\sum_k \gamma_{k,2} \right) - \\ \sum_k \nu_{k,1}\gamma_{k,1} - \sum_k \nu_{k,2}\gamma_{k,2} - \sum_k \mu_{k,1} W_{k,1} - \sum_k \mu_{k,2} W_{k,2} + \xi \sum_k W_{k,1}. \end{aligned} \tag{25}$$

where λ_k, β_1, β_2 and ξ are the Lagrange multipliers associated respectively with constraints $\mathbf{C1}$, $\mathbf{C2}$, $\mathbf{C3}$ and $\mathbf{C6}$ of Problem 1, and where $\nu_{k,1}, \nu_{k,2}, \mu_{k,1}, \mu_{k,2}$ are the the Lagrange multipliers associated with the positivity constraints of $\gamma_{k,1}, \gamma_{k,2}, W_{k,1}, W_{k,2}$ respectively. In the expression of C_k, a technical difficulty arises from the fact that function $\gamma_{k,i}\mathbb{E}\left[\log\left(1 + g_{k,i}\frac{W_{k,i}}{\gamma_{k,i}} Z\right)\right]$ is not differentiable at point $\gamma_{k,i} = 0$. One can easily overcome this issue by replacing the non-negativity constraint $\gamma_{k,i} \geq 0$ by the strict positivity constraint $\gamma_{k,i} \geq \epsilon_0$, for an arbitrary $\epsilon_0 > 0$. However, as this point is essentially technical, we simply put $\epsilon_0 = 0$ with slight lack of rigor. This assumption will simplify the presentation without changing the results. The complete proof that does not make this simplifying assumption can be found in [28]. We now apply the Lagrange-Karush-Kuhn-Tucker conditions to characterize the optimal vector $\mathbf{x}_A$. Taking the derivative of (25) with respect to $W_{k,i}$ and $\gamma_{k,i}$ $(i = 1, 2)$ leads to

$$1 - \lambda_k g_{k,i} \mathbb{E}\left[\frac{Z}{1 + g_{k,i}\frac{W_{k,i}}{\gamma_{k,i}} Z}\right] - \mu_{k,i} + \xi\delta_i = 0 \tag{26}$$

$$-\lambda_k \mathbb{E}\left[\log\left(1 + g_{k,i}\frac{W_{k,i}}{\gamma_{k,i}} Z\right) - \frac{g_{k,i}\frac{W_{k,i}}{\gamma_{k,i}} Z}{1 + g_{k,i}\frac{W_{k,i}}{\gamma_{k,i}} Z}\right] + \beta_i - \nu_{k,i} = 0 \tag{27}$$

where $\delta_i = 1$ if $i = 1$ and $\delta_i = 0$ if $i = 2$. We can easily show that the constraint $R_k \leq C_k$ must hold with equality, and is always active in the sense that the Lagrange multiplier λ_k associated with this constraint is strictly positive. Identifying parameter λ_k in (26) and (27) yields $f\left(g_{k,i}\frac{W_{k,i}}{\gamma_{k,i}}\right) = \frac{g_{k,i}(\beta_i-\nu_{k,i})}{1-\mu_{k,i}+\xi\delta_i}$, where f is the function defined by (8). Replacing the value of $g_{k,i}\frac{W_{k,i}}{\gamma_{k,i}}$ in (26) by $f^{-1}\left(\frac{g_{k,i}(\beta_i-\nu_{k,i})}{1-\mu_{k,i}+\xi\delta_i}\right)$ directly provides the following equation:

$$1-\mu_{k,i}+\xi\delta_i = \lambda_k g_{k,i} F\left(\frac{g_{k,i}(\beta_i-\nu_{k,i})}{1-\mu_{k,i}+\xi\delta_i}\right), \tag{28}$$

where F is the function defined by (12). Define $\mathcal{A}_i = \{k/\nu_{k,i}=0\}$. In other words, $\mathcal{A}_1$ is the set of users of Cell A being assigned non zero share of the band $\mathcal{I}$, and $\mathcal{A}_2$ is the set of users of Cell A being assigned non zero share of $\mathcal{P}_A$. By complementary slackness, we may write on the opposite $\overline{\mathcal{A}_i} = \{k/\gamma_{k,i}=0\}$ where $\overline{E}$ denotes the complementary set of any set $E \subset \{1,\dots K^A\}$. After some algebra, it can be shown that $\nu_{k,i}=0$ implies $\mu_{k,i}=0$. Thus,

$$\forall k \in \mathcal{A}_i,\quad \frac{g_{k,i}}{1+\xi\delta_i}F\left(\frac{g_{k,i}}{1+\xi\delta_i}\beta_i\right) = \lambda_k^{-1}\,. \tag{29}$$

