Resource Allocation for Downlink Cellular OFDMA Systems: Part II—Practical Algorithms and Optimal Reuse Factor

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Abstract

In a companion paper (see Resource Allocation for Downlink Cellular OFDMA Systems: Part I — Optimal Allocation), we characterized the optimal resource allocation in terms of power control and subcarrier assignment, for a downlink sectorized OFDMA system impaired by multicell interference. In our model, the network is assumed to be one dimensional (linear) for the sake of analysis. We also assume that a certain part of the available bandwidth is likely to be reused by different base stations while that the other part of the bandwidth is shared in an orthogonal way between these base stations. The optimal resource allocation characterized in Part I is obtained by minimizing the total power spent by the network under the constraint that all users' rate requirements are satisfied. It is worth noting that when optimal resource allocation is used, any user receives data either in the reused bandwidth or in the protected bandwidth, but not in both (except for at most one pivot-user in each cell). We also proposed an algorithm that determines the optimal values of users' resource allocation parameters.

As a matter of fact, the optimal allocation algorithm proposed in Part I requires a large number of operations. In the present paper, we propose a distributed practical resource allocation algorithm with low complexity. We study the asymptotic behavior of both this simplified resource allocation algorithm and the optimal resource allocation algorithm of Part I as the number of users in each cell tends to infinity. Our analysis allows to prove that the proposed simplified algorithm is asymptotically optimal *i.e.*, it achieves the same asymptotic transmit power as the optimal algorithm as the number of users in each cell tends to infinity. As a byproduct of our analysis, we characterize the optimal value of the frequency

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reuse factor. Simulations sustain our claims and show that substantial performance improvements are obtained when the optimal value of the frequency reuse factor is used.

Index Terms

OFDMA, Multicell Resource Allocation, Distributed Resource Allocation, Asymptotic Analysis.

I. Introduction

In a companion paper [1], we introduced the problem of joint power control and subcarrier assignment in the downlink of a one-dimensional sectorized two-cell OFDMA system. Resource allocation parameters have been characterized in such a way that *i)* the total transmit power of the network is minimum and *ii)* all users' rate requirements are satisfied. Similarly to [2], we investigate the case where the channel state information at the Base Station (BS) side is limited to some channel statistics. However, contrary to [2], our model assumes that the available bandwidth is divided into two bands: the first one is reused by different base stations (and is thus subject to multicell interference) while the second one is shared in an orthogonal way between the adjacent base stations (and is thus protected from multicell interference). The number of subcarriers in each band is directly related to the frequency reuse factor. We also assume that each user is likely to modulate subcarriers in each of these two bands and thus we do not assume *a priori* a geographical separation of users modulating in the two different bands. The solution to the above resource allocation problem is given in the first part of this work. This solution turns out to be "binary": except for at most one pivot-user, users in each cell must be divided into two groups, the nearest users modulating subcarriers only in the reused band and the farthest users modulating subcarriers only in the protected band. An algorithm that determines the optimal values of users' resource allocation parameters is also proposed in the first part.

It is worth noting that this optimal allocation algorithm is still computationally demanding, especially when the number of users in each cell is large. One of the computationally costliest operations involved in the optimal allocation is the determination of the pivot-user in each cell. In the present paper, we propose a distributed simplified resource allocation algorithm with low computational complexity, and we discuss its performance as compared to the optimal resource allocation algorithm of Part I. This simplified algorithm assumes a pivot-distance that is fixed in advance prior to the resource allocation process. Of course, this predefined pivot-distance should be relevantly chosen. For that sake, we show that when the fixed pivot-distance of the simplified algorithm is chosen according to a certain asymptotic analysis of the optimal allocation scheme, the performance of the simplified algorithm is close to the optimal one, provided that the number of users in the network is large enough. Therefore, following the approach of [2], we propose to characterize the limit of the total transmit power which results from the optimal resource allocation policy as the number of users in each cell tends to infinity. Several existing works on resource allocation resorted to this kind of asymptotic analysis, principally in order to get tractable formulations of the optimization problem that can be solved analytically. For example, the asymptotic analysis was used in [3] and [4] in the context of downlink and uplink single cell OFDMA systems respectively, as well as in [5] in the context of *Code Division Multiple Access* (CDMA) systems with fading channels. Another application of the asymptotic analysis can be found in [6]. The authors of the cited work addressed the optimization of the sum rate performance in a multicell network. In this context, the authors proposed a decentralized algorithm that maximizes an upperbound on the network sum rate. Interestingly, this upperbound is proved to be tight in the asymptotic regime when the number of users per cell is allowed to grow to infinity. However, the proposed algorithm does not guaranty fairness among the different users.

In this paper, we use the asymptotic analysis in order to obtain a compact form of the (asymptotic) power transmitted by the network for the optimal resource allocation algorithm, and we use this result to propose relevant values of the fixed pivot-distance associated with the simplified allocation algorithm. We prove in particular that when this fixed pivot-distance is chosen equal to the asymptotic optimal pivot-distance, then the power transmitted when using the proposed simplified resource allocation is asymptotically equivalent to the minimum power associated with the optimal algorithm. This limiting expression no longer depends on the particular network configuration, but on an asymptotic, or "average", state of the network. More precisely, the asymptotic transmit power depends on the average rate requirement and on the density of users in each cell. It also depends on the value α of the frequency reuse factor. As a byproduct of our asymptotic analysis, we are therefore able to determine an optimal value of the latter reuse factor. This optimal value is defined as the value of α which minimizes the asymptotic power.

The rest of this paper is organized as follows. In Section II we recall the system model as well as the joint resource allocation problem. In Section III, we propose a novel suboptimal distributed resource allocation algorithm. Section IV is devoted to the asymptotic analysis of the performance of this simplified allocation algorithm as well as the performance of the optimal resource allocation scheme of Part I when the number of users tends to infinity. Theorem 1 characterizes the asymptotic behavior of the optimal joint allocation scheme. The results of this theorem are used in Subsection IV-D in order to determine relevant values of the fixed pivot-distances associated with the simplified allocation algorithm. Provided that these relevant values are used, Proposition 2 states that the simplified algorithm is asymptotically optimal. Section VI addresses the selection of the best frequency reuse factor. Finally, Section VII is devoted to the numerical illustrations of our results.

II. SYSTEM MODEL AND PREVIOUS RESULTS

A. System Model

We consider a sectorized downlink OFDMA cellular network. We focus on two neighboring one-dimensional (linear) cells, say Cell A and Cell B, as illustrated by Figure 1. Denote by D the radius of

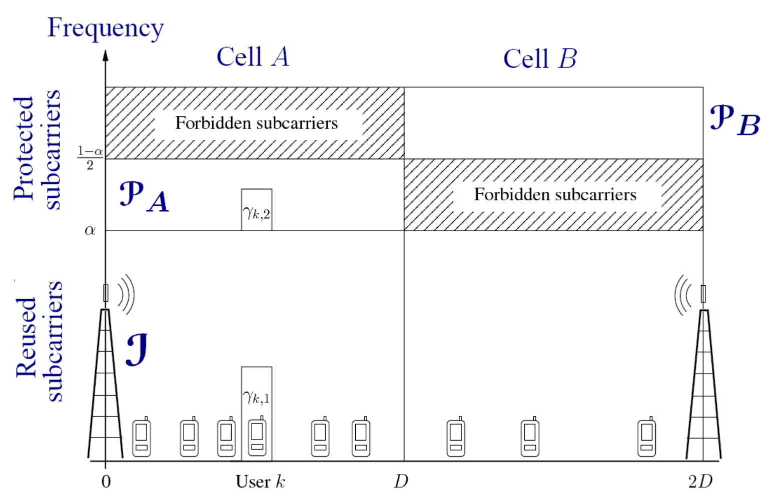

Figure 1. Two-Cell System model

each cell. We denote by K^A the number of users of Cell A and by K^B the number of users of Cell B. The total number of available subcarriers in the system is denoted by N. For a given user $k \in 1, 2, \ldots, K^c$ in Cell c ($c \in \{A, B\}$), we denote by x_k the distance that separates him/her from BS c, and by $\mathcal{N}_k$ the set of indices corresponding to the subcarriers modulated by k. $\mathcal{N}_k$ is a subset of $\{0, 1, \ldots, N-1\}$. The signal received by user k at the nth subcarrier ($n \in \mathcal{N}_k$) and at the mth OFDM block is given by

$$y_k(n, m) = H_k(n, m) s_k(n, m) + w_k(n, m), \tag{1}$$

where $s_k(n, m)$ represents the data symbol transmitted by BS c. Process $w_k(n, m)$ is an additive noise which encompasses the thermal noise and the possible multicell interference. Coefficient $H_k(n, m)$ is the frequency response of the channel at the subcarrier n and the OFDM block m. Random variables $H_k(n, m)$ are assumed Rayleigh distributed with variance $\rho_k^c = E[|H_k(n, m)|^2]$. Channel coefficients are supposed to be perfectly known at the receiver side, and unknown at the BS side. We assume that ρ_k vanishes with the distance x_k based on a given path loss model. The set of available subcarriers is partitioned into three subsets: $\mathcal{I}$ containing the reused subcarriers shared by the two cells; $\mathcal{P}_A$ and $\mathcal{P}_B$ containing the protected subcarriers only used by users in Cell A and B respectively. The *reuse factor* α is defined as the ratio between the number of reused subcarriers and the total number of subcarriers:

$$\alpha = \frac{\text{card}(\mathcal{I})}{N}$$

so that $\mathcal{I}$ contains αN subcarriers. If user k modulates a subcarrier $n \in \mathcal{I}$, the additive noise contains both thermal noise of variance σ^2 and interference. Therefore, the variance σ_k^2 of this noise-plus-interference process depends on k and coincides with $\sigma_k^2 = \mathbb{E}\left[|\tilde{H}_k(n,m)|^2\right] Q_1^B + \sigma^2$, where $\tilde{H}_k(n,m)$ represents the channel between BS B and user k of Cell A at frequency n and OFDM block m, and where $Q_1^B = \sum_{k=1}^{K^B} \gamma_{k,1}^B P_{k,1}^B$ is the average power transmitted by BS B in the interference bandwidth $\mathcal{I}$. The remaining $(1-\alpha)N$ subcarriers are shared by the two cells, Cell A and B , in an orthogonal way. If user k modulates such a subcarrier $n \in \mathcal{P}_c$, the additive noise $w_k(n,m)$ contains only thermal noise. In other words, subcarrier n does not suffer from multicell interference. Then we simply write $\mathbb{E}[|w_k(n,m)|^2] = \sigma^2$. The resource allocation parameters for user k are: $P_{k,1}^c$ the power transmitted on each of the subcarriers of the non protected band $\mathcal{I}$ allocated to him, $\gamma_{k,1}^c$ his share of $\mathcal{I}$, $P_{k,2}^c$ the power transmitted on each of the subcarriers of the protected band $\mathcal{P}_c$ allocated to him and $\gamma_{k,2}^c$ his share of $\mathcal{P}_c$. In other words,

$$\gamma_{k,1}^c = \text{card}(\mathcal{I} \cap \mathcal{N}_k)/N \qquad \gamma_{k,2}^c = \text{card}(\mathcal{P}_c \cap \mathcal{N}_k)/N \ .$$

As a consequence, $\sum_{k=1}^{K^c} \gamma_{k,1}^c = \alpha$ and $\sum_{k=1}^{K^c} \gamma_{k,2}^c = \frac{1-\alpha}{2}$ for each cell c. Moreover, let $g_{k,1}$ (resp. $g_{k,2}$) be the channel Gain to Noise Ratio (GNR) in band $\mathcal{I}$ (resp. $\mathcal{P}_c$), namely $g_{k,1} = \rho_k/\sigma_k^2$ (resp. $g_{k,2} = \rho_k/\sigma^2$). "Setting a resource allocation for cell c" means setting a value for parameters $\{\gamma_{k,1}^c, \gamma_{k,2}^c, P_{k,1}^c, P_{k,2}^c\}_{k=1\dots K^c}$.

