Large Complex Correlated Wishart Matrices: Fluctuations and Asymptotic Independence at the Edges.

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Abstract

We study the asymptotic behavior of eigenvalues of large complex correlated Wishart matrices at the edges of the limiting spectrum. In this setting, the support of the limiting eigenvalue distribution may have several connected components. Under mild conditions for the population matrices, we show that for every generic positive edge of that support, there exists an extremal eigenvalue which converges almost surely toward that edge and fluctuates according to the Tracy-Widom law at the scale $N^{2/3}$. Moreover, given several generic positive edges, we establish that the associated extremal eigenvalue fluctuations are asymptotically independent. Finally, when the leftmost edge is the origin (hard edge), the fluctuations of the smallest eigenvalue are described by mean of the Bessel kernel at the scale N^2.

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1 Introduction

Correlated Wishart matrices and more generally empirical covariance matrices are ubiquitous models in applied mathematics. After Marčenko and Pastur's seminal contribution [48], a systematic study of their large dimension properties has been undertaken (see for instance [6, 56] and the many references therein), which found many applications in multivariate statistics , e.g. [1], electrical engineering, e.g. [26], mathematical finance [44, 52], etc.

Now that many global properties of their spectrum are well-understood (cf. [3, 4, 5, 53, 62]), attention has shifted to local properties (cf. [8, 31, 20], etc.) and their underlying universal phenomenas (cf. [43] and references therein).

The main contribution of this article is to provide a local analysis of the spectrum of large complex correlated Wishart matrices near the edges of the limiting support: It is well-known that such random Hermitian matrices have a real spectrum whose limiting support may display several disjoint intervals. Beside the behavior of the largest and smallest random eigenvalues, we investigate here the fluctuations of the eigenvalues that converge to any end-point of the limiting support. These eigenvalues are referred to as **extremal eigenvalues**, for which we shall provide a precise definition later.

The model. Let $\mathbf{X}_N$ be a $N\times n$ matrix with independent and identically distributed (i.i.d.) standard complex Gaussian entries $\mathcal{N}_{\mathbb{C}}(0,1)$, and let $\mathbf{\Sigma}_N$ be a $n\times n$ deterministic positive definite Hermitian matrix. The random matrix of interest here is the $N\times N$ matrix

$$\mathbf{M}_N=\frac{1}{N}\mathbf{X}_N\mathbf{\Sigma}_N\mathbf{X}_N^* \,. \tag{1.1}$$

It has N non-negative eigenvalues $0 \leqslant x_1 \leqslant \ldots \leqslant x_N$, but which may be of different nature: $\min(n, N)$ of them are non-negative random (i.e. non-deterministic) eigenvalues, whilst the other $N - \min(n, N)$ eigenvalues are deterministic and equal to zero. A companion matrix of interest is the $n \times n$ sample covariance matrix

$$\widetilde{\mathbf{M}}_N = \frac{1}{N}\mathbf{\Sigma}_N^{1/2}\mathbf{X}_N^*\mathbf{X}_N\mathbf{\Sigma}_N^{1/2}, \tag{1.2}$$

which models the empirical covariance of a sample of N independent observations

$$\left\{\mathbf{\Sigma}_N^{1/2}[\mathbf{X}_N^*]_k\ , \quad 1 \leqslant k \leqslant N\right\}$$

where $[\mathbf{X}_N^*]_k$ stands for the k-th column of $\mathbf{X}_N^*$, with population covariance matrix $\mathbf{\Sigma}_N$. Indeed, matrices $\mathbf{M}_N$ and $\widetilde{\mathbf{M}}_N$ share the same non-null eigenvalues with the associated multiplicities.

We shall consider the asymptotic regime where $n = n(N)$, $N \to \infty$ and

$$\lim_{N\to\infty} \frac{n}{N} = \gamma \in (0, \infty). \tag{1.3}$$

This regime will be simply referred to as $N \to \infty$ in the sequel.

The random matrix $\mathbf{M}_N$ can also be interpreted as a multiplicative deformation of the Laguerre Unitary Ensemble (LUE) and is related to multiple Laguerre polynomials. A close matrix model is the additive deformation of the Gaussian Unitary Ensemble (GUE), also known as GUE with an external source; it involves multiple Hermite polynomials instead. For further information, see [17] and references therein. Capitaine and Péché [25] recently studied the fluctuations of extremal eigenvalues for this model.

We now briefly review the literature and present our contribution.

Global regime. Denote by μ_N the empirical distribution of the eigenvalues of $\mathbf{M}_N$, also called spectral measure (or distribution) of $\mathbf{M}_N$ in the sequel. Namely,

$$\mu_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$$

where δ_x is the Dirac measure at point x. In the uncorrelated case where $\mathbf{\Sigma}_N = I_n$, it is well-known [48] that μ_N almost surely (a.s.) converges weakly toward the Marčenko-Pastur (MP) distribution of parameter γ,

$$\mu_{\text{MP}}^{\gamma}(\mathrm{d}x) = (1-\gamma)^+\delta_0 + \frac{1}{2\pi x}\sqrt{(\mathfrak{b}-x)\big(x-\mathfrak{a}\big)}\,\mathbf{1}_{[\mathfrak{a},\mathfrak{b}]}(x)\mathrm{d}x, \tag{1.4}$$

where $x^+ = \max(x, 0)$ and the endpoints of its support read $\mathfrak{a} = (1-\sqrt{\gamma})^2$ and $\mathfrak{b} = (1+\sqrt{\gamma})^2$.

In the general case where $\mathbf{\Sigma}_N$ is not the identity, say with eigenvalues $0 < \lambda_1 \leqslant \cdots \leqslant \lambda_n$, a similar result holds true [62] under the additional assumption that the spectral measure

$$\nu_N = \frac{1}{n}\sum_{j=1}^n \delta_{\lambda_j} \tag{1.5}$$

of $\mathbf{\Sigma}_N$ converges weakly toward a limiting distribution ν. In the latter case, the limit μ of μ_N only depends on the limiting parameters γ and ν but is no longer explicit; this dependence

$\mu = \mu(\gamma, \nu)$ will be indicated when needed. However, its Cauchy-Stieltjes transform satisfies an explicit fixed-point equation from which many properties of μ can be inferred. For example, it is known that if $\nu(\{0\}) = 0$, then

$$\mu(\mathrm{d}x) = (1-\gamma)^{+}\delta_0 + \rho(x)\mathrm{d}x, \tag{1.6}$$

where $\rho(x)$ is a non-negative and continuous function on $(0, +\infty)$. Depending on the properties of γ and ν, the support of $\rho(x)\mathrm{d}x$ may have several connected components, see Section 2 for more precise informations. Alternatively, one can describe $\mu(\gamma, \nu)$ in terms of the free multiplicative convolution of M̌P distribution (1.4) with ν, see [70]. From now we shall refer to the support of $\rho(x)\mathrm{d}x$ as the **bulk** and to the endpoints of its connected components as the **edges**. Also, a positive edge is called **soft edge** and the terminology **hard edge** is here used when the edge is the origin.

Left and right edges. We say that an edge $\mathfrak{a}$ is a left edge, resp. $\mathfrak{b}$ is a right edge, if for every $\delta > 0$ small enough,

$$\int_{\mathfrak{a}}^{\mathfrak{a}+\delta} \rho(x)\mathrm{d}x > 0, \qquad \text{resp.} \quad \int_{\mathfrak{b}-\delta}^{\mathfrak{b}} \rho(x)\mathrm{d}x > 0\ .$$

The leftmost edge can be a soft edge or a hard edge depending on the value of γ, as explained in Section 2. Of course, any other left edge and any right edge are soft edges.

Local regime: Behavior at the rightmost edge. If $\mathbf{\Sigma}_N$ is the identity, Geman [35] proved the a.s. convergence of the largest eigenvalue $x_{\max}$ of $\mathbf{M}_N$ to the right edge of M̌P's bulk $\mathfrak{b} = (1+\sqrt{\gamma})^2$, for independent, not necessarily Gaussian, real entries of $\mathbf{X}_N$. Johansson [40] established Tracy-Widom fluctuations for $x_{\max}$ at the scale $N^{2/3}$ for complex Gaussian entries; Johnstone [41] established a similar result for real Gaussian entries. Subsequent works [58, 59, 65, 72] then relaxed the Gaussian assumption, illustrating a phenomenon of universality.

If $\mathbf{\Sigma}_N$ is a finite-rank perturbation of the identity, the limiting eigenvalue distribution is still given by M̌P distribution (1.4). Baik and Silverstein [9] studied the limiting behaviour of $x_{\max}$ for general entries. In the complex Gaussian case, Baik, Ben Arous and Péché [8] thoroughly described the fluctuations of the largest eigenvalues at the right edge and unveiled a remarkable phase transition phenomenon (referred to as BBP phase transition in the sequel). They established that the convergence and fluctuations of $x_{\max}$ are actually highly sensitive to the way ν_N converges to δ_1. More precisely, depending on the strength of the perturbation, they established that deformed Tracy-Widom fluctuations near the right edge $\mathfrak{b}$ at the scale $N^{2/3}$ can arise, and that $x_{\max}$ may also converge outside the bulk with Gaussian-like[1] fluctuations at the scale $N^{1/2}$; in the latter case $x_{\max}$ is referred to as an outlier. Thus, depending on the way ν_N converges toward its limit, the universality phenomenon may break down. Finally, Bloemendal and Virág [20, 19] and Mo [51] extended the results in [8] for real Gaussian entries, see also [18] for further extensions.

For general $\mathbf{\Sigma}_N$'s and complex Gaussian matrices, El Karoui [31] ($n/N \leqslant 1$) and then Onatski [55] ($n/N > 1$) followed the approach developed in [8] to establish Tracy-Widom

[1]By Gaussian-like, we mean that the largest eigenvalue of $\mathbf{M}_N$, when correctly centered et rescaled and when associated to a large perturbation of the identity $\mathbf{\Sigma}_N$ of finite multiplicity k, asymptotically converges to the distribution of the largest eigenvalue of a fixed $k \times k$ GUE.

fluctuations for $x_{\max}$, under mild conditions concerning $\mathbf{\Sigma}_N$'s spectral measure ν_N provided that the rightmost edge satisfies some regularity condition. The Gaussian assumption has recently been relaxed by Bao et al. [10] (the random variables remaining complex) and the complex one, by Lee and Schnelli [45] who handle the real Gaussian case and also the real non gaussian case for diagonal $\mathbf{\Sigma_N}$'s. Knowles and Yin [42] extend [45] to general $\mathbf{\Sigma_N}$'s (see also the comment on universality in Section 3.2).

Local regime: Behavior at the leftmost edge. When $\mathbf{\Sigma}_N$ is the identity, Bai and Yin [7] established the a.s. convergence of the smallest eigenvalue $x_{\min}$ of $\mathbf{M}_N$ to MP's left edge $\mathfrak{a} = (1-\sqrt{\gamma})^2$, see also [6, Chapter 5]. The nature of the fluctuations of $x_{\min}$ dramatically changes whether $\gamma = 1$ (hard edge) or $\gamma \neq 1$ (soft edge). In the soft edge case, the fluctuations remain of a Tracy-Widom nature, see Borodin and Forrester [23] and further extensions by Feldheim and Sodin [33]. In the hard edge case, the fluctuations of $x_{\min}$ arise at the scale N^2; if $n = N$ then the limiting distribution follows the exponential law as shown by Edelman [30] (cf. [66] for further extensions), while if $n = N + \alpha$ with α independent of N, then the limiting distribution has been described by Forrester [34] with the help of Bessel kernels , see Section 3 for a precise definition. The Gaussian assumption has been relaxed by Ben Arous and Péché [12].

To the best knowledge of the authors, no result for the fluctuations at the leftmost edge in the general $\mathbf{\Sigma}_N$ case is available in the literature.

Local regime: When ν is the weighted sum of two Dirac measures. When $\mathbf{\Sigma}_N$ has exactly two fixed eigenvalues, each with multiplicity of order N, a full asymptotic analysis is known for the correlation kernel $\mathrm{K}_N(x, y)$ associated with the eigenvalues of $\mathbf{M}_N$ (see Sections 4.2 and 4.3). More precisely, around each edge a local uniform convergence for $\mathrm{K}_N(x, y)$ has been obtained, using the connection to multiple Laguerre polynomials, by Lysov and Wielonsky [47] and Mo [50]. This provides a first step toward Tracy-Widom fluctuations.

Local regime: Asymptotic independence. When $\mathbf{\Sigma}_N$ is the identity and $\gamma > 1$ (and also in the case of the GUE), Basor, Chen and Zhang [11] proved that $x_{\min}$ and $x_{\max}$, properly rescaled, are asymptotically independent as $N \to \infty$. Their approach heavily relies on orthogonal polynomials techniques, which are not available for complex correlated Wishart matrices. Using different techniques, the asymptotic independence for the GUE's smallest and largest eigenvalues was also obtained by Bianchi et al. [16] and Bornemann [21].

Again, it seems there is no result concerning the asymptotic independence for the extremal eigenvalues, even for the smallest and largest eigenvalues, in the general $\mathbf{\Sigma}_N$ case.

Main results. Recall the asymptotic regime (1.3) of interest. We first state the main assumptions related to matrix $\mathbf{M}_N$ (cf. (1.1)) and then informally state the main results of the paper; pointers to the precise definitions and statements are provided in the next paragraph.

Assumption 1. The entries of $\mathbf{X}_N$ are i.i.d. standard complex Gaussian random variables.

Assumption 2. The following properties hold true:

1. The spectral measure ν_N of $\mathbf{\Sigma}_N$ weakly converges toward a limiting probability distribution ν as $N \to \infty$.

2. The eigenvalues $0 < \lambda_1 \leqslant \ldots \leqslant \lambda_n$ of $\mathbf{\Sigma}_N$ stay in a compact subset of $(0, +\infty)$ which is independent of N, namely,

$$\liminf_{N\to\infty} \lambda_1 > 0 \ , \qquad \limsup_{N\to\infty} \lambda_n < +\infty \ . \tag{1.7}$$

In particular, $\nu(\{0\}) = 0$.

Another important assumption is the fact that the considered edges need to be **regular**. By this, we mean an edge which satisfies the regularity condition of Definition 2.5. This condition essentially rules out pathological behaviors at edges, e.g. when the limiting eigenvalue density does not vanish like a square root. It however enables the appearance of outliers.

Theorem 1. *Let Assumptions 1 and 2 hold true. Then*

(a) *Extremal eigenvalues: Given a regular right (resp. left) edge, there are perfectly located maximal (resp. minimal) eigenvalues which converge a.s. toward this edge as $N \to \infty$; these eigenvalues are called extremal eigenvalues.*

(b) *Tracy-Widom fluctuations: Given a regular right (resp. left) soft edge, the associated extremal eigenvalue, properly rescaled, converges in law to the Tracy-Widom distribution (resp. reversed Tracy-Widom distribution) at the scale $N^{2/3}$.*

(c) *Asymptotic independence: Given a finite family of regular soft edges, the associated extremal eigenvalues, properly rescaled, are asymptotically independent as $N \to \infty$.*

(d) *Hard edge fluctuations: In the case where $\gamma = 1$, the bulk displays a hard edge at 0. If $n = N + \alpha$ with $\alpha \in \mathbb{Z}$ independent of N, then the fluctuations of the smallest eigenvalue, properly rescaled, are described by mean of the Bessel kernel with parameter $\alpha \in \mathbb{N}$ at the scale N^2.*

Close to our work is the recent paper by Capitaine and Péché [25] where the fluctuations of the extremal eigenvalues for the additive deformation of the GUE are established, that is the counterpart of Part (b) of Theorem 1, together with Gaussian-like fluctuations for outliers and fluctuations of the eigenvalue process at cusp points (i.e. when two bulks merge together) with the appearance of the Pearcey process. As the involved techniques are extremely model-dependent, the technical difficulties are substantially different for the model under study. The study of the fluctuations of the eigenvalue process at a cusp point for large complex correlated Whishart matrices will appear elsewhere [39].

Let us now briefly comment on Theorem 1.

In Part (a), we rely on results by Silverstein et al. [3, 4, 63] on the support of limiting spectral distributions and on fine asymptotic properties of the empirical spectrum to define regular edges and to properly express the convergence of extremal eigenvalues.

In Part (b), we first obtain an asymptotic Fredholm determinantal representation of the extremal eigenvalues' distribution and then perform an asymptotic analysis of the associated kernels to prove convergence toward the Airy kernel. The latter analysis is based on a steepest descent analysis involving contours deformations. Contrary to the analysis performed by Baik-Ben Arous-Péché [8], El Karoui [31] and Péché-Capitaine [25] who work out explicit deformed contours, our analysis relies on a more abstract argument where the existence of appropriate contours is obtained by mean of the maximum principle for subharmonic functions. This argument has the advantage to work for every regular right or left edge (and also for cusp points, cf. [39]) up to minor modifications. Let us also stress that we do not follow the same strategy as in [8, 31] concerning the involved operators convergence.

In Part (c), our proof of the asymptotic independence builds upon the operator-theoretic approach developed by Bornemann [21] in the context of the GUE. We actually show that a weaker mode of convergence for the involved operators than the one required in [21] is sufficient to establish the asymptotic independence; it has the advantage to be compatible with the previous asymptotic analysis.

Part (d) also relies on an asymptotic analysis of the rescaled kernel. It is based on an appropriate representation of the Bessel kernel as a double complex integral.

Organization of the paper. In Section 2, we provide a precise description for the bulk and the extremal eigenvalues and introduce the notion of regular edge. The precise statement of Part (a) of Theorem 1 is provided in Theorem 2 and proved.

In Section 3, we state our results concerning the fluctuations of the extremal eigenvalues and their asymptotic independence. Parts (b), (c) and (d) of Theorem 1 are respectively stated in Theorem 3, Theorem 4 and Theorem 5. We also recall there the definition of the Tracy-Widom distribution and the hard edge distribution described by mean of the Bessel kernel (Sections 3.1 & 3.3). We close the section with an asymptotic study of the condition number of large correlated Wishart matrices, a discussion on non regular edges, on spikes phenomena and provide some graphical illustrations.

Section 4 is devoted to the proof for Theorem 3 (Tracy-Widom fluctuations). Section 5 is devoted to the proof of Theorem 4 (asymptotic independence for extremal eigenvalues). Finally, Section 6 is devoted to the proof of Theorem 5 (hard edge fluctuations).

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2 Bulk description, regularity and extremal eigenvalues

In this section, we introduce the notion of **regular soft edges** (cf. Definition 2.5) and **extremal eigenvalues** (cf. Theorem 2), the main properties of which are gathered in Propositions 2.11 and 2.12. Theorem 2 provides a precise statement for Theorem 1-(a). Before this, we provide a precise description of the bulk, mainly based on [63].

2.1 Description of the limiting bulk

In [48], Marčenko and Pastur characterized the Cauchy-Stieltjes transform[2] of the limiting distribution $\mu = \mu(\gamma, \nu)$ of the eigenvalues of $\mathbf{M}_N$ as $N \to \infty$,

$$m(z) = \int \frac{1}{z - \lambda} \mu(\mathrm{d}\lambda), \qquad z \in \mathbb{C}_+ = \{z \in \mathbb{C} : \ \mathrm{Im}(z) > 0\},$$

as the unique solution $m \in \mathbb{C}_- = \{z \in \mathbb{C} : \ \mathrm{Im}(z) < 0\}$ of the fixed-point equation

$$m = \left(z - \gamma \int \frac{\lambda}{1 - m\lambda} \nu(\mathrm{d}\lambda) \right)^{-1} \quad \text{for any } z \in \mathbb{C}_+. \tag{2.1}$$

Recall that, by Assumption 1, $\gamma = \lim n/N \in (0, +\infty)$, the probability measure ν is the limiting eigenvalue distribution of $\mathbf{\Sigma}_N$ and its compact support is included in $(0, +\infty)$. In particular, $\nu(\{0\}) = 0$.

In [63], Silverstein and Choi showed that

$$\mu(\mathrm{d}x) = (1 - \gamma)^+ \delta_0 + \rho(x)\mathrm{d}x, \tag{2.2}$$

where ρ is a non-negative and continuous function on $(0, \infty)$ which is analytic wherever it is positive. Moreover, following a procedure already described by Marčenko and Pastur, they showed rigorously how to extract from the fixed point equation above a characterization of the support of μ, and thus of $\rho(x)\mathrm{d}x$. Specifically, the function $m(z)$ has an explicit inverse on $m(\mathbb{C}_+)$ given by

$$g(z) = \frac{1}{z} + \gamma \int \frac{\lambda}{1 - z\lambda} \nu(\mathrm{d}\lambda) \tag{2.3}$$

and this inverse extends analytically to a neighborhood of $\mathbb{C}_- \cup D$ where D is the open subset of the real line

$$D = \big\{ x \in \mathbb{R} : \quad x \neq 0, \quad x^{-1} \notin \mathrm{Supp}(\nu) \big\}. \tag{2.4}$$

Except in the proof of Proposition 2.7 below, we shall confine the notation g to the restriction of this function to D. On any interval I of $\mathbb{R} \setminus \mathrm{Supp}(\mu)$, the function m exists, is real and is decreasing (as a Cauchy-Stieltjes transform). Consequently, its inverse also exists and is decreasing on $m(I)$. Silverstein and Choi showed that g is this inverse, and that $\mathbb{R} \setminus \mathrm{Supp}(\mu)$ coincides with the values of $g(x)$ where this function is decreasing on D:

Proposition 2.1 (Silverstein & Choi [63]). *For any $x \in \mathbb{R} \setminus \mathrm{Supp}(\mu)$, let $p = m(x)$. Then $p \in D$, $x = g(p)$, and $g'(p) < 0$. Conversely, let $p \in D$ such that $g'(p) < 0$. Then $x = g(p) \in \mathbb{R} \setminus \mathrm{Supp}(\mu)$ and $p = m(x)$.*

Remark 2.2. This proposition has the following practical importance: In order to find $\mathrm{Supp}(\mu)$, plot the function g on D; whenever g is decreasing ($g'(x) < 0$), remove the corresponding points $g(x)$ from the vertical axis. What is left is precisely $\mathrm{Supp}(\mu)$.

As an example, a plot of the function g is provided in Figure 1 along with $\mathrm{Supp}(\mu)$ in the case where ν is the weighted sum of two Dirac measures and $\gamma < 1$.

The soft edges of the bulk are described more precisely by the next proposition.

[2] Note that our definition of the Cauchy-Stieltjes transform differs by a sign from the one in [48] but will turn out to be more convenient in the sequel.

Proposition 2.3 (Silverstein & Choi [63]). *Any soft left edge $\mathfrak{a}$ satisfies one of the two following properties:*

(a) *There exists a unique $\mathfrak{c} \in D$ such that $\mathfrak{a} = g(\mathfrak{c})$, $g'(\mathfrak{c}) = 0$ and $g''(\mathfrak{c}) < 0$.*

(b) *There exists a unique $\mathfrak{c} \in \partial D$ such that $(\mathfrak{c}, \mathfrak{c} + \varepsilon) \subset D$ for some $\varepsilon > 0$ small enough, the function g is decreasing on $(\mathfrak{c}, \mathfrak{c} + \varepsilon)$, and $\mathfrak{a} = \lim_{x \downarrow \mathfrak{c}} g(x)$. In that case, we write $\mathfrak{a} = g(\mathfrak{c})$.*

Conversely, for any point $\mathfrak{c}$ satisfying one of these properties, $\mathfrak{a} = g(\mathfrak{c})$ is a soft left edge.