On the other hand, if $\nu_{k,i}>0$, KKT conditions lead to

$$\forall k \in \overline{\mathcal{A}_i},\quad \frac{g_{k,i}}{1+\xi\delta_i}F\left(\frac{g_{k,i}}{1+\xi\delta_i}\beta_i\right) < \lambda_k^{-1} \tag{30}$$

To prove that inequality (30) holds, one needs to separate the two possible cases $W_{k,i}=0$ and $W_{k,i}>0$. *i)* If $W_{k,i}=0$, equation (27) leads to $\beta_i=\nu_{k,i}$. Thus, (28) is equivalent to $1-\mu_{k,i}+\xi\delta_i=\lambda_k g_{k,i}$, which implies that $\frac{g_{k,i}}{1+\xi\delta_i}\le\lambda_k^{-1}$ since $\mu_{k,i}\ge 0$. Noticing that $F\left(\frac{g_{k,i}}{1+\xi\delta_i}\beta_i\right)<1$ and multiplying this inequality by the previous one, we obtain the desired equation (30). *ii)* If $W_{k,i}>0$, complementary slackness condition $\mu_{k,i}W_{k,i}=0$ along with equation (28) lead to $\mu_{k,i}=0=1+\xi\delta_i-\lambda_k g_{k,i}F\left(\frac{g_{k,i}(\beta_i-\nu_{k,i})}{1-\mu_{k,i}+\xi\delta_i}\right)$. As function $F(x)$ is strictly decreasing, $F\left(\frac{g_{k,i}}{1+\xi\delta_i}\beta_i\right) < F\left(\frac{g_{k,i}(\beta_i-\nu_{k,i})}{1-\mu_{k,i}+\xi\delta_i}\right) = \frac{1+\xi\delta_i}{\lambda_k g_{k,i}}$. We thus obtain inequality (30) as well.

To summarize, every global solution to our optimization problem can thus be characterized by the following set of conditions:

1) For every $k\in\mathcal{A}_i$:

$$\frac{g_{k,i}}{1+\xi\delta_i}F\left(\frac{g_{k,i}}{1+\xi\delta_i}\beta_i\right)=\lambda_k^{-1},\quad \frac{W_{k,i}}{\gamma_{k,i}} = g_{k,i}^{-1}f^{-1}\left(\frac{g_{k,i}}{1+\xi\delta_i}\beta_i\right) \tag{31}$$

2) For every $k\in\bar{\mathcal{A}}_i$:

$$\frac{g_{k,i}}{1+\xi\delta_i}F\left(\frac{g_{k,i}}{1+\xi\delta_i}\beta_i\right)<\lambda_k^{-1},\quad W_{k,i}=0 \tag{32}$$

3)

$$\forall k\;\; C_k=R_k,\qquad \sum_k \gamma_{k,1}=\alpha,\qquad \sum_k\gamma_{k,2}=\frac{1-\alpha}{2},\;\; \xi\left(\sum_k W_{k,1}-\mathcal{Q}\right)=0\,.$$

We determine now which users are in $\mathcal{A}_1$ and which are in $\mathcal{A}_2$. For that sake, the following conjecture will be revealed useful in the sequel. Define $h(x) = \frac{x(F^{-1}(x))'}{F^{-1}(x)}$.

Conjecture 1. *Function $f(x)$ is strictly convex. Function $h(x)$ is non increasing on the interval $(0,1)$.*

In order to validate the above conjecture, Figures 8 and 9 represent the second respectively derivative of f which is obviously positive, and the first derivative of h, which is obviously negative on $(0,1)$. We