B. Joint Resource Allocation for Cells A and B

Assume that each user k has a rate requirement of R_k nats/s/Hz. In the first Part of this work [1], our aim was to jointly optimize the resource allocation for the two cells which i) allows to satisfy all target rates R_k of all users, and ii) minimizes the power used by the two base stations in order to achieve these rates. For each cell $c \in \{A, B\}$, denote by $\bar{c}$ the adjacent cell ($\overline{A} = B$ and $\overline{B} = A$). The ergodic capacity associated with a user k in Cell c is given by

$$C_k = \gamma_{k,1}^c \mathbb{E}\left[\log\left(1 + g_{k,1}(Q_1^{\bar{c}}) P_{k,1}^c Z\right)\right] + \gamma_{k,2}^c \mathbb{E}\left[\log\left(1 + g_{k,2} P_{k,2}^c Z\right)\right] \ , \qquad (2)$$

where Z is a standard Chi-Square distributed random variable with two degrees of freedom, and where coefficient $g_{k,1}(Q_1^{\bar{c}})$ is given by

$$g_{k,1}(Q_1^{\bar{c}}) = \frac{\rho_k}{\mathbb{E}\left[|\tilde{H}_k(n,m)|^2\right] Q_1^{\bar{c}} + \sigma^2}, \tag{3}$$

where $\tilde{H}_k(n,m)$ represents the channel between BS $\bar{c}$ and user k of Cell c at frequency n and OFDM block m. Coefficient $g_{k,1}(Q_1^{\bar{c}})$ represents the signal to interference plus noise ratio in the interference band $\mathcal{I}$. We assume that users are numbered from the nearest to the BS to the farthest. As in [1], the following problem will be referred to as the joint resource allocation problem for Cells A and B: Minimize the total power spent by both base stations $Q_T^{(K)} = \sum_{c=A,B} \sum_{k=1}^{K^c} (\gamma_{k,1}^c P_{k,1}^c + \gamma_{k,2}^c P_{k,2}^c)$ with respect to $\{\gamma_{k,1}^c, \gamma_{k,2}^c, P_{k,1}^c, P_{k,2}^c\}_{\substack{c=A,B \\ k=1\ldots K^c}}$ under the following constraint that all users' rate requirements R_k are satisfied *i.e.*, for each user k in any cell c, $R_k \leq C_k$. The solution to this problem has been determined in the first part of this work [1]. As a noticeable point, the results of [1] indicate the existence in each cell of a pivot-user that separates two groups of users: the "protected" users and the "non protected" users. The following proposition states this binary property of the solution.

Proposition 1 ([1])**.** *Any global solution to the joint resource allocation problem is "binary"* i.e., *there exists a user L^c in each Cell c such that $\gamma_{k,2} = 0$ for closest users $k < L^c$, and $\gamma_{k,1} = 0$ for farthest users $k > L^c$.*

In the sequel, we denote by $d^{c,(K)}$ the position of the pivot-user L^c in Cell c *i.e.*, $d^{c,(K)} = x_{L^c}$. A resource allocation algorithm is also proposed in [1]. This algorithm turns out to have a high computational complexity and the determination of the optimal value of the pivot-distance $d^{c,(K)}$ turns out to be one of the costliest operations involved in this algorithm. This is why we propose in the follwing section of the present paper a suboptimal simplified allocation algorithm that assumes a predefined pivot-distance.

III. Practical Resource Allocation Algorithm

A. Motivations and Main idea

Proposition 1 provides the general form of the optimal resource allocation, showing in particular the existence of pivot-users L^A, L^B in both Cells A, B, separating the users who modulate in band $\mathcal{I}$ from the users who modulate in bands $\mathcal{P}^A$ and $\mathcal{P}^B$. As a matter of fact, the determination of pivot-users L^A, L^B is one of the costliest operations of this optimal allocation (see [1] for a detailed computational complexity analysis). Thus, it would be convenient to propose an allocation procedure for which the pivot-position would be **fixed in advance** to a constant rather than systematically computed/optimized. We propose a simplified resource allocation algorithm based on this idea. Furthermore, we prove that when the value of the fixed pivot-distances is relevantly chosen, the proposed algorithm is asymptotically optimal as the number of users increases. In other words, the total power spent by the network for large K when using our suboptimal algorithm does not exceed the minimum power that would have been spent by using the optimal resource allocation. The proposed algorithm is based on the following idea.

Recall the definition of $d^{A,(K)}$ and $d^{B,(K)}$ as the respective position of the optimal pivot-users L^A and L^B defined by Proposition 1. As the optimal pivot-positions $d^{A,(K)}$ and $d^{B,(K)}$ are difficult to compute explicitly and depend on the particular rates and users' positions, we propose to replace $d^{A,(K)}$ and $d^{B,(K)}$ with predefined values d^A_{subopt} and d^B_{subopt} fixed before the resource allocation process. In our suboptimal algorithm, all users in Cell c whose distance to the BS is less than d^c_{subopt} modulate in the interference band $\mathcal{I}$. Users farther than d^c_{subopt} modulate in the protected band $\mathcal{P}^c$. Of course, we still need to determine the pivot-distances d^A_{subopt} and d^B_{subopt}. **A procedure that permits the relevant selection of $\boldsymbol{d^A_{\text{subopt}}}$, $\boldsymbol{d^B_{\text{subopt}}}$ is given in Section IV-C**.

B. *Detailed Description*

Assume that the values of d^A_{subopt} and d^B_{subopt} have been fixed beforehand prior to the resource allocation process. For each Cell c, define by $\mathcal{K}^c_I$ the subset of $\{1,\dots K^c\}$ corresponding to the users whose distance to BS c is less than d^c_{subopt}. Define by $\mathcal{K}^c_P$ the set of users whose distance to BS c is larger than d^c_{subopt}.

1) Resource allocation for protected users: Focus for instance on Cell A. For each $k \in \mathcal{K}^A_P$, we arbitrarily set $\gamma^A_{k,1} = P^A_{k,1} = 0$ *i.e.*, user k is forced to modulate in the protected band $\mathcal{P}^A$ only. For such users, the remaining resource allocation parameters $\gamma^A_{k,2}, P^A_{k,2}$ are obtained by solving the following classical single cell problem w.r.t. $(\gamma^A_{k,2}, P^A_{k,2})_{k\in\mathcal{K}^A_P}$:

"Minimize the transmitted power $\sum_{k\in\mathcal{K}^A_P} \gamma^A_{k,2} P^A_{k,2}$ under rate constraint $R_k < C_k$ for each $k \in \mathcal{K}^A_P$".

The above problem is a simple particular case of the single cell problem addressed in [1]. Define the functions $f(x) = \frac{\mathbb{E}[\log(1+xZ)]}{\mathbb{E}\left[\frac{Z}{1+xZ}\right]} - x$ and $C(x) = \mathbb{E}[\log(1+f^{-1}(x)Z)]$ on $\mathbb{R}_+$. The solution is given by

$$P^A_{k,2} = g^{-1}_{k,2} f^{-1}(g_{k,2}\tilde{\beta}_2)$$
$$\gamma^A_{k,2} = \frac{R_k}{\mathbb{E}\left[\log\left(1+g_{k,2}P^A_{k,2}Z\right)\right]}\ ,$$

where parameter $\tilde{\beta}_2$ is obtained by writing that constraint $\sum_k \gamma_{k,2}^A = \frac{1-\alpha}{2}$ holds or equivalently, $\tilde{\beta}_2$ is the unique solution to:

$$\sum_{k\in\mathcal{K}_P^A} \frac{R_k}{C(g_{k,2}\tilde{\beta}_2)} = \frac{1-\alpha}{2} \, .$$

We proceed similarly for Cell B.

2) Resource allocation for interfering users: We now focus on users $k \in \mathcal{K}_I^c$ for each cell $c = A, B$. For such users, we arbitrarily set $\gamma_{k,2}^c = P_{k,2}^c = 0$ *i.e.*, users in $\mathcal{K}_I^c$ are forced to modulate in the interference band $\mathcal{I}$ only, for each cell c. The remaining resource allocation parameters $\gamma_{k,1}^A, P_{k,1}^A, \gamma_{k,1}^B, P_{k,1}^B$ are obtained by solving the following simplified multicell problem.

Problem 1. [Multicell] *Minimize* $\sum_{c=A,B}\sum_{k\in\mathcal{K}_I^c} \gamma_{k,1}^c P_{k,1}^c$ *w.r.t.* $(\gamma_{k,1}^A, P_{k,1}^A, \gamma_{k,1}^B, P_{k,1}^B)_k$ *under the following constraints for each cell* $c \in \{A, B\}$*:*

$$\mathbf{C1}: \forall c,\ \forall k \in \mathcal{K}_I^c, R_k \le C_k \qquad \mathbf{C2}: \forall c, \sum_{k\in\mathcal{K}_I^c} \gamma_{k,1}^c = \alpha \qquad \mathbf{C3}: \gamma_{k,1}^c \ge 0 \, .$$

Clearly, the above Problem can be interpreted as a particular case of the initial resource allocation (Problem 2 in [1]) addressed in Section II-B of the present paper. The main difference is that the initial multicell problem jointly involves the resource allocation parameters in three bands $\mathcal{I}$, $\mathcal{P}^A$ and $\mathcal{P}^B$ whereas the present problem only optimizes the resource allocation parameters corresponding to band $\mathcal{I}$, while arbitrarily setting the others to zero. Therefore, the results of Part I [1], Theorem 2 of [1] in particular, can directly be used to determine the global solution to Problem 1.

Remark 1 (**Feasibility).** *Recall that the initial joint resource allocation Problem (Problem 2 in [1]) described in Section II-B in the present paper was always feasible. Intuitively, this was due to the fact that any user was likely to modulate in the protected band if needed, so that any rate requirement* R_k *was likely to be satisfied by simply increasing the power in the protected band. In the present case, the protected band is by definition forbidden to users in* $\mathcal{K}_I^c$*. Theoretically speaking, Problem 1 might not be feasible due to multicell interference. Fortunately, we will see this case does not happen, at least for a sufficiently large number of users, if the values of the pivot-distances* d_{subopt}^A *and* d_{subopt}^B *are well chosen. This point will be discussed in more detail in Section V.*

Define $Q_1^c = \sum_{k\in\mathcal{K}_I^c} \gamma_{k,1}^c P_{k,1}^c$ as the average power transmitted by BS c in the interference bandwidth $\mathcal{I}$.

By straightforward application of Theorem 2, we obtain that for each Cell c and for each user $k \in \mathcal{K}_I^c$,

$$P_{k,1}^c = g_{k,1}^{-1}(Q_1^{\bar{c}}) f^{-1}(g_{k,1}(Q_1^{\bar{c}})\tilde{\beta}_1^c) \tag{4}$$

$$\gamma_{k,1}^c = \frac{R_k}{\mathbb{E}\left[\log\left(1 + g_{k,1}(Q_1^{\bar{c}})P_{k,1}^c Z\right)\right]}\,, \tag{5}$$

where for each $c = A, B$ and for a fixed value of $Q_1^{\bar{c}}$, parameters $(\tilde{\beta}_1^c, Q_1^c)$ are the unique solution to the following system of equations:

$$\sum_{k\in\mathcal{K}_I^c} \frac{R_k}{C(g_{k,1}(Q_1^{\bar{c}})\tilde{\beta}_1^c)} = \alpha \tag{6}$$

$$Q_1^c = \sum_{k\in\mathcal{K}_I^c} R_k \frac{g_{k,1}^{-1}(Q_1^{\bar{c}}) f^{-1}(g_{k,1}(Q_1^{\bar{c}})\tilde{\beta}_1^c)}{C(g_{k,1}(Q_1^{\bar{c}})\tilde{\beta}_1^c)}\,. \tag{7}$$

Note that the first equation is nothing else that the constraint **C2**: $\sum_k \gamma_{k,1}^c = \alpha$. The second equation is nothing else than the definition $Q_1^c = \sum_{k\in\mathcal{K}_I^c} \gamma_{k,1}^c P_{k,1}^c$. We now prove that the system of four equations (6)-(7) for $c = A, B$ admits a unique solution $\tilde{\beta}_1^A, Q_1^A, \tilde{\beta}_1^B, Q_1^B$ and we provide a simple algorithm allowing to determine this solution.

Focus on a given Cell c and consider any fixed value $Q_1^{\bar{c}}$. Denote by $\tilde{I}^c(Q_1^{\bar{c}})$ the rhs of equation (7) where $\tilde{\beta}_1^c$ is defined as the unique solution to (6). Clearly, the couple (Q_1^A, Q_1^B) is a fixed point of the vector valued function $\tilde{\mathbf{I}}(Q_1^A, Q_1^B) = (\tilde{I}^A(Q_1^B), \tilde{I}^B(Q_1^A))$.

$$(Q_1^A, Q_1^B) = \tilde{\mathbf{I}}(Q_1^A, Q_1^B)\,. \tag{8}$$

As a matter of fact, it can be shown that such a fixed point of $\tilde{\mathbf{I}}$ is unique. This claim can be proved using the approach previously proposed by [12].