Similarly, any (soft) right edge $\mathfrak{b}$ of the measure μ satisfies one of the two following properties:

(a) *There exists a unique $\mathfrak{d} \in D$ such that $\mathfrak{b} = g(\mathfrak{d})$, $g'(\mathfrak{d}) = 0$ and $g''(\mathfrak{d}) > 0$.*

(b) *There exists a unique $\mathfrak{d} \in \partial D$ such that $(\mathfrak{d} - \varepsilon, \mathfrak{d}) \subset D$ for some $\varepsilon > 0$ small enough, the function g is decreasing on $(\mathfrak{d} - \varepsilon, \mathfrak{d})$, and $\mathfrak{b} = \lim_{x \uparrow \mathfrak{d}} g(x)$. In that case, we write $\mathfrak{b} = g(\mathfrak{d})$.*

Conversely, for any point $\mathfrak{d}$ satisfying one of these properties, $\mathfrak{b} = g(\mathfrak{d})$ is a right edge of the measure μ.

Hence, any soft edge of the bulk coincides with a unique extremum $\mathfrak{c}$ of the function g and it reads $g(\mathfrak{c})$. These extrema may or may not be attained on D. In case they are, the second derivative of g is never equal to zero there, and it has been proved in [63] that the density vanishes like a square root at the associated edges. We shall see later that the Tracy-Widom fluctuations appear in this case. A right edge $\mathfrak{b} = g(\mathfrak{d})$ together with its preimage $\mathfrak{d}$ are plotted in Figure 1.

The next proposition provides additional information on the bulk that will be useful in the sequel. Its proof is in Appendix A.

Proposition 2.4. *Let Assumption 2 hold true. Let $\mathfrak{a}$ be the leftmost edge of the bulk. The following facts hold true:*

(a) *If $\gamma > 1$, then $\mathfrak{a} > 0$. Moreover, the function $g(x)$ increases from zero to $\mathfrak{a}$ then decreases from $\mathfrak{a}$ to $-\infty$ as x increases from $-\infty$ to zero. In particular, $\mathfrak{a}$ is the unique maximum of g on $(-\infty, 0)$.*

(b) *If $\gamma \leqslant 1$, the function g is negative and decreasing on $(-\infty, 0)$.*

(c) *If $\gamma < 1$, then $\mathfrak{a} > 0$. Moreover, if we set $\eta = \inf \operatorname{Supp}(\nu) > 0$, then $\mathfrak{a} = g(\mathfrak{c})$ is the supremum of g on $(1/\eta, \infty)$. In addition, g increases to $\mathfrak{a}$ on $(1/\eta, \mathfrak{c})$ whenever this interval is non empty, then decreases from $\mathfrak{a}$ to zero on $(\mathfrak{c}, \infty)$.*

Let $\mathfrak{b} = g(\mathfrak{d})$ be a right edge of the bulk. Then the following facts hold true:

(d) *$[\mathfrak{d}, \infty) \not\subset D$.*

(e) *Assume $\mathfrak{b}$ is the rightmost edge of the bulk. For any $\gamma \in (0,\infty)$, if we set $\xi = \sup \mathrm{Supp}(\nu) < \infty$, then g decreases from infinity to $\mathfrak{b}$ on $(0,\mathfrak{d})$ and increases on $(\mathfrak{d}, 1/\xi)$ if this last interval is not empty. In particular, $\mathfrak{d}$ is the unique extremum of g on $(0,1/\xi)$.*

Fact (a) shows that when $\gamma > 1$, the study of g on $(-\infty, 0)$ allows to locate the leftmost edge $\mathfrak{a}$ and this edge only. Facts (a) and (b) show that if $\gamma \leqslant 1$ then it suffices to study g on $D \cap (0,\infty)$ to locate the edges of the bulk. In particular, if $\gamma < 1$, Fact (c) shows that the location of $\mathfrak{a}$ is provided by the study of g on $(1/\eta, \infty)$. This is illustrated by Figure 1, where $\mathfrak{a}$ is the rightmost maximum of the function g. Fact (d) shows that when $\mathfrak{b} = g(\mathfrak{d})$ is a right edge of the bulk, then $\mathfrak{d}$ cannot belong to the unbounded connected component of D in $(0,\infty)$. Finally, the behavior of g described by (e) is illustrated on Figure 1 by the plot of this function on the interval $(0, 1/3)$.

2.2 Regularity condition and its consequences

So far, we have thoroughly described the edges of the limiting eigenvalue distribution. Remember however that BBP phase transition [8] may occur regardless of the limiting spectral distribution (which is always $\check{\mathrm{M}}$P distribution in [8]). As we shall see later, the notion of **regular endpoint** captures a joint condition on the limiting spectral distribution μ and on the convergence $\nu_N \to \nu$, which will guarantee Tracy-Widom fluctuations (cf. Theorem 3).

Definition 2.5. (Regular edge) Recall that the λ_i's are the eigenvalues of matrix $\mathbf{\Sigma}_N$; a soft edge $\mathfrak{a} = g(\mathfrak{c})$ is *regular* if

$$\liminf_{N\to\infty} \min_{j=1}^{n} |\mathfrak{c} - \lambda_j^{-1}| > 0 \ . \tag{2.5}$$

In particular, $\mathfrak{c} \in D$.

Remark 2.6. The following facts will illustrate the range of the definition.

(a) If $\mathfrak{a} = g(\mathfrak{c})$ is a regular soft edge, then the weak convergence $\nu_N \to \nu$ stated in Assumption 2 rules out the options labelled (b) in Proposition 2.3.

(b) If $\mathfrak{a}$ is an endpoint satisfying one of the options labelled (a) in Proposition 2.3, and if, furthermore, the distance $\mathrm{dist}(\lambda_i, \mathrm{Supp}(\nu))$ satisfies

$$\max_{1\leqslant j\leqslant n} \mathrm{dist}(\lambda_j, \mathrm{Supp}(\nu)) \xrightarrow[N\to\infty]{} 0 \ ,$$

then $\mathfrak{a}$ is a regular endpoint of $\mathrm{Supp}(\mu)$. However, this last condition is not necessary. Further comments will be made in Section 3.1 below.

(c) If $\gamma > 1$, then the leftmost edge is regular (for a proof of this fact, simply write the leftmost edge as $g(\mathfrak{c})$, then Proposition 2.4–(a) shows that $\mathfrak{c} < 0$, which immediatly implies (2.5)).

Let $\gamma_N = n/N$ and consider now the probability measure $\mu(\gamma_N, \nu_N)$, which is the unique solution of the fixed point equation (2.1) associated with the data γ_N, ν_N. It is a finite-N deterministic equivalent of the spectral measure of $\mathbf{M}_N$. Associated to $\mu(\gamma_N, \nu_N)$ is the function

$$g_N(z) = \frac{1}{z} + \gamma_N \int \frac{\lambda}{1 - z\lambda} \nu_N(\mathrm{d}\lambda) = \frac{1}{z} + \frac{1}{N} \sum_{j=1}^{n} \frac{\lambda_j}{1 - z\lambda_j}, \tag{2.6}$$

(cf. (2.3)). Similarly to $\mu(\gamma,\nu)$, the measure $\mu(\gamma_N,\nu_N)$ has a density on $(0,\infty)$ and its support can also be characterized with the help of Proposition 2.1 (simply replace g by g_N). We furthermore have the following proposition:

Proposition 2.7. *Let Assumption 2 hold true. Let $g(\mathfrak{c})$ be a regular soft edge. Then, for N large enough,*

(a) *g_N is analytic in a complex neighborhood of $\mathfrak{c}$ which is independent of N.*

(b) *g_N converges to g uniformly on the compact sets of this neighborhood, and so does its k-th order derivative $g_N^{(k)}$ to $g^{(k)}$, for any $k \geqslant 1$.*

(c) *There exists a sequence of real numbers $\mathfrak{c}_N$, unique up to a finite number of terms, such that $\mathfrak{c}_N \to \mathfrak{c}$, $g'_N(\mathfrak{c}_N) = 0$, and $g_N^{(k)}(\mathfrak{c}_N) \to g^{(k)}(\mathfrak{c})$ as $N \to \infty$ for any k.*

This proposition shows in particular that when a soft edge $g(\mathfrak{c})$ is regular, there is a sequence $g_N(\mathfrak{c}_N)$ of endpoints of $\operatorname{Supp}(\mu(\gamma_N,\nu_N))$ that converge to $g(\mathfrak{c})$, and $\mathfrak{c}_N$ satisfies

$$\liminf_{N\to\infty} \min_{j=1}^{n} |\mathfrak{c}_N - \lambda_j^{-1}| > 0. \tag{2.7}$$

Proof. Set $\eta = \min(|\mathfrak{c}|/2, \liminf_N \min_j |\lambda_j^{-1} - \mathfrak{c}|)$, and let $B = B(\mathfrak{c}, \eta/2)$ be the open ball with center $\mathfrak{c}$ and radius $\eta/2$. Since

$$\frac{\lambda_j}{|1 - z\lambda_j|} = \frac{1}{|\lambda_j^{-1} - z|} \leqslant \frac{1}{|\lambda_j^{-1} - \mathfrak{c}| - |z - \mathfrak{c}|} \leqslant \frac{3}{\eta}$$

for $z \in B$ and for all N large, the functions g_N are analytic and uniformly bounded on B for all N large. This establishes (a) in particular. Moreover, this yields that the family of analytic functions g_N is uniformly bounded on B. Thus, by Montel's theorem, the family g_N is normal. It follows from the convergences $\gamma_N \to \gamma$ and $\nu_N \to \nu$ provided by Assumption 1 that g_N converges pointwise to g on B. Consequently, g_N converges to g uniformly on the compact subsets of B, and the same is true for the convergence of the $g_N^{(k)}$ to $g^{(k)}$ by [60, Th. 10.28]. Turning to (c), notice that $\mathfrak{c}$ is a zero of g' by the regularity assumption, see Remark 2.6-(a). Since g'_N converges to g' uniformly on the compact sets of B and g' is analytic there, Hurwitz's theorem shows that g'_N has a zero $\mathfrak{c}_N$ that converges to the zero $\mathfrak{c}$ of g' and that this zero is unique provided N is large enough. Moreover this zero is real since $g'_N(\bar{z}) = \overline{g'_N(z)}$. Write $|g_N^{(k)}(\mathfrak{c}_N) - g^{(k)}(\mathfrak{c})| \leqslant |g_N^{(k)}(\mathfrak{c}_N) - g^{(k)}(\mathfrak{c}_N)| + |g^{(k)}(\mathfrak{c}_N) - g^{(k)}(\mathfrak{c})|$. Since for any k, $g_N^{(k)}$ converge uniformly to g^k on the compact subsets of B, the first term at the right hand side vanishes as $N \to \infty$. The second term vanishes as $N \to \infty$ by the continuity of $g^{(k)}$. This establishes (c). □

2.3 Extremal eigenvalues and their convergence

Our purpose is now to locate the eigenvalues of $\mathbf{M}_N$ that converge to a prescribed edge, or equivalently those of $\widetilde{\mathbf{M}}_N$ (denoted by $\tilde{x}_1 \leqslant \cdots \leqslant \tilde{x}_n$). The idea is the following: Given an interval (u,v) outside $\operatorname{Supp}(\mu(\gamma_N,\nu_N))$, its preimage $(m(v), m(u))$ by g then lies in between two groups of λ_j^{-1}'s, provided N is sufficiently large. Thus, there is a unique integer $\phi(N)$ for which $\lambda_{\phi(N)+1}^{-1} < m(v) < m(u) < \lambda_{\phi(N)}^{-1}$. This $\phi(N)$ defines the deterministic index for which $x_{\phi(N)}$ converges a.s. toward the prescribed edge. Figure 1 illustrates this phenomenon. The following proposition formalizes this.

Remark 2.8. (Convention) In the remaining, we shall systematically use the notational convention $\lambda_0 = \tilde{x}_0 = 0$ and $\lambda_{n+1} = \tilde{x}_{n+1} = \infty$.

Proposition 2.9 (Bai & Silverstein [3, 4])**.** *Let Assumptions 1 and 2 hold true. Assume that* $[u, v]$ *with* $u > 0$ *lies in an open interval outside* $\mathrm{Supp}(\mu(\gamma_N, \nu_N))$ *for* N *large enough and recall the definition (2.1) of the fixed-point solution* m*. Then the following facts hold true:*

(a) *If* $\gamma > 1$*, then* $\tilde{x}_{n-N+1} \to \mathfrak{a}$ *almost surely as* $N \to \infty$*, where* $\mathfrak{a} > 0$ *is the leftmost edge of the bulk.*

(b) *In any of the two cases i)* $\gamma \leqslant 1$ *or ii)* $\gamma > 1$ *and* $[u, v] \not\subset [0, \mathfrak{a}]$*, it holds that* $m(v) > 0$*. Let* $\phi(N)$ *be the integer defined as*

$$\lambda_{\phi(N)+1} > m(v)^{-1} \quad \text{and} \quad \lambda_{\phi(N)} < m(u)^{-1}. \tag{2.8}$$

Then

$$\mathbb{P}\Big(\tilde{x}_{\phi(N)+1} > v, \quad \tilde{x}_{\phi(N)} < u \quad \text{for all large } N\Big) = 1 \ . \tag{2.9}$$

Remark 2.10. In [4], the result was established for matrices $\mathbf{X}_N$ taken from a doubly infinite array of i.i.d. random variables with finite fourth moment. If the entries are Gaussian, one can relax the doubly infinite array assumption and establish Proposition 2.9 by using the completely different tools of [46].

We are now in position to properly state and prove part *(a)* of Theorem 1.

Theorem 2 (Extremal eigenvalues)**.** *Let Assumptions 1 and 2 hold true*[3]*.*

(a) *If* $\gamma > 1$ *and* $\mathfrak{a}$ *is the leftmost edge of the bulk, then set* $\varphi(N) = n - N + 1$*. Otherwise, let* $\mathfrak{a} = g(\mathfrak{c})$ *be a regular soft left edge and let* $\varphi(N) = \min\{j : \lambda_j^{-1} < \mathfrak{c}\}$*. Then, almost surely,*

$$\lim_{N\to\infty} \tilde{x}_{\varphi(N)} = \mathfrak{a} \quad \text{and} \quad \liminf_{N\to\infty} (\mathfrak{a} - \tilde{x}_{\varphi(N)-1}) > 0 \ .$$

(b) *Let* $\mathfrak{b} = g(\mathfrak{d})$ *be a regular right edge and let* $\phi(N) = \max\{j : \lambda_j^{-1} > \mathfrak{d}\}$*. Then, almost surely*

$$\lim_{N\to\infty} \tilde{x}_{\phi(N)} = \mathfrak{b} \quad \text{and} \quad \liminf_{N\to\infty} (\tilde{x}_{\phi(N)+1} - \mathfrak{b}) > 0 \ .$$

Eigenvalues $\tilde{x}_{\varphi(N)}$ *and* $\tilde{x}_{\phi(N)}$ *are called extremal eigenvalues.*

Proof. We shall only prove the result for a right edge $\mathfrak{b}$. By Proposition 2.7, we can choose a compact neighborhood B of $\mathfrak{d}$ such that g_N and g'_N uniformly converge to g and g'. Let p, q, r, s be real numbers such that $p < q < r < s < \mathfrak{d}$, $[p, s] \subset B$, and $g'(x) < 0$ for $x \in [p, s]$. This last condition is made possible by the fact that $\mathfrak{b}$ is a right edge of $\mathrm{Supp}(\mu)$ (cf. Figure 1). Let $u = g(r)$ and $v = g(q)$. Since g_N and g'_N converge uniformly to g and g' respectively on $[p, s]$, it holds that $g'_N(x) < 0$ on $[p, s]$, and $[u, v] \subset [g_N(s), g_N(p)]$ for all N large. Proposition 2.1 applied to $\mu(\gamma_N, \nu_N)$ shows then that $[u, v]$ lies in an open set outside $\mathrm{Supp}(\mu(\gamma_N, \nu_N))$ for all N sufficiently large.

[3] In view of Remark 2.10, one can relax the Gaussianity assumption in Theorem 2 and replace it by the fact that $\mathbf{X}_N$'s entries are extrated from a doubly infinite array of i.i.d. random variables.

Now the integer $\phi(N)$ defined in the statement is characterized by the inequalities

$$\lambda_{\phi(N)+1}^{-1} \;<\; \mathfrak{d} \;<\; \lambda_{\phi(N)}^{-1} \;.$$

Since no λ_j^{-1}'s belong to B for N large enough, we can equivalently write

$$\lambda_{\phi(N)+1}^{-1} \;<\; q = m(v) \;<\; r = m(u) \;<\; \lambda_{\phi(N)}^{-1}$$

which is (2.8). By Proposition 2.9, we get (2.9).

Since $v > \mathfrak{b}$, we have $\liminf_N (\tilde{x}_{\phi(N)+1} - \mathfrak{b}) > 0$ with probability one. Moreover, we know that a.s., the number of $\tilde{x}_i$ in $[\mathfrak{b} - \varepsilon, \mathfrak{b}]$ is non zero for any $\varepsilon > 0$ and for all large N. Making $r \uparrow \mathfrak{d}$, we get $u = g(r) \downarrow \mathfrak{b}$. Since $\tilde{x}_{\phi(N)} < u$ a.s. for all large N, we get that $\tilde{x}_{\phi(N)} \to \mathfrak{b}$ a.s. when $N \to \infty$. $\square$

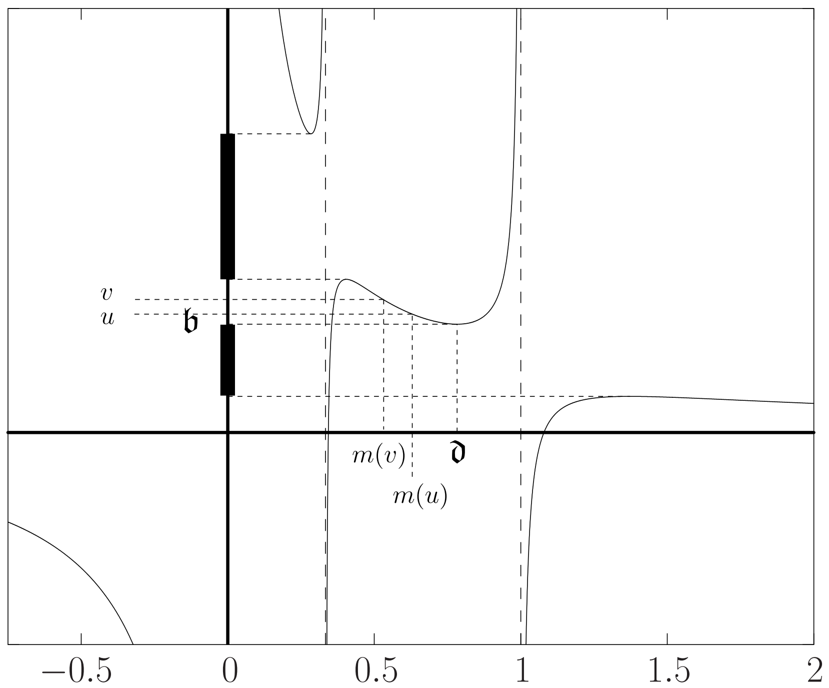

Figure 1: Plot of $g : D \to \mathbb{R}$ for $\gamma = 0.1$ and $\nu = 0.7\delta_1 + 0.3\delta_3$. In this case, $D = (-\infty, 0) \cup (0, \frac{1}{3}) \cup (\frac{1}{3}, 1) \cup (1, \infty)$. The two thick segments on the vertical axis represent $\mathrm{Supp}(\mu)$. The right edge $\mathfrak{b}$ of the measure μ satisfies property (a) of Proposition 2.3.

2.4 Summary of the properties of regular edges

For the reader's convenience and constant use in the sequel, we gather in the two following propositions some of the most important properties of regular edges introduced above. Recall the convention in Remark 2.8.

Proposition 2.11 (Left regular soft edges)**.** *Let Assumption 2 hold true. Let $\mathfrak{a}$ be a left edge.*

(a) *Consider first the case where* $\mathfrak{a}$ *is the leftmost edge.*

  - *If* $\gamma > 1$ *then* $\mathfrak{a} = g(\mathfrak{c}) > 0$ *with* $\mathfrak{c} < 0$ *and* $\mathfrak{a}$ *is a regular soft edge.*
  - *If* $\gamma < 1$ *then* $\mathfrak{a} = g(\mathfrak{c}) > 0$ *with* $\mathfrak{c} > 0$*;* $\mathfrak{a}$ *is a soft edge but its regularity is a priori not granted.*

(b) *Assume now that* $\mathfrak{a}$ *is a regular left soft edge. Then*

$$\mathfrak{a} = g(\mathfrak{c}) \quad \textit{with} \quad \left\{ \begin{array}{lcl} g'(\mathfrak{c}) & = & 0 \\ g''(\mathfrak{c}) & < & 0 \end{array} \right. \quad \textit{and} \quad \left\{ \begin{array}{ll} \mathfrak{c} < 0 & \textit{if } \mathfrak{a} \textit{ is the leftmost edge and } \gamma > 1, \\ \mathfrak{c} > 0 & \textit{otherwise.} \end{array} \right. .$$

For N *large enough, there exists a unique sequence* $\mathfrak{c}_N$ *such that* $g_N'(\mathfrak{c}_N) = 0$ *and*

$$\mathfrak{c}_N \xrightarrow[N\to\infty]{} \mathfrak{c} \ , \quad g_N^{(k)}(\mathfrak{c}_N) \xrightarrow[N\to\infty]{} g^{(k)}(\mathfrak{c}) \quad \textit{for any } k \geqslant 0 \ ,$$

where by $g_N^{(0)}, g^{(0)}$ *we mean* g_N, g*. Finally, there exists a deterministic sequence* $(\varphi(N))$ *such that almost surely,*

$$\lim_{N\to\infty} \tilde{x}_{\varphi(N)} = \mathfrak{a} \quad \textit{and} \quad \liminf_{N\to\infty} (\mathfrak{a} - \tilde{x}_{\varphi(N)-1}) > 0 \ .$$

Proposition 2.12 (Right regular soft edges)**.** *Let Assumption 2 hold true and assume that* $\mathfrak{b}$ *is a regular right soft edge. Then*

$$\mathfrak{b} = g(\mathfrak{d}) \quad \textit{with} \quad \left\{ \begin{array}{lcl} g'(\mathfrak{d}) & = & 0 \\ g''(\mathfrak{d}) & > & 0 \end{array} \right. \quad \textit{and} \quad \mathfrak{d} > 0 \ .$$

For N *large enough, there exists a unique sequence* $\mathfrak{d}_N$ *such that* $g_N'(\mathfrak{d}_N) = 0$ *and*

$$\mathfrak{d}_N \xrightarrow[N\to\infty]{} \mathfrak{d} \ , \quad g_N^{(k)}(\mathfrak{d}_N) \xrightarrow[N\to\infty]{} g^{(k)}(\mathfrak{d}) \quad \textit{for any } k \geqslant 0 \ .$$

Finally, there exists a deterministic sequence $(\phi(N))$ *such that almost surely,*

$$\lim_{N\to\infty} \tilde{x}_{\phi(N)} = \mathfrak{b} \quad \textit{and} \quad \liminf_{N\to\infty} (\tilde{x}_{\phi(N)+1} - \mathfrak{b}) > 0 \ .$$

3 Fluctuations around the edges

In this section, we state the main results of the paper, namely the fluctuations of the extremal eigenvalues and their asymptotic independence. Parts (b), (c) and (d) of Theorem 1 are respectively formalized in Theorem 3 (Section 3.1), Theorem 4 (Section 3.2) and Theorem 5 (Section 3.3). We also provide a discussion on non-regular edges and spikes phenomena with graphical illustrations.