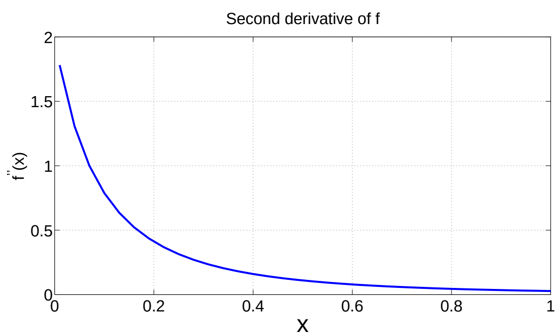

Figure 8. Second derivative of function f

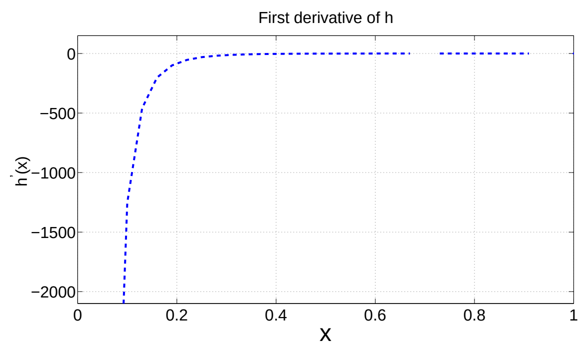

Figure 9. First derivative of function h

show now that equations (29) and (30) are sufficient to prove that the following lemma holds.

Lemma 1. *Any global solution to Problem 1 is "binary"* i.e., *there exists a user L in Cell A such that $\gamma_{k,2} = 0$ for closest users $k < L$, and $\gamma_{k,1} = 0$ for farthest users $k > L$.*

Proof: Now define $L = \min \mathcal{A}_2$ as the closest user to the base station among all users modulating in the protected band $\mathcal{P}_A$. By definition of L, we have $\gamma_{1,2} = \cdots = \gamma_{L-1,2} = 0$ which is equivalent to the first part of the desired result. Now we prove the second part *i.e.*, $\gamma_{L+1,1} = \cdots = \gamma_{K^A,1} = 0$. To simplify notations, we define for each user k, $\tilde{g}_{k,1} = \frac{g_{k,1}}{1+\xi}$. By definition, $L \in \mathcal{A}_2$. By immediate application of the above KKT conditions, $g_{L,2}F(g_{L,2}\beta_2) = \lambda_k^{-1} \geq \tilde{g}_{L,1}F\left(\tilde{g}_{L,1}\beta_1\right)$. As F is decreasing, we obtain $\beta_2 < \frac{1}{g_{L,2}}F^{-1}\left(\frac{\tilde{g}_{L,1}}{g_{L,2}}F\left(\tilde{g}_{L,1}\beta_1\right)\right)$. Now consider a second user $k \geq L+1$ and assume by contradiction that $k \in \mathcal{A}_1$. Using the same arguments, it is straightforward to show that $\beta_2 > \frac{1}{g_{k,2}}F^{-1}\left(\frac{\tilde{g}_{k,1}}{g_{k,2}}F\left(\tilde{g}_{k,1}\beta_1\right)\right)$. Putting all pieces together, $\frac{1}{g_{k,2}}F^{-1}\left(\frac{\tilde{g}_{k,1}}{g_{k,2}}F\left(\tilde{g}_{k,1}\beta_1\right)\right) < \frac{1}{g_{L,2}}F^{-1}\left(\frac{\tilde{g}_{L,1}}{g_{L,2}}F\left(\tilde{g}_{L,1}\beta_1\right)\right)$. We now prove that the above inequality cannot hold when $k > L$. To that end, we introduce the following notations. Define $x = \tilde{g}_{L,1}\beta_1$, $r = \frac{\rho_k}{\rho_L}$, $t = \frac{\sigma_L^2}{\sigma_k^2}$ and $s = \frac{\sigma^2}{\sigma_L^2(1+\xi)}$. Using these notations, the above inequality reduces to

$$\frac{1}{r}F^{-1}\left(stF\left(rtx\right)\right) < F^{-1}\left(sF\left(x\right)\right) \; . \tag{33}$$

Note that in the above inequality, all variables r, s, t are strictly less than one. We now prove with the help of Conjecture 1 that the above inequality leads to a contradiction. In fact, Conjecture 1 states that function $f(x)$ is strictly convex. As $f(x)$ is also strictly increasing, its inverse f^{-1} is strictly concave strictly increasing. Therefore, for every $t < 1$ and for every $y > 0$, $f^{-1}(ty) > tf^{-1}(y)$. Using the definition of function $F(x)$, it is straightforward to show that the latter inequality leads to

$$\forall (r,s,t) \in (0,1)^3, \quad \frac{1}{r}F^{-1}(stF(trx)) > \frac{1}{r}F^{-1}(sF(rx)) \tag{34}$$

for each real x. As function $h(x) = \frac{x(F^{-1}(x))'}{F^{-1}(x)}$ is non increasing on $(0,1)$, it can be shown after some algebra [28] that function $r \to \frac{1}{r}F^{-1}(sF(rx))$ is decreasing on $(0,1)$. As a consequence,

$$\forall (r,s) \in (0,1)^2, \quad \frac{1}{r}F^{-1}(sF(rx)) \geq F^{-1}(sF(x)) \, . \tag{35}$$

Clearly, (34) and (35) contradict inequality (33). This proves the desired lemma. ■

Lemma 1 establishes the "binary" property of any global solution to Problem 1. One still needs to prove that equations (9), (10) and (11) hold. Fortunately, these equations result directly from combining the above claim with equations (31) and (32).