Lemma 1. *Function $\tilde{\mathbf{I}}$ is such that the following properties hold.*

1) *Positivity:* $\tilde{\mathbf{I}}(Q^A, Q^B) > 0$.
2) *Monotonicity: If* $Q^A \geq Q^{A'}, Q^B \geq Q^{B'}$, *then* $\tilde{\mathbf{I}}(Q^A, Q^B) \geq \tilde{\mathbf{I}}(Q^{A'}, Q^{B'})$.
3) *Scalability: for all* $t > 1$, $t\tilde{\mathbf{I}}(Q^A, Q^B) > \tilde{\mathbf{I}}(tQ^A, tQ^B)$.

The proof of Lemma 1 is provided in Appendix B. It uses arguments which are very similar to those of [11]. Function $\tilde{\mathbf{I}}$ is then a *standard interference function*, using the terminology of [12]. Therefore, as stated in [12], such a function $\tilde{\mathbf{I}}$ admits at most one fixed point. On the other hand, the existence of a fixed point is ensured by the feasibility of Problem 1 and by the fact that (8) holds for any global solution. In other words, if Problem 1 is feasible, then function $\tilde{\mathbf{I}}$ does admit a fixed point and this fixed point is unique. Putting all pieces together, there exists a unique solution to (8), which can be obtained thanks to a simple fixed point algorithm. In practice, resource allocation in band $\mathcal{I}$ can be achieved by the following procedure.

Ping-pong algorithm for interfering users

1) Initialization: $Q_1^B = 0$.
2) Cell A: Given the current value of the power Q_1^B transmitted by base station B in the interference bandwidth, compute $\tilde{\beta}_1^A, Q_1^A$ as the unique solution to (6)-(7) with $c = A$.
3) Cell B: Given the current value of Q_1^A, compute $\tilde{\beta}_1^B, Q_1^B$ by (6)-(7).
4) Go back to step 2 until convergence.
5) Define resource allocation parameters by (4)-(5).

Comments

1) **Convergence of the ping-pong algorithm.** We stated earlier that Problem 1 is either feasible or infeasible, depending on the value of $(d^A_{\text{subopt}}, d^B_{\text{subopt}})$. If Problem 1 is feasible, then the ping-pong algorithm converges. If this problem is infeasible, the the ping-pong algorithm diverges. One of the main purposes of Section IV-C is to provide relevant values of $(d^A_{\text{subopt}}, d^B_{\text{subopt}})$ such that convergence of the ping-pong algorithm holds for sufficiently large number K of users.
2) Note that the only information needed by Base Station c about Cell $\overline{c}$ is the current value of the power $Q_1^{\overline{c}}$ transmitted by Base Station $\overline{c}$ in the interference band $\mathcal{I}$. This value can *i)* either be measured by Base Station c at each iteration of the ping-pong algorithm, or *ii)* it can be communicated to it by Base Station $\overline{c}$ over a dedicated link. In the first case, no message passing is required, and in the second case only few information is exchanged between the base stations. The ping-pong algorithm can thus be implemented in a distributed fashion.

C. Complexity Analysis

We showed earlier that allocation for protected users can be reduced to the determination in each cell of the value of $\tilde{\beta}_2^c$, which is the unique solution to the equation $\sum_{k\in\mathcal{K}_P^A} \frac{R_k}{C(g_{k,2}\tilde{\beta}_2^c)} = \frac{1-\alpha}{2}$. We argued in [1] that solving this kind of equations requires a computational complexity proportional to the number of terms in the lhs of the equation, which is itself of order $O(K)$. Using similar arguments, we can show that each iteration of the ping-pong algorithm for non protected users can be performed with a complexity of order $O(K)$. Let J designate the number of iterations needed till convergence. The overall computational complexity of the ping-pong algorithm, and hence of the simplified resource allocation scheme as well, is thus of the order of $O(JK)$. Our simulations showed that the ping-pong algorithm converges relatively quickly in most of the cases. Indeed, no more than $J = 15$ iterations were needed in almost all the simulations settings to reach convergence within a very reasonable accuracy. The complexity of the simplified algorithm is to be compared with the computational complexity of the optimal algorithm which was shown in [1] to be of the order of $O(MK\log_2 K)$, where M is the number of points inside a certain 2D search grid.

IV. Asymptotic Optimality of the Simplified Resource Allocation Scheme

The aim of this section is to evaluate the performance of the proposed simplified algorithm. The relevant performance metric in the context of this paper is the total power that must be transmitted by the base stations. Since the simplified algorithm assumes predefined pivot-distances $(d^A_{\text{subopt}}, d^B_{\text{subopt}})$ fixed prior to the resource allocation process, the performance of the proposed algorithm depends on the choice of these fixed pivot-distances. One must therefore determine what relevant value should be selected for $(d^A_{\text{subopt}}, d^B_{\text{subopt}})$. A possible method is addressed in this section and consists in studying the case where the number of users tends to infinity.

A. Main Tools: Asymptotic analysis

We study first the performance of the **optimal** allocation algorithm proposed in Part I [1] when the number of users in each cell tends to infinity. From the results of this asymptotic study, we conclude the asymptotic behaviour of the optimal pivot-distances $\left(d^{A,(K)}, d^{B,(K)}\right)$. It turns out that when the number K of users increases, the optimal pivot-distances as well as the total transmitted power no longer depend on the particular cell configuration, but on an asymptotic state of the network, such as the average rate requirement and the density of users in each cell. Thanks to this result, we can now choose the fixed pivot-distances associated with the simplified algorithm to be equal to the asymptotic pivot-distances. In this case, one can show that the performance gap between the simplified and the optimal allocation schemes vanishes for high numbers of users. We introduce now the mathematical assumptions and tools that we use for defining the asymptotic regime.

1) Notations and Basic Assumptions: In the sequel, we denote by B the total bandwidth of the system in Hz. We consider the asymptotic regime where the number of users in each cell tends to infinity. We denote by $r_k = BR_k$ the data rate requirement of user k in nats/s, and we recall that R_k is the data rate requirement of user k in nats/s/Hz. Notice that the total rate $\sum_{k=1}^{K^c} r_k$ which should be delivered by BS c tends to infinity as well. Thus, we need to let the bandwidth B grow to infinity in order to satisfy the growing data rate requirement. Recalling that $K = K^A + K^B$ denotes the total number of users in both cells, the asymptotic regime will be characterized by $K \to \infty, B \to \infty$ and $K/B \to t$ where t is a positive real number. We assume on the other hand that K^c/K ($c \in \{A, B\}$) tends to some positive constant as K tends to infinity. Without restrictions, this constant is assumed in the sequel to be equal to 1/2 *i.e.*, the number of users becomes equivalent in each cell. In order to simplify the proofs of our results, we assume without restriction that for each k, the rate requirement r_k is upper-bounded by a certain constant $r_{\max}$, $r_k \le r_{\max}$, where $r_{\max}$ can be chosen as large as needed, and that users of each cell are located in the interval $[\epsilon, D]$ where $\epsilon > 0$ can be chosen as small as needed. Recall that x_k denotes the position of each user k *i.e.,* the distance between the user and the BS. The variance of the channel gain of user k will be written as $\rho_k = \rho(x_k)$ where $\rho(x)$ models the path loss. Typically, function $\rho(x)$ has the form $\rho(x) = \lambda x^{-s}$ where λ is a certain gain and where s is the path-loss coefficient, $s \ge 2$. In the sequel, we denote by $g_2(x) = \frac{\rho(x)}{\sigma^2}$ the received gain to noise ratio in the protected bandwidth, for a user at position x. This way, $g_2(x_k) = g_{k,2}$. Similarly, we define for each user k in cell A, $g_1(x_k, Q_1^B) = g_{k,1}(Q_1^B)$. More generally, $g_1(x, \mathcal{Q})$ denotes the gain-to-interference-plus-noise ratio in the interference bandwidth at position x when the interfering cell is transmitting with power $\mathcal{Q}$ in band $\mathcal{I}$. Functions $g_1(x, .)$ and $g_2(x)$ are assumed to be continuous functions of x. It is worth noting that for each x, $g_2(x) = g_1(x, 0)$. Finally, recall that coefficient $\gamma_{k,1}^c$ (resp. $\gamma_{k,2}^c$) is defined as the ratio between the part of the interference bandwidth $\mathcal{I}$ (resp. protected bandwidth $\mathcal{P}_c$) and the total bandwidth. Thus, $\gamma_{k,1}^c$ and $\gamma_{k,2}^c$ tend to zero as the total bandwidth B tends to infinity for each k.

2) Statistical Tools and Main Ideas of the Asymptotic Study: Theorem 2 of Part I [1] reduces the determination of the whole set of resource allocation parameters in both cells to the determination of ten unknown parameters $\{Q_1^c, \beta_i^c, L^c, \xi^c\}_{c=A,B,\, i=1,2}$. Parameter Q_1^c in particular represents the power transmitted by Cell c in the non protected band $\mathcal{I}$. Consider now one of the two Cells $c \in \{A, B\}$, and denote by $\overline{c}$ the second (adjacent) cell. In the sequel, we use the notation $Q_1^{c,(K)}$ (resp. $Q_2^{c,(K)}$) instead of Q_1^c (resp. Q_2^c) to designate the power transmitted by BS c in the non protected band $\mathcal{I}$ (resp. the protected band $\mathcal{P}_c$) when the optimal solution characterized by Proposition 1 is used.

$$Q_1^{c,(K)} = \sum_{k=1}^{L^c} \gamma_{k,1}^c P_{k,1}^c \tag{9}$$

$$Q_2^{c,(K)} = \sum_{k=L^c}^{K^c} \gamma_{k,2}^c P_{k,2}^c \, . \tag{10}$$

The new notation $Q_1^{c,(K)}, Q_2^{c,(K)}$ is used to indicate the dependency of the results on the number of users K. For the same reason, parameters $L^c, \beta_1^c, \beta_2^c, \xi^c$ will be denoted in the sequel by $L^{c,(K)}, \beta_1^{c,(K)}, \beta_2^{c,(K)}, \xi^{c,(K)}$ respectively. Our goal now is to characterize the behavior of the resource allocation strategy as $K, B \to \infty$ and, in particular, the behavior of powers $Q_1^{c,(K)}$, $Q_2^{c,(K)}$. By straightforward application of Theorem 2 of Part I, $Q_1^{c,(K)} = \sum_{k=1}^{L^c} \gamma_{k,1}^c P_{k,1}^c$ can be written as

$$Q_1^{c,(K)} = \sum_{k<L^{c,(K)}} R_k \mathcal{F}(x_k, \beta_1^{c,(K)}, Q_1^{\bar{c},(K)}, \xi^{c,(K)}) + W_{L^{c,(K)},1}^c \,, \tag{11}$$

where $W_{L^{c,(K)},1}^c = \gamma_{L^{c,(K)},1}^c P_{L^{c,(K)},1}^c$ denotes the power transmitted to the pivot-user $L^{c,(K)}$ in the interference band $\mathcal{I}$, and where function $\mathcal{F}$ is defined by

$$\mathcal{F}(x, \beta, \mathcal{Q}, \xi) = \frac{f^{-1}\left(\frac{g_1(x,\mathcal{Q})}{1+\xi}\beta\right)}{g_1(x,\mathcal{Q}) C\left(\frac{g_1(x,\mathcal{Q})}{1+\xi}\beta\right)} \tag{12}$$

for each $x, \beta, \mathcal{Q}$. The first term in the rhs of (11) represents the total power allocated to all users $k < L^{c,(K)}$. It is quite intuitive that the power allocated to one user $W_{L^{c,(K)},1}^c$ is negligible when compared to the power allocated to all users $k < L^{c,(K)}$. In fact, it can easily be shown that the first term of (11) is bounded as $K \to \infty$ whereas $W_{L^{c,(K)},1}^c$ tends to zero. In the sequel, we use notation $W_{L^{c,(K)},1}^c = o_K(1)$, where $o_K(1)$ stands for any term which converges to zero as $K \to \infty$. In order to study the limit of this expression as K tends to infinity, we introduce for each one of the two cells the following measure $\nu^{c,(K)}$ defined on the Borel sets of $\mathbb{R}_+ \times \mathbb{R}_+$ as follows

$$\nu^{c,(K)}(I, J) = \frac{1}{K^c} \sum_{k=1}^{K^c} \delta_{r_k, x_k}(I, J) \tag{13}$$

where I and J are any intervals of $\mathbb{R}_+$ and where δ_{r_k,x_k} is the Dirac measure at point (r_k, x_k). In order to have more insights on the meaning of this tool, it is useful to remark that $\nu^{c,(K)}(I, J)$ is equal to

$$\nu^{c,(K)}(I, J) = \frac{\text{number of users located in } J \text{ and requiring a rate (in nats/s) in interval } I}{\text{total number of users}}.$$

Thus, measure $\nu^{c,(K)}$ can be interpreted as the distribution of the set of couples (r_k, x_k) of Cell c. The introduction of the above measure simplifies considerably the asymptotic study of the transmit power. Indeed, replacing R_k (in nats/s/Hz) by $\frac{r_k \text{ (nats/s)}}{B}$ in equation (11), we obtain

$$\begin{aligned} Q_1^{c,(K)} &= \frac{1}{B} \sum_{k<L^{c,(K)}} r_k \mathcal{F}(x_k, \beta_1^{c,(K)}, Q_1^{\bar{c},(K)}, \xi^{c,(K)}) + o_K(1) \\ &= \frac{K^c}{B} \iint_{\Delta_1^{c,(K)}} r \mathcal{F}(x, \beta_1^{c,(K)}, Q_1^{\bar{c},(K)}, \xi^{c,(K)}) d\nu^{c,(K)}(r, x) + o_K(1) \,, \end{aligned} \tag{14}$$

where integration is considered with respect to the set $\Delta_1^{c,(K)} = [0, r_{\max}] \times [\epsilon, d^{c,(K)}]$, where $d^{c,(K)} = x_{L^{c,(K)}}$ is the position of pivot-user $L^{c,(K)}$ and where ϵ can be chosen, as stated earlier in this section, as small as needed. It is quite intuitive that the asymptotic power $\lim_{K\to\infty} Q_1^{c,(K)}$ can be obtained from (14) by replacing $\frac{K^c}{B} = \frac{K}{B} \times \frac{K^c}{K}$ by $t \times \frac{1}{2}$ and the distribution $\nu^{c,(K)}$ by the *asymptotic distribution* ν^c of couples (r_k, x_k) as K tends to infinity. The existence and the definition of this asymptotic distribution is provided by the following assumption.