As an application, we obtain in Section 3.4 new results for the asymptotic behavior of the condition number of complex correlated Wishart matrices.

3.1 Tracy-Widom fluctuations at the regular soft edges

We first introduce the Tracy-Widom distribution. The Airy function Ai is the unique solution of the differential equation $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$ which satisfies the asymptotic behavior

$$\mathrm{Ai}(x) = \frac{1}{2\sqrt{\pi}x^{1/4}} e^{-\frac{2}{3}x^{3/2}} \big(1 + o(1)\big)\,, \qquad x \to +\infty\,.$$

With a slight abuse of notation, denote by $\mathrm{K_{Ai}}$ the integral operator associated with the Airy kernel

$$\mathrm{K_{Ai}}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}(y)\mathrm{Ai}'(x)}{x-y}\,. \tag{3.1}$$

A real-valued random variable X is said to have Tracy-Widom distribution if

$$\mathbb{P}(X \leqslant s) = \det\big(I - \mathrm{K_{Ai}}\big)_{L^2(s,\infty)}, \qquad s \in \mathbb{R},$$

where the right hand side stands for the Fredholm determinant of the restriction to $L^2(s,\infty)$ of the operator $\mathrm{K_{Ai}}$ (see also Section 4.2). Tracy and Widom [67] established the famous representation

$$\det\big(I - \mathrm{K_{Ai}}\big)_{L^2(s,\infty)} = \exp\left(-\int_s^\infty (x-s)q(x)^2 \mathrm{d}x\right),$$

where q is the Hastings-McLeod solution of the Painlevé II equation, namely the unique solution of $q''(x) = 2q(x)^3 + xq(x)$ with boundary condition $q(x) \sim \mathrm{Ai}(x)$ as $x \to \infty$.

We are now in position to state our result concerning the Tracy-Widom fluctuations. Recall that g_N has been introduced in (2.6).

Theorem 3. *Let Assumptions 1 and 2 hold true.*

(a) *Let $\mathfrak{a}$ be a left regular soft edge, and $\tilde{x}_{\varphi(N)}$ and $(\mathfrak{c}_N)_N$ be as in Proposition 2.11. Set*

$$\mathfrak{a}_N = g_N(\mathfrak{c}_N), \qquad \sigma_N = \left(\frac{2}{-g_N''(\mathfrak{c}_N)}\right)^{1/3}.$$

Then, for every $s \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{P}\Big(N^{2/3}\sigma_N\big(\mathfrak{a}_N - \tilde{x}_{\varphi(N)}\big) \leqslant s\Big) = \det\big(I - \mathrm{K_{Ai}}\big)_{L^2(s,\infty)}. \tag{3.2}$$

(b) *Let $\mathfrak{b}$ be a right regular soft edge, and $\tilde{x}_{\phi(N)}$ and $(\mathfrak{d}_N)_N$ be as in Proposition 2.12. Set*

$$\mathfrak{b}_N = g_N(\mathfrak{d}_N), \qquad \delta_N = \left(\frac{2}{g_N''(\mathfrak{d}_N)}\right)^{1/3}.$$

Then, for every $s \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{P}\Big(N^{2/3}\delta_N\big(\tilde{x}_{\phi(N)} - \mathfrak{b}_N\big) \leqslant s\Big) = \det\big(I - \mathrm{K_{Ai}}\big)_{L^2(s,\infty)}. \tag{3.3}$$

The proof is deferred to Section 4 and an outline is provided in Section 4.1.

Connexion with El Karoui's result.

Let us first comment the last theorem in the light of El Karoui's result [31], see also Onatski's work [55]. If we assume that
$$\liminf_{N\to\infty} \mathfrak{d}_N \lambda_n < 1, \tag{3.4}$$
then, as a consequence of the analysis provided in Section 2, the sequence $(\mathfrak{d}_N)_N$ is associated with the rightmost edge $\mathfrak{b}$ and the associated extremal eigenvalue has to be the largest eigenvalue of $\widetilde{\mathbf{M}}_N$ (or equivalently of $\mathbf{M}_N$). Moreover, (3.4) implies that $\mathfrak{b}$ is regular, so that Theorem 3 applies. This is the result of El Karoui announced in the introduction, which he actually proves in a more general setting.

Indeed, in [31] the weak convergence of ν_N toward some limiting probability distribution and the convergence of n/N to some limit were not assumed; it is only assumed that n/N stays in a bounded set of $(0, 1]$ (actually of $(0, +\infty)$ after [55]) together with (3.4). Let us mention that under these only assumptions, by compactness one can always extract converging subsequences for ν_N and n/N so that our result applies along a subsequence.

Notice also that the condition (3.4) is stronger than our regularity condition, since $\mathfrak{b}$ can be regular with $\liminf_N \mathfrak{d}_N\lambda_n > 1$. In this case, the extremal eigenvalue associated with the rightmost edge is no longer the largest eigenvalue of $\widetilde{\mathbf{M}}_N$; this entails the presence of outliers, as we shall explain in the next paragraph. Our result then states that the largest eigenvalue which actually converges to the rightmost edge $\mathfrak{b}$ fluctuates for large N according to the Tracy-Widom law.

Non-regular edges and spikes phenomena.

In Remark 2.6-(b), we explained that when a soft edge reads $\mathfrak{b} = g(\mathfrak{d})$ with $\mathfrak{d} \notin \partial D$, and when the Hausdorff distance between $\mathrm{Supp}(\nu_N)$ and $\mathrm{Supp}(\nu)$ converges to zero, then the endpoint $\mathfrak{b}$ is regular. Still assuming that $\mathfrak{d} \notin \partial D$, let us now assume instead that
$$\nu_N = \frac{k}{n}\delta_\zeta + \tilde{\nu}_N$$
where k is a fixed positive integer, $\zeta > 0$ is fixed and lies outside $\mathrm{Supp}(\nu)$, and the Hausdorff distance between $\mathrm{Supp}(\tilde{\nu}_N)$ and $\mathrm{Supp}(\nu)$ converges to zero. The eigenvalue ζ of $\mathbf{\Sigma}_N$ with multiplicity k is often called a **spike**. Assume without loss of generality that $\mathfrak{b}$ is a right edge and that that $1/\zeta$ belongs to the same connected component of D as $\mathfrak{d}$. Three situations that we describe without formal proofs are of interest:

1. The spike ζ satisfies $g'(1/\zeta) < 0$. This can only happen if $1/\zeta < \mathfrak{d}$, as shown in Figure 2. In that case, ζ produces k **outliers**, i.e., eigenvalues of $\widetilde{\mathbf{M}}_N$ which converge to a value outside the bulk, see [9, 14]. In terms of the support of $\mu(\gamma_N, \nu_N)$, the location of these outliers corresponds to a small interval in $\mathrm{Supp}(\mu(\gamma_N, \nu_N))$ (see Figure 2) which is absent from $\mathrm{Supp}(\mu(\gamma, \nu))$. The width of this new interval is of order $N^{-1/2}$.

   Since $1/\zeta < \mathfrak{d}$, the regularity condition still holds for $\mathfrak{b}$, and Tracy-Widom fluctuations around $\mathfrak{b}_N = g_N(\mathfrak{d}_N)$ will be observed.

   Let us say a few words on the fluctuations of the outliers. Notice that ζ incurs the presence of a local minimum and a new local maximum in g_N which are absent from g, see Figures 1 and 2. Considering e.g. the minimum reached at, say $\mathfrak{d}'_N$, one can show that $|1/\zeta - \mathfrak{d}'_N|$ is of order $N^{-1/2}$. In particular, the regularity assumption (2.7) is not satisfied for $\mathfrak{d}'_N$. In fact, it is known that when they are scaled by $N^{1/2}$, the k outliers asymptotically fluctuate up to a multiplicative constant as the eigenvalues of a $k \times k$ matrix taken from the GUE ensemble, see [8, 2, 13] among others.

2. The spike ζ satisfies $g'(1/\zeta) > 0$. The case where $1/\zeta > \mathfrak{d}$ is shown on Figure 3. Here, the spike ζ does not create an outlier and the regularity condition on $\mathfrak{b}$ is still satisfied. Tracy-Widom fluctuations around $\mathfrak{b}_N = g_N(\mathfrak{d}_N)$ will be also observed here.

3. The spike depends generally on N and satisfies $1/\zeta \to \mathfrak{d}$ as $N \to \infty$. Here, we are at the crossing point of the phase transition discovered in [8] between the "Tracy Widom regime" and the "GUE regime". More specifically, under an additional condition (see (B.2)) we shall briefly outline in Appendix B that at the scale $N^{2/3}$ the asymptotic fluctuations are described by the so-called deformed Tracy-Widom law whose distribution function F_k is defined in [8, Eq. 17]. One can also be interested in the regime where $k = k(N) \to \infty$ as $N \to \infty$. In the setting of additive perturbations of Wigner matrices, this situation been considered by Péché when $k/N \to 0$, and she proved Tracy-Widom fluctuations arise, see [57, Theorem 1.5]. We do not pursue in this direction here.

All these arguments can be straightforwardly generalized to the case where a finite number of different spikes are present.

As explained in the third point above and in Appendix B , we can tackle the situation where an edge satisfies a weak kind of non-regularity. Nevertheless, our approach breaks down in the case of a limiting measure ν for which Proposition 2.3-(b) occurs.

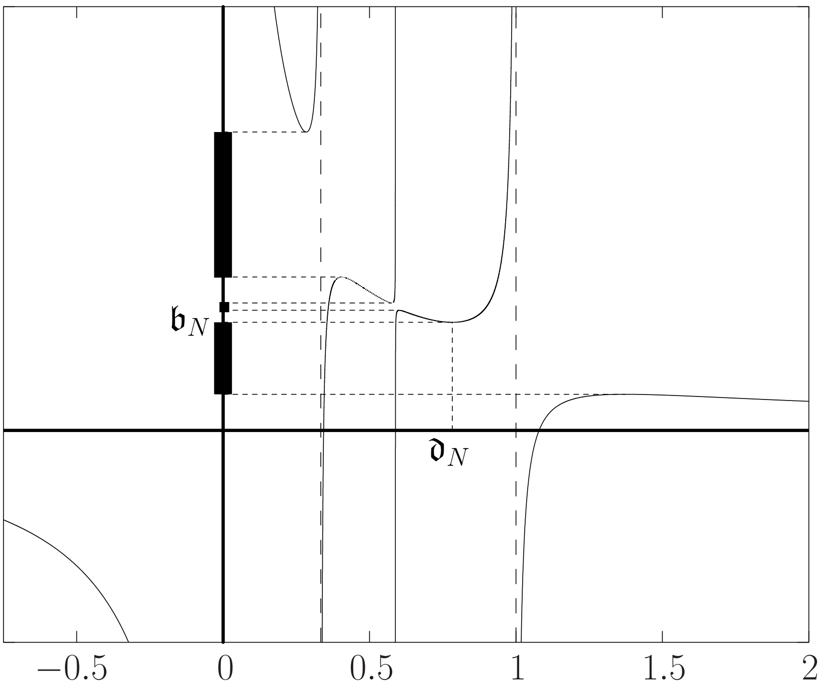

Figure 2: Plot of $g_N(x)$ for $n = 300$, $\gamma_N = 0.1$ and $\nu_N = \frac{1}{300}\delta_{1.7} + \frac{209}{300}\delta_1 + \frac{90}{300}\delta_3$. The spike $\zeta = 1.7$ produces an outlier. Asymptote at $1/\zeta$ not shown for better visibility.

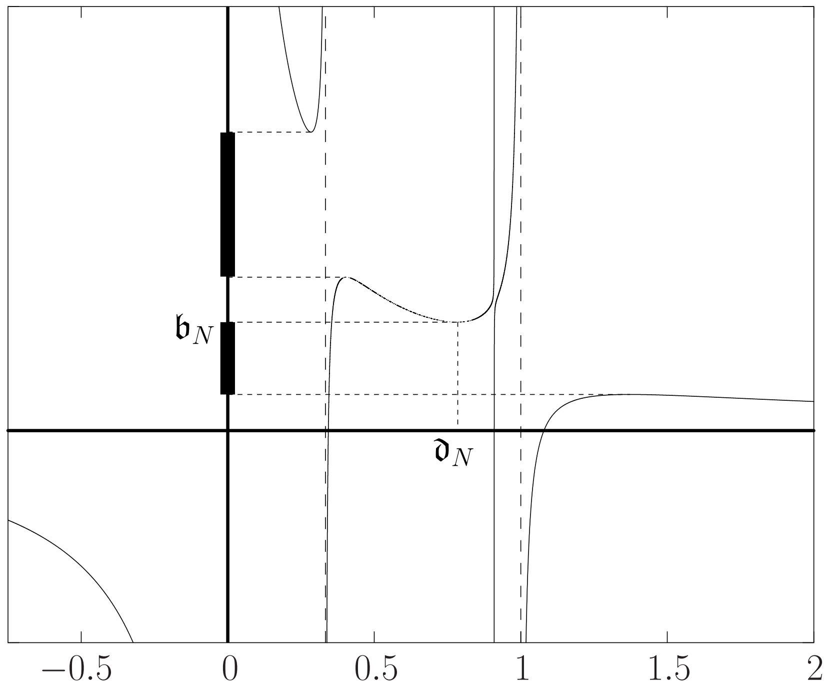

Figure 3: Plot of $g_N(x)$ for $n = 300$, $\gamma_N = 0.1$ and $\nu_N = \frac{1}{300}\delta_{1.1} + \frac{209}{300}\delta_1 + \frac{90}{300}\delta_3$. The spike $\zeta = 1.1$ does not produce an outlier. Asymptote at $1/\zeta$ not shown for better visibility.

3.2 Asymptotic independence

Our next result states that the fluctuations of the extremal eigenvalues associated with any finite number of regular soft edges are asymptotically independent.

Theorem 4. *Let Assumptions 1 and 2 hold true and let I and J be finite sets of indices. Denote by $(\mathfrak{a}_i)_{i\in I}$ left regular soft edges and by $(\mathfrak{b}_j)_{j\in J}$ right regular soft edges.*

Let $\tilde{x}_{\varphi_i(N)}$ and $\mathfrak{c}_{i,N}$ be associated to $\mathfrak{a}_i$ as in Proposition 2.11, and denote by

$$\mathfrak{a}_{i,N} = g_N(\mathfrak{c}_{i,N}), \qquad \sigma_{i,N} = \left(\frac{2}{-g''_N(\mathfrak{c}_{i,N})}\right)^{1/3},$$

Similarly, let $\tilde{x}_{\phi_j(N)}$ and $\mathfrak{d}_{j,N}$ be associated to $\mathfrak{b}_j$ as in Proposition 2.12, and denote by

$$\mathfrak{b}_{j,N} = g_N(\mathfrak{d}_{j,N}), \qquad \delta_{j,N} = \left(\frac{2}{g''_N(\mathfrak{d}_{j,N})}\right)^{1/3},$$

Then, for every real numbers $(s_i)_{i\in I}$, $(t_j)_{j\in J}$, we have

$$\lim_{N\to\infty} \mathbb{P}\left(N^{2/3}\sigma_{i,N}\left(\mathfrak{a}_{i,N} - x_{\varphi_i(N)}\right) \leqslant s_i, \quad N^{2/3}\delta_{j,N}\left(x_{\phi_j(N)} - \mathfrak{b}_{j,N}\right) \leqslant t_j, \quad i \in I\,,\ j \in J\right)$$
$$= \prod_{i\in I} \det\left(I - \mathrm{K}_{\mathrm{Ai}}\right)_{L^2(s_i,\infty)} \prod_{j\in J} \det\left(I - \mathrm{K}_{\mathrm{Ai}}\right)_{L^2(t_j,\infty)}.$$

We prove Theorem 4 in Section 5. Our strategy is to build on the operator-theoretic proof of Bornemann in the case of the smallest and largest eigenvalues of the GUE [21]; it essentially amounts to prove that the off-diagonal entries of a two by two operator valued matrix decay to zero in the trace class norm. In our setting, the problem involves a larger operator valued matrix and we show that obtaining the decay to zero for the off-diagonal entries in the Hilbert-Schmidt norm is actually sufficient. We establish the latter by using the estimates established in Section 4.

A comment on universality. The results presented in this paper rely on the fact that the entries of $\mathbf{X}_N$ are complex Gaussian random variables, a key assumption in order to take advantage of the determinantal structure of the eigenvalues of the model under study. A recent work [42] by Knowles and Yin enables to transfer the results presented here (except the hard edge fluctuations, see Theorem 5 below) to the case of complex, but not necessarily Gaussian, random variables. Indeed, by combining the local convergence to the limiting distribution established in [42] together with Theorems 3 and 4 , one obtains Tracy-Widom fluctuations and asymptotic independence in this more general setting, provided that the entries of matrix $\mathbf{X}_N$ fulfill some moment condition. This also provides a similar generalization of our Proposition 3.2 describing the asymptotic behavior for the condition number of $\mathbf{M}_N$ when $\gamma > 1$. Let us stress that the case of real Gaussian random variables (except the largest one covered in [45]), of important interest in statistical applications, remains open.

3.3 Fluctuations at the hard edge

Proposition 2.4 shows that when the leftmost edge is a hard edge, $\gamma = 1$ (actually, one can show that this is an equivalence). In order to study the smallest random eigenvalue fluctuations at the hard edge, we restrict ourselves to the case where $n = N + \alpha$, where $\alpha \in \mathbb{Z}$ is independent of N. Thus, the smallest random eigenvalue of $\mathbf{M}_N$ is

$$x_{\min} = \begin{cases} x_1 = \tilde{x}_{\alpha+1} & \text{if } \alpha \geqslant 0, \\ x_{1-\alpha} = \tilde{x}_1 & \text{if } \alpha < 0. \end{cases}$$

We shall prove that the fluctuations of $x_{\min}$ around the origin are described by mean of the Bessel kernel with parameter α, that we introduce now.

The Bessel function of the first kind J_α with parameter $\alpha \in \mathbb{Z}$ is defined by

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{2n}, \qquad x > 0. \tag{3.5}$$

Note that when $\alpha < 0$, the first $|\alpha|$ terms in the series vanish since the Gamma function Γ has simple poles on the non-positive integers. Denote by $\mathrm{K}_{\mathrm{Be},\alpha}$ the Bessel kernel

$$\mathrm{K}_{\mathrm{Be},\alpha}(x,y) = \frac{\sqrt{y}J_\alpha(\sqrt{x})J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)}\,, \tag{3.6}$$

and by extension $\mathrm{K}_{\mathrm{Be},\alpha}$ the associated integral operator. Given a non-negative real-valued random variable X, the following probability distribution will be of particular interest

$$\mathbb{P}(X \geqslant s) = \det\left(I - \mathrm{K}_{\mathrm{Be},\alpha}\right)_{L^2(0,s)}, \qquad s > 0,$$

where the right hand side stands for the Fredholm determinant of the restriction to $L^2(0,s)$ of the integral operator $\mathrm{K}_{\mathrm{Be},\alpha}$. When $\alpha = 0$ this is actually the distribution of an exponential law of parameter 1, namely $\det\big(I - \mathrm{K}_{\mathrm{Be},0}\big)_{L^2(0,s)} = e^{-s}$. Also of interest is the alternative representation due to Tracy and Widom [68]:

$$\det\big(I - \mathrm{K}_{\mathrm{Be},\alpha}\big)_{L^2(0,s)} = \exp\left(-\frac{1}{4}\int_0^s (\log s - \log x) q(x)^2 \mathrm{d}x\right),$$

where q is the solution of a differential equation which is reducible to a particular case of the Painlevé V equation (involving α in its parameters) and boundary condition $q(x) \sim J_\alpha(\sqrt{x})$ as $x \to 0$.

Let us now state our result for the fluctuations around the hard edge.

Theorem 5. *Let Assumptions 1 and 2 hold true; assume moreover that $n = N + \alpha$, where $\alpha \in \mathbb{Z}$ is independent of N. Set*

$$\sigma_N = \frac{4}{N}\sum_{j=1}^{n}\frac{1}{\lambda_j}. \tag{3.7}$$

Then, for every $s > 0$, we have

$$\lim_{N\to\infty} \mathbb{P}\Big(N^2 \sigma_N\, x_{\min} \geqslant s\Big) = \det\big(I - \mathrm{K}_{\mathrm{Be},\alpha}\big)_{L^2(0,s)}. \tag{3.8}$$

In particular, if $N = n$, then we have for every $s > 0$

$$\lim_{N\to\infty} \mathbb{P}\Big(N^2 \sigma_N\, x_{\min} \geqslant s\Big) = e^{-s}. \tag{3.9}$$

Remark 3.1. The assumption that ν_N converges weakly toward some limit ν is actually not used in the proof of Theorem 5. Namely, this result holds true under Assumption 1 and Assumption 2-2 only.

We provide a proof for Theorem 5 in Section 6. It is also based on a asymptotic analysis for the rescaled kernel; the key observation here is that when an edge is the hard edge, the associated critical point $\mathfrak{c}$ should be located at infinity (when embedding the complex plane into the Riemann sphere).

3.4 Application: Condition numbers

The condition number of the matrix $\mathbf{M}_N$ with eigenvalues $0 \leqslant x_1 \leqslant \cdots \leqslant x_N$ is defined by

$$\kappa_N = \frac{x_N}{x_1},$$

provided it is finite, that is $n/N \geqslant 1$. If $n/N < 1$, one may instead consider the condition number associated to $\widetilde{\mathbf{M}}_N$ defined as $\tilde{\kappa}_N = \tilde{x}_n/\tilde{x}_1$. The study of condition numbers is important in numerical linear algebra [71, 38] and random matrix theory has already provided interesting theoretical [30, 11] and applied [49, 15] results. As a consequence of our former results, we provide an asymptotic study for κ_N (one can easily derive similar results for $\tilde{\kappa}_N$).

Notation: We use the notation $\xrightarrow{\mathcal{D}}$ for the convergence in distribution of random variables.