APPENDIX B

PROOF OF PROPOSITION 1 AND DETERMINATION OF L, β_1, β_2 AND ξ

Step 1: General form of the solution and existence of L, β_1, β_2, ξ.

Theorem 1 provides the general form of any global solution to Problem 1 and proves that any optimal resource allocation is binary *i.e.*, there exists a certain pivot-integer L such that $\gamma_{k,2}^A = 0$ for $k < L$ and $\gamma_{k,1}^A = 0$ for $k > L$. Furthermore, it proves that there exist three parameters β_1, β_2 and ξ such that equations (9), (10) and (11) hold. As explained above, β_1, β_2, ξ are the Lagrange multipliers associated with constraints **C2**, **C3** and **C6** respectively. Now, the remaining task is first to determine the values of L, β_1, β_2, ξ, and second, to prove the uniqueness of the global solution to Problem 1.

Step 2: Determination of L, β_1, β_2 for a fixed value of ξ.

To simplify, first assume that the value of Lagrange multiplier ξ is fixed. We determine L, β_1, β_2 as functions of ξ. Recall from step 1 that user L is defined as the only user who is likely to modulate in both bands $\mathcal{I}$ and $\mathcal{P}_A$. Parameters $\gamma_{L,1}^A, \gamma_{L,2}^A$ respectively provide the part of the band $\mathcal{I}$ and $\mathcal{P}_A$ which is modulated by user L. A first equation is obtained by writing that $C_L = R_L$ *i.e.*, the rate constraint **C1** holds with equality. Recall that C_L is defined by (6) as $\gamma_{L,1}^A \mathbb{E}\left[\log(1 + g_{L,1}P_{L,1}^A Z)\right] +$

$\gamma_{L,2}^A \mathbb{E}\left[\log(1+g_{L,2}P_{L,2}^A\xi)\right]$. Plugging the expression (11) of parameters $\gamma_{L,1}^A, P_{L,1}^A, \gamma_{L,2}^A, P_{L,2}^A$ into this expression, equality $C_L/R_L = 1$ becomes

$$\left[\alpha - \sum_{k<L} \frac{R_k}{C\left(\frac{g_{k,1}}{1+\xi}\beta_1\right)}\right] \frac{C\left(\frac{g_{L,1}}{1+\xi}\beta_1\right)}{R_L} + \left[\frac{1-\alpha}{2} - \sum_{k>L} \frac{R_k}{C(g_{k,2}\beta_2)}\right] \frac{C(g_{L,2}\beta_2)}{R_L} = 1 \tag{36}$$

where $C(x)$ is the function defined by (13) for each $x \geq 0$ as $C(x) = \mathbb{E}[\log(1+f^{-1}(x)Z)]$. In equation (36), both terms enclosed inside the brackets coincide with $\gamma_{L,1}^A$ and $\gamma_{L,2}^A$ respectively. As function $C(x)$ is increasing from 0 to ∞ on $\mathbb{R}_+$, constraints $\gamma_{L,1}^A \geq 0$ and $\gamma_{L,2}^A \geq 0$ hold only if $\beta_1/(1+\xi) \geq a_{L-1}^A$ and $\beta_2 \geq b_L^A$ where for each l, a_l^A and b_l^A the unique positive numbers such that:

$$\sum_{k=1}^{l} \frac{R_k}{C\left(g_{k,1}a_l^A\right)} = \alpha \quad \text{and} \quad \sum_{k=l+1}^{K^A} \frac{R_k}{C(g_{k,2}b_l^A)} = \frac{1-\alpha}{2},$$

with $a_0^A = b_{K^A}^A = 0$ by convention. Note that a_l^A is an increasing sequence while b_l^A is a decreasing sequence. Furthermore, in order that (36) holds, both (nonnegative) terms should be less than one. Thus, $\alpha - \sum_{k\leq L} \frac{R_k}{C\left(\frac{g_{k,1}}{1+\xi}\beta_1\right)} \leq 0$ and $\frac{1-\alpha}{2} - \sum_{k\geq L} \frac{R_k}{C(g_{k,2}\beta_2)} \leq 0$. As a consequence, $\beta_1/(1+\xi) \leq a_L^A$ and $\beta_2 \leq b_{L-1}^A$. Finally,

$$\left(\frac{\beta_1}{1+\xi}, \beta_2\right) \in [a_{L-1}^A, a_L^A] \times [b_L^A, b_{L-1}^A] \,. \tag{37}$$