Assumption 1. *As K tends to infinity, measure $\nu^{c,(K)}$ converges weakly to a measure ν^c.*

We refer to [7] for the materials on the convergence of measures. In order to have some insight on the behavior of equation (14) in the asymptotic regime, imagine for the sake of simplicity that sequences $d^{A,(K)}, d^{B,(K)}, Q_1^{A,(K)}, Q_1^{B,(K)}, \beta_1^{A,(K)}, \beta_1^{B,(K)}, \xi^{A,(K)}, \xi^{B,(K)}$ are convergent and that they converge respectively to $d^A, d^B, Q_1^A, Q_1^B, \beta_1^A, \beta_1^B, \xi^A, \xi^B$. This assumption is of course arbitrary for the moment, but it allows to better understand the main ideas of our asymptotic analysis. More rigorous considerations on the convergence of these sequences will be discussed later on. Ignoring at first such technical issues, it is intuitive from equation (14) that $Q_1^{c,(K)}$ converges to a constant Q_1^c defined by

$$Q_1^c = \frac{t}{2} \iint_{\Delta_1^c} r\mathcal{F}(x, \beta_1^c, Q_1^{\bar{c}}, \xi^c) d\nu^c(r, x) \,, \tag{15}$$

where $\Delta_1^c = [0, r_{\max}] \times [\epsilon, d^c]$. In other words, we manage to express the limit of the power $Q_1^{c,(K)}$ transmitted by station c in the interference band as a function of the asymptotic cell configuration. In order to further simplify the above expression, it is also realistic to assume that measure ν^c is the measure product of a limit rate distribution times a limit location distribution. Assumption 2 below is motivated by the observation that in practice, the rate requirement r_k of a given user is usually not related to the position x_k of the user in each cell.

Assumption 2. *Measure ν^c is such that $d\nu^c(r, x) = d\zeta^c(r) \times d\lambda^c(x)$ where ζ^c is the limit distribution of rates and λ^c is the limit distribution of the users' locations. Here $\times$ denotes the product of measures.*

Measures ζ and λ respectively correspond to the distributions of the rates and the positions of the users within one cell. For instance, the value $\bar{r}^c = \frac{t}{2} \int_0^{r_{\max}} r \, d\zeta^c(r)$ represents the average rate requirement per channel use in Cell c. We furthermore assume that measures λ^A and λ^B are absolutely continuous with respect to the Lebesgue measure on $[\epsilon, D]$. Using Assumption 2, equation (15) becomes

$$Q_1^c = \bar{r}^c \int_{\epsilon}^{d^c} \mathcal{F}(x, \beta_1^c, Q_1^{\bar{c}}, \xi^c) \, d\lambda^c(x). \tag{16}$$

Of course, a similar result can be obtained for $Q_2^{c,(K)}$ *i.e.,* the power transmitted by base station c in the protected band $\mathcal{P}^c$. To that end, we simply note that function $g_2(x)$ satisfies $g_2(x) = g_1(x, 0)$. Using similar tools, the expression of $Q_2^{c,(K)}$ given by (24) converges as $K \to \infty$ toward

$$Q_2^c = \bar{r}^c \int_{d^c}^{D} \mathcal{F}(x, \beta_2^c, 0, 0) \, d\lambda^c(x). \tag{17}$$

Equations (16) and (17) respectively provide the limits of $Q_1^{c,(K)}$ and $Q_2^{c,(K)}$ as a function of some parameters $d^c, \beta_1^c, \beta_2^c$ and $Q_1^{\bar{c}}$ (assumed for the moment to be the limits of $d^{c,(K)}, \beta_1^{c,(K)}, \beta_2^{c,(K)}$ and $Q_1^{\bar{c},(K)}$ as long as such limits exist). These unknown parameters still need to be characterized. Therefore, we must determine a system of equations which is satisfied by these parameters. This task is done by Theorem 1 given below.

B. Asymptotic Performance of the Optimal Resource Allocation

Define the following function $\mathcal{G}(x,\beta,\mathcal{Q},\xi) = \frac{1}{C\left(\frac{g_1(x,\mathcal{Q})}{1+\xi}\beta\right)}$ for each $x,\beta,\mathcal{Q},\xi$. The proof of the following result is provided in Appendix A.

Theorem 1. *Assume that $K = K^A + K^B \to \infty$ in such a way that $K/B \to t > 0$ and $K^A/K \to 1/2$. Assume that the optimal solution for the joint resource allocation problem (Problem 2 in [1]) is used for each K. The total power spent by the network $Q_T^{(K)} = \sum_{c=A,B} \sum_{k=1}^{K^c} (\gamma_{k,1}^c P_{k,1}^c + \gamma_{k,2}^c P_{k,2}^c)$ converges to a constant Q_T. The limit Q_T has the following form:*

$$Q_T = \sum_{c=A,B} \bar{r}^c \left(\int_\epsilon^{d^c} \mathcal{F}(x,\beta_1^c,Q_1^{\bar{c}},\xi^c)\, d\lambda^c(x) + \int_{d^c}^{D} \mathcal{F}(x,\beta_2^c,0,0)\, d\lambda^c(x) \right) , \qquad (18)$$

where for each $c = A, B$, the following system of equations in variables $d^c, \beta_1^c, \beta_2^c, \xi^c$ is satisfied:

$$\bar{r}^c \int_\epsilon^{d^c} \mathcal{G}(x,\beta_1^c,Q_1^{\bar{c}},\xi^c)\, d\lambda^c(x) = \alpha \qquad (19)$$

$$\bar{r}^c \int_{d^c}^{D} \mathcal{G}(x,\beta_2^c,0,0)\, d\lambda^c(x) = \frac{1-\alpha}{2} \qquad (20)$$

$$\frac{g_1(d^c,Q_1^{\bar{c}})}{1+\xi^c} F\left(\frac{g_1(d^c,Q_1^{\bar{c}})}{1+\xi^c}\beta_1^c\right) = g_2(d^c) F(g_2(d^c)\beta_2^c) \qquad (21)$$

$$\bar{r}^c \int_\epsilon^{d^c} \mathcal{F}(x,\beta_1^c,Q_1^{\bar{c}},\xi^c)\, d\lambda^c(x) = Q_1^c \, . \qquad (22)$$

Moreover, for each $c = A, B$ and for any arbitrary fixed value $(\tilde{Q}_1^A, \tilde{Q}_1^B)$, the system of equations (19)-(20)-(21)-(22) admits at most one solution $(d^c, \beta_1^c, \beta_2^c, \xi^c)$.

As a consequence, when optimal multicell resource allocation is used, the total power spent by the network converges to a constant which can be evaluated through the results of Theorem 1. This result allows to evaluate the asymptotic power spent by the network as a function of the reuse factor α, the average rate requirement $\bar{r}$ and the asymptotic distribution of users in each cell λ.

Now that the asymptotic performance of the optimal allocation scheme has been studied, the value of the fixed pivot-distances $d_{\text{subopt}}^A, d_{\text{subopt}}^B$ associated with the simplified allocation algorithm can be relevantly chosen to be equal in each Cell c to the asymptotic pivot distance d^c defined by Theorem 1.

C. Determination of the fixed pivot-distances $d^A_{subopt}, d^B_{subopt}$ for the simplified allocation scheme

We stated earlier in Section III that the suboptimal algorithm replaces the optimal value $d^{c,(K)}$ of the pivot-distance in each Cell c with a fixed value d^c_{subopt}. Intuitively, if d^A_{subopt} and d^B_{subopt} are chosen such that $d^{A,(K)} \simeq d^A_{\text{subopt}}$ and $d^{B,(K)} \simeq d^B_{\text{subopt}}$ for large K, the performance of our algorithm shall be close to the optimal one as K increases. Therefore, we must determine an asymptotically optimal pair of pivot-distances (d^A, d^B). To that end we propose the following procedure.

Note first by referring to Theorem 1 that the value of d^A, d^B can be easily determined once the relevant values of Q_1^A and Q_1^B have been determined. The remaining task is thus the determination of the value of (Q_1^A, Q_1^B). To that end, we propose to perform an exhaustive search on (Q_1^A, Q_1^B).

i) For each point $(\tilde{Q}_1^A, \tilde{Q}_1^B)$ on a certain 2D search grid, solve the system (19)-(20)-(21)-(22) introduced by Theorem 1 for both $c = A, B$. Theorem 1 states that this system admits at most one solution for any arbitrary fixed value $(\tilde{Q}_1^A, \tilde{Q}_1^B)$. If the investigated point $(\tilde{Q}_1^A, \tilde{Q}_1^B)$ of the grid is such that the system (19)-(20)-(21)-(22) does admit a solution, we can obtain this solution denoted by $d^c(\tilde{Q}_1^A, \tilde{Q}_1^B)$, $\beta_1^c(\tilde{Q}_1^A, \tilde{Q}_1^B)$, $\beta_2^c(\tilde{Q}_1^A, \tilde{Q}_1^B)$, $\xi^c(\tilde{Q}_1^A, \tilde{Q}_1^B)$ thanks to a simple procedure inspired by the *single-cell* procedure proposed in Part I [1] for finite number of users:

- Solve the system (19)-(20)-(21)-(22$'$) formed by replacing the equality in equation (22) of system (19)-(20)-(21)-(22) by the following inequality

$$\bar{r}^c \int_{\epsilon}^{d^c} \mathcal{F}(x, \beta_1^c, Q_1^{\bar{c}}, \xi^c) d\lambda^c(x) \leq \tilde{Q}_1^c \,. \tag{22$'$}$$

  The existence and the uniqueness of the solution to this new system for an arbitrary $(\tilde{Q}_1^A, \tilde{Q}_1^B) \in \mathbb{R}_+^2$ can be proved by extending, to the case of infinite number of users, Proposition 1 which was provided in [1] for the case of finite number of users.
- If the resulting power $\bar{r}^c \int_{\epsilon}^{d^c} \mathcal{F}(x, \beta_1^c, Q_1^{\bar{c}}, \xi^c) d\lambda^c(x)$ transmitted in the interference band $\mathcal{P}_c$ is equal to $\tilde{Q}_1^c$, then the resulting value of $d^c(\tilde{Q}_1^A, \tilde{Q}_1^B)$ coincides with the unique solution to system (19)-(20)-(21)-(22) . Once again, this claim can be proved by extending Proposition 1 of [1] to the case of infinite number of users.
- If the power $\bar{r}^c \int_{\epsilon}^{d^c} \mathcal{F}(x, \beta_1^c, Q_1^{\bar{c}}, \xi^c) d\lambda^c(x)$ is less than $\tilde{Q}_1^c$, then $d^c(\tilde{Q}_1^A, \tilde{Q}_1^B)$ is clearly not a solution to system (19)-(20)-(21)-(22) , as equality (22) does not hold. In this case, it can be easily shown that system (19)-(20)-(21)-(22) has no solution. The point $(\tilde{Q}_1^A, \tilde{Q}_1^B)$ is thus eliminated.

ii) Compute the total power

$$Q_T(\tilde{Q}_1^A, \tilde{Q}_1^B) = \sum_{c=A,B} \sum_k \gamma_{k,1}^c P_{k,1}^c + \gamma_{k,2}^c P_{k,2}^c$$

that would be transmitted if the values of Q_1^A and Q_1^B introduced by Theorem 1 were respectively equal to $\tilde{Q}_1^B$ and $\tilde{Q}_1^A$.

iii) The final value of d^A, d^B is given by $d^A(Q_1^A, Q_1^B)$, $d^B(Q_1^A, Q_1^B)$, the value associated with (Q_1^A, Q_1^B) the argument of the minimum power transmitted by the network:

$$(Q_1^A, Q_1^B) = \arg \min_{(\tilde{Q}_1^A, \tilde{Q}_1^B)} Q_T(\tilde{Q}_1^A, \tilde{Q}_1^B) \ .$$

iv) Finally, we choose

$$d^A_{\text{subopt}} = d^A \text{ and } d^B_{\text{subopt}} = d^B.$$

Note that the same procedure provides as a byproduct the limit Q_T of the total transmit power as $Q_T = Q_T(Q_1^A, Q_1^B)$.