Proposition 3.2. *Let Assumptions 1 and 2 hold true and $\gamma > 1$. Let $\mathfrak{a}$ be the leftmost edge, assume it is regular and let $(\mathfrak{c}_N)_N$ and $\mathfrak{c}$ be as in Proposition 2.11. Let $\mathfrak{b}$ be the rightmost edge, assume it is regular and let $(\mathfrak{d}_N)_N$ and $\mathfrak{d}$ be as in Proposition 2.12. Denote by*

$$\mathfrak{a}_N = g_N(\mathfrak{c}_N)\ , \quad \sigma_N = \left(\frac{2}{-g_N''(\mathfrak{c}_N)}\right)^{1/3} \quad \text{and} \quad \mathfrak{b}_N = g_N(\mathfrak{d}_N)\ , \quad \delta_N = \left(\frac{2}{g_N''(\mathfrak{d}_N)}\right)^{1/3} .$$

Assume moreover that $x_1 \to \mathfrak{a}$ and $x_N \to \mathfrak{b}$ a.s. Then

$$\kappa_N \xrightarrow[N\to\infty]{a.s.} \frac{\mathfrak{b}}{\mathfrak{a}} \quad \text{and} \quad N^{2/3}\left(\kappa_N - \frac{\mathfrak{b}_N}{\mathfrak{a}_N}\right) \xrightarrow[N\to\infty]{\mathcal{D}} \frac{X}{\delta\mathfrak{a}} + \frac{\mathfrak{b}Y}{\sigma\mathfrak{a}^2}$$

where X and Y are two independent Tracy-Widom distributed random variables, and where

$$\sigma = \left(\frac{2}{-g''(\mathfrak{c})}\right)^{1/3} = \lim_{N\to\infty} \sigma_N \quad \text{and} \quad \delta = \left(\frac{2}{g''(\mathfrak{d})}\right)^{1/3} = \lim_{N\to\infty} \delta_N\ .$$

Remark 3.3. The condition that $x_1 \to \mathfrak{a}$ and $x_N \to \mathfrak{b}$ a.s. imposes that neither x_N nor x_1 are outliers, otherwise their fluctuations (together with those of κ_N) would be of order $N^{1/2}$ and a different asymptotic analysis (somewhat easier) should be conducted. We do not pursue in this direction here.

Proof. Only the convergence in distribution requires an argument. Write

$$\begin{aligned} N^{2/3}\left(\kappa_N - \frac{\mathfrak{b}_N}{\mathfrak{a}_N}\right) &= N^{2/3}\left(\frac{x_N}{x_1} - \frac{\mathfrak{b}_N}{\mathfrak{a}_N}\right) = N^{2/3}\left(\frac{\mathfrak{a}_N x_N - \mathfrak{b}_N x_1}{x_1\mathfrak{a}_N}\right) \\ &= \frac{N^{2/3}}{x_1\mathfrak{a}_N}\left\{\mathfrak{a}_N\left(x_N - \mathfrak{b}_N\right) - \mathfrak{b}_N\left(x_1 - \mathfrak{a}_N\right)\right\} \\ &= \frac{1}{x_1\delta_N} N^{2/3}\delta_N\left(x_N - \mathfrak{b}_N\right) + \frac{\mathfrak{b}_N}{x_1\mathfrak{a}_N\sigma_N} N^{2/3}\sigma_N\left(\mathfrak{a}_N - x_1\right) . \end{aligned}$$

Using the asymptotically independent Tracy-Widom fluctuations of $N^{2/3}\delta_N(x_N - \mathfrak{b}_N)$ and $N^{2/3}\sigma_N(\mathfrak{a}_N - x_1)$ (cf. Theorems 3 and 4) together with the a.s. convergence $x_1 \to \mathfrak{a}$ and the convergences $\mathfrak{a}_N \to \mathfrak{a}$, $\mathfrak{b}_N \to \mathfrak{b}$, $\delta_N \to \delta$ and $\sigma_N \to \sigma$ (cf. Prop. 2.7), one can conclude using Slutsky's lemma [69, Lemma 2.8]. □

We now handle the case where $\gamma = 1$.

Proposition 3.4. *Let Assumptions 1 and 2 hold true and let $n = N + \alpha$ where $\alpha \in \mathbb{N}$ is independent of N. Let*

$$\sigma_N = \frac{4}{N}\sum_{j=1}^{n}\frac{1}{\lambda_j} \quad \text{and} \quad \sigma = 4\int\frac{1}{x}\,\mathrm{d}\nu(x) = \lim_{N\to\infty}\sigma_N\ .$$

Assume that a.s. $x_N \to \mathfrak{b}$ for some $\mathfrak{b} > 0$. Then

$$\frac{1}{N^2}\kappa_N \xrightarrow[N\to\infty]{\mathcal{D}} \frac{\mathfrak{b}\sigma}{X}$$

where X is a random variable with distribution

$$\mathbb{P}\left(X \geqslant s\right) = \det\left(I - \mathrm{K}_{\mathrm{Be},\alpha}\right)_{L^2(0,s)}\ , \quad s > 0\ .$$

Proof. Write

$$\frac{\kappa_N}{N^2} = \frac{\sigma_N(x_N - \mathfrak{b})}{N^2\sigma_N x_1} + \frac{\sigma_N \mathfrak{b}}{N^2\sigma_N x_1} \ .$$

Since by assumption $x_N - \mathfrak{b} \to 0$ a.s. and by Theorem 5 $(N^2\sigma_N x_1)^{-1} \to X^{-1}$ in distribution, where X has the distribution specified in the statement, we have

$$\frac{\sigma_N(x_N - \mathfrak{b})}{N^2\sigma_N x_1} \xrightarrow[N\to\infty]{\mathcal{D}} 0 \ .$$

By Slutsky's lemma, $N^{-2}\kappa_N$ then converges toward $\mathfrak{b}\sigma X^{-1}$ in distribution. $\square$

Remark 3.5. Interestingly, in the square case where $\gamma = 1$, the fluctuations of the largest eigenvalue x_N (either of order $N^{1/2}$ if x_N is an outlier or of order $N^{2/3}$ in the Tracy-widom regime) have no influence on the fluctuations of κ_N as these are imposed by the limiting distribution of x_1 at the hard edge.

4 Proof of Theorem 3: Tracy-Widom fluctuations

This section is devoted to the proof of Theorem 3.

4.1 Outline of the proof

Step 1 (preparation): As in [8] and [31], the starting point to establish Tracy-Widom fluctuations is that the random eigenvalues of $\mathbf{M}_N$ or $\widetilde{\mathbf{M}}_N$ form a determinantal point process, so that the gap probabilities can be expressed as Fredholm determinants of an integral operator K_N with kernel $\mathrm{K}_N(x,y)$. We provide all the necessary material from operator theory in Section 4.2. In Section 4.3 we first recall the double contour integral formula for $\mathrm{K}_N(x,y)$ obtained in [8, 55]. Next, we show using Theorem 2 that one can represent the cumulative distribution functions for the extremal eigenvalues as Fredholm determinants involving K_N asymptotically. As a consequence, proving the Tracy-Widom fluctuations boils down to establish the appropriate convergence of rescaled versions $\widetilde{\mathrm{K}}_N(x,y)$ of the kernel $\mathrm{K}_N(x,y)$ toward the Airy kernel. To this end, we split $\widetilde{\mathrm{K}}_N(x,y)$ into two parts, $\mathrm{K}_N^{(0)}(x,y)$ and $\mathrm{K}_N^{(1)}(x,y)$, each involving different integration contours.

Step 2 (contours deformations): Anticipating the forthcoming asymptotic analysis, we focus in Section 4.4 on right edges and prove the existence of appropriate integration contours coming with $\mathrm{K}_N^{(0)}(x,y)$ and $\mathrm{K}_N^{(1)}(x,y)$; the case of a left edge is deferred to Section 4.6. To obtain appropriate explicit contours is usually the hard part in the asymptotic analysis, see in particular [31]. Here, we instead provide a non-constructive proof for the existence of appropriate contours by mean of the maximum principle for subharmonic functions, and which has the advantage to work for every regular edge up to minor modifications.

Step 3 (asymptotic analysis): Still focusing on the right edge setting, we prove in Section 4.5.1 that $\mathrm{K}_N^{(0)}(x,y)$ does not contribute in the large N limit. Moreover, we prove the convergence of kernel $\mathrm{K}_N^{(1)}(x,y)$ to the Airy kernel in an apropriate sense and then complete the proof of Theorem 3-(b). For this last step, we use a different approach than in [8, 31]: Instead of relying on a factorization trick and the Hölder inequality to obtain the trace class convergence, we use an argument involving the regularized Fredholm determinant $\det_2$ to show the convergence of the Fredholm determinants. Finally, in Section 4.6, we adapt the arguments to the left edge setting and complete the proof of Theorem 3.

4.2 Operators, Fredholm determinants and determinantal processes

Trace class operators and Fredholm determinants. We provide hereafter a few elements of operator theory; classical references are [27, 37, 64]. Consider a compact linear operator A acting on a separable Hilbert space $\mathcal{H}$ (we write $\mathrm{A} \in L(\mathcal{H})$), and denote by $(s_n)_{n=1}^{\infty}$ the **singular values** of A repeated according to their multiplicities, i.e. the eigenvalues of $(\mathrm{AA}^*)^{1/2}$. The set

$$\mathcal{J}_1 = \left\{ \mathrm{A} \in L(\mathcal{H})\,,\ \sum_{n=1}^{\infty} s_n < \infty \right\}$$

is the (sub-)algebra of **trace class operators** endowed with the norm $\|\mathrm{A}\|_1 = \sum_{n=1}^{\infty} s_n$; $(\mathcal{J}_1, \|\cdot\|_1)$ is complete. If $\mathrm{A} \in \mathcal{J}_1$ with eigenvalues $(a_n)_{n=1}^{\infty}$ (repeated according to their multiplicities), then the **trace** and the **Fredholm determinant** of A:

$$\mathrm{Tr}(\mathrm{A}) = \sum_{n=1}^{\infty} a_n \quad \text{and} \quad \det(I - \mathrm{A}) = \prod_{n=1}^{\infty} (1 - a_n)\ ,$$

are well-defined and finite (Lidskii trace theorem) . The maps $\mathrm{A} \mapsto \mathrm{Tr}(\mathrm{A})$ and $\mathrm{A} \mapsto \det(I - \mathrm{A})$ are continuous on $(\mathcal{J}_1, \|\cdot\|_1)$. If both AB and BA are trace class, then we have the useful identity

$$\det(I - \mathrm{AB}) = \det(I - \mathrm{BA}). \tag{4.1}$$

Similarly, let

$$\mathcal{J}_2 = \left\{ \mathrm{A} \in L(\mathcal{H})\,,\ \sum_{n=1}^{\infty} s_n^2 < \infty \right\}$$

be the (sub)algebra of **Hilbert-Schmidt operators** endowed with the norm $\|A\|_2 = \left\{\sum_{n=1}^{\infty} s_n^2\right\}^{1/2}$. The set $(\mathcal{J}_2, \|\cdot\|_2)$ is complete. If $\mathrm{A} \in \mathcal{J}_2$ with eigenvalues $(a_n)_{n=1}^{\infty}$ (repeated according to their multiplicities), then the **regularized 2-determinant** of A,

$$\det{}_2(I - \mathrm{A}) = \prod_{n=1}^{\infty} (1 - a_n) e^{a_n}\ , \tag{4.2}$$

is well-defined and finite. The map $\mathrm{A} \mapsto \det_2(I - \mathrm{A})$ is continuous on $(\mathcal{J}_2, \|\cdot\|_2)$.

The inclusion $\mathcal{J}_1 \subset \mathcal{J}_2$ is straightforward. Moreover, the Hölder inequality $\|\mathrm{AB}\|_1 \leqslant \|\mathrm{A}\|_2 \|\mathrm{B}\|_2$ yields that if A, B are Hilbert-Schmidt then both AB and BA are trace class. The following simple property will play a key role in the sequel:

Proposition 4.1. *Let* $\mathrm{A} \in \mathcal{J}_1$ *then*

$$\det{}_2(I - \mathrm{A}) = \det(I - \mathrm{A}) e^{\mathrm{Tr}(\mathrm{A})}\ .$$

As a consequence, if the operators $\mathrm{A}_n, \mathrm{A} \in \mathcal{J}_1$ *are such that* $\mathrm{Tr}(\mathrm{A}_n) \to \mathrm{Tr}(\mathrm{A})$ *and* $\|\mathrm{A}_n - \mathrm{A}\|_2 \to 0$ *as* $n \to \infty$*, then*

$$\det(I - \mathrm{A}_n) \xrightarrow[n \to \infty]{} \det(I - \mathrm{A})\ .$$

Integral operators. When working on $\mathcal{H} = L^2(\mathbb{R})$, we identify a given kernel $(x,y) \mapsto \mathrm{K}(x,y)$ with its associated integral operator $\mathrm{K}f = \int \mathrm{K}(\cdot,y)f(y)\,\mathrm{d}y$ acting on $L^2(\mathbb{R})$, provided the latter makes sense. Let $J \subset \mathbb{R}$ be a Borel set and $\mathbf{1}_J$ be the orthogonal projection of $L^2(\mathbb{R})$ onto $L^2(J)$. The restriction $\mathrm{K}|_J$ of K to $L^2(J)$ is defined by

$$\mathrm{K}|_J f(x) = \mathbf{1}_J(x) \int_J \mathrm{K}(x,y)f(y)\,\mathrm{d}y, \qquad f \in L^2(J),$$

and is associated to the kernel $(x,y) \mapsto \mathbf{1}_J(x)\mathrm{K}(x,y)\mathbf{1}_J(y)$, namely $\mathrm{K}|_J = \mathbf{1}_J\mathrm{K}\mathbf{1}_J$. In order to keep track of these projections when dealing with Fredholm determinants, we shall often write $\det(I-\mathrm{K})_{L^2(J)}$ for $\det(I - \mathbf{1}_J\mathrm{K}\mathbf{1}_J)$.

Given a measurable kernel $\mathrm{K} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, the associated integral operator K on $L^2(\mathbb{R})$ is Hilbert-Schmidt if and only if

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{K}(x,y)^2 \mathrm{d}x\mathrm{d}y < \infty \ ,$$

and in this case we have

$$\|\mathrm{K}\|_2 = \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{K}(x,y)^2 \mathrm{d}x\mathrm{d}y\right)^{1/2} . \tag{4.3}$$

We finally recall (cf. [37, Th. 8.1]) that if $\mathrm{K} : [a,b]\times[a,b] \to \mathbb{R}$ is a continuous kernel whose associated operator $\mathbf{1}_{(a,b)}\mathrm{K}\mathbf{1}_{(a,b)}$ is trace class[4] on $L^2(\mathbb{R})$, then

$$\mathrm{Tr}\left(\mathbf{1}_{(a,b)}\mathrm{K}\mathbf{1}_{(a,b)}\right) = \int_a^b \mathrm{K}(x,x)\,\mathrm{d}x \ . \tag{4.4}$$

Convention: From now, the trace Tr and the Hilbert-Schmidt norm $\|\cdot\|_2$ will always refer to the Hilbert space $L^2(\mathbb{R})$.

Determinantal point process. Real random variables $x_1, \dots, x_m$ are said to form a determinantal point process with kernel $\mathrm{K} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ (and Lebesgue measure for reference measure) if its gap probabilities are expressed as Fredholm determinants, namely for any Borel set $J \subset \mathbb{R}$ we have

$$\mathbb{P}\Big(\sharp\big\{1 \leqslant k \leqslant m : \ x_k \in J\big\} = 0\Big) = \det\big(I - \mathrm{K}\big)_{L^2(J)},$$

provided that the right hand side makes sense; the latter stands for the Fredholm determinant of the restriction to $L^2(J)$ of the integral operator with kernel $\mathrm{K}(x,y)$.

4.3 The kernel of a correlated Wishart matrix and its properties

The next proposition will be of fundamental use in this paper.

Proposition 4.2. *Let Assumption 1 holds true. Then, for every N, the $\min(n,N)$ random eigenvalues of $\widetilde{\mathbf{M}}_N$ (and equivalently of $\mathbf{M}_N$) form a determinantal point process associated with the kernel*

$$\mathrm{K}_N(x,y) = \frac{N}{(2i\pi)^2}\oint_\Gamma \mathrm{d}z \oint_\Theta \mathrm{d}w\, e^{-Nx(z-q)+Ny(w-q)} \frac{1}{w-z}\left(\frac{z}{w}\right)^N \prod_{j=1}^n \left(\frac{w-\lambda_j^{-1}}{z-\lambda_j^{-1}}\right), \tag{4.5}$$

[4] See for instance [37, Theorem 8.2] for sufficient conditions on K to be trace class.

where the real $q \in (0, \lambda_n^{-1})$ is a free parameter and we recall that the λ_i's are the eigenvalues of $\mathbf{\Sigma}_N$. Γ and Θ are disjoint closed contours, both oriented counterclockwise, such that Γ encloses the λ_j^{-1}'s and lies in $\{z \in \mathbb{C} : \operatorname{Re} z > q\}$, whereas Θ encloses the origin and lies in $\{z \in \mathbb{C} : \operatorname{Re} z < q\}$.

By convention, all the contours we shall consider will be assumed to be simple and oriented counterclockwise. The integration contours are shown in Figure 4.

This proposition can be found in [8] ($n/N \leqslant 1$) where it is attributed to Johansson, and in [55] ($n/N > 1$). Notice that since the pioneering work of Brézin and Hikami [24], many such double integral representations appeared for determinantal point processes.

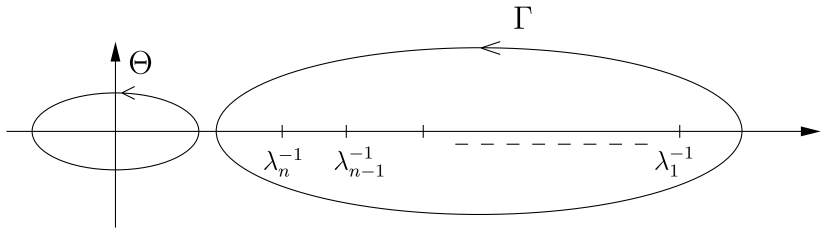

Figure 4: The contours of integration

Remark 4.3. The assumption over q, i.e. $q \in (0, \lambda_n^{-1})$, ensures that K_N with kernel (4.5) is trace class on $L^2(\mathbb{R})$. In the sequel, we shall only need K_N to be **locally trace class**, that is trace class on $L^2(J)$ for every compact subset $J \subset \mathbb{R}$. As an important consequence, we can choose $q \in \mathbb{R}$ with no further restriction. In fact, let $q \in (0, \lambda_n^{-1})$, $q' \in \mathbb{R}$ and $J \subset \mathbb{R}$ a compact set, then the multiplication operator $\mathrm{E} : f(x) \mapsto e^{(q'-q)Nx} f(x)$ and its inverse E^{-1} are trace class on $L^2(J)$. Write $\mathrm{K}_N = \mathrm{K}_N \mathrm{E}^{-1}\mathrm{E}$ and use (4.1) to get

$$\det(I - \mathrm{K}_N)_{L^2(J)} = \det(I - \mathrm{E}\mathrm{K}_N\mathrm{E}^{-1})_{L^2(J)} .$$

The kernel of $\mathrm{E}\mathrm{K}_N\mathrm{E}^{-1}$ is simply obtained by (4.5) where q has been replaced by q' and our claim follows.

Asymptotic determinantal representation for the law of extremal eigenvalues. Recall that to prove Tracy-Widom fluctuations for the maximal eigenvalue $\tilde{x}_n$ of $\widetilde{\mathbf{M}}_N$, a classical way to proceed is to identify the events $\{N^{2/3}\sigma_N(\tilde{x}_n - \mathfrak{b}_N) \leqslant s\} = \{\text{no } \tilde{x}_i\text{'s in } (\mathfrak{b}_N + s/(N^{2/3}\sigma_N), \infty)\}$, to use the determinantal representation

$$\mathbb{P}\Big(N^{2/3}\sigma_N(\tilde{x}_n - \mathfrak{b}_N) \leqslant s\Big) = \det(I - \mathrm{K}_N)_{L^2(\mathfrak{b}_N + s/(N^{2/3}\sigma_N), \infty)} ,$$

and to prove the convergence of operator K_N to the Airy operator K_{Ai} after the rescaling $x \mapsto \mathfrak{b}_N + x/(N^{2/3}\sigma_N)$ for the trace class topology. This would yield the desired result since the Fredholm determinant is continuous for that topology.

Since the probabilities of interest $\mathbb{P}(N^{2/3}\sigma_N(\tilde{x}_{\phi(N)} - \mathfrak{b}_N) \leqslant s)$ and $\mathbb{P}(N^{2/3}\sigma_N(\mathfrak{a}_N - \tilde{x}_{\varphi(N)}) \leqslant s)$ can no longer be expressed as gap probabilities in general, we provide below an asymptotic Fredholm determinant representation as $N \to \infty$ for these.

Proposition 4.4. *Consider the setting of Theorem 3 and recall that by convention $\tilde{x}_0 = 0$ and $\tilde{x}_{n+1} = +\infty$. Then the following facts hold true:*

(a) *For every $\varepsilon > 0$ small enough and for every sequence $(\eta_N)_N$ of positive numbers satisfying $\lim_N \eta_N = +\infty$,*

$$\mathbb{P}\Big(\eta_N\big(\mathfrak{a}_N - \tilde{x}_{\varphi(N)}\big) \leqslant s\Big) = \det\big(I - \mathrm{K}_N\big)_{L^2(\mathfrak{a}_N-\varepsilon,\,\mathfrak{a}_N - s/\eta_N)} + o(1) \tag{4.6}$$

as $N \to \infty$.

(b) *For every $\varepsilon > 0$ small enough and for every sequence $(\eta_N)_N$ of positive numbers satisfying $\lim_N \eta_N = +\infty$,*

$$\mathbb{P}\Big(\eta_N\big(\tilde{x}_{\phi(N)} - \mathfrak{b}_N\big) \leqslant s\Big) = \det\big(I - \mathrm{K}_N\big)_{L^2(\mathfrak{b}_N+s/\eta_N,\,\mathfrak{b}_N+\varepsilon)} + o(1) \tag{4.7}$$

as $N \to \infty$.