Consider the case where $\gamma_{L,1}^A$, $\gamma_{L,2}^A$ are both nonzero. It can easily be seen from the KKT conditions derived in Appendix A that

$$\frac{g_{L,1}}{1+\xi} F\left(\frac{g_{L,1}}{1+\xi}\beta_1\right) = g_{L,2}F(g_{L,2}\beta_2) \,, \tag{38}$$

where F is the function defined by (12). Now using (37) in the above equation along with the fact that $F(.)$ is a decreasing function, one can easily see that L can be defined as

$$L = \min\left\{ l = 1 \ldots K^A / \frac{g_{l,1}}{1+\xi} F\left(g_{l,1}a_l^A\right) \leq g_{l,2}F\left(g_{l,2}b_l^A\right)\right\}. \tag{39}$$

In practice, the search for L can be achieved by dichotomy, computing a_l^A and b_l^A only for a limited number of values of l. Once L is fixed, it is straightforward to show that the system formed by equation (38) and (36) admits a unique solution (β_1, β_2). This is due to the fact that functions $C(.)$ and $F(.)$ are monotone. Lagrange multiplier β_1, β_2 can thus be obtained using classical root search tools. As a remark, we note the existence of a rather pathological case, which we do not address in details because of its limited importance. To obtain equation (38) we assumed that $\gamma_{L,1}^A$ and $\gamma_{L,2}^A$ are strictly positive. If this is not the case, say $\gamma_{L,1}^A = 0$, it turns out that the system (36)-(38) has no solution. However, L can still be obtained by (39) and β_1, β_2 can be easily obtained from (36) which lead to $\beta_1 = (1+\xi)a_L^A$, $\beta_2 = b_{L-1}^A$.

For the sake of simplicity, we will still refer to (β_1, β_2) as the unique solution to system (36)-(38), with slight language abuse, keeping in mind that we just put $\beta_1 = (1+\xi)a_L^A$, $\beta_2 = b_{L-1}^A$ in the pathological case where such a solution does not exist. This convention will be used throughout the paper without restriction.

Step 3: Determination of ξ.

So far, we proved that for a fixed value of ξ, the optimal resource allocation is unique and follows equations (9), (10) and (11), where $L = L(\xi)$ is given by (39) and $(\beta_1, \beta_2) = (\beta_1(\xi), \beta_2(\xi))$ is the unique solution to system (36)-(38). The remaining task is now to determine ξ. Before addressing this point, it is worth providing some insights on the impact of ξ or equivalently, on the role of the low nuisance constraint **C6** on the resource allocation. Recall that ξ is the Lagrange multiplier associated with constraint **C6**. From an intuitive point of view, a large value of ξ means in some sense that constraint **C6** is severely restraining, whereas $\xi = 0$ means that constraint **C6** has no role and could have been deleted without modifying the solution to Problem 1. It turns out that increasing ξ has the effect of decreasing the total power $Q_1^A = \sum_k \gamma_{k,1}^A P_{k,1}^A$ which is transmitted in the interference band. This statement can be proved as follows. First, we observe from equation (39) that parameter $L = L(\xi)$ is a non increasing function of ξ. Second, it is straightforward to show that for each k, $P_{k,1}^A$ is a decreasing function of ξ. Indeed, equation (9) implies that it is the composition of an increasing function $f^{-1}(x)$ and a decreasing function $\xi \mapsto \beta_1(\xi)/(1+\xi)$ (decreasingness of $\beta_1(\xi)/(1+\xi)$ is obtained after some algebra from (36) and (37)). Third, $W_{k,1}^A = P_{k,1}^A R_k / \mathbb{E}\left[\log(1 + g_{k,1} P_{k,1}^A Z)\right]$ is an increasing function of $P_{k,1}^A$. It is thus a decreasing function of ξ as a composition of an increasing and a decreasing function $P_{k,1}^A$. Therefore, the presence of an active constraint **C6** has a double impact on the resource allocation: *i)* it decreases the number L of users who modulate in the interference band $\mathcal{I}$, and *ii)* it decreases the power $W_{k,1}$ of each user in this band. We now determine ξ. First we propose to compute the resource allocation assuming $\xi = 0$. If the corresponding value of Q_1^A is such that $Q_1^A \leq \mathcal{Q}$, then the procedure stops: KKT conditions are met. Otherwise, this means that constraint **C6** should be active: $\xi > 0$. From complementary slackness condition, **C6** should be met with equality : one should determine ξ such that $Q_1^A = \sum_k \gamma_{k,1}^A P_{k,1}^A$ coincides with $\mathcal{Q}$:

$$\sum_{k \leq L} \gamma_{k,1}^A P_{k,1}^A = \mathcal{Q}, \tag{40}$$

where $\gamma_{k,1}^A, P_{k,1}^A$ are defined by (9) and where $L = L(\xi)$, $\beta_1 = \beta_1(\xi)$, $\beta_2 = \beta_2(\xi)$ have been defined previously. As mentioned above, Q_1^A is a decreasing function of ξ so that the solution ξ to equation $Q_1^A = \mathcal{Q}$ is unique.

APPENDIX C
PROOF OF THEOREM 2

Notations. In the sequel, $\mathbf{x}$ represents a vector of multicell allocation parameters such that $\mathbf{x} = [\mathbf{x}_A{}^T, \mathbf{x}_B{}^T]^T$ where $\mathbf{x}_A = [(\mathbf{W}^A)^T, (\boldsymbol{\gamma}^A)^T]^T$ and $\mathbf{x}_B = [(\mathbf{W}^B)^T, (\boldsymbol{\gamma}^B)^T]^T$ and where for each $c = A, B$, $\mathbf{W}^c = [W^c_{1,1}, W^c_{1,2}, \ldots, W^c_{K^c,1}, W^c_{K^c,2}]^T$ and $\boldsymbol{\gamma} = [\gamma^c_{1,1}, \gamma^c_{1,2}, \ldots, \gamma^c_{K^c,1}, \gamma^c_{K^c,2}]^T$. We respectively denote by $Q_1(\mathbf{x}_c) = \sum_k W^c_{k,1}$ and $Q_2(\mathbf{x}_c) = \sum_k W^c_{k,2}$ the powers transmitted by Base Station c in the interference band $\mathcal{I}$ and in the protected band $\mathcal{P}_c$. When resource allocation $\mathbf{x}$ is used, the total power transmitted by the network is equal to $Q(\mathbf{x}) = \sum_c Q_1(\mathbf{x}_c) + Q_2(\mathbf{x}_c)$.

Recall that Problem 2 is nonconvex. It cannot be solved using classical convex optimization methods. Denote by $\mathbf{x}^* = [\mathbf{x}^*_A{}^T, \mathbf{x}^*_B{}^T]^T$ any global solution to Problem 2.

Characterizing $\mathbf{x}^*$ via single cell results.

From $\mathbf{x}^*$ we construct a new vector $\mathbf{x}$ which is as well a global solution and which admits a “binary” form: for each Cell c, $\gamma^c_{k,1} = 0$ if $k > L^c$ and $\gamma^c_{k,2} = 0$ if $k < L^c$, for a certain pivot-integer L^c. For each Cell c, vector $\mathbf{x}_A$ is defined as a global solution to the *single cell* allocation Problem 1 when

a) the admissible nuisance constraint $\mathcal{Q}$ is set to $\mathcal{Q} = Q_1(\mathbf{x}^*_A)$,
b) the gain-to-interference-plus-noise-ratio in band $\mathcal{I}$ is set to $g_{k,1} = g_{k,1}\left(Q_1(\mathbf{x}^*_B)\right)$.

Vector $\mathbf{x}_B$ is defined similarly, by simply exchanging A and B in the above definition. Denote by $\mathbf{x} = [\mathbf{x}_A{}^T, \mathbf{x}_B{}^T]^T$ the resource allocation obtained by the above procedure. The following Lemma holds.

Lemma 2. *Resource allocation parameters* $\mathbf{x}$ *and* $\mathbf{x}^*$ *coincide:* $\mathbf{x} = \mathbf{x}^*$.