Comments

It is clear from our previous discussion that the above procedure for computing (d^A, d^B) can be done in advance prior to resource allocation. This is essentially due to the fact that the asymptotically optimal pair of pivot-distances (d^A, d^B) does not depend on the particular cell configuration, but on an asymptotic or "average" state of the network. The procedure can be run for instance before base stations are brought into operation. It can also be done once in a while as the asymptotic distribution of the users and the average rate requirement $\bar{r}$ can be subject to changes: but these changes occur after long periods of time. Therefore, the number of operations needed for the computation of (d^A, d^B) is not a major concern because it does not affect the computational complexity of resource allocation.

D. Asymptotic Performance of the Simplified Algorithm

Denote by $Q_{subopt}^{(K)}$ the total power transmitted when our simplified allocation algorithm is applied. Recall that $Q_T^{(K)}$ designates the total power transmitted by the network when the optimal resource allocation associated with the joint resource allocation problem (Problem 2 of [1]) is used.

Proposition 2. *The following equality holds:*

$$\lim_{K\to\infty} Q_{subopt}^{(K)} = \lim_{K\to\infty} Q_T^{(K)} \ .$$

Proposition 2 can be proved using the same arguments as the ones used in Appendix A. The detailed proof is omitted. The above Proposition states that the proposed suboptimal algorithm tends to be optimal w.r.t. the joint resource allocation problem, as the number of users increases. Therefore, our algorithm is at the same time much simpler than the initial optimal resource allocation algorithm of [1], and has similar performance at least for a sufficient number of users in each cell. Section VII will furthermore indicate that even for a moderate number of users, our suboptimal algorithm is actually nearly optimal.

V. On the Convergence of the Simplified Allocation Algorithm

As stated before, the simplified algorithm performs the resource allocation in each Cell c independently for the protected $\mathcal{K}_P^c$ and the non protected $\mathcal{K}_I^c$ users, which are separated by the predefined pivot-distance d_{subopt}^c. Resource allocation for the non protected users is done by the iterative and distributed ping-pong algorithm described in Section III. It was stated in Section III that the convergence of the ping-pong algorithm is ensured by the feasibility of the the problem of resource allocation for the non protected users $\{\mathcal{K}_I^A, \mathcal{K}_I^B\}$ (Problem 1). If Problem 1 is feasible, the ping-pong algorithm converges. If Problem 1 is infeasible, the ping-pong algorithm diverges. It was also stated in Section III that Problem 1 may not be feasible if arbitrary values of the pivot-distances d_{subopt}^A and d_{subopt}^B are used. Fortunately, feasibility of the latter problem will not be an issue if the value of d_{subopt}^A and d_{subopt}^B are relevantly chosen as described by the procedure introduced in Section IV-D. Indeed, it can be shown in this case that at least for large K, the set $\mathcal{K}_I^c$ will contain the users who would anyway have been restricted to the interference band $\mathcal{I}$ if the optimal resource allocation of Part I [1] was used. More precisely, it can be shown that there exists a value K_0 of K beyond which Problem 1 is always feasible. The proof of this statement is omitted from this paper but is provided in [13] and is based on sensitivity analysis of perturbed optimization problems [14]. It is worth mentioning that in our simulations, Problem 1 was feasible in almost all the settings of the system, even for a moderate number of users per cell as small as 25.

VI. Selection of the Best Reuse Factor

The selection of a relevant value α allowing to optimize the network performance is of crucial importance as far as cellular network design is concerned. The definition of an *optimal* reuse factor requires however some care. The first intuition would consist in searching for the value of α which minimizes the total power $Q_T^{(K)} = Q_T^{(K)}(\alpha)$ transmitted by the network, for a finite number of users K. However, $Q_T^{(K)}(\alpha)$ depends on the particular target rates and the particular positions of users. In practice, the reuse factor should be fixed prior to the resource allocation process and its value should be independent of the particular cells configurations. A solution adopted by several works in the literature consists in performing system level simulations and choosing the corresponding value of α that results in the best average performance. In this context, we cite [8], [9] and [10] without being exclusive. In this paper, we are interested in providing analytical methods that permit to choose a relevant value of the reuse factor. This is why we propose to select the value α_{opt} of the reuse factor as

$$\alpha_{\text{opt}} = \arg\min_{\alpha} \lim_{K\to\infty} Q_T^{(K)}(\alpha) \; .$$

Recall that the limiting power $Q_T = \lim_{K\to\infty} Q_T^{(K)}$ is given by equation (18). In practice, we propose to compute the value of $Q_T = Q_T(\alpha)$ for several values of α on a grid in the interval $[0, 1]$. For each value of α on the grid, $Q_T(\alpha)$ can be obtained using the procedure presented in subsection IV-C. Note also that complexity issues are of few importance, as the optimization is done prior to the resource allocation process. It does not affect the complexity of the global resource allocation procedure. We shall see in Section VII that significant gains are obtained when using the optimized value of the reuse factor instead of an arbitrary value.

VII. Simulations

We first begin by presenting the technical parameters of the system model. In our simulations, we considered a Free Space Loss model (FSL) characterized by a path loss exponent $s = 2$ as well as the so-called Okumura-Hata (O-H) model for open areas [15] with a path loss exponent $s = 3$. The carrier frequency is $f_0 = 2.4GHz$. At this frequency, path loss in dB is given by $\rho_{dB}(x) = 20\log_{10}(x) + 100.04$ in the case where $s = 2$, where x is the distance in kilometers between the BS and the user. In the case $s = 3$, $\rho_{dB}(x) = 30\log_{10}(x) + 97.52$. The signal bandwidth B is equal to 5 MHz and the thermal noise power spectral density is equal to $N_0 = -170$ dBm/Hz. Each cell has a radius $D = 500$m.

Asymptotically optimal pivot-distance and frequency reuse factor: We first apply the results of of

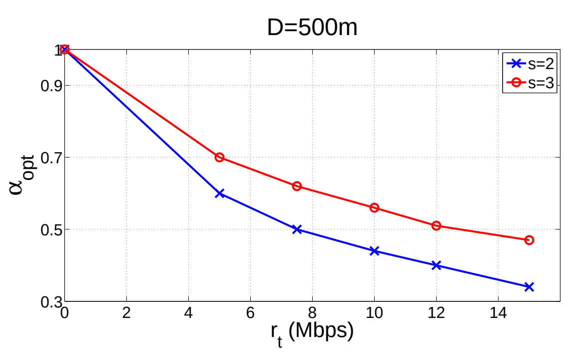

Figure 2. Optimal reuse factor vs. sum rate

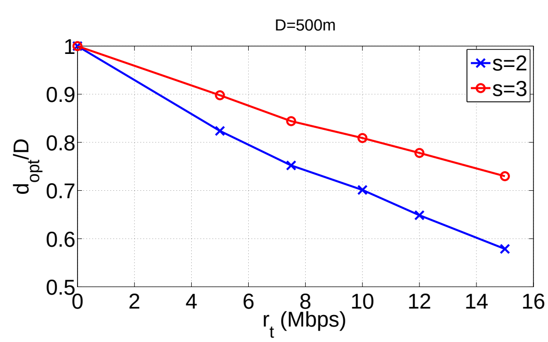

Figure 3. Optimal pivot-distance vs. sum rate

Sections IV and VI in order to obtain the values of the asymptotically optimal pivot-distances d^A, d^B

and the asymptotically optimal reuse factor α_{opt}. These values are necessary for the implementation of the simplified allocation algorithm proposed in Section III. Each of the two cells is assumed to have in the asymptotic regime the same uniform distribution of users: $\lambda^A = \lambda^B = \lambda$ where $d\lambda(x) = dx/D$. The average rate requirement in each cell is assumed to be the same, too: $\bar{r}^A = \bar{r}^B = \bar{r}$, where $\bar{r}^c$ is defined in Subsection IV-A2 as the average data rate in Cell c measured in bits/sec/Hz. In this case, the optimal pivot-distance is the same in each cell *i.e.*, $d^A = d^B$. Define $d_{\text{opt}} = d^A = d^B$. The value of d_{opt} and α_{opt} was obtained using the method depicted by Subsection IV-C and Section VI respectively. Denote by r_t the total data rate of all the users of a sector measured in bits/sec ($r_t = \bar{r} * B$). Figure 2 and Figure 3 plot respectively α_{opt} and the normalized pivot-distance d_{opt}/D as functions of the total rate r_t for two values of the path loss exponent: $s = 2$ and $s = 3$. Note from Figure 2 that α_{opt} and d_{opt} are both decreasing functions of r_t. This result is expected, given that higher values of r_t will lead to higher transmit powers and consequently to higher levels of interference. More users will need thus to be "protected" from the higher interference. For that purpose, the pivot-position must be closer to the base station and a larger part of the available bandwidth must be reserved for the protected bands $\mathcal{P}_A$ and $\mathcal{P}_B$. Note also that, in the case $s = 3$, "less protection" is needed than in the case where $s = 2$. In other words, $d_{\text{opt}}(s = 3) > d_{\text{opt}}(s = 2)$ and $\alpha_{\text{opt}}(s = 3) > \alpha_{\text{opt}}(s = 2)$. This observation can be explained by the fact that, when the path loss exponent is higher, the interference produced by the adjacent base station will undergo more fading than in the case when the path loss exponent is lower.

Simplified resource allocation: In Section III, we proposed a suboptimal allocation algorithm characterized by its reduced computational complexity compared to the optimal allocation algorithm depicted in [1]. This algorithm assumes fixed pivot-distances d^A_{subopt}, d^B_{subopt}. Here, we study the performance of this algorithm when d^A_{subopt} and d^B_{subopt} are chosen according to the procedure provided in Section IV-C *i.e.*, $d^A_{\text{subopt}} = d_{\text{opt}}$ and $d^B_{\text{subopt}} = d_{\text{opt}}$, where d_{opt} is the asymptotically optimal pivot-distance defined earlier in this section. In order to study the performance of this algorithm, we need to compare, for a large number of system settings, $Q^{(K)}_{\text{subopt}}$ the total transmit power that must be spent when applying the simplified algorithm, with $Q^{(K)}_T$ the total transmit power that must be spent when the optimal resource allocation scheme of Part I [1] is applied. The results must then be averaged in order to obtain performance measurements that are independent of the particular system setting. For that sake, we consider that in each cell users are randomly distributed and that the distance separating each user from the base station is a random variable with a uniform distribution on the interval $[0, D]$. On the other hand, we assume without restriction that all users have the same target rate, and that the number of users is the same for the two cells $K^A = K^B$. Define $\mathbf{X}$ as the vector containing the positions of all the users in the system *i.e.*, $\mathbf{X} = (x_1, x_2, \ldots, x_{K^c})_{c=A,B}$. Recall that $\forall k$, x_k is a random variable with a uniform distribution on $[0, D]$. For each realization of $\mathbf{X}$, define $Q_T^{(K)}(\mathbf{X}, \alpha)$ as the total transmit power that results from applying the optimal joint resource allocation scheme of Part I with the value of the reuse factor fixed to α. Define $Q_T^{(K)}(\mathbf{X}) = \min_\alpha Q_T^{(K)}(\mathbf{X}, \alpha)$. In the same way, denote by $Q_{\text{subopt}}^{(K)}(\mathbf{X})$ the total transmit power that results from applying the simplified resource allocation scheme of Section III with the value of the reuse factor fixed to α_{opt} defined in Section VI. For each realization of the random vector $\mathbf{X}$, the values of $Q_T^{(K)}(\mathbf{X})$ and $Q_{\text{subopt}}^{(K)}(\mathbf{X})$ were calculated and then averaged to obtain $\mathbb{E}_{\mathbf{X}}[Q_T^{(K)}(\mathbf{X})]$ and $\mathbb{E}_{\mathbf{X}}[Q_{\text{subopt}}^{(K)}(\mathbf{X})]$ respectively. In figure 4, we plot for a range of values of the sum rate r_t measured in