Proof. We only prove (b), proof of (a) being similar. Observe that Theorem 2–(b) and the convergence $\mathfrak{b}_N \to \mathfrak{b}$ yield together the existence of $\varepsilon > 0$ small enough such that

$$\mathbb{P}\Big(\eta_N\big(\tilde{x}_{\phi(N)} - \mathfrak{b}_N\big) \leqslant s\Big) = \mathbb{P}\Big(\eta_N\big(\tilde{x}_{\phi(N)} - \mathfrak{b}_N\big) \leqslant s\ ,\ \tilde{x}_{\phi(N)+1} \geqslant \mathfrak{b}_N + \varepsilon\Big) + o(1) \tag{4.8}$$

as $N \to \infty$. Now, ε being fixed, use the determinantal representation to write

$$\det\big(I - \mathrm{K}_N\big)_{L^2(\mathfrak{b}_N+s/\eta_N\,,\,\mathfrak{b}_N+\varepsilon)} = \mathbb{P}\left(\sharp\Big\{\ell \leqslant k \leqslant n:\ \mathfrak{b}_N + s/\eta_N \leqslant \tilde{x}_k \leqslant \mathfrak{b}_N + \varepsilon\Big\} = 0\right) \tag{4.9}$$

where $\ell = n - \min(N, n) + 1$. Recall the notational convention in Remark 2.8; we obtain by splitting along disjoint events

$$\begin{aligned}
&\mathbb{P}\left(\sharp\Big\{\ell \leqslant k \leqslant n:\ \mathfrak{b}_N + s/\eta_N \leqslant \tilde{x}_k \leqslant \mathfrak{b}_N + \varepsilon\Big\} = 0\right)\\
&= \quad \mathbb{P}\Big(\eta_N\big(\tilde{x}_{\phi(N)} - \mathfrak{b}_N\big) \leqslant s\ ,\ \tilde{x}_{\phi(N)+1} \geqslant \mathfrak{b}_N + \varepsilon\Big)\\
&\quad + \quad \mathbb{P}\Big(\tilde{x}_\ell \geqslant \mathfrak{b}_N + \varepsilon\Big)\\
&\quad + \sum_{k=\ell,\, k\neq\phi(N)}^{n} \mathbb{P}\Big(\tilde{x}_k \leqslant \mathfrak{b}_N + s/\eta_N\ ,\ \tilde{x}_{k+1} \geqslant \mathfrak{b}_N + \varepsilon\Big).
\end{aligned} \tag{4.10}$$

Since we have the upper bounds

$$\begin{aligned}
\sum_{k=\ell}^{\phi(N)-1} \mathbb{P}\Big(\tilde{x}_k \leqslant \mathfrak{b}_N + s/\eta_N\ ,\ \tilde{x}_{k+1} \geqslant \mathfrak{b}_N + \varepsilon\Big) &\leqslant \mathbb{P}\Big(\tilde{x}_{\phi(N)} \geqslant \mathfrak{b}_N + \varepsilon\Big)\ ,\\
\sum_{k=\phi(N)+1}^{n} \mathbb{P}\Big(\tilde{x}_k \leqslant \mathfrak{b}_N + s/\eta_N\ ,\ \tilde{x}_{k+1} \geqslant \mathfrak{b}_N + \varepsilon\Big) &\leqslant \mathbb{P}\Big(\tilde{x}_{\phi(N)+1} \leqslant \mathfrak{b}_N + s/\eta_N\Big),
\end{aligned}$$

we obtain from (4.9), (4.10), Th. 2–(b) and the convergence $\mathfrak{b}_N \to \mathfrak{b}$ that

$$\det\big(I - \mathrm{K}_N\big)_{L^2(\mathfrak{b}_N+s/\eta_N\,,\,\mathfrak{b}_N+\varepsilon)} = \mathbb{P}\Big(\eta_N\big(\tilde{x}_{\phi(N)} - \mathfrak{b}_N\big) \leqslant s\ ,\ \tilde{x}_{\phi(N)+1} \geqslant \mathfrak{b}_N + \varepsilon\Big) + o(1). \tag{4.11}$$

Finally, (4.7) follows by combining (4.8) and (4.11). □

Rescaling and splitting the kernel K_N**.** We introduce hereafter the rescaled kernel $\widetilde{\mathrm{K}}_N$ and provide an alternative integral representation with new contours. The aim is to prepare the forthcoming asymptotic analysis for right regular edges.

Let $\mathfrak{b}$ be a soft regular right edge. By Proposition 2.12, there exist $\mathfrak{d} > 0$ such that

$$\mathfrak{b} = g(\mathfrak{d}), \qquad g'(\mathfrak{d}) = 0, \qquad g''(\mathfrak{d}) > 0, \tag{4.12}$$

and an associated sequence $(\mathfrak{d}_N)$ such that $g_N^{(k)}(\mathfrak{d}_N) \to g^{(k)}(\mathfrak{d})$. Denote by

$$\mathfrak{b}_N = g_N(\mathfrak{d}_N), \qquad \delta_N = \Big(\frac{2}{g_N''(\mathfrak{d}_N)}\Big)^{1/3}, \tag{4.13}$$

then in particular

$$g_N'(\mathfrak{d}_N) = 0, \qquad \lim_{N\to\infty} \mathfrak{d}_N = \mathfrak{d}, \qquad \lim_{N\to\infty} \mathfrak{b}_N = \mathfrak{b}, \qquad \lim_{N\to\infty} \delta_N = \Big(\frac{2}{g''(\mathfrak{d})}\Big)^{1/3}. \tag{4.14}$$

In particular $\mathfrak{c}_N$, $g_N''(\mathfrak{c}_N)$ and σ_N are positive numbers for every N large enough, and $(\sigma_N)_N$ is a bounded sequence.

It follows from the definition of the extremal eigenvalue $\tilde{x}_{\phi(N)}$, see Theorem 2, and Proposition 4.4 that for every $\varepsilon > 0$ small enough

$$\mathbb{P}\Big(N^{2/3}\delta_N\big(\tilde{x}_{\phi(N)} - \mathfrak{b}_N\big) \leqslant s\Big) = \det\big(I - \mathrm{K}_N\big)_{L^2(\mathfrak{b}_N + s/(N^{2/3}\delta_N),\, \mathfrak{b}_N+\varepsilon)} + o(1) \tag{4.15}$$

as $N\to\infty$. By a change of variable, we can write

$$\det\big(I - \mathrm{K}_N\big)_{L^2(\mathfrak{b}_N + s/(N^{2/3}\delta_N),\, \mathfrak{b}_N+\varepsilon)} = \det\big(I - \mathbf{1}_{(s,\,\varepsilon N^{2/3}\delta_N)}\widetilde{\mathrm{K}}_N\mathbf{1}_{(s,\,\varepsilon N^{2/3}\delta_N)}\big)_{L^2(s,\infty)}, \tag{4.16}$$

where the scaled integral operator $\widetilde{\mathrm{K}}_N$ has kernel

$$\widetilde{\mathrm{K}}_N(x,y) = \frac{1}{N^{2/3}\delta_N}\mathrm{K}_N\Big(\mathfrak{b}_N + \frac{x}{N^{2/3}\delta_N}, \mathfrak{b}_N + \frac{y}{N^{2/3}\delta_N}\Big) \tag{4.17}$$

and where $\mathrm{K}_N(x,y)$ was introduced in (4.5). Consider the map

$$f_N(z) = -\mathfrak{b}_N(z - \mathfrak{d}_N) + \log(z) - \frac{1}{N}\sum_{i=1}^{n}\log(1-\lambda_i z). \tag{4.18}$$

Remark 4.5. In order to fully define f_N, one needs to specify the determination of the logarithm. This will be done when needed. Notice however that functions $\operatorname{Re} f_N$, $\exp(f_N)$ and the derivatives $f_N^{(k)}$ are always well-defined.

By taking $q = \mathfrak{d}_N$ in (4.5), which is possible according to Remark 4.3, we have

$$\mathrm{K}_N(x,y) = \frac{N}{(2i\pi)^2}\oint_\Gamma \mathrm{d}z \oint_\Theta \mathrm{d}w\, e^{-Nx(z-\mathfrak{d}_N)+Ny(w-\mathfrak{d}_N)}\frac{1}{w-z}\Big(\frac{z}{w}\Big)^N\prod_{j=1}^{n}\Big(\frac{1-\lambda_j w}{1-\lambda_j z}\Big), \tag{4.19}$$

where we recall that the contour Γ encloses the λ_j^{-1}'s whereas the contour Θ encloses the origin and is disjoint from Γ. It then follows from the definition (4.17) of $\widetilde{\mathrm{K}}_N$ that

$$\widetilde{\mathrm{K}}_N(x,y) = \frac{N^{1/3}}{(2i\pi)^2\delta_N}\oint_\Gamma \mathrm{d}z\oint_\Theta \mathrm{d}w\,\frac{1}{w-z}e^{-N^{1/3}x\frac{(z-\mathfrak{d}_N)}{\delta_N}+N^{1/3}y\frac{(w-\mathfrak{d}_N)}{\delta_N}+Nf_N(z)-Nf_N(w)}. \tag{4.20}$$

The key observation here is the identity

$$f'_N(z) = g_N(z) - g_N(\mathfrak{d}_N), \tag{4.21}$$

which follows from (2.6) and (4.13). As a byproduct, (4.14) yields that $\mathfrak{d}_N$ is a root of multiplicity two for f'_N, and more precisely

$$f'_N(\mathfrak{d}_N) = f''(\mathfrak{d}_N) = 0, \qquad f_N^{(3)}(\mathfrak{d}_N) = g''_N(\mathfrak{d}_N) > 0. \tag{4.22}$$

The aim is to perform a saddle point analysis for f_N around its critical point $\mathfrak{d}_N$. To this end, we deform the contours Γ and Θ in a way that they pass near $\mathfrak{d}_N$.

If $\mathfrak{d}_N$ is smaller than all the λ_j^{-1}'s, as it is the case in [31] when dealing with the maximal eigenvalue, then go directly to Section 4.4, set $\Gamma^{(1)} = \Gamma$, $\mathrm{K}_N^{(1)} = \widetilde{\mathrm{K}}_N$ and disregard every statement related to $\Gamma^{(0)}$.

If not, then we proceed in two steps. First, we split Γ into two disjoint contours $\Gamma^{(0)}$ and $\Gamma^{(1)}$ as shown on Figure 5: the contour $\Gamma^{(0)}$ encloses the λ_j^{-1}'s which are smaller than $\mathfrak{d}_N$, while $\Gamma^{(1)}$ encloses the λ_j^{-1}'s which are larger that $\mathfrak{d}_N$. Notice that Proposition 2.4–(d) applied to the measure ν_N shows that the set $\{j\,,\ 1 \leqslant j \leqslant n\ :\ \lambda_j^{-1} > \mathfrak{d}_N\}$ is not empty. Therefore, the contour $\Gamma^{(1)}$ is always well-defined.

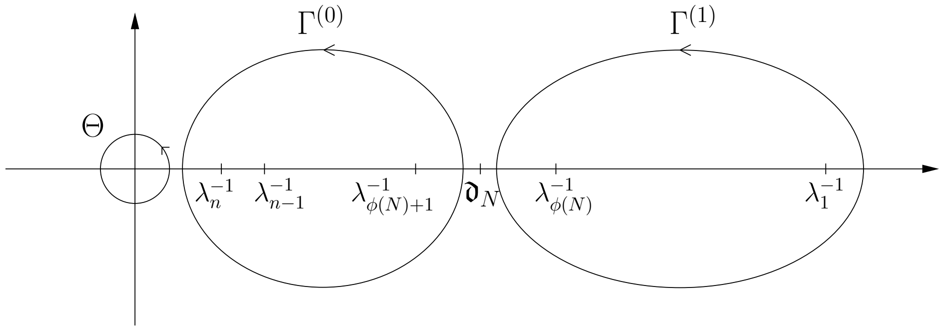

Figure 5: The new contours $\Gamma^{(0)}$ and $\Gamma^{(1)}$.

We now introduce for $\alpha \in \{0, 1\}$ the kernels

$$\mathrm{K}_N^{(\alpha)}(x, y) = \frac{N^{1/3}}{(2i\pi)^2 \delta_N} \oint_{\Gamma^{(\alpha)}} \mathrm{d}z \oint_{\Theta} \mathrm{d}w \, \frac{1}{w - z} e^{-N^{1/3} x \frac{(z - \mathfrak{d}_N)}{\delta_N} + N^{1/3} y \frac{(w - \mathfrak{d}_N)}{\delta_N} + N f_N(z) - N f_N(w)}\ , \tag{4.23}$$

then it follows from the residue theorem that

$$\widetilde{\mathrm{K}}_N(x, y) = \mathrm{K}_N^{(0)}(x, y) + \mathrm{K}_N^{(1)}(x, y)\ , \tag{4.24}$$

and a similar identity for the associated operators.

In the second step, we modify the contour Θ in order for it to surround $\Gamma^{(0)}$ while remaining at the left of $\mathfrak{d}_N$, cf. Figure 6. This can be done with no harm for the kernel $\mathrm{K}_N^{(1)}$. As for $\mathrm{K}_N^{(0)}$, this modification for the contours yields a residue term, coming with the singularity $(w - z)^{-1}$ of the integrand. The latter residue term equals

$$\frac{N^{1/3}}{2i\pi \delta_N} \oint_{\Gamma^{(0)}} e^{N^{1/3} \frac{(y - x)}{\delta_N} (z - \mathfrak{d}_N)} \mathrm{d}z$$

and thus identically vanishes since the integrand is analytic.

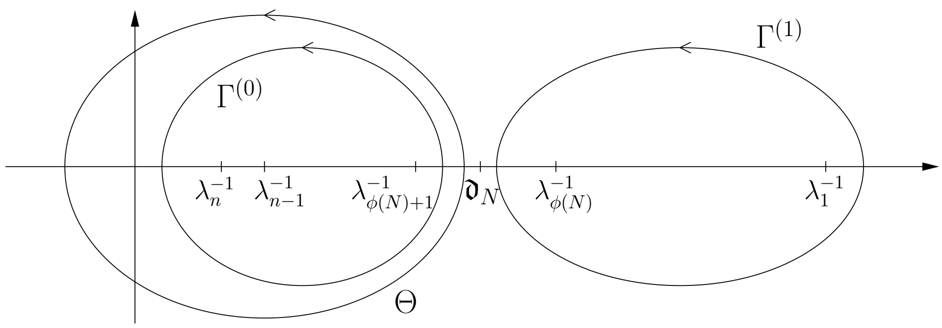

Figure 6: The new contours for the kernel $\widetilde{\mathrm{K}}_N$.

4.4 Contours deformations and subharmonic functions: The right edge case

We now provide the existence of deformations for the contours $\Gamma^{(0)}$, $\Gamma^{(1)}$ and Θ which are appropriate for the asymptotic analysis. These new contours will be referred to as $\Upsilon^{(0)}$, $\Upsilon^{(1)}$ and $\widetilde{\Theta}$.

Proposition 4.6. *For every $\rho > 0$ small enough, there exists a contour $\Upsilon^{(0)}$ independent of N and two contours $\Upsilon^{(1)} = \Upsilon^{(1)}(N)$ and $\widetilde{\Theta} = \widetilde{\Theta}(N)$ which satisfy for every N large enough the following properties.*

(1) (a) $\Upsilon^{(0)}$ *encircles the λ_j^{-1}'s smaller than $\mathfrak{d}_N$,*

(b) $\Upsilon^{(1)}$ *encircles all the λ_j^{-1}'s larger than $\mathfrak{d}_N$,*

(c) $\widetilde{\Theta}$ *encircles all the λ_j^{-1}'s smaller than $\mathfrak{d}_N$ and the origin.*

(2) (a) $\Upsilon^{(1)} = \Upsilon_* \cup \Upsilon^{(1)}_{res}$ *where*

$$\Upsilon_* = \{\mathfrak{d}_N + te^{\pm i\pi/3} : \ t \in [0, \rho]\}.$$

(b) $\widetilde{\Theta} = \widetilde{\Theta}_* \cup \widetilde{\Theta}_{res}$ *where*

$$\widetilde{\Theta}_* = \{\mathfrak{d}_N - te^{\pm i\pi/3} : \ t \in [0, \rho]\}.$$

(3) *There exists $K > 0$ independent of N such that*

(a) $\mathrm{Re}\big(f_N(z) - f_N(\mathfrak{d}_N)\big) \leqslant -K$ *for all $z \in \Upsilon^{(0)}$*

(b) $\mathrm{Re}\big(f_N(z) - f_N(\mathfrak{d}_N)\big) \leqslant -K$ *for all $z \in \Upsilon^{(1)}_{res}$*

(c) $\mathrm{Re}\big(f_N(w) - f_N(\mathfrak{d}_N)\big) \geqslant K$ *for all $w \in \widetilde{\Theta}_{res}$*

(4) *There exists $d > 0$ independent of N such that*

$$\inf\big\{|z - w| : \ z \in \Upsilon^{(0)},\, w \in \widetilde{\Theta}\big\} \geqslant d \ ,$$
$$\inf\big\{|z - w| : \ z \in \Upsilon_*,\, w \in \widetilde{\Theta}_{res}\big\} \geqslant d \ ,$$
$$\inf\big\{|z - w| : \ z \in \Upsilon^{(1)}_{res},\, w \in \widetilde{\Theta}_*\big\} \geqslant d \ ,$$
$$\inf\big\{|z - w| : \ z \in \Upsilon^{(1)}_{res},\, w \in \widetilde{\Theta}_{res}\big\} \geqslant d \ .$$

(5) (a) *The contours* $\Upsilon^{(1)}$ *and* $\widetilde{\Theta}$ *lie in a bounded subset of* $\mathbb{C}$ *independent of* N*,*
(b) *The lengths of* $\Upsilon^{(1)}$ *and* $\widetilde{\Theta}$ *are uniformly bounded in* N*.*

Note that both the contours $\Upsilon^{(1)}$ and $\widetilde{\Theta}$ pass through the critical point $\mathfrak{d}_N$.

In order to provide a proof for Proposition 4.6, we first establish a few lemmas. We recall that $B(z,\rho)$ for $z\in\mathbb{C}$ and $\rho>0$ stands for the open ball of $\mathbb{C}$ with center z and radius ρ.

Recall that $0<\inf_N \lambda_n^{-1}\leqslant \sup_N \lambda_1^{-1}<\infty$ by Assumption 2. By the regularity assumption, namely $\liminf_N \min_{j=1}^n |\mathfrak{d}-\lambda_j^{-1}|>0$, there exists $\varepsilon>0$ such that $\lambda_j^{-1}\in(0,+\infty)\setminus B(\mathfrak{d},\varepsilon)$ for every $1\leqslant j\leqslant n$ and every N large enough. Denote by $\mathcal{K}$ the compact set

$$\mathcal{K}=\left(\left[\inf_N\frac{1}{\lambda_n},\sup_N\frac{1}{\lambda_1}\right]\setminus B(\mathfrak{d},\varepsilon)\right)\cup\{0\}\ . \tag{4.25}$$

Notice that by construction $\{x\in\mathbb{R}:\ x^{-1}\in\mathrm{Supp}(\nu_N)\}\subset\mathcal{K}$ for every N large enough, and also that $\{x\in\mathbb{R}:\ x^{-1}\in\mathrm{Supp}(\nu)\}\subset\mathcal{K}$ because of the weak convergence $\nu_N\to\nu$.

Recall the definition (4.18) of f_N and introduce its asymptotic counterpart:

$$f(z)=-\mathfrak{b}(z-\mathfrak{d})+\log(z)-\gamma\int\log(1-xz)\nu(\mathrm{d}x)\ . \tag{4.26}$$

Notice that whereas f and f_N are defined up to a determination of the complex logarithm,

$$\mathrm{Re}\, f(z)=-\mathfrak{b}\,\mathrm{Re}(z-\mathfrak{d})+\log|z|-\gamma\int\log|1-xz|\nu(\mathrm{d}x) \tag{4.27}$$

and $\mathrm{Re}\, f_N$ are well-defined. The following properties of $\mathrm{Re}\, f$ and $\mathrm{Re}\, f_N$ around $\mathfrak{d}$ and $\mathfrak{d}_N$ will be of constant use in the sequel.

Lemma 4.7. *Let Assumption 2 hold true and let* $\mathcal{K}$ *be as in* (4.25)*. Then*

(a) *The function* $\mathrm{Re}\, f_N$ *converges locally uniformly to* $\mathrm{Re}\, f$ *on* $\mathbb{C}\setminus\mathcal{K}$*. Moreover,*

$$\lim_{N\to\infty}\mathrm{Re}\, f_N(\mathfrak{d}_N)=\mathrm{Re}\, f(\mathfrak{d}). \tag{4.28}$$

(b) *There exists* $\rho_0>0$ *and* $\Delta=\Delta(\rho_0)>0$ *independent of* N *such that for every* N *large enough,* $B(\mathfrak{d}_N,\rho)\subset\mathbb{C}\setminus\mathcal{K}$ *for every* $\rho\in(0,\rho_0]$ *and, whatever the analytic representation of* f_N *on* $B(\mathfrak{d}_N,\rho)$*,*

$$\begin{aligned}\left|f_N(z)-f_N(\mathfrak{d}_N)-g_N''(\mathfrak{d}_N)(z-\mathfrak{d}_N)^3/6\right| &\leqslant \Delta|z-\mathfrak{d}_N|^4\ ,\\ \left|\mathrm{Re}(f_N(z)-f_N(\mathfrak{d}_N))-g_N''(\mathfrak{d}_N)\,\mathrm{Re}[(z-\mathfrak{d}_N)^3]/6\right| &\leqslant \Delta|z-\mathfrak{d}_N|^4\end{aligned}$$

for all $z\in B(\mathfrak{d}_N,\rho_0)$ *.*

(c) *There exists* $\rho_0>0$ *and* $\Delta=\Delta(\rho_0)>0$ *such that* $B(\mathfrak{d},\rho_0)\subset\mathbb{C}\setminus\mathcal{K}$ *and for all* $z\in B(\mathfrak{d},\rho_0)$*,*

$$\left|\mathrm{Re}(f(z)-f(\mathfrak{d}))-g''(\mathfrak{d})\,\mathrm{Re}[(z-\mathfrak{d})^3]/6\right|\ \leqslant\ \Delta|z-\mathfrak{d}|^4\ .$$

Proof. Fix an open ball B of $\mathbb{C}\setminus\mathcal{K}$. By definition of $\mathcal{K}$, one can chose a determination of the logarithm such that f_N is well-defined and holomorphic there for N large enough. Indeed, there exists an analytic determination of the logarithm on every simply connected domain of $\mathbb{C}\setminus\{0\}$. Use the same determination for f, which is then also well-defined and holomorphic on B. By weak convergence of ν_N to ν, f_N converges pointwise to f on B. Similarly as in the proof of Proposition 2.7, the sequence of holomorphic functions $(f_N)_N$ is uniformly bounded on B and thus has compact closure by the Montel theorem, which upgrades the pointwise convergence $f_N \to f$ to the uniform one on B. The uniform convergence of $\operatorname{Re} f_N$ to $\operatorname{Re} f$ on B follows since $|\operatorname{Re} f_N(z) - \operatorname{Re} f(z)| \leqslant |f_N(z) - f(z)|$ for all $z \in B$. Now since $\mathfrak{d}_N \to \mathfrak{d}$ and $\mathfrak{d}_N, \mathfrak{d} \in \mathbb{C}\setminus\mathcal{K}$ for all N large enough by the regularity assumption, (4.28) follows from the local uniform convergence $\operatorname{Re} f_N \to \operatorname{Re} f$ on $\mathbb{C}\setminus\mathcal{K}$ and (a) is proved.

It follows from Proposition 2.7 that for $\rho_0 > 0$ small enough and every N large enough we have $B(\mathfrak{d}_N, \rho_0) \subset B(\mathfrak{d}, 2\rho_0) \subset \mathbb{C}\setminus\mathcal{K}$. Using the same determination of the log as previously yields that f_N is well-defined and holomorphic on $B(\mathfrak{d}_N, \rho_0)$. Since (4.14) and (4.21) yield $f_N'(\mathfrak{d}_N) = f_N''(\mathfrak{d}_N) = 0$, $f_N^{(3)}(\mathfrak{d}_N) = g_N''(\mathfrak{d}_N) > 0$ and $f_N^{(4)} = g_N^{(3)}$ for all N large enough, we can perform a Taylor expansion for f_N around $\mathfrak{d}_N$ in order to get

$$\left|f_N(z) - f_N(\mathfrak{d}_N) - g_N''(\mathfrak{d}_N)(z-\mathfrak{d}_N)^3/6\right| \leqslant \frac{|z-\mathfrak{d}_N|^4}{24} \max_{w\in B(\mathfrak{d},2\rho_0)} \left|g_N^{(3)}(w)\right|$$

provided that $z \in B(\mathfrak{d}_N, \rho_0)$. Proposition 2.7 moreover provides that $g_N^{(3)}$ converges uniformly on $B(\mathfrak{d}, 2\rho_0)$ to $g^{(3)}$ which is bounded there. We therefore get the existence of $\Delta = \Delta(\rho_0)$ independent of N for which the first inequality in Part (b) of the proposition is satisfied. The inequality for the real part directly follows and Part (b) of the proposition is proved; so is Part (c) by using similar arguments.