Proof: It is straightforward to show that $\mathbf{x}$ is a feasible point for the joint multicell Problem 2 in the sense that constraints **C1-C5** of Problem 2 are met. This is the consequence of the low nuisance constraint $Q_1(\mathbf{x}_c) \leq Q_1(\mathbf{x}^*_c)$ which ensures that the interference which is *produced* by each base station when using the new allocation $\mathbf{x}$ is no bigger than the interference produced when the initial allocation $\mathbf{x}^*$ is used. Second, it is straightforward to show that $\mathbf{x}$ is a global solution to the multicell Problem 2. Indeed, the power $Q_1(\mathbf{x}_c) + Q_2(\mathbf{x}_c)$ spent by Base Station c is necessarily less than the initial power $Q_1(\mathbf{x}^*_c) + Q_2(\mathbf{x}^*_c)$ *by definition* of the minimization Problem 1. Thus $Q(\mathbf{x}) \leq Q(\mathbf{x}^*)$. Of course, as $\mathbf{x}^*$ has been chosen itself as a global minimum of Q, the latter inequality should hold with equality: $Q(\mathbf{x}) = Q(\mathbf{x}^*)$. Therefore, $\mathbf{x}^*$ and $\mathbf{x}$ are both global solutions to the multicell Problem 2. As an immediate consequence, inequality $Q_1(\mathbf{x}_c) + Q_2(\mathbf{x}_c) \leq Q_1(\mathbf{x}^*_c) + Q_2(\mathbf{x}^*_c)$ holds with equality in both Cells c:

$$Q_1(\mathbf{x}_c) + Q_2(\mathbf{x}_c) = Q_1(\mathbf{x}^*_c) + Q_2(\mathbf{x}^*_c)\ . \tag{41}$$

Clearly, $\mathbf{x}_A^*$ is a feasible point for Problem 1 when setting constant $\Omega = Q_1(\mathbf{x}_A^*)$ and $g_{k,1} = g_{k,1}(Q_1(\mathbf{x}_B^*))$. Indeed constraint $\mathbf{C6}$ is equivalent to $Q_1(\mathbf{x}_A^*) \leq \Omega$ and is trivially met (with equality) by definition of Ω. Since the objective function $Q_1(\mathbf{x}_A^*) + Q_2(\mathbf{x}_A^*)$ coincides with the global minimum as indicated by (41), $\mathbf{x}_A^*$ is a global minimum for the single cell Problem 1. By Theorem 1, this single cell problem admits a unique global minimum $\mathbf{x}_A$. Therefore, $\mathbf{x}_A^* = \mathbf{x}_A$. By similar arguments, $\mathbf{x}_B^* = \mathbf{x}_B$. ■

Using the above Lemma along with Theorem 1, we conclude that any global solution $\mathbf{x}^*$ to the joint multicell Problem 2 satisfies equations (18), (19) and (20), where parameters $L^c, \beta_1^c, \beta_2^c, \xi^c$ for $c = A, B$ in the latter equations can be defined as in Appendix B using values $g_{k,1} = g_{k,1}(Q_1(\mathbf{x}_{\bar{c}}^*))$ and $\Omega = Q_1(\mathbf{x}_c^*)$. The proof of Theorem 2 is thus complete.

REFERENCES

[1] M. Chiang, *Geometric Programming for Communication Systems*, Foundations and Trends in Communications and Information Theory, Volume 2, Issue 1-2, 2005.

[2] M. Wiczanowski, S. Stanczak and H. Boche, *Providing quadratic convergence of decentralized power control in wireless networks - the method of min-max functions*, IEEE Transactions on Signal Processing, 2007.

[3] D. Gesbert, M. Kountouris, *Joint Power Control and User Scheduling in Multi-cell Wireless Networks: Capacity Scaling Laws,* submitted to IEEE Trans. On Information Theory, September 2007.

[4] C. Lengoumbi, Ph. Godlewski and Ph. Martins, *Dynamic Subcarrier Reuse with Rate Guaranty in a Downlink Multicell OFDMA System*, The 17th Annual IEEE International Symposium on Personal, Indoor and Mobile Radio Communications, 2006.

[5] S. Hammouda, S. Tabbane and Ph. Godlewski, *Improved Reuse Partitioning and Power Control for Downlink Multi-cell OFDMA Systems*, International Workshop on Broadband Wireless Access for ubiquitous Networking, September, 2006.

[6] S. Pietrzyk and G.J.M. Janssen, *Radio resource allocation for cellular networks based on OFDMA with QoS guarantees*, IEEE Global Telecommunications Conference GLOBECOM '04, Dec. 2004.

[7] J. Li, H. Kim, Y. Lee and Y. Kim, *A novel broadband wireless OFDMA scheme for downlink in cellular communications*, IEEE Wireless Communications and Networking Conference WCNC, March 2003.