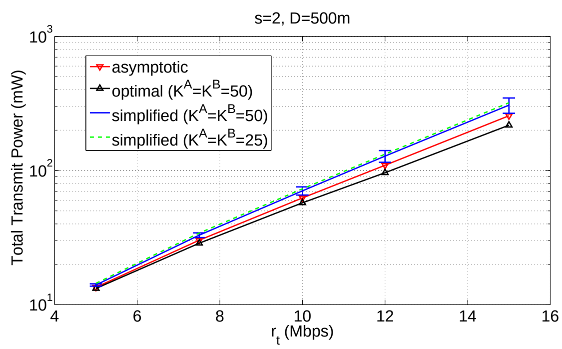

Figure 4. Optimal and suboptimal transmit power vs. sum rate

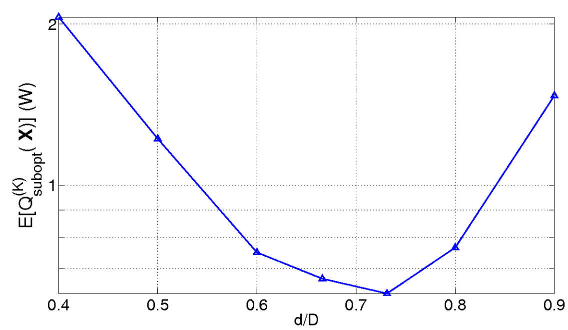

Figure 5. Transmit power vs. the pivot-distance d for the simplified allocation scheme ($r_t = 10Mbps$, $K^c = 50$)

bits/sec ($r_t = \sum_{k=1}^{K^c} R_k B$) the values of $\mathbb{E}_{\mathbf{X}}[Q_T^{(K)}(\mathbf{X})]$ and $\mathbb{E}_{\mathbf{X}}[Q_{\text{subopt}}^{(K)}(\mathbf{X})]$ in two cases: $K^c = 25$ and $K^c = 50$. The error bars in the figure represents the variance of the random variable $Q_{\text{subopt}}^{(K)}(\mathbf{X})$ in the case $K^c = 50$. In the same figure, the corresponding values of the asymptotic transmit power Q_T defined by Theorem 1 are also plotted. This figure shows that, even for a reasonable number of users equal to 25 in each cell, the transmit power needed when we apply the suboptimal algorithm is very close to the power needed when we apply the optimal resource allocation scheme. The gap between the two powers is of course even smaller for $K^c = 50$. This result validates Proposition 2 which states that our proposed suboptimal resource allocation scheme is asymptotically optimal. Figure 5 is dedicated to illustrate the sensitivity of the simplified allocation scheme with respect to the pivot-distance d_{subopt} in the case $K^c = 50$. For that sake, the figure plots the total transmit power resulting from applying the simplified scheme as a function of d_{subopt}. The minimum in the figure corresponds to the asymptotically optimal pivot distance $d_{\text{subopt}} = d_{\text{opt}}$. We note that using values different from d_{opt} increases the suboptimality of the simplified scheme. Let us go back to Figure 4. The latter figure shows that over the range of the considered values of the total data rate r_t, the total transmit power $\mathbb{E}_{\mathbf{X}}[Q_T^{(K)}(\mathbf{X})]$ for $K^c = 50$ is practically equal to the asymptotic power Q_T. This result suggests that, for a number of users equal to 50 in each cell, the system is already in its asymptotic regime. In order to validate the latter affirmation, one still needs to investigate the value of the mean square error $(Q_T^{(K)} - Q_T)^2$ as well. This is done by Figure 6 which plots $\frac{\mathbb{E}_{\mathbf{X}}(Q_T^{(K)}(\mathbf{X})-Q_T)^2}{Q_T^2}$, the mean square error normalized by Q_T^2.

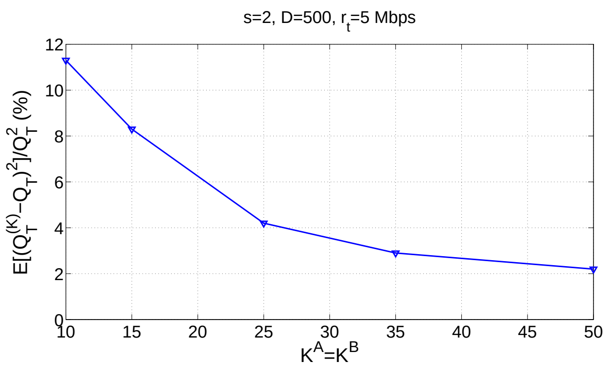

Figure 6. $\frac{\mathbb{E}_{\mathbf{X}}(Q_T^{(K)}(\mathbf{X})-Q_T)^2}{Q_T^2}$ vs. number of users per cell

VIII. Conclusions

In this pair of papers, the resource allocation problem for sectorized downlink OFDMA systems has been studied in the context of a partial reuse factor $\alpha \in [0, 1]$. In the first part of this work, the general solution to the (nonconvex) optimization problem has been provided. It has been proved that the solution admits a simple form and that the initial tedious problem reduces to the identification of a limited number of parameters. As a noticeable property, it has been proved that the optimal resource allocation policy is "binary": there exists a pivot-distance to the BS such that users who are farther than this distance should only modulate protected subcarriers, while closest users should only modulate reused subcarriers. A resource allocation algorithm has been also proposed.

In the second part, we proposed a suboptimal resource allocation algorithm which avoids the costly search for parameters such as the optimal pivot-distance. In the proposed procedure, the optimal pivot-distance is simply replaced by a fixed value. In order to provide a method to relevantly select this fixed pivot-distance, the asymptotic behavior of the optimal resource allocation has been studied as the number of users tends to infinity. In the case where the fixed pivot-distance associated with the simplified algorithm is chosen to be equal to the asymptotically optimal pivot-distance, it has been shown that our simplified resource allocation algorithm is asymptotically equivalent to the optimal one as the number of users increases. Simulations proved the relevancy of our algorithm even for a small number of users. Using the results of the asymptotic study, the optimal value of the reuse factor has been characterized. It is defined as the value of α which minimizes the asymptotic value of the minimum transmit power. Our simulations proved that substantial improvements in terms of spectral efficiency can be expected when using the relevant value of the reuse factor.

APPENDIX A
PROOF OF THEOREM 1

Theorem 1 characterizes the asymptotic behaviour of the minimal transmit power resulting from applying the optimal resource allocation when the number of users K tends to infinity. It is thus useful at this point to recall the theorem given in the first part of this work which characterizes the optimal allocation for finite values of K. Define the function $F(x) = \mathbb{E}\left[\frac{Z}{1+f^{-1}(x)Z}\right]$. For each cell $c = A, B$ and for each $l = 1 \dots K^c$, define by a_l^c and b_l^c the unique positive numbers such that $\sum_{k=1}^{l} \frac{R_k}{C(g_{k,1}a_l^c)} = \alpha$ and $\sum_{k=l+1}^{K^c} \frac{R_k}{C(g_{k,2}b_l^c)} = \frac{1-\alpha}{2}$, with $a_0^c = b_{K^c}^c = 0$ by convention.

Theorem 2 ([1]).

(A) *Any global solution to the joint resource allocation problem has the following form. For each Cell c, there exists an integer $L^c \in \{1, \dots, K^c\}$, and there exist four positive numbers β_1^c, β_2^c, ξ^c, $Q_1^{\bar{c}}$ such that*

1) *For each $k < L^c$,*

$$\left.\begin{array}{l} P_{k,1}^c = g_{k,1}(Q_1^{\bar{c}})^{-1} f^{-1}\left(\dfrac{g_{k,1}(Q_1^{\bar{c}})}{1+\xi^c}\beta_1^c\right) \\ \gamma_{k,1}^c = \dfrac{R_k}{\mathbb{E}\left[\log\left(1 + g_{k,1}(Q_1^{\bar{c}})P_{k,1}^c Z\right)\right]} \end{array}\right| \begin{array}{l} P_{k,2}^c = 0 \\ \gamma_{k,2}^c = 0 \end{array} \tag{23}$$

2) *For each $k > L^c$,*

$$\begin{array}{l} P_{k,1}^c = 0 \\ \gamma_{k,1}^c = 0 \end{array}\left| \begin{array}{l} P_{k,2}^c = g_{k,2}^{-1} f^{-1}(g_{k,2}\beta_2^c) \\ \gamma_{k,2}^c = \dfrac{R_k}{\mathbb{E}\left[\log\left(1 + g_{k,2}P_{k,2}^c Z\right)\right]} \end{array}\right. \tag{24}$$

3) *For $k = L^c$*

$$\left.\begin{array}{l} P_{k,1}^c = g_{k,1}(Q_1^{\bar{c}})^{-1} f^{-1}\left(\dfrac{g_{k,1}(Q_1^{\bar{c}})}{1+\xi^c}\beta_1\right) \\ \gamma_{k,1}^c = \alpha - \displaystyle\sum_{l=1}^{k-1} \gamma_{l,1}^c \end{array}\right| \begin{array}{l} P_{k,2}^c = g_{k,2}^{-1} f^{-1}(g_{k,2}\beta_2^c) \\ \gamma_{k,2}^c = \dfrac{1-\alpha}{2} - \displaystyle\sum_{l=k+1}^{K^c} \gamma_{l,2}^c. \end{array} \tag{25}$$

(B) *For each $c = A, B$, the system $\mathcal{S}^c(Q_1^A, Q_1^B)$ formed by the following four equations is satisfied.*

$$L^c = \min\left\{ l = 1 \dots K^c / \frac{g_{l,1}(Q_1^{\bar{c}})}{1+\xi^c} F\left(\frac{g_{l,1}(Q_1^{\bar{c}})}{1+\xi^c} a_l\right) \leq g_{l,2} F\left(g_{l,2} b_l\right) \right\} \tag{26}$$

$$\frac{g_{L^c,1}(Q_1^{\bar{c}})}{1+\xi^c} F\left(\frac{g_{L^c,1}(Q_1^{\bar{c}})}{1+\xi^c}\beta_1^c\right) = g_{L^c,2} F(g_{L^c,2}\beta_2^c) \tag{27}$$

$$\gamma_{L^c,1}^c C\left(\frac{g_{L^c,1}(Q_1^{\bar{c}})}{1+\xi^c}\beta_1^c\right) + \gamma_{L^c,2}^c C(g_{L,2}\beta_2^c) = R_{L^c} \tag{28}$$

$$\sum_k^{L^c} \gamma_{k,1}^c P_{k,1}^c = Q_1^c\,, \tag{29}$$

where the values of $\gamma_{k,1}^c$ and $P_{k,1}^c$ in (29) are the functions of $(\beta_1^c, \beta_2^c, \xi^c)$ defined by equation (23).

(C) *Furthermore, for each $c = A, B$ and for any arbitrary values $\tilde{Q}_1^A$ and $\tilde{Q}_1^B$, the system of equations $\mathcal{S}^c(\tilde{Q}_1^A, \tilde{Q}_1^B)$ admits at most one solution $(L^c, \beta_1^c, \beta_2^c, \xi^c)$.*

In subsection IV-A2, we obtained that for each cell $c = A, B$,

$$Q_1^{c,(K)} = \frac{K^c}{B}\iint_{\Delta_1^{c,(K)}} r\mathcal{F}(x, \beta_1^{c,(K)}, Q_1^{\bar{c},(K)}, \xi^{c,(K)})d\nu^{c,(K)}(r,x) + o_K(1) \tag{30}$$

$$Q_2^{c,(K)} = \frac{K^c}{B}\iint_{\Delta_2^{c,(K)}} r\mathcal{F}(x, \beta_2^{c,(K)}, 0, 0)d\nu^{c,(K)}(r,x) + o_K(1) \tag{31}$$

where $\Delta_1^{c,(K)} = [0,\rho] \times [\epsilon, d^{c,(K)}]$ and $\Delta_2^{c,(K)} = [0,\rho] \times [d^{c,(K)}, D]$ and where $d^{c,(K)}$ is the pivot-distance *i.e.*, the position of user $L^{c,(K)}$. Our aim is to prove that $Q_T^{(K)} = \sum_c Q_1^{c,(K)} + Q_2^{c,(K)}$ converges as $K \to \infty$, and to characterize the limit. For each cell $c \in \{A, B\}$, sequence $d^{c,(K)}$ is bounded by definition ($d^{c,(K)} \in [0, D]$). Consider a subsequence ϕ_K such that $(d^{A,(\phi_K)}, d^{B,(\phi_K)})$ converges to a certain limit, say (d^A, d^B). We prove that in this case, all quantities $Q_1^{c,(\phi_K)}$, $Q_2^{c,(\phi_K)}$, $\beta_1^{c,(\phi_K)}$, $\beta_2^{c,(\phi_K)}$, $\xi^{c,(\phi_K)}$ converge to some values Q_1^c, Q_2^c, β_1^c, β_2^c, ξ^c which we shall characterize. Focus for instance on sequence $\beta_2^{c,(\phi_K)}$. Recalling that $\gamma_{L^{c,(K)},2}^c$ tends to zero as $K \to \infty$ ($\gamma_{L^{c,(K)},2}^c = o_K(1)$) and replacing each $\gamma_{k,2}^c$ with expression (23) $\gamma_{k,2}^c = R_k / C(g_{k,2}^c \beta_2^{c,(K)})$, we obtain immediately

$$\frac{1}{B}\sum_{k > L^{c,(K)}} r_k \mathcal{G}(x, \beta_2^{c,(K)}, 0, 0) + o_K(1) = \frac{1-\alpha}{2}\,, \tag{32}$$

where we defined

$$\mathcal{G}(x, \beta, \mathcal{Q}, \xi) = \frac{1}{C\left(\frac{g_1(x,\mathcal{Q})}{1+\xi}\beta\right)} \tag{33}$$

for each $x, \beta, \mathcal{Q}, \xi$. In the asymptotic regime, we obtain the following lemma.