□

We now provide a qualitative analysis for the map $\operatorname{Re} f$. First, we study the behavior of $\operatorname{Re} f(z)$ as $|z| \to \infty$. To do so, we introduce the sets

$$\Omega_- = \left\{z \in \mathbb{C} : \operatorname{Re} f(z) < \operatorname{Re} f(\mathfrak{d})\right\}, \quad \Omega_+ = \left\{z \in \mathbb{C} : \operatorname{Re} f(z) > \operatorname{Re} f(\mathfrak{d})\right\}, \tag{4.29}$$

and prove the following.

Lemma 4.8. *Both Ω_+ and Ω_- have a unique unbounded connected component. Moreover, given any $\alpha \in (0, \pi/2)$, there exists $R > 0$ large enough such that*

$$\Omega_-^R = \left\{z \in \mathbb{C} : \quad |z| > R, \quad -\frac{\pi}{2} + \alpha < \arg(z) < \frac{\pi}{2} - \alpha\right\} \subset \Omega_-, \tag{4.30}$$

$$\Omega_+^R = \left\{z \in \mathbb{C} : \quad |z| > R, \quad \frac{\pi}{2} + \alpha < \arg(z) < \frac{3\pi}{2} - \alpha\right\} \subset \Omega_+. \tag{4.31}$$

Proof. Recall the expression (4.27) of $\operatorname{Re} f(z)$ which yields that $\operatorname{Re} f(z) = -\mathfrak{b}\operatorname{Re}(z-\mathfrak{d}) + O(\log|z|)$ as $|z| \to \infty$. Since $\mathfrak{b} > 0$, it follows that for any fixed $\alpha \in (0, \pi/2)$ there exists $R > 0$ large enough such that

$$\Omega_-^R \subset \Omega_-, \qquad \Omega_+^R \subset \Omega_+. \tag{4.32}$$

Next, we compute for any $A \in \mathbb{R}\setminus\{0\}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re} f(t+iA) = -\mathfrak{b} + \frac{t}{t^2+A^2} + \gamma\int \frac{(x^{-1}-t)}{(x^{-1}-t)^2+A^2}\,\nu(\mathrm{d}x).$$

Since $\mathfrak{b} > 0$ and $\mathrm{Supp}(\nu)$ is a compact subset of $(0, +\infty)$, there exists $A_0 > 0$ such that for any A satisfying $|A| \geqslant A_0$ the map $t \mapsto \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Re} f(t + iA)$ is negative, namely $t \mapsto \operatorname{Re} f(t + iA)$ is decreasing. Assume there exists an other unbounded connected component of Ω_-, different from the one containing Ω_-^R. By (4.32), this unbounded connected component then lies in $\mathbb{C} \setminus (\Omega_-^R \cup \Omega_+^R)$ and thus there exists z_0 in this component satisfying $|\mathrm{Im}(z_0)| \geqslant A_0$. Since the half line $\{\mathrm{Re}(z_0) + t + i\mathrm{Im}(z_0) : \; t \geqslant 0\}$ then belongs to Ω_- and eventually hits Ω_-^R, we obtain a contradiction. The same arguments apply to Ω_+. □

Next, we describe the behavior of $\operatorname{Re} f$ at the neighborhood of $\mathfrak{d}$. Taking advantage of Lemma 4.7-(c) which encodes that $\operatorname{Re} f(z) - \operatorname{Re} f(\mathfrak{d})$ behaves like $\operatorname{Re}[(z-\mathfrak{d})^3]$ around $\mathfrak{d}$, we describe in the following lemma subdomains of $\Omega_\pm$ of interest.

Lemma 4.9. *There exist $\eta > 0$ and $\theta > 0$ small enough such that, if we set*

$$\Delta_k = \Big\{ z \in \mathbb{C} : \quad 0 < |z - \mathfrak{d}| < \eta, \quad \Big| \arg(z - \mathfrak{d}) - k\frac{\pi}{3} \Big| < \theta \Big\}$$

for $-2 \leqslant k \leqslant 3$, then

$$\Delta_{2k+1} \subset \Omega_- \; , \quad \Delta_{2k} \subset \Omega_+ \; , \quad k \in \{-1, 0, 1\}.$$

The regions Δ_k are shown on Figure 7.

Proof. Recall Lemma 4.7-(c) and let $\eta < \rho_0$ as defined there. Then

$$\big| \operatorname{Re} f(z) - \operatorname{Re} f(\mathfrak{d}) - g''(\mathfrak{d}) \operatorname{Re}[(z-\mathfrak{d})^3]/6 \big| \leqslant \Delta(\rho_0) |z - \mathfrak{d}|^4$$

for every $z \in B(\mathfrak{d}, \eta)$. Notice that $\operatorname{Re}[(z-\mathfrak{d})^3] = (-1)^k$ if $z = \mathfrak{d} + e^{ik\pi/3}$ for consecutive integers k. Since $g''(\mathfrak{d}) > 0$, the lemma follows by choosing η small enough. □

We denote by Ω_{2k+1} the connected component of Ω_- which contains Δ_{2k+1}. Similarly, Ω_{2k} stands for the connected component of Ω_+ which contains Δ_{2k}. We now describe these sets by using the maximum principle for subharmonic functions, in the same spirit as in [29, Sec. 6.1], see also [28, Section 2.4.2], although the setting is more involved here; such a use of the maximum principle has been communicated to us by Steven Delvaux.

Recall that if G is an open subset of $\mathbb{C}$, a function $u : G \to \mathbb{R} \cup \{-\infty\}$ is **subharmonic** if u is upper semicontinuous, i.e. $\{z \in G\, , \; u(z) < \alpha\}$ is open for every $\alpha \in \mathbb{R}$, and, for every closed disk $\overline{B}(z, \delta)$ contained in G, we have the inequality

$$u(z) \;\leqslant\; \frac{1}{2\pi} \int_0^{2\pi} u(z + \delta e^{i\theta}) \mathrm{d}\theta \; .$$

A function $u : G \to \mathbb{R} \cup \{+\infty\}$ is **superharmonic** if $-u$ is subharmonic; in particular it is lower semicontinuous. Moreover, if $u : G \to \mathbb{C}$ is subharmonic, it satisfies a maximum principle: For any bounded domain (i.e. connected open set) $U \subset \mathbb{C}$ where u is subharmonic, if for some $\kappa \in \mathbb{R}$ it holds that

$$\limsup_{z \to \zeta, \; z \in U} u(z) \;\leqslant\; \kappa \; , \quad \zeta \in \partial U \, ,$$

then $u \leqslant \kappa$ on U. Similarly, superharmonic functions satisfy a minimum principle.

The use of the maximum principle for subharmonic functions is made possible here because of the following observation.

Lemma 4.10. *The function* $\operatorname{Re} f$ *is subharmonic on* $\mathbb{C} \setminus \{x \in \mathbb{R} : \ x^{-1} \in \operatorname{Supp}(\nu)\}$ *and superharmonic on* $\mathbb{C} \setminus \{0\}$*.*

Proof. It will be enough to establish the result for the map

$$z \mapsto \log|z| - \gamma \int \log|1 - xz| \, \nu(dx) = \log|z| - \gamma \int \log|z - x| \, \tau(dx) - \gamma \int \log x \, \nu(dx)$$

where the compactly supported probability measure τ is the image of ν by $x \mapsto x^{-1}$. The assumptions on ν imply that $\log x$ is ν-integrable. Now, it is a standard fact from potential theory that, given a positive Borel measure η on $\mathbb{C}$ with compact support, the map $z \mapsto \int \log|z - x| \, \eta(\mathrm{d}x)$ is subharmonic on $\mathbb{C}$ and harmonic on $\mathbb{C} \setminus \operatorname{Supp}(\eta)$, see e.g. [61, Chapter 0]. Consequently, $z \mapsto \log|z|$ is harmonic on $\mathbb{C} \setminus \{0\}$ and subharmonic on $\mathbb{C}$, and $z \mapsto \gamma \int \log|z - x| \, \tau(dx)$ is harmonic on $\mathbb{C} \setminus \operatorname{Supp}(\tau)$ and subharmonic on $\mathbb{C}$. The result follows. □

Equipped with Lemma 4.10, we can obtain more information concerning the connected components of $\Omega_\pm$.

Lemma 4.11. *The following hold true.*

1. *If* Ω_* *is a connected component of* Ω_+*, then* Ω_* *is open and, if* Ω_* *is moreover bounded, there exists* $x \in \operatorname{Supp}(\nu)$ *such that* $x^{-1} \in \Omega_*$*.*
2. *Let* Ω_* *be a connected component of* Ω_- *with non-empty interior.*
   (a) *If* Ω_* *is bounded, then* $0 \in \Omega_*$*.*
   (b) *If* Ω_* *is bounded, then its interior is connected.*
   (c) *If* $0 \notin \Omega_*$*, then the interior of* Ω_* *is connected.*

Proof. Let us show 1. We set $\alpha = \operatorname{Re} f(\mathfrak{d})$. Since $\operatorname{Re} f(z) \to -\infty$ as $|z| \to 0$, then $0 \notin \{z \in \mathbb{C}, \operatorname{Re} f > \alpha\}$. Hence

$$\{z \in \mathbb{C} : \ \operatorname{Re} f > \alpha\} = \{z \in \mathbb{C} \setminus \{0\} : \ \operatorname{Re} f > \alpha\} \, .$$

But since $\operatorname{Re} f$ is superharmonic on $\mathbb{C} \setminus \{0\}$, $\{z \in \mathbb{C} \setminus \{0\} : \ \operatorname{Re} f > \alpha\}$ is an open set on $\mathbb{C}$. As a consequence, all his connected components are open, hence the desired result. In particular, Ω_* is open and $\partial\Omega_* \subset \partial\Omega_+$; hence $\operatorname{Re} f \leqslant \operatorname{Re} f(\mathfrak{d})$ on $\partial\Omega_*$. If Ω_* is moreover bounded, then we have $\operatorname{Re} f > \operatorname{Re} f(\mathfrak{d})$ on the bounded domain Ω_* and $\operatorname{Re} f \leqslant \operatorname{Re} f(\mathfrak{d})$ on its boundary. Since subharmonic functions satisfy a maximum principle, $\operatorname{Re} f$ cannot be subharmonic on the whole set Ω_* and (1) follows from Lemma 4.10.

We now turn to 2(a). We argue by contradiction and assume that Ω_* is a bounded connected component of Ω_- which does not contain the origin. The fact that Ω_* has non-empty interior implies that at least one of the sets $\Omega_* \cap \{\operatorname{Im}(z) > 0\}$ and $\Omega_* \cap \{\operatorname{Im}(z) < 0\}$ is non-empty. Consider the set

$$\Omega_*^{\mathrm{sym}} = \left\{ z \in \mathbb{C} : \ \overline{z} \in \Omega_* \right\}$$

and notice it is also a connected component of Ω_- because of the symmetry $\operatorname{Re} f(z) = \operatorname{Re} f(\overline{z})$. Without loss of generality, assume that $\Omega_* \cap \{\operatorname{Im}(z) > 0\} \neq \emptyset$ (otherwise switch the role of Ω_* and Ω_*^{sym} in what follows). Since $\operatorname{Re} f$ is subharmonic on $\mathbb{C} \setminus \mathcal{K}$, Ω_- is open and so are its connected components, in particular Ω_*, and then $\Omega_* \cap \{\mathrm{Im}(z) > 0\}$. Now $\mathrm{Re}\, f$ being continuous on $\mathbb{C} \setminus \mathcal{K}$ by Lemma 4.10, we have:

$$\mathrm{Re}\, f(z) = \mathrm{Re}\, f(\mathfrak{d}), \qquad z \in \partial\Omega_* \setminus \mathcal{K}. \tag{4.33}$$

Let us fix $\varepsilon_0 > 0$ such that $\Omega_* \cap \{\mathrm{Im}(z) \geqslant \varepsilon_0\} \neq \emptyset$ and pick $z_0 \in \Omega_*$ satisfying $\mathrm{Im}(z_0) \geqslant \varepsilon_0$ and $\mathrm{Re}\, f(z_0) < \mathrm{Re}\, f(\mathfrak{d})$. Our goal is to construct a bounded domain which contains z_0 but not the origin and where $\mathrm{Re}\, f > \mathrm{Re}\, f(z_0)$ on its boundary. Indeed, this would lead to a contradiction via the minimum principle for superharmonic functions since $\mathrm{Re}\, f$ is superharmonic on $\mathbb{C}\setminus\{0\}$ as stated in Lemma 4.10.

First, notice that if $\mathrm{dist}(\Omega_*, \mathbb{R}) > 0$ then $\mathrm{Re}\, f$ is harmonic on Ω_*, $\mathrm{Re}\, f = \mathrm{Re}\, f(\mathfrak{d})$ on $\partial\Omega_*$ and $\mathrm{Re}\, f < \mathrm{Re}\, f(\mathfrak{d})$ on Ω_* which is a bounded domain. But this contradicts the minimum principle for (super)harmonic functions, and thus $\mathrm{dist}(\Omega_*, \mathbb{R}) = 0$. Because $\mathrm{dist}(\Omega_*, \mathbb{R}) = 0$ and $\Omega_* \cap \{\mathrm{Im}(z) > 0\}$ is open and non-empty, for every $\varepsilon > 0$ small enough $\Omega_* \cap \{\mathrm{Im}(z) = \varepsilon\} = U + i\varepsilon$ where U is a non-empty open subset of the real line. Thus, we can write

$$\Omega_* \cap \{\mathrm{Im}(z) = \varepsilon\} \quad = \quad \bigcup_{j \in J} \big(u^{(j)}_{\min}(\varepsilon), u^{(j)}_{\max}(\varepsilon)\big) + i\varepsilon$$

where J is a countable set satisfying $\mathrm{Card}(J) \geqslant 1$, and the $u^{(j)}_{\min}(\varepsilon)$'s and $u^{(j)}_{\min}(\varepsilon)$'s are real numbers such that any open intervals $(u^{(j_1)}_{\min}(\varepsilon), u^{(j_1)}_{\max}(\varepsilon))$ and $(u^{(j_2)}_{\min}(\varepsilon), u^{(j_2)}_{\max}(\varepsilon))$ are disjoint whenever $j_1 \neq j_2$. Notice that by symmetry,

$$\Omega_*^{\mathrm{sym}} \cap \{\mathrm{Im}(z) = -\varepsilon\} \quad = \quad \bigcup_{j \in J} \big(u^{(j)}_{\min}(\varepsilon), u^{(j)}_{\max}(\varepsilon)\big) - i\varepsilon\ .$$

By construction, for every $j \in J$, both $u^{(j)}_{\min}(\varepsilon) + i\varepsilon$ and $u^{(j)}_{\max}(\varepsilon) + i\varepsilon$ belong to $\partial\Omega_* \setminus \mathbb{R}$. In particular, by (4.33) and the symmetry $\mathrm{Re}\, f(z) = \mathrm{Re}\, f(\overline{z})$,

$$\mathrm{Re}\, f\big(u^{(j)}_{\min}(\varepsilon) \pm i\varepsilon\big) \;=\; \mathrm{Re}\, f\big(u^{(j)}_{\max}(\varepsilon) \pm i\varepsilon\big) \;=\; \mathrm{Re}\, f(\mathfrak{d}), \qquad j \in J. \tag{4.34}$$

Since by assumption $0 \notin \Omega_* \subset \Omega_-$, there exists $\delta > 0$ such that $B(0, \delta) \cap \Omega_* = \emptyset$ otherwise $0 \in \partial\Omega_*$, but in this case, the boundary condition $\mathrm{Re}\, f(z) = \mathrm{Re}\, f(\mathfrak{d})$ would be violated near zero as $\mathrm{Re}\, f(z) \to -\infty$ for $|z| \to 0$. As Ω_* is moreover bounded by assumption, $|u^{(j)}_{\min}(\varepsilon)|$ and $|u^{(j)}_{\max}(\varepsilon)|$ stay in a compact subset of $(0, +\infty)$ independent from ε and $j \in J$ as $\varepsilon \to 0$. As a consequence, we can choose $\varepsilon \in (0, \varepsilon_0)$ small enough so that

$$\max\left(\frac{\varepsilon^2}{u^{(j)}_{\min}(\varepsilon)^2}, \frac{\varepsilon^2}{u^{(j)}_{\max}(\varepsilon)^2}\right) \;<\; \min\Big(\mathrm{Re}\, f(\mathfrak{d}) - \mathrm{Re}\, f(z_0), \frac{1}{2}\Big)\,, \quad j \in J\,. \tag{4.35}$$

If we moreover consider for any $j \in J$ the open rectangle

$$\mathcal{R}_j(\varepsilon) = \Big\{u + iv \in \mathbb{C} : \quad u^{(j)}_{\min}(\varepsilon) < u < u^{(j)}_{\max}(\varepsilon), \quad |v| < \varepsilon\Big\},$$

then we can also assume that ε is small enough so that $0 \notin \mathcal{R}_j(\varepsilon)$ for every $j \in J$.

Let $j \in J$ and $\eta \in \mathbb{R}$ be such that $|\eta| \leqslant \varepsilon$. Denote by $z_\varepsilon = u^{(j)}_{\min}(\varepsilon) + i\varepsilon$ and $z_\eta = u^{(j)}_{\min}(\varepsilon) + i\eta$. Since $|1 - xz_\eta| \leqslant |1 - xz_\varepsilon|$ for every $x \in \mathbb{R}$, it follows that

$$\int \log|1 - xz_\eta|\, \nu(\mathrm{d}x) \;\leqslant\; \int \log|1 - xz_\varepsilon|\, \nu(\mathrm{d}x)$$

and, together with (4.34), that

$$\operatorname{Re} f(z_\eta) \;\geqslant\; \operatorname{Re} f(z_\varepsilon) + \log\left|\frac{z_\eta}{z_\varepsilon}\right| \;=\; \operatorname{Re} f(\mathfrak{d}) + \log\left|\frac{z_\eta}{z_\varepsilon}\right|. \tag{4.36}$$

Next, we have

$$\begin{aligned}\log\left|\frac{z_\eta}{z_\varepsilon}\right| \;&=\; \frac{1}{2}\log\left(\frac{u_{\min}^{(j)}(\varepsilon)^2+\eta^2}{u_{\min}^{(j)}(\varepsilon)^2+\varepsilon^2}\right) \;=\; \frac{1}{2}\log\left(1-\frac{\varepsilon^2-\eta^2}{u_{\min}^{(j)}(\varepsilon)^2+\varepsilon^2}\right)\\ &\geqslant\; \frac{1}{2}\log\left(1-\frac{\varepsilon^2}{u_{\min}^{(j)}(\varepsilon)^2}\right) \;\geqslant\; -\frac{\varepsilon^2}{u_{\min}^{(j)}(\varepsilon)^2}\,,\end{aligned} \tag{4.37}$$

where for the last inequality we used that $\log(1-x) \geqslant -2x$ for any $x \in [0,1/2]$. By combining (4.35)–(4.37), we have shown that

$$\operatorname{Re} f\big(u_{\min}^{(j)}(\varepsilon)+i\eta\big) > \operatorname{Re} f(z_0)\,, \quad |\eta| \leqslant \varepsilon\,, \quad j \in J\,. \tag{4.38}$$

The same line of arguments also shows that

$$\operatorname{Re} f\big(u_{\max}^{(j)}(\varepsilon)+i\eta\big) > \operatorname{Re} f(z_0)\,, \quad |\eta| \leqslant \varepsilon\,, \quad j \in J\,. \tag{4.39}$$

Now, consider the set

$$\widetilde{\Omega}_* = \Big\{z \in \Omega_* : \operatorname{Im}(z) \geqslant \varepsilon\Big\} \cup \Big\{z \in \Omega_*^{\mathrm{sym}} : \operatorname{Im}(z) \leqslant -\varepsilon\Big\} \cup \Big(\bigcup_{j\in J} \mathcal{R}_j(\varepsilon)\Big)$$

and notice it is a bounded open set containing z_0 (since $\operatorname{Im}(z_0) \geqslant \varepsilon_0 > \varepsilon$), but which may not be connected, and which does not contain the origin. Let $\widetilde{\Omega}_*(z_0)$ be the connected component of $\widetilde{\Omega}_*$ which contains z_0. Since $0 \notin \widetilde{\Omega}_*(z_0)$, $\operatorname{Re} f$ is superharmonic on the bounded domain $\widetilde{\Omega}_*(z_0)$. It follows from (4.33), (4.38), (4.39) and the symmetry $\operatorname{Re} f(z) = \operatorname{Re} f(\overline{z})$ that $\operatorname{Re} f > \operatorname{Re} f(z_0)$ on $\partial\widetilde{\Omega}_*(z_0)$. This yields a contradiction with the minimum principle for superharmonic functions and 2(a) follows.

We now turn to 2(b) and again argue by contradiction. Let Ω_* be connected component of Ω_- such that its interior $\operatorname{int}(\Omega_*)$ is not connected. Notice that since $\operatorname{Re} f$ is continuous on $\mathbb{C} \setminus \mathcal{K}$, we have $\operatorname{int}(\Omega_*) \setminus \mathcal{K} = \Omega_* \setminus \mathcal{K}$ and in particular (4.33) yields

$$\operatorname{Re} f(z) = \operatorname{Re} f(\mathfrak{d}), \qquad z \in \partial \operatorname{int}(\Omega_*) \setminus \mathcal{K}.$$

If Ω_* is bounded, then by 2(a) we have $0 \in \Omega_*$ and moreover, since $\operatorname{Re} f(z) \to -\infty$ as $z \to 0$, $0 \in \operatorname{int}(\Omega_*)$. Let Ω_*' be a connected component of $\operatorname{int}(\Omega_*)$ which does not contain the origin. It is then a bounded domain on which $\operatorname{Re} f < \operatorname{Re} f(\mathfrak{d})$ and $\operatorname{Re} f = \operatorname{Re} f(\mathfrak{d})$ on $\partial\Omega_*' \setminus \mathcal{K}$. By picking $z_0 \in \Omega_*' \cap \{\operatorname{Im}(z) > 0\}$ and by performing the same construction than in the proof of (2a) but replacing Ω_* by Ω_*', we obtain a bounded domain $\widetilde{\Omega}_*'(z_0)$ containing z_0 in its interior, on which $\operatorname{Re} f$ is superharmonic, and such that $\operatorname{Re} f > \operatorname{Re} f(z_0)$ on its boundary. The minimum principle for superharmonic functions shows that this is impossible and 2(b) follows.

To prove 2(c), assume now that $0 \notin \Omega_*$, so that Ω_* is necessarily unbounded by 2(a). By using that $\operatorname{int}(\Omega_*) \setminus \mathcal{K} = \Omega_* \setminus \mathcal{K}$ where $\mathcal{K}$ is a compact set, that Ω_- has a unique unbounded connected component by Lemma 4.8, and that by assumption $\operatorname{int}(\Omega_*)$ is not connected, it follows that at least one connected component of $\mathrm{int}(\Omega_*)$, say Ω'_*, is bounded. Since by assumption $0 \notin \Omega'_*$, the same argument than in the proof of 2(b) yields a contradiction and 2(c) is proved.