[8] L. Yan, Z. Wenan, and S. Junde, *An adaptive subcarrier, bit and power allocation algorithm for multicell OFDM systems*, Canadian Conference on Electrical and Computer Engineering CCECE, May 2003.

[9] G. Li, H. Liu, *Downlink dynamic resource allocation for multicell OFDMA system*, IEEE 58th Vehicular Technology Conference VTC, Oct. 2003.

[10] H. Kwon, W. I. Lee, and B. G. Lee, *Low-Overhead Resource Allocation with Load Balancing in Multi-cell OFDMA Systems*, IEEE 61st Vehicular Technology Conference VTC, Mai 2005.

[11] M. Pischella and J-C. Belfiore, *Power Control in Distributed Cooperative OFDMA Cellular Networks*, IEEE Transactions on Wireless Communications, vol. 7, no. 3, March 2008.

[12] S. Gault and W. Hachem and P. Ciblat, *Performance Analysis of an OFDMA Transmission System in a Multi-Cell Environment*, IEEE Transactions on Communications, num. 12, vol. 55, pp. 2143-2159, December, 2005.

[13] T. Thanabalasingham, S. V. Hanly, L. L. H. Andrew and J. Papandriopoulos, *Joint Allocation of Subcarriers and Transmit Powers in a Multiuser OFDM Cellular Network*, IEEE International Conference on Communications ICC '06, vol. 1, June 2006.

[14] R. D. Yates, *A Framework for Uplink Power Control in Cellular Radio Systems*, IEEE Journal on Selected Areas in Communications, vol. 13, no. 7, September 1995.

[15] WiMAX Forum, *Mobile WiMAX - Part II: A Comparative Analysis*, available at http://www.wimaxforum.org/.

[16] IEEE 802.16-2004, *Part 16: Air interface for fixed broadband wireless access systems*, IEEE Standard for Local and Metropolitan Area Networks, Oct. 2004.

[17] A. D. Wyner, *Shanon theoretic approach to a Gaussian cellular multiple-access channel*, IEEE Trans. Inform. Theory, vol. 40, pp. 1713-1727, Nov. 1994.

[18] O. Somekh and S. Shamai, *Shannon-theoretic approach to Gaussian cellular multi-access channel with fading*, IEEE Trans. Inform. Theory, vol. 46, pp. 1401-1425, July 2000.

[19] C. Zhou, P. Zhang, M. L. Honig and S. Jordan, *Two-cell power allocation for downlink CDMA*, IEEE Transactions on wireless Communications, vol. 3, no. 6, pp. 2256-2266, Nov. 2004.

[20] S. Mukherjee and H. Viswanathan, *Resource allocation strategies for linear symmetric wireless networks with relays*, IEEE International Conference on Communications ICC, 2002.

[21] N. Ksairi, P. Bianchi, P. Ciblat and W. Hachem, *Resource allocation for the downlink of OFDMA cellular networks and Optimization of the Reuse factor*, International Symposium on Information Theory and its Applications (ISITA), Auckland (New Zealand), Dec. 2008.

[22] D. Tse and P. Visawanath, *Fundamentals of wireless communication*, Cambridge University Press, 2005.

[23] *Flash-OFDM, OFDM based all-IP wireless technology*, IEEE C802.20-03/16, www.flarion.com.

[24] S. Plass, A. Dammann, and S. Kaiser. *Analysis of coded OFDMA in a downlink multi-cell scenario*. In Proceedings 9th International OFDM Workshop (InOWo 2004), Dresden, Germany, pages 22-26, Sept. 2004.

[25] S. Plass, X. G. Doukopoulos, R. Legouable, *Investigations on Link-Level Inter-Cell Interference in OFDMA Systems*, Communications and Vehicular Technology, 2006 Symposium on , vol., no., pp.49-52, 23-23 Nov. 2006.

[26] P. Deuflhard, *Newton Methods for Nonlinear Problems: Affine Invariance and Adaptive Algorithms*, Springer, 2005.

[27] COST Action 231, *Digital Mobile Radio toward Future Generation Systems, final report*, Tech. Rep., European Communities, EUR 18957, 1999.

[28] N. Ksairi, P. Bianchi, P. Ciblat and W. Hachem, *Resource Allocation for Downlink Cellular OFDMA Systems: Technical Report*, Tech. Rep., May 2009. Available at http://www.tsi.enst.fr/∼bianchi/Resource_allocation_technical_report.pdf .