Lemma 2. *As $K \to \infty$, sequence $\beta_2^{c,(\phi_K)}$ converges to the unique solution β_2^c to the following equation:*

$$\frac{t}{2}\int_0^\rho \int_{d^c}^D r\mathcal{G}(x, \beta_2^c, 0, 0)d\nu^c(r,x) = \frac{1-\alpha}{2}\,. \tag{34}$$

Proof: Existence and uniqueness of the solution to (34) is straightforward since function $\beta \mapsto \mathcal{G}(x,\beta,\mathcal{Q},\xi)$ is strictly decreasing from ∞ to 0 on $\mathbb{R}_+$. We remark that sequence $\beta_2^{c,(\phi_K)}$ is bounded *i.e.,* $\beta_2^{c,(\phi_K)} \leq \kappa$ for a certain constant κ. In order to prove this claim, assume that there exists a subsequence $\beta_2^{c,\phi_{\zeta(K)}}$ which converges to infinity. This hypothesis implies that the subsequence given by the lhs of (32) for K of the form $K=\zeta(K')$ converges to zero as $K'\to\infty$. This is in contradiction with (32) which states that the latter sequence converges to $\frac{1-\alpha}{2}$. Using similar arguments, it can be shown that $\beta_2^{c,(\phi_K)}$ is lower bounded by a certain $\epsilon'>0$ *i.e.,* $\epsilon' < \beta_2^{c,(\phi_K)} < \kappa$. Denote by β_2 any accumulation point of $\beta_2^{c,(\phi_K)}$ and define $\beta_2^{c,(\theta_K)}$ a subsequence of $\beta_2^{c,(\phi_K)}$ (*i.e.,* θ_K coincides with $\phi_{\zeta(K)}$ for a certain function ζ) which converges to β_2. We prove that β_2 is given by (34). Define $G(r,x,y)=r\mathcal{G}(x,y,0,0)$. We show that the difference

$$\Delta_K = \left|\int_0^\rho\int_{d^{c,(\theta_K)}}^D G(r,x,\beta_2^{c,(\theta_K)})d\nu^{c,(\theta_K)}(r,x) - \int_0^\rho\int_{d^c}^D G(r,x,\beta_2^{c,(\theta_K)})d\nu^c(r,x)\right|$$

tends to zero as $K\to\infty$. By the triangular inequality,

$$\begin{aligned}\Delta_K \leq & \left|\int_0^\rho\int_{d^{c,(\theta_K)}}^D G(r,x,\beta_2^{c,(\theta_K)})d\nu^{c,(\theta_K)}(r,x) - \int_0^\rho\int_{d^c}^D G(r,x,\beta_2^{c,(\theta_K)})d\nu^{c,(\theta_K)}(r,x)\right| \\ & + \left|\int_0^\rho\int_{d^c}^D G(r,x,\beta_2)d\nu^{c,(\theta_K)}(r,x) - \int_0^\rho\int_{d^c}^D G(r,x,\beta_2^{c,(\theta_K)})d\nu^c(r,x)\right| \\ & + \int_0^\rho\int_{d^c}^D \left|G(r,x,\beta_2^{c,(\theta_K)}) - G(r,x,\beta_2)\right| d\nu^{c,(\theta_K)}(u,x)\,.\end{aligned}$$

Respectively denote by $\Delta_{K,1}$, $\Delta_{K,2}$, $\Delta_{K,3}$ the first, second and third terms of the above equation. We first study $\Delta_{K,1}$. Clearly, function $G(u,x,\beta)$ is bounded on $[0,\rho]\times[\epsilon,D]\times[\epsilon',\kappa]$. Denote by ξ an upper bound. Then, $\Delta_{K,1}\leq \xi\nu_c^{\theta_K}(I_K)$, where $I_K=[0,\rho]\times[d^{c,(\theta_K)},d^c]$ (or $I_K=[0,\rho]\times[d^c,d^{c,(\theta_K)}]$ if $d^c<d^{c,(\theta_K)}$). Recall that $d^{c,(\theta_K)}$ converges to d^c by definition, so that $\nu^c(I_K)=\zeta^c([0,\rho])\lambda^c([d^{c,(\theta_K)},d^c])$ converges to zero as long as measure λ^c has no mass point at d^c. Since $\nu^{c,(\theta_K)}$ converges weakly to ν^c, it is straightforward to show that $\nu^{c,(\theta_K)}(I_K)$, and thus $\Delta_{K,1}$, tend to zero. Now focus on $\Delta_{K,2}$. The first term $\int\int G(r,x,\beta_2)d\nu^{c,(\theta_K)}(r,x)$ which composes $\Delta_{K,2}$ converges to $\int\int G(r,x,\beta_2)d\nu^c(r,x)$ by the weak convergence of $\nu^{c,(\theta_K)}$ to ν^c. The second term $\int\int G(r,x,\beta_2^{c,(\theta_K)})d\nu^c(r,x)$ converges to the same limit by Lebesgue's dominated convergence Theorem. Thus, $\Delta_{K,2}$ tends to zero. In order to prove that $\Delta_{K,3}$ tends to zero, we remark that $\sup\left|\frac{\partial G(x,r,\beta)}{\partial\beta}\right|<\infty$, where the supremum is taken w.r.t. $(x,r,\beta)\in[0,\rho]\times[\epsilon,D]\times[\epsilon',\kappa]$. Denote by C the latter supremum. We easily obtain $\left|G(r,x,\beta_2^{c,(\theta_K)})-G(r,x,\beta_2)\right|\leq C\left|\beta_2^{c,(\theta_K)}-\beta_2\right|$, so that $\Delta_{K,3}\leq C\left|\beta_2^{c,(\theta_K)}-\beta_2\right|\nu_c^{\theta_K}([0,\rho]\times[d^c,D])$. Since $\nu_c^{\theta_K}$ is a probability measure, $\Delta_{K,3}\leq C\left|\beta_2^{c,(\theta_K)}-\beta_2\right|$. Thus $\Delta_{K,3}$ tends to zero as K tends to infinity. Putting all pieces together, Δ_K tends to zero. Using (32), $\frac{t}{2}\int_0^\rho\int_{d^c}^D G(r,x,\beta_2^{c,(\theta_K)})d\nu^c(r,x)$ converges to $\frac{1-\alpha}{2}$. By continuity arguments,

$\beta_2 = \lim_K \beta_2^{c,(\theta_K)}$ satisfies (34). Thus $\beta_2^{c,(\phi_K)}$ is a bounded sequence such that any accumulation point is equal to β_2 defined by (34). Thus $\lim_K \beta_2^{c,(\phi_K)} = \beta_2$. ■

Using Lemma 2, we may now characterize the limit of (31) as $K \to \infty$. Using the fact that $\lim_K \beta_2^{c,(\phi_K)} = \beta_2^c$ and $\lim_K d^{c,(\phi_K)} = d^c$ along with some technical arguments which are similar to the ones used in the proof of Lemma 2, we obtain

$$Q_2^{c,(\phi_K)} = \frac{K^c}{B} \int_0^\rho \int_{d^c}^D r\mathcal{F}(x, \beta_2^c, 0, 0) d\nu^{c,(\phi_K)}(r, x) + o_K(1) \tag{35}$$

where β_2^c is the unique solution to (34). As $\nu^{c,(\phi_K)}$ converges weakly to ν^c, $Q_2^{c,(\phi_K)}$ converges to

$$Q_2^c = \frac{t}{2} \int_0^\rho \int_{d^c}^D r\mathcal{F}(x, \beta_2, 0, 0) d\nu^c(r, x) \ . \tag{36}$$

The same approach can be used to analyze the behavior of sequences $Q_1^{c,(\phi_K)}$ and $\beta_1^{c,(\phi_K)}$ for each $c = A, B$. After similar derivations, we obtain the following result. As $K \to \infty$, sequence ($\beta_1^{A,(\phi_K)}$, $Q_1^{A,(\phi_K)}$, $\xi^{A,(\phi_K)}$, $\beta_1^{B,(\phi_K)}$, $Q_1^{B,(\phi_K)}$, $\xi^{B,(\phi_K)}$) converges to the *unique* solution $(\beta_1^A, Q_1^A, \xi^A, \beta_1^B, Q_1^B, \xi^B)$ to the following system of six equations:

$$\left.\begin{array}{l} Q_1^c = \dfrac{t}{2} \displaystyle\int_0^\rho \int_\epsilon^{d^c} r\mathcal{F}(x, \beta_1^c, Q_1^{\bar{c}}, \xi^c) \, d\nu^c(r, x) \\ \dfrac{t}{2} \displaystyle\int_0^\rho \int_\epsilon^{d^c} r\mathcal{G}(x, \beta_1^c, Q_1^{\bar{c}}, \xi^c) \nu^c(r, x) = \alpha \\ \dfrac{g_1(d^c, Q_1^{\bar{c}})}{1+\xi^c} F\left(\dfrac{g_1(d^c, Q_1^{\bar{c}})}{1+\xi^c} \beta_1^c\right) = g_2(d^c) F(g_2(d^c)\beta_2^c) \end{array}\right\} \quad c = A, B \ , \tag{37}$$

where β_2^c and d^c are the limits of $\beta_2^{c,(\phi_K)}$ and $d^{c,(\phi_K)}$ respectively. We discuss now the existence and the uniqueness of the solution to the above system of equation. For that sake, recall the definition of functions $\mathcal{F}$ and $\mathcal{G}$ given by (12) and (33) respectively. Note that $\mathcal{F}(x, \beta, \mathcal{Q}, \xi) = \mathcal{F}(x, \frac{\beta}{1+\xi}, \mathcal{Q}, 0)$, and that $\mathcal{G}(x, \beta, \mathcal{Q}, \xi) = \mathcal{G}(x, \frac{\beta}{1+\xi}, \mathcal{Q}, 0)$. Define $\tilde{\beta}_1^c = \frac{\beta_1^c}{1+\xi^c}$ for $c \in \{A, B\}$. By applying this new notation, The first two equations of system (37) give place to the following system of four equations:

$$\left.\begin{array}{l} Q_1^c = \dfrac{t}{2} \displaystyle\int_0^\rho \int_\epsilon^{d^c} r\mathcal{F}(x, \tilde{\beta}_1^c, Q_1^{\bar{c}}, 0) \, d\nu^c(r, x) \\ \dfrac{t}{2} \displaystyle\int_0^\rho \int_\epsilon^{d^c} r\mathcal{G}(x, \tilde{\beta}_1^c, Q_1^{\bar{c}}, 0) \nu^c(r, x) = \alpha \end{array}\right\} \quad c = A, B \ . \tag{38}$$

The existence and the uniqueness of the solution $(\tilde{\beta}_1^c, Q_1^c)_{c=A,B}$ to the system (38) was thoroughly studied in [2]. Applying the results of [2] in our context, we conclude that $(\frac{\beta_1^c}{1+\xi^c}, Q_1^c)_{c=A,B} = (\tilde{\beta}_1^c, Q_1^c)_{c=A,B}$ is unique. We turn now back to the third equation of system (37) to get the following equality $\xi^c = \frac{g_1(d^c, Q_1^{\bar{c}}) F\left(g_1(d^c, Q_1^{\bar{c}}) \frac{\beta_1^c}{1+\xi^c}\right)}{g_2(d^c) F(g_2(d^c)\beta_2^c)} - 1$. The latter equation proves the uniqueness of ξ^c for $c = A, B$. The uniqueness of β_1^c follows directly from the same equation.