□

Recall that the Δ_k's are defined in Lemma 4.9, that Δ_{-1}, Δ_1 and Δ_3 are in Ω_- and Δ_{-2}, Δ_0 and Δ_2 are in Ω_+. Recall also that the Ω_k's are the associated connected components containing the Δ_k's. We use the previous lemmas to describe the sets Ω_k's.

Lemma 4.12. *The following hold true.*

1. *The sets Ω_1 and Ω_{-1} are equal, with a connected interior and unbounded. In particular, for every $0 < \alpha < \pi/2$ there exists $R > 0$ such that*

$$\Big\{ z \in \mathbb{C} : \quad |z| > R, \quad -\frac{\pi}{2} + \alpha < \arg(z) < \frac{\pi}{2} - \alpha \Big\} \subset \Omega_1 \, . \tag{4.40}$$

2. *The sets Ω_2 and Ω_{-2} are equal, open, connected and unbounded. In particular there exists $R > 0$ such that*

$$\Big\{ z \in \mathbb{C} : \quad |z| > R, \quad \frac{\pi}{2} + \alpha < \arg(z) < \frac{3\pi}{2} - \alpha \Big\} \subset \Omega_2 \, . \tag{4.41}$$

3. *The interior of Ω_3 is connected and there exists $\delta > 0$ such that $B(0, \delta) \subset \Omega_3$.*

Proof. We first prove 2. Since Ω_2 is by definition a connected subset of Ω_+, Lemma 4.11 (1) yields that it is open. Next, we show by contradiction that Ω_2 is unbounded. If Ω_2 is bounded, then Lemma 4.11.1 shows there exists $x \in \mathrm{Supp}(\nu)$ such that $x^{-1} \in \Omega_2$. If $x^{-1} < \mathfrak{d}$ (resp. $x^{-1} > \mathfrak{d}$), then it follows from the symmetry $\mathrm{Re}\, f(\overline{z}) = \mathrm{Re}\, f(z)$ that Ω_2 completely surrounds Ω_3 (resp. Ω_1), see for instance Fig. 7. Moreover, Lemma 4.9 implies that Ω_3 (resp. Ω_1) has non-empty interior. As a consequence, Ω_3 (resp. Ω_1) is a bounded connected component of Ω_- which does not contain the origin, and Lemma 4.11-2(a) shows this is impossible. The symmetry $\mathrm{Re}\, f(\overline{z}) = \mathrm{Re}\, f(z)$ moreover provides that Ω_{-2} is also unbounded, and (2) follows from the inclusion (4.31) and the fact that Ω_+ has a unique unbounded connected component, see Lemma 4.8.

We now prove 1. Since Ω_2 is unbounded and symmetric around the real axis, then Ω_1 does not contain the origin and it follows from Lemma 4.11-2(a),2(c) that Ω_1 is unbounded and has a connected interior. Then, (1) follows from symmetry $\mathrm{Re}\, f(\overline{z}) = \mathrm{Re}\, f(z)$, the inclusion (4.30) and the fact that Ω_- has a unique unbounded connected component (cf. Lemma 4.8).

Finally, since Ω_3 is bounded as a byproduct of Lemma 4.12-2, it has a connected interior (Lemma 4.11-2(b)) and contains the origin (Lemma 4.11-2(a)). Moreover, since $\mathrm{Re}\, f(z) \to -\infty$ as $z \to 0$, (3) follows. □

We are finally in position to prove Proposition 4.6.

Proof of Proposition 4.6. Given any $\rho > 0$ small enough, it follows from the convergence of $\mathfrak{d}_N$ to $\mathfrak{d}$ that for all N_0 large enough the points $\mathfrak{d}_{N_0} + \rho e^{i\pi/3}$ and $\mathfrak{d}_{N_0} + \rho e^{-i\pi/3}$ belong to Δ_1 and Δ_{-1} respectively. Thus both points belong to Ω_1 by Lemma 4.12-1. As a consequence, we can complete the path $\{\mathfrak{d}_{N_0} + t e^{\pm i\pi/3} : \ t \in [0, \rho]\}$ into a (closed) contour with a path $\Upsilon^{(1)}_{res}(N_0)$ lying in the interior of Ω_1 (see Figure 7). Since $\Upsilon^{(1)}_{res}(N_0)$ lies in the interior of Ω_1,

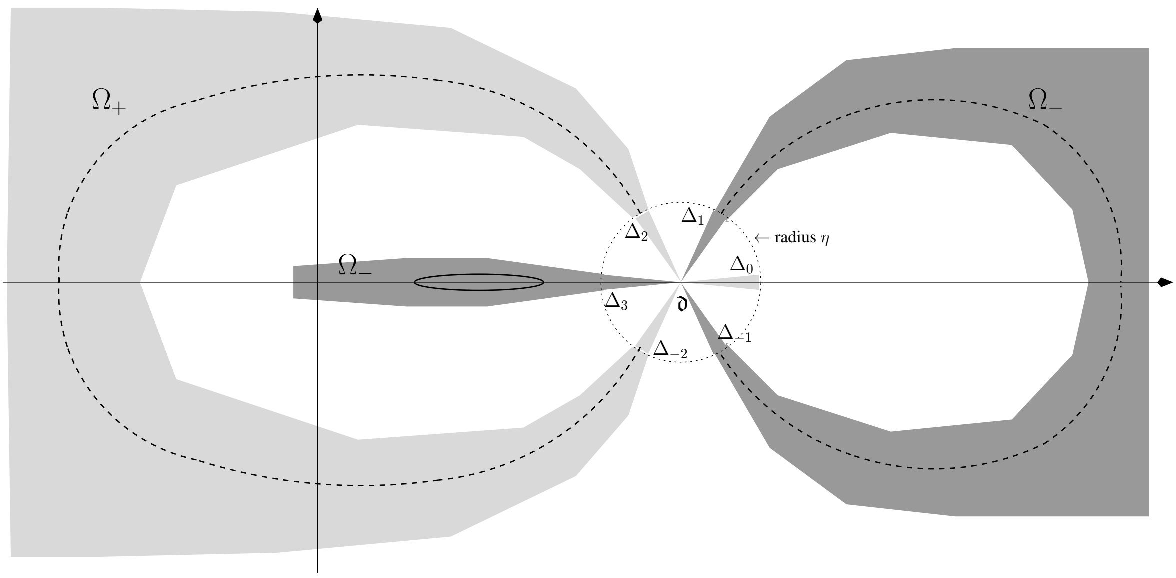

Figure 7: Preparation of the saddle point analysis for a right edge. The dotted path at the right is $\Upsilon^{(1)}_{res}(N_0)$. The dotted path at the left is its counterpart for $\widetilde{\Theta}$. The closed contour at the left of $\mathfrak{d}$ is $\Upsilon^{(0)}$.

the convergence $\mathfrak{d}_N \to \mathfrak{d}$ moreover yields that we can perform the same construction for all $N \geqslant N_0$ with $\Upsilon^{(1)}_{res}(N)$ in a closed tubular neighborhood $\mathcal{T} \subset \Omega_1$ of $\Upsilon^{(1)}_{res}(N_0)$. By Lemma 4.12-1 again, we can moreover choose $\Upsilon^{(1)}_{res}(N_0)$ in a way that it has finite length and only crosses the real axis at a real number lying on the right of $\mathcal{K}$. By construction, this yields that the set $\mathcal{T}$ is compact and that the $\Upsilon^{(1)}_{res}(N)$'s can be chosen with a uniformly bounded length as long as $N \geqslant N_0$. Since $\Omega_1 \subset \Omega_-$ there exists $K > 0$ such that $\operatorname{Re} f(z) \leqslant \operatorname{Re} f(\mathfrak{d}) - 3K$ on $\mathcal{T}$. Since moreover $\operatorname{Re} f_N$ uniformly converges to $\operatorname{Re} f$ on $\mathcal{T}$ and $\operatorname{Re} f_N(\mathfrak{d}_N) \to \operatorname{Re} f(\mathfrak{d})$ according to Lemma 4.7-(a), we can choose N_0 large enough such that $\operatorname{Re} f_N \leqslant \operatorname{Re} f + K$ on $\mathcal{T}$ and $\operatorname{Re} f(\mathfrak{d}) \leqslant \operatorname{Re} f_N(\mathfrak{d}_N) + K$. This finally yields that $\operatorname{Re}(f_N(z) - f_N(\mathfrak{d}_N)) \leqslant -K$ for all $z \in \mathcal{T}$ and proves the existence of a contour $\Upsilon^{(1)}$ satisfying the requirements of Proposition 4.6, except for the point (4). Similarly, the same conclusion for $\widetilde{\Theta}$ follows from the same lines but by using Ω_2 instead of Ω_1 and Lemma 4.12-2.

As a consequence of Lemma 4.12-3, there exists a contour in the interior of Ω_3 surrounding $\{x \in \mathcal{K} :\ 0 < x < \mathfrak{d}\}$ but staying in $\{z \in \mathbb{C} :\ \operatorname{Re} z > 0\}$ and which intersects exactly twice the real axis in $\mathbb{R} \setminus \mathcal{K}$ with finite length, see Figure 7. Using again Lemma 4.7-(a), the existence of $\Upsilon^{(0)}$ with the properties provided in the statement of Proposition 4.6 follows.

Finally, the item (4) of Proposition 4.6 is clearly satisfied by construction since the sets Ω_- and Ω_+ are disjoint, and the proof of the proposition is therefore complete. □

4.5 Asymptotic analysis for the right edges and proof of Theorem 3-(b)

Recall that $\widetilde{\mathrm{K}}_N = \mathrm{K}^{(0)}_N + \mathrm{K}^{(1)}_N$. We now analyze the asymptotic behavior of $\mathrm{K}^{(0)}_N$ in the next section and then investigate $\mathrm{K}^{(1)}_N$ in Section 4.5.2.

4.5.1 Asymptotic analysis for $\mathrm{K}_N^{(0)}$

Recall the definition (4.23) of the kernel $\mathrm{K}_N^{(0)}$ and its associated contours $\Gamma^{(0)}$ and Θ, cf. Figure 6. The aim of this section is to establish the following statement, which asserts that $\mathrm{K}_N^{(0)}$ will have no impact on the asymptotic analysis in the large N limit.

Proposition 4.13. *Let Assumptions 1 and 2 hold true, then for every $\varepsilon > 0$ small enough*

$$\lim_{N\to\infty} \left\|\mathbf{1}_{(s,\,\varepsilon N^{2/3}\delta_N)}\mathrm{K}_N^{(0)}\mathbf{1}_{(s,\,\varepsilon N^{2/3}\delta_N)}\right\|_2 = 0\,, \tag{4.42}$$

$$\lim_{N\to\infty} \mathrm{Tr}\left(\mathbf{1}_{(s,\,\varepsilon N^{2/3}\delta_N)}\mathrm{K}_N^{(0)}\mathbf{1}_{(s,\,\varepsilon N^{2/3}\delta_N)}\right) = 0\,. \tag{4.43}$$

Notation: If a contour Γ is parametrized by $\gamma : I \to \Gamma$ for some interval $I \subset \mathbb{R}$, then for every map $h : \Gamma \to \mathbb{C}$ we set

$$\int_\Gamma h(z)|\mathrm{d}z| = \int_I h\circ\gamma(t)\,|\gamma'(t)|\mathrm{d}t$$

when it does make sense. In particular, $\oint_\Gamma |\mathrm{d}z|$ is the length of the contour Γ.

Proof. Recall that by definition of $\mathrm{K}_N^{(0)}(x,y)$, see (4.23), we have

$$\mathrm{K}_N^{(0)}(x,y) = \frac{N^{1/3}}{(2i\pi)^2\delta_N}\oint_{\Gamma^{(0)}}\mathrm{d}z\oint_\Theta \mathrm{d}w\,\frac{1}{w-z}e^{-N^{1/3}x(z-\mathfrak{d}_N)/\delta_N+N^{1/3}y(w-\mathfrak{d}_N)/\delta_N+Nf_N(z)-Nf_N(w)}, \tag{4.44}$$

where Θ and $\Gamma^{(0)}$ are as in Figure 6. We now deform the contours Θ and $\Gamma^{(0)}$ so that $\Theta = \widetilde{\Theta}$ and $\Gamma^{(0)} = \Upsilon^{(0)}$ where $\widetilde{\Theta}$ and $\Upsilon^{(0)}$ are given by Proposition 4.6. As a consequence of Proposition 4.6-(4), we have the upper bound

$$\left|\mathrm{K}_N^{(0)}(x,y)\right| \leqslant \frac{N^{1/3}}{d(2\pi)^2\delta_N}\oint_{\Upsilon^{(0)}} e^{-N^{1/3}x\,\mathrm{Re}(z-\mathfrak{d}_N)/\delta_N+N\,\mathrm{Re}(f_N(z)-f_N(\mathfrak{d}_N))}|\mathrm{d}z| \times\oint_{\widetilde{\Theta}} e^{N^{1/3}y\,\mathrm{Re}(w-\mathfrak{d}_N)/\delta_N-N\,\mathrm{Re}(f_N(w)-f_N(\mathfrak{d}_N))}|\mathrm{d}w|. \tag{4.45}$$

Recall that $\Upsilon^{(0)}$ does not depend on N. By Proposition 4.6-(5b), the contour $\widetilde{\Theta}$ lies in a compact set. Hence there exists $L > 0$ independent of N such that $|\mathrm{Re}(z-\mathfrak{c}_N)| \leqslant L$ for $z \in \Upsilon^{(0)}$ or $z \in \widetilde{\Theta}$. Together with Proposition 4.6 (3a), we obtain that for all $x \geqslant s$

$$\oint_{\Upsilon^{(0)}} e^{-N^{1/3}x\,\mathrm{Re}(z-\mathfrak{d}_N)/\delta_N+N\,\mathrm{Re}(f_N(z)-f_N(\mathfrak{d}_N))}|\mathrm{d}z| \leqslant e^{-NK+\frac{N^{1/3}}{\delta_N}(L(x-s)+L|s|)}\oint_{\Upsilon^{(0)}}|\mathrm{d}z|\,. \tag{4.46}$$

Similarly, by splitting $\widetilde{\Theta}$ into $\widetilde{\Theta}_{res}$ and $\widetilde{\Theta}_*$, we get from Proposition 4.6-(3c) for every $y \geqslant s$

$$\oint_{\widetilde{\Theta}} e^{N^{1/3}y\,\mathrm{Re}(w-\mathfrak{d}_N)/\delta_N-N\,\mathrm{Re}(f_N(w)-f_N(\mathfrak{d}_N))}|\mathrm{d}w| \leqslant e^{N^{1/3}L(y-s)/\delta_N+N^{1/3}L|s|/\delta_N}\left(e^{-NK}\int_{\widetilde{\Theta}_{res}}|\mathrm{d}w| + \int_{\widetilde{\Theta}_*} e^{-N\,\mathrm{Re}(f_N(w)-f_N(\mathfrak{d}_N))}|\mathrm{d}w|\right). \tag{4.47}$$

The definition of $\widetilde{\Theta}_*$ and Lemma 4.7-(b) then yield

$$\begin{aligned}\int_{\widetilde{\Theta}_*} e^{-N\,\mathrm{Re}(f_N(w)-f_N(\mathfrak{d}_N))}|\mathrm{d}w| &\leqslant \int_{\widetilde{\Theta}_*} e^{-Ng_N''(\mathfrak{d}_N)\,\mathrm{Re}(w-\mathfrak{d}_N)^3+N\Delta|w-\mathfrak{d}_N|^4}|\mathrm{d}w| \\ &\leqslant \int_{\widetilde{\Theta}_*} e^{-Ng_N''(\mathfrak{d}_N)\,\mathrm{Re}(w-\mathfrak{d}_N)^3+N\rho\Delta|w-\mathfrak{d}_N|^3}|\mathrm{d}w| \\ &= 2\int_0^\rho e^{-Nt^3(g_N''(\mathfrak{d}_N)-\rho\Delta)}\mathrm{d}t \qquad \leqslant \quad 2\rho \end{aligned} \tag{4.48}$$

provided that ρ is chosen small enough so that $g_N''(\mathfrak{d}_N)-\rho\Delta>0$.

By combining (4.45)–(4.48), we thus obtained that there exist constants $C_0, C_1>0$ independent of N such that for every $x,y\geqslant s$ and every N large enough

$$\left|\mathrm{K}_N^{(0)}(x,y)\right| \leqslant C_0\, e^{-C_1N+\frac{N^{1/3}}{\delta_N}2L(x+y)} . \tag{4.49}$$

Since by (4.3),

$$\left\|\mathbf{1}_{(s,\,\varepsilon N^{2/3}\delta_N)}\mathrm{K}_N^{(0)}\mathbf{1}_{(s,\,\varepsilon N^{2/3}\delta_N)}\right\|_2 = \left(\int_s^{\varepsilon N^{2/3}\delta_N}\int_s^{\varepsilon N^{2/3}\delta_N} \mathrm{K}_N^{(0)}(x,y)^2\mathrm{d}x\mathrm{d}y\right)^{1/2},$$

we obtain from (4.49) the rough estimate

$$\left\|\mathbf{1}_{(s,\,\varepsilon N^{2/3}\delta_N)}\mathrm{K}_N^{(0)}\mathbf{1}_{(s,\,\varepsilon N^{2/3}\delta_N)}\right\|_2 \leqslant C_0(\varepsilon N^{2/3}\delta_N - s)e^{-N(C_1-4\varepsilon L)}$$

from which (4.42) follows provided that we chose ε small enough. Similarly, by (4.4)

$$\mathrm{Tr}\left(\mathbf{1}_{(s,\,\varepsilon N^{2/3}\delta_N)}\mathrm{K}_N^{(0)}\mathbf{1}_{(s,\,\varepsilon N^{2/3}\delta_N)}\right) = \int_s^{\varepsilon N^{2/3}\delta_N} \mathrm{K}_N^{(0)}(x,x)\mathrm{d}x,$$

and (4.49) yields the estimate

$$\begin{aligned}\left|\,\mathrm{Tr}\left(\mathbf{1}_{(s,\,\varepsilon N^{2/3}\delta_N)}\mathrm{K}_N^{(0)}\mathbf{1}_{(s,\,\varepsilon N^{2/3}\delta_N)}\right)\right| &\leqslant \int_s^{\varepsilon N^{2/3}\delta_N} |\mathrm{K}_N^{(0)}(x,x)|\mathrm{d}x \\ &\leqslant C_0(\varepsilon N^{2/3}\sigma_N - s)e^{-N(C_1-4\varepsilon L)}\end{aligned}$$

which proves (4.43) as soon as ε is small enough. Proof of Proposition 4.13 is therefore complete.

□

4.5.2 Asymptotic analysis for $\mathrm{K}_N^{(1)}$ and proof of Theorem 3-(b)

We now investigate the convergence of $\mathrm{K}_N^{(1)}$ toward K_{Ai} and thereafter complete the proof of Theorem 3-(b).

Proposition 4.14. *For every $\varepsilon>0$ small enough, we have*

$$\lim_{N\to\infty}\left\|\mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)}(\mathrm{K}_N^{(1)}-\mathrm{K}_{\mathrm{Ai}})\mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)}\right\|_2 = 0 , \tag{4.50}$$

$$\lim_{N\to\infty}\mathrm{Tr}\left(\mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)}(\mathrm{K}_N^{(1)}-\mathrm{K}_{\mathrm{Ai}})\mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)}\right) = 0 . \tag{4.51}$$

First, we represent the Airy kernel as a double complex integral. To do so, we introduce for some $\delta > 0$, which will be specified later, the contours

$$\Gamma^\infty = \Big\{ \mathfrak{d}_N + \delta e^{i\pi\theta} : \theta \in [-\pi/3, \pi/3] \Big\} \cup \Big\{ \mathfrak{d}_N + te^{\pm i\pi/3} : t \in [\delta, \infty) \Big\} , \tag{4.52}$$

$$\Theta^\infty = \Big\{ \mathfrak{d}_N + \delta e^{i\pi\theta} : \theta \in [2\pi/3, 4\pi/3] \Big\} \cup \Big\{ \mathfrak{d}_N - te^{\pm i\pi/3} : t \in [\delta, \infty) \Big\} , \tag{4.53}$$

and prove the following.

Lemma 4.15. *For every $\delta > 0$ and $x, y \in \mathbb{R}$, we have*

$$\mathrm{K}_{\mathrm{Ai}}(x,y) = \frac{N^{1/3}}{(2i\pi)^2 \delta_N} \oint_{\Gamma^\infty} \mathrm{d}z \oint_{\Theta^\infty} \mathrm{d}w \frac{1}{w-z} e^{-N^{1/3}\frac{x(z-\mathfrak{d}_N)}{\delta_N} + \frac{N}{6} g_N''(\mathfrak{d}_N)(z-\mathfrak{d}_N)^3} \times e^{N^{1/3}\frac{y(w-\mathfrak{d}_N)}{\delta_N} - \frac{N}{6} g_N''(\mathfrak{d}_N)(w-\mathfrak{d}_N)^3} .$$

Proof. First, it easily follows from the differential equation satisfied by the Airy function, namely $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$, and an integration by part that

$$\mathrm{K}_{\mathrm{Ai}}(x,y) = \int_0^\infty \mathrm{Ai}(x+u)\mathrm{Ai}(y+u)\mathrm{d}u. \tag{4.54}$$

The Airy function admits the following complex integral representation (see e.g. [54, p.53])

$$\mathrm{Ai}(x) = -\frac{1}{2i\pi} \oint_{\Xi} e^{-xz+z^3/3}\mathrm{d}z = \frac{1}{2i\pi} \oint_{\Xi'} e^{xw-w^3/3}\mathrm{d}w, \tag{4.55}$$

where Ξ and Ξ' are disjoint unbounded contours, and Ξ goes from $e^{i\pi/3}\infty$ to $e^{-i\pi/3}\infty$ whereas Ξ' goes from $e^{-2i\pi/3}\infty$ to $e^{2i\pi/3}\infty$. By plugging (4.55) into (4.54) and by using the Fubini theorem, we obtain

$$\begin{aligned} \mathrm{K}_{\mathrm{Ai}}(x,y) &= -\frac{1}{(2i\pi)^2} \oint_{\Xi} \mathrm{d}z \oint_{\Xi'} \mathrm{d}w e^{-xz+z^3/3+yw-w^3/3} \int_0^\infty e^{u(w-z)}\mathrm{d}u , \\ &= \frac{1}{(2i\pi)^2} \oint_{\Xi} \mathrm{d}z \oint_{\Xi'} \mathrm{d}w \frac{1}{w-z} e^{-xz+z^3/3+yw-w^3/3} , \end{aligned} \tag{4.56}$$

since $\mathrm{Re}(w-z) < 0$ for all $z \in \Xi$ and $w \in \Xi'$. Lemma 4.15 then follows after the changes of variables $z \mapsto N^{1/3}(z-\mathfrak{d}_N)/\delta_N$ and $w \mapsto N^{1/3}(w-\mathfrak{d}_N)/\delta_N$, the mere definition $\delta_N^3 = 2/g_N''(\mathfrak{d}_N)$ and an appropriate deformation of the contours. □

We now turn to the proof of Proposition 4.14.