So far, we have proved the uniqueness of the solution to the system (37) of equation. As for the convergence of sequences ($\beta_1^{A,(\phi_K)}$, $Q_1^{A,(\phi_K)}$, $\xi^{A,(\phi_K)}$, $\beta_1^{B,(\phi_K)}$, $Q_1^{B,(\phi_K)}$, $\xi^{B,(\phi_K)}$) to this unique solution, its proof is omitted here due to the lack of space, but follows the same ideas as the proof of convergence of ($\beta_2^{A,(\phi_K)}$, $Q_2^{A,(\phi_K)}$) and ($\beta_2^{B,(\phi_K)}$, $Q_2^{B,(\phi_K)}$) provided above.

So far, me managed to prove that for any convergent subsequence $(d^{A,(\phi_K)}, d^{B,(\phi_K)}) \to (d^A, d^B)$, the set of parameters ($Q_1^{c,(\phi_K)}$, $Q_2^{c,(\phi_K)}$, $\beta_1^{c,(\phi_K)}$, $\beta_2^{c,(\phi_K)}$, $\xi^{c,(\phi_K)})_{c=A,B}$ converges to some values Q_1^c, Q_2^c, β_1^c, β_2^c, ξ^c which are completely characterized by the system of equations (34), (36) and (37), as functions (d^A, d^B). Using decomposition $\nu^c = \zeta^c \times \lambda^c$, the system formed by equations (34), (36) and (37) is equivalent to the system (19)-(20)-(21)-(22) provided in Theorem 1. At this point, we thus proved that at least for some subsequences ϕ_K defined as above, the subsequence $Q_T^{c,(\phi_K)}$ converges to a limit which has the form given by Theorem 1. The remaining task is to prove that $Q_T^{(K)}$ is a convergent sequence.

First, note that $Q_T^{(K)}$ is a bounded sequence. Indeed, $Q_T^{(K)}$ is defined as the minimum power that can be transmitted by the network to satisfy the rate requirements. By definition, $Q_T^{(K)}$ is thus less than the power obtained when using the naive solution which consists in forcing each base station to transmit only in the protected band ($\gamma_{k,1}^c$ is forced to zero for each user k of each cell c). Now it can easily be shown that when $K \to \infty$, the power associated with this naive solution converges to a constant. As a consequence, one can determine an upperbound on $Q_T^{(K)}$ which does not depend on K.

Second, assume for instance that Q_T and Q'_T are two accumulation points of sequence $Q_T^{(K)}$. By contradiction, assume that $Q_T < Q'_T$. Extract for instance a certain subsequence of $Q_T^{(K)}$ which converges to Q_T. Inside this subsequence, one can further extract a subsequence, say θ_K, such that

$$Q_T^{(\theta_K)} \longrightarrow Q_T, \qquad d^{c,(\theta_K)} \longrightarrow d^c, \;\; \forall c = A, B$$

where d^A and d^B are some constants both (just use the fact that $d^{c,(K)}$ is bounded for each c). Clearly, Q_T can be written as in (18), where parameters β_1^c, β_2^c, d^c, Q_1^c, ξ^c satisfy the system of equations (19)-(20)-(21)-(22) . We now consider the following *suboptimal* resource allocation policy for finite numbers of users K^A and K^B. In each cell $c \in \{A, B\}$, users k whose distance x_k to their BS is less than d^c are forced to modulate in the interference band $\mathcal{I}$ only, while users k which are farther than d^c are forced to modulate in the protected band $\mathcal{P}_c$ only. In other words, for each user k in cell c, we impose

$$[\mathbf{C}'] \begin{cases} x_k < d^c & \Rightarrow \gamma_{k,2}^c = P_{k,2}^c = 0 \\ x_k \geq d^c & \Rightarrow \gamma_{k,1}^c = P_{k,1}^c = 0 \end{cases} \quad \forall c = A, B\,. \tag{39}$$

Particular values of the (nonzero) resource allocation parameters $\gamma_{k,i}^c, P_{k,i}^c$ are obtained by minimizing the classical joint multicell resource allocation problem (Problem 2 in [1]), only including the additional constraint $[\mathbf{C}']$. As a new constraint has been added, it is clear that the total power transmitted by the network, say $Q_T^{(K),*}$, is always larger than the total power $Q_T^{(K)}$ achieved by the optimal resource allocation, for any K. On the other hand, using the same asymptotic tools as previously, it can be shown after some algebra that

$$\lim_K Q_T^{(\theta_K),*} = \lim_K Q_T^{(\theta_K)} = Q_T \ .$$

In other words, this suboptimal solution performs as good as the optimal one when K has the form $K = \theta_{K'}$ for some K'. Although we omit the proof, this observation is rather intuitive. Indeed for such $K = \theta_{K'}$, the optimal values of the pivot-distances converge to the arbitrary ones d^A, d^B. Even more importantly, it can be shown that the total power $Q_T^{(K),*}$ spent when using the suboptimal procedure converges as $K \to \infty$. Therefore,

$$\lim_K Q_T^{(K),*} = Q_T \ .$$

Now consider a subsequence ψ_K such that $\lim_K Q_T^{(\psi_K)} = Q'_T > Q_T$, and compare our suboptimal allocation policy to the optimal one for the K's of the form $K = \psi_{K'}$. As $\lim_K Q_T^{(\psi_K)} > \lim_K Q_T^{(\psi_K),*}$, there exist a certain $\epsilon > 0$ and there exists a certain K_0 such that for any $K > K_0$,

$$Q_T^{(\psi_K)} > Q_T^{(\psi_K),*} + \epsilon \ .$$

The above inequality contradicts the fact that $Q_T^{(\psi_K)}$ is the global solution to the joint multicell resource allocation problem (Problem 2 in [1]). Therefore, Q'_T necessarily coincides with Q_T. This proves that $Q_T^{(K)}$ converges to Q_T. To complete the proof of Theorem 1, one still needs to prove that for any fixed value of $(Q_1^A, Q_1^B) \in \mathbb{R}_+^2$, the system formed by equations (19)-(20)-(21)-(22) admits at most one solution. The main ideas of this proof were evoked in the proof of Proposition 1 of [1]. However, the complete proof is omitted due to lack of space.

Appendix B

Proof of Lemma 1

The proof is an adaptation to the context of our system model of a proof provided in [11].

1) Positivity: the proof of this property is straightforward by definition of function $\tilde{\mathbf{I}}$.

2) Monotonicity: We focus first on Cell A. Recall that function $\tilde{I}^A(Q^B)$ was defined by means of the system formed by equations (4), (5), (6) and (7), with the value of Q_1^B in the latter equations fixed to Q^B. A careful look at these equations reveals, with the aid of the results of [1], that $\tilde{I}^A(Q^B)$ is actually the minimum value of the transmit power $\sum_{k\in\mathcal{K}_I^A} \gamma_{k,1}^A P_{k,1}^A$ resulting from the following single cell problem. The proof of the above claim can be obtained by direct application of the results of [1]. Denote by $W_{k,1}^A = \gamma_{k,1}^A P_{k,1}^A$ the average power transmitted to user k in band $\mathcal{I}$.

[Single cell problem] *"Minimize the transmit power $\sum_{k\in\mathcal{K}_I^A} W_{k,1}^A$ under the rate constraint $R_k \le C_k$ for each $k \in \mathcal{K}_I^A$ when the power transmitted by Base Station B in band $\mathcal{I}$ is fixed to Q^B".*

Denote by $\{\gamma_{k,1}^{A,*}(Q^B),\, W_{k,1}^{A,*}(Q^B)\}_k$ the global solution to the above single cell optimization. Note that $\tilde{I}^A(Q^B) = \sum_{k\in\mathcal{K}_I^A} W_{k,1}^{A,*}(Q^B)$. Note also that for each user $k \in \mathcal{K}_I^A$, the following equation holds

$$\gamma_{k,1}^{A,*}(Q^B)\mathbb{E}\left[\log\left(1 + g_{k,1}(Q^B)\frac{W_{k,1}^{A,*}(Q^B)}{\gamma_{k,1}^{A,*}(Q^B)}Z\right)\right] = R_k. \tag{40}$$

Now, consider the case when the power transmitted by Base Station B in band $\mathcal{I}$ is fixed to $Q^{B\prime} \le Q^B$. Moreover, assume that we fix the value of the sharing factor $\gamma_{k,1}^A$ of any user k to the value $\gamma_{k,1}^{A,*}(Q^B)$ which was optimal when the interference level was equal to Q^B. Denote by $W_{k,1}^A$ the value of the power that must be transmitted to user k in this case in order to satisfy his/her rate requirement *i.e.*,

$$\gamma_{k,1}^{A,*}(Q^B)\mathbb{E}\left[\log\left(1 + g_{k,1}(Q^{B\prime})\frac{W_{k,1}^{A}}{\gamma_{k,1}^{A,*}(Q^B)}Z\right)\right] = R_k. \tag{41}$$

Note that function $x \mapsto \mathbb{E}\left[\log\left(1 + xZ\right)\right]$ is increasing on $\mathbb{R}_+$. Using this property along with equations (40) and (41) and the fact that $g_{k,1}(Q^B) \le g_{k,1}(Q^{B\prime})$, we have $W_{k,1}^{A,*}(Q^B) \ge W_{k,1}^A$. Now, since $\gamma_{k,1}^{A,*}(Q^B)$ is not necessarily optimal when the interference level is equal to $Q^{B\prime}$, $\sum_k W_{k,1}^A \ge \sum_k W_{k,1}^{A,*}(Q^{B\prime})$. Consequently, $\sum_k W_{k,1}^{A,*}(Q^B) \ge \sum_k W_{k,1}^{A,*}(Q^{B\prime})$, where $\{\gamma_{k,1}^{A,*}(Q^{B\prime}),\, W_{k,1}^{A,*}(Q^{B\prime})\}_k$ is the global solution to the above single cell problem when the interference level is equal to $Q^{B\prime}$. We conclude that $\tilde{I}^A(Q^B) \ge \tilde{I}^A(Q^{B\prime})$. In the same way, one can show that $\tilde{I}^B(Q^A) \ge \tilde{I}^B(Q^{A\prime})$ when $Q^A \ge Q^{A\prime}$. Combining the latter two results for Cell A and B, we prove the monotonicity property of the vector-valued function $\tilde{\mathbf{I}}(Q^A, Q^B)$.

3) Scalability: Consider first Cell A and let $\{\gamma_{k,1}^{A,*}(Q^B),\, W_{k,1}^{A,*}(Q^B)\}_k$ be defined as above as the global solution to the single cell problem when the power transmitted by Base Station B in band $\mathcal{I}$ is fixed to Q^B. Now, consider the case when the power transmitted by Base Station B in band $\mathcal{I}$ is fixed to tQ^B, where $t > 1$. Assume also that we force the value of the sharing factor $\gamma_{k,1}^A$ of any user k to be equal to $\gamma_{k,1}^{A,*}(Q^B)$ which was optimal when the interference level was equal to Q^B. Denote by $W_{k,1}^A$ the value of the power that must be transmitted to user k in this case in order to satisfy his/her rate requirement *i.e.*,

$$\gamma_{k,1}^{A,*}(Q^B)\mathbb{E}\left[\log\left(1 + g_{k,1}(tQ^B)\frac{W_{k,1}^{A}}{\gamma_{k,1}^{A,*}(Q^B)}Z\right)\right] = R_k. \tag{42}$$

It can be easily verified that $\frac{1}{t}g_{k,1}(Q^B) < g_{k,1}(tQ^B)$. Using the latter inequality along with equations (40) and (42) and the fact that function $x \mapsto \mathbb{E}\left[\log\left(1 + xZ\right)\right]$ is increasing on $\mathbb{R}_+$, we have $tW_{k,1}^{A,*}(Q^B) > W_{k,1}^A$. Since $\gamma_{k,1}^{A,*}(Q^B)$ is not M necessarily optimal when the interference level is equal to tQ^B, $\sum_k W_{k,1}^A \ge \sum_k MW_{k,1}^{A,*}(tQ^B)$. Consequently, $t\sum_k W_{k,1}^{A,*}(Q^B) \ge \sum_k W_{k,1}^{A,*}(tQ^B)$. We conclude that $t\tilde{I}^A(Q^B) \geq \tilde{I}^A(tQ^B)$. Similarly, we can easily show that $t\tilde{I}^B(Q^A) \geq \tilde{I}^B(tQ^A)$. Combining the latter two inequalities, we can prove the scalability property of the vector-valued function $\tilde{\mathbf{I}}(Q^A, Q^B)$.

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