Proof of Proposition 4.14. Recall that

$$\mathrm{K}_N^{(1)}(x,y) = \frac{N^{1/3}}{(2i\pi)^2 \delta_N} \oint_{\Gamma^{(1)}} \mathrm{d}z \oint_{\Theta} \mathrm{d}w \, \frac{1}{w-z} e^{-N^{1/3}\frac{x(z-\mathfrak{d}_N)}{\delta_N} + N^{1/3}\frac{y(w-\mathfrak{d}_N)}{\delta_N} + Nf_N(z) - Nf_N(w)} . \tag{4.57}$$

The key step in the analysis is to deform the contours $\Gamma^{(1)}$ and Θ into $\Upsilon^{(1)}$ and $\widetilde{\Theta}$ of Proposition 4.6, but since the later intersect in $\mathfrak{d}_N$, we need to slightly modify them.

Let ρ_0 be fixed so that Lemma 4.7 holds true, fix $\rho \leqslant \rho_0$ and recall the definitions of

$$\Upsilon^{(1)} = \Upsilon_* \cup \Upsilon^{(1)}_{res} \quad \text{and} \quad \widetilde{\Theta} = \widetilde{\Theta}_* \cup \widetilde{\Theta}_{res} \tag{4.58}$$

as provided by Proposition 4.6. Since $\Upsilon_* \cap \widetilde{\Theta}_* = \{\mathfrak{d}_N\}$, we deform them to make them disjoint. Set

$$\delta = N^{-1/3} \tag{4.59}$$

and from now until the end of the proof, denote (with a slight abuse of notation)

$$\begin{aligned} \Upsilon_* &= \left\{\mathfrak{d}_N + \delta e^{i\theta} : \ \theta \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]\right\} \cup \left\{\mathfrak{d}_N + t e^{\pm i \frac{\pi}{3}} : \ t \in [\delta, \rho]\right\} \ , & (4.60)\\ &:= \Upsilon_{*,1} \cup \Upsilon_{*,2} & (4.61)\\ \widetilde{\Theta}_* &= \left\{\mathfrak{d}_N + \delta e^{i\theta} : \ \theta \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]\right\} \cup \left\{\mathfrak{d}_N - t e^{\pm i \frac{\pi}{3}} : \ t \in [\delta, \rho]\right\} \ . & (4.62) \end{aligned}$$

Notice in particular that this deformation provides now the control

$$\min\left\{|w-z| : \ z \in \Upsilon_*, \ w \in \widetilde{\Theta}_*\right\} \geqslant \delta \ .$$

Now, let $\Gamma^{(1)} = \Upsilon^{(1)}$ and $\Theta = \widetilde{\Theta}$. We can also express the Airy contours Γ^∞ and Θ^∞ as

$$\begin{aligned} \Gamma^\infty &= \Upsilon_* \cup \Gamma^\infty_{res} \qquad \text{with} \qquad \Gamma^\infty_{res} = \left\{\mathfrak{d}_N + t e^{\pm i\pi/3} : \ t \in [\rho, \infty)\right\} \ , \\ \Theta^\infty_{res} &= \widetilde{\Theta}_* \cup \Theta^\infty_{res} \qquad \text{with} \qquad \Theta^\infty_{res} = \left\{\mathfrak{d}_N - t e^{\pm i\pi/3} : \ t \in [\rho, \infty)\right\} \ . \end{aligned}$$

It follows from Proposition 4.6-(4) and the definition of the contours that there exists d' such that for any

$$(\Xi, \Xi') \in \{(\Upsilon_*, \widetilde{\Theta}_{res}), (\Upsilon^{(1)}_{res}, \widetilde{\Theta}_*), (\Upsilon^{(1)}_{res}, \widetilde{\Theta}_{res}), (\Gamma_*, \Theta^\infty_{res}), (\Gamma^\infty_{res}, \widetilde{\Theta}_*), (\Gamma^\infty_{res}, \Theta^\infty_{res})\}$$

we have

$$\min\left\{|w-z| : \ z \in \Xi, \ w \in \Xi'\right\} \geqslant d' \ .$$

As a consequence, by using (4.57), (4.59), Lemma 4.15 and by splitting contours into their different components, we obtain that

$$\left|\mathrm{K}^{(1)}_N(x,y) - \mathrm{K}_{\mathrm{Ai}}(x,y)\right| \leqslant \frac{N^{2/3}}{(2\pi)^2 \delta_N} E_0 + \frac{N^{1/3}}{d'(2\pi)^2 \delta_N}\Big(E_1 + E_2 + E_3 + E_4 + E_5 + E_6\Big) \tag{4.63}$$

where, setting for convenience

$$F_N(x,z) = e^{-N^{1/3}\frac{x(z-\mathfrak{d}_N)}{\delta_N} + N(f_N(z) - f_N(\mathfrak{d}_N))} \ , \qquad F_{\mathrm{Ai}}(x,z) = e^{-N^{1/3}\frac{x(z-\mathfrak{d}_N)}{\delta_N} + \frac{N}{6} g''_N(\mathfrak{d}_N)(z-\mathfrak{d}_N)^3} \ ,$$
$$G_N(y,w) = e^{N^{1/3}\frac{y(w-\mathfrak{d}_N)}{\delta_N} - N(f_N(w) - f_N(\mathfrak{d}_N))} \ , \qquad G_{\mathrm{Ai}}(y,w) = e^{N^{1/3}\frac{y(w-\mathfrak{d}_N)}{\delta_N} - \frac{N}{6} g''_N(\mathfrak{d}_N)(w-\mathfrak{d}_N)^3} ,$$

we introduced

$$E_0 = \int_{\Upsilon_*} |\mathrm{d}z| \int_{\widetilde{\Theta}_*} |\mathrm{d}w| \left| F_N(x,z)G_N(y,w) - F_{\mathrm{Ai}}(x,z)G_{\mathrm{Ai}}(y,w) \right| , \tag{4.64}$$

$$E_1 = \left(\int_{\Upsilon_*} \left| F_N(x,z) \right| |\mathrm{d}z| \right) \left(\int_{\widetilde{\Theta}_{res}} \left| G_N(y,w) \right| |\mathrm{d}w| \right) , \tag{4.65}$$

$$E_2 = \left(\int_{\Upsilon_{res}} \left| F_N(x,z) \right| |\mathrm{d}z| \right) \left(\int_{\widetilde{\Theta}_*} \left| G_N(y,w) \right| |\mathrm{d}w| \right) , \tag{4.66}$$

$$E_3 = \left(\int_{\Upsilon_{res}} \left| F_N(x,z) \right| |\mathrm{d}z| \right) \left(\int_{\widetilde{\Theta}_{res}} \left| G_N(y,w) \right| |\mathrm{d}w| \right) , \tag{4.67}$$

$$E_4 = \left(\int_{\Upsilon_*} \left| F_{\mathrm{Ai}}(x,z) \right| |\mathrm{d}z| \right) \left(\int_{\Theta_{res}^{\infty}} \left| G_{\mathrm{Ai}}(y,w) \right| |\mathrm{d}w| \right) , \tag{4.68}$$

$$E_5 = \left(\int_{\Gamma_{res}^{\infty}} \left| F_{\mathrm{Ai}}(x,z) \right| |\mathrm{d}z| \right) \left(\int_{\widetilde{\Theta}_*} \left| G_{\mathrm{Ai}}(y,w) \right| |\mathrm{d}w| \right) , \tag{4.69}$$

$$E_6 = \left(\int_{\Gamma_{res}^{\infty}} \left| F_{\mathrm{Ai}}(x,z) \right| |\mathrm{d}z| \right) \left(\int_{\Theta_{res}^{\infty}} \left| G_{\mathrm{Ai}}(y,w) \right| |\mathrm{d}w| \right) . \tag{4.70}$$

Convention: In the rest of the proof, $C, C_0, C_1, \dots$ stand for positive constants which are independent on N or x, y, but which may change from one line to an other.

Step 1: Estimates for E_0. We rely on the following elementary inequality,

$$\begin{aligned} |e^u - e^v| &= e^{\mathrm{Re}(v)} |e^{(u-v)} - 1| \\ &\leqslant e^{\mathrm{Re}(v)} \sum_{k \geqslant 1} \frac{|u-v|^k}{k!} \leqslant |u-v| e^{\mathrm{Re}(v) + |u-v|} , \end{aligned} \tag{4.71}$$

which holds for every $u, v \in \mathbb{C}$. By combining this inequality for

$$u = N(f_N(z) - f_N(\mathfrak{d}_N)) - N(f_N(w) - f_N(\mathfrak{d}_N)) , \quad v = \frac{N g_N''(\mathfrak{d}_N)}{6} \left\{ (z - \mathfrak{d}_N)^3 - (w - \mathfrak{d}_N)^3 \right\}$$

together with Lemma 4.7-(b), we obtain

$$\begin{aligned} & |F_N(x,z)G_N(y,w) - F_{\mathrm{Ai}}(x,z)G_{\mathrm{Ai}}(y,w)| \\ &\leqslant \Delta N \left(|z - \mathfrak{d}_N|^4 + |w - \mathfrak{d}_N|^4 \right) e^{-N^{1/3} \frac{x \,\mathrm{Re}(z - \mathfrak{d}_N)}{\delta_N} + N^{1/3} \frac{y \,\mathrm{Re}(w - \mathfrak{d}_N)}{\delta_N}} \\ &\qquad \times e^{\frac{N g_N''(\mathfrak{d}_N)}{6} \mathrm{Re}(z - \mathfrak{d}_N)^3 + N\Delta |z - \mathfrak{d}_N|^4 - \frac{N g_N''(\mathfrak{d}_N)}{6} \mathrm{Re}(w - \mathfrak{d}_N)^3 + N\Delta |w - \mathfrak{d}_N|^4} \end{aligned}$$

provided that $z, w \in B(\mathfrak{d}_N, \rho)$. This yields with (4.64)

$$E_0 \leqslant \Delta \int_{\Upsilon_*} N|z-\mathfrak{d}_N|^4 e^{-N^{1/3}\frac{x\,\mathrm{Re}(z-\mathfrak{d}_N)}{\delta_N}} e^{\frac{Ng_N''(\mathfrak{d}_N)}{6}\mathrm{Re}(z-\mathfrak{d}_N)^3+N\Delta|z-\mathfrak{d}_N|^4}|\mathrm{d}z|$$
$$\times \int_{\widetilde{\Theta}_*} e^{N^{1/3}\frac{x\,\mathrm{Re}(w-\mathfrak{d}_N)}{\delta_N}} e^{-\frac{Ng_N''(\mathfrak{d}_N)}{6}\mathrm{Re}(w-\mathfrak{d}_N)^3+N\Delta|w-\mathfrak{d}_N|^4}|\mathrm{d}w|$$
$$+\Delta \int_{\Upsilon_*} e^{-N^{1/3}\frac{x\,\mathrm{Re}(z-\mathfrak{d}_N)}{\delta_N}} e^{\frac{Ng_N''(\mathfrak{d}_N)}{6}\mathrm{Re}(z-\mathfrak{d}_N)^3+N\Delta|z-\mathfrak{d}_N|^4}|\mathrm{d}z|$$
$$\times \int_{\widetilde{\Theta}_*} N|w-\mathfrak{d}_N|^4 e^{N^{1/3}\frac{x\,\mathrm{Re}(w-\mathfrak{d}_N)}{\delta_N}} e^{-\frac{Ng_N''(\mathfrak{d}_N)}{6}\mathrm{Re}(w-\mathfrak{d}_N)^3+N\Delta|w-\mathfrak{d}_N|^4}|\mathrm{d}w|. \qquad (4.72)$$

We first handle the integrals over the contour $\Upsilon_* = \Upsilon_{*,1}\cup\Upsilon_{*,2}$, see (4.61), and consider separately the two different portions of the contour. First, let $z\in\Upsilon_{*,1}$ and recall that $x\geqslant s$ by assumption. Since

$$\frac{\delta}{2} \;\leqslant\; \mathrm{Re}(z-\mathfrak{d}_N) \;\leqslant\; |z-\mathfrak{d}_N| \;\leqslant\; \delta \quad \text{ and } \quad \delta = \frac{1}{N^{1/3}}\ ,$$

we have $|z-\mathfrak{d}_N|^4 = N^{-4/3}$ and the following estimates

$$e^{-N^{1/3}\frac{x\,\mathrm{Re}(z-\mathfrak{d}_N)}{\delta_N}} \;\leqslant\; e^{-\frac{x-s}{2\delta_N}+\frac{|s|}{\delta_N}}\ ,$$
$$e^{Ng_N''(\mathfrak{d}_N)\,\mathrm{Re}(z-\mathfrak{d}_N)^3+N\Delta|z-\mathfrak{d}_N|^4} \;\leqslant\; e^{g_N''(\mathfrak{d}_N)+\frac{\Delta}{N^{1/3}}}\ .$$

This immediately yields

$$\int_{\Upsilon_{*,1}} N|z-\mathfrak{d}_N|^4 e^{-N^{1/3}\frac{x\,\mathrm{Re}(z-\mathfrak{d}_N)}{\delta_N}} e^{\frac{Ng_N''(\mathfrak{d}_N)}{6}\mathrm{Re}(z-\mathfrak{d}_N)^3+N\Delta|z-\mathfrak{d}_N|^4}|\mathrm{d}z|$$
$$\leqslant \quad \frac{1}{N^{1/3}}e^{-\frac{x-s}{2\delta_N}+\frac{|s|}{\delta_N}}e^{g_N''(\mathfrak{d}_N)+\frac{\Delta}{N^{1/3}}}\left(\frac{2\pi}{3N^{1/3}}\right) \quad\leqslant\quad \frac{C}{N^{2/3}}e^{-\frac{x-s}{2\delta_N}} \qquad (4.73)$$

where $2\pi/3N^{1/3}$ accounts for the length of $\Upsilon_{*,1}$. Similarly

$$\int_{\Upsilon_{*,1}} e^{-N^{1/3}\frac{x\,\mathrm{Re}(z-\mathfrak{d}_N)}{\delta_N}} e^{\frac{Ng_N''(\mathfrak{d}_N)}{6}\mathrm{Re}(z-\mathfrak{d}_N)^3+N\Delta|z-\mathfrak{d}_N|^4}|\mathrm{d}z| \quad\leqslant\quad \frac{C}{N^{1/3}}e^{-\frac{x-s}{2\delta_N}}\ . \qquad (4.74)$$

Consider now the situation where $z\in\Upsilon_{*,2}$. In this case,

$$\mathrm{Re}(z-\mathfrak{d}_N)=\frac{t}{2}\quad,\quad \mathrm{Re}(z-\mathfrak{d}_N)^3=-t^3\quad,\quad |z-\mathfrak{d}_N|^4=t^4\ ,$$

with $N^{-1/3}\leqslant t\leqslant\rho$ and thus

$$e^{-N^{1/3}\frac{x\,\mathrm{Re}(z-\mathfrak{d}_N)}{\delta_N}} \;\leqslant\; e^{-tN^{1/3}\frac{x-s}{2\delta_N}+N^{1/3}\frac{|s|t}{2\delta_N}} \;\leqslant\; e^{-\frac{x-s}{2\delta_N}+N^{1/3}\frac{|s|t}{2\delta_N}}\ ,$$
$$e^{Ng_N''(\mathfrak{d}_N)\,\mathrm{Re}(z-\mathfrak{d}_N)^3+N\Delta|z-\mathfrak{d}_N|^4} \;\leqslant\; e^{-N(g_N''(\mathfrak{d}_N)-\rho\Delta)t^3}\ .$$

Assuming that we chose ρ small enough so that $g''(\mathfrak{d})-\rho\Delta>0$ and recalling that $g_N''(\mathfrak{d}_N)\to g''(\mathfrak{d})$, this provides for every N large enough the inequalities

$$\int_{\Upsilon_{*,2}} N|z-\mathfrak{d}_N|^4 e^{-N^{1/3}\frac{x\,\mathrm{Re}(z-\mathfrak{d}_N)}{\delta_N}} e^{\frac{Ng_N''(\mathfrak{d}_N)}{6}\mathrm{Re}(z-\mathfrak{d}_N)^3+N\Delta|z-\mathfrak{d}_N|^4}|\mathrm{d}z|$$
$$\leqslant\; 2e^{-\frac{x-s}{2\delta_N}}\int_{N^{-1/3}}^{\rho} Nt^4 e^{N^{1/3}\frac{|s|t}{2\delta_N}-N(g_N''(\mathfrak{d}_N)-\rho\Delta)t^3}\mathrm{d}t$$
$$\leqslant\; \frac{2}{N^{2/3}}e^{-\frac{x-s}{2\delta_N}}\int_1^\infty u^4 e^{\frac{|s|u}{2\delta_N}-(g_N''(\mathfrak{d}_N)-\rho\Delta)u^3}\mathrm{d}u \quad\leqslant\quad \frac{C}{N^{2/3}}e^{-\frac{x-s}{2\delta_N}}\ . \qquad (4.75)$$

Similarly,

$$\int_{\Upsilon_{*,2}} e^{-N^{1/3}\frac{x\,\mathrm{Re}(z-\mathfrak{d}_N)}{\delta_N}} e^{N\frac{g_N''(\mathfrak{d}_N)}{6}\mathrm{Re}(z-\mathfrak{d}_N)^3+N\Delta|z-\mathfrak{d}_N|^4}|\mathrm{d}z| \leqslant \frac{C}{N^{1/3}}e^{-\frac{x-s}{2\delta_N}} . \tag{4.76}$$

Gathering (4.73)-(4.76), we finally obtain estimates over the whole contour Υ_* :

$$\begin{aligned} &\int_{\Upsilon_*} N|z-\mathfrak{d}_N|^4 e^{-N^{1/3}\frac{x\,\mathrm{Re}(z-\mathfrak{d}_N)}{\delta_N}} e^{\frac{Ng_N''(\mathfrak{d}_N)}{6}\mathrm{Re}(z-\mathfrak{d}_N)^3+N\Delta|z-\mathfrak{d}_N|^4}|\mathrm{d}z| \leqslant \frac{C}{N^{2/3}}e^{-\frac{x-s}{2\delta_N}} , \\ &\int_{\Upsilon_*} e^{-N^{1/3}\frac{x\,\mathrm{Re}(z-\mathfrak{d}_N)}{\delta_N}} e^{\frac{Ng_N''(\mathfrak{d}_N)}{6}\mathrm{Re}(z-\mathfrak{d}_N)^3+N\Delta|z-\mathfrak{d}_N|^4}|\mathrm{d}z| \leqslant \frac{C}{N^{1/3}}e^{-\frac{x-s}{2\delta_N}} . \end{aligned} \tag{4.77}$$

The same line of arguments also yields equivalent estimates for the integrals over $\widetilde{\Theta}^*$. Namely,

$$\begin{aligned} &\int_{\widetilde{\Theta}_*} N|w-\mathfrak{d}_N|^4 e^{N^{1/3}\frac{y\,\mathrm{Re}(w-\mathfrak{d}_N)}{\delta_N}} e^{-\frac{Ng_N''(\mathfrak{d}_N)}{6}\mathrm{Re}(w-\mathfrak{d}_N)^3+N\Delta|w-\mathfrak{d}_N|^4}|\mathrm{d}w| \leqslant \frac{C}{N^{2/3}}e^{-\frac{y-s}{2\delta_N}} , \\ &\int_{\widetilde{\Theta}_*} e^{N^{1/3}\frac{y\,\mathrm{Re}(w-\mathfrak{d}_N)}{\delta_N}} e^{-\frac{Ng_N''(\mathfrak{d}_N)}{6}\mathrm{Re}(w-\mathfrak{d}_N)^3+N\Delta|w-\mathfrak{d}_N|^4}|\mathrm{d}w| \leqslant \frac{C}{N^{1/3}}e^{-\frac{y-s}{2\delta_N}} . \end{aligned} \tag{4.78}$$

Combining (4.77)-(4.78), we have shown that

$$E_0 \leqslant \frac{C}{N}e^{-\frac{x+y-2s}{2\delta_N}} . \tag{4.79}$$

Step 2: Estimates for the remaining E_i's. Using the same estimates as in Step 1, we can prove that

$$\int_{\Upsilon_*} |F_N(x,z)|\,|\mathrm{d}z| \leqslant \frac{C}{N^{1/3}}e^{-\frac{x-s}{2\delta_N}} , \qquad \int_{\Upsilon_*} \Big|F_{\mathrm{Ai}}(x,z)\Big||\mathrm{d}z| \leqslant \frac{C}{N^{1/3}}e^{-\frac{x-s}{2\delta_N}} , \tag{4.80}$$

$$\int_{\widetilde{\Theta}_*} \Big|G_N(y,w)\Big||\mathrm{d}w| \leqslant \frac{C}{N^{1/3}}e^{-\frac{y-s}{2\delta_N}} , \qquad \int_{\widetilde{\Theta}_*} \Big|G_{\mathrm{Ai}}(y,w)\Big||\mathrm{d}w| \leqslant \frac{C}{N^{1/3}}e^{-\frac{y-s}{2\delta_N}} . \tag{4.81}$$

The definitions of the paths and Proposition 4.6 yield that there exists $L > 0$ independent of N such that

$$|\,\mathrm{Re}(z-\mathfrak{c}_N)| \leqslant L , \qquad z \in \Upsilon_* \cup \widetilde{\Theta}_* \cup \Upsilon_{res} \cup \widetilde{\Theta}_{res}.$$

This estimate, together with Proposition 4.6 (3b), (3c) and (5c) yields that for every $x, y \geqslant s$

$$\int_{\Upsilon_{res}} \Big|F_N(x,z)\Big||\mathrm{d}z| \leqslant Ce^{-NK+N^{1/3}L\frac{x-s}{\delta_N}+N^{1/3}\frac{L|s|}{\delta_N}} , \tag{4.82}$$

$$\int_{\widetilde{\Theta}_{res}} \Big|G_N(y,w)\Big||\mathrm{d}w| \leqslant Ce^{-NK+N^{1/3}L\frac{y-s}{\delta_N}+N^{1/3}\frac{L|s|}{\delta_N}} . \tag{4.83}$$

Combining (4.80)-(4.83), we readily obtain

$$E_1 + E_2 + E_3 \leqslant Ce^{-C_1N+C_2N^{1/3}\frac{x+y}{\delta_N}} .$$

We now handle

$$\int_{\Gamma_{res}^\infty} \Big|F_{\mathrm{Ai}}(x,z)\Big||\mathrm{d}z| \quad \text{and} \quad \int_{\Theta_{res}^\infty} \Big|G_{\mathrm{Ai}}(y,w)\Big||\mathrm{d}w| .$$

We have

$$\begin{aligned}\int_{\Gamma_{res}^{\infty}} \Big|F_{\mathrm{Ai}}(x,z)\Big||\mathrm{d}z| &= \int_{\Theta_{res}^{\infty}} \Big|G_{\mathrm{Ai}}(y,w)\Big||\mathrm{d}w| \\ &= 2\int_{\rho}^{\infty} e^{-\frac{N^{1/3}xt}{2\delta_N}-\frac{Ng_N''(\mathfrak{d}_N)}{6}t^3}\mathrm{d}t \leqslant 2\int_{\rho}^{\infty} e^{\frac{N^{1/3}|s|t}{2\delta_N}-\frac{Ng_N''(\mathfrak{d}_N)}{6}t^3}\mathrm{d}t\ .\end{aligned}$$

Let now N large enough so that

$$3\frac{g_N''(\mathfrak{d}_N)N}{6}\rho^2-\frac{N^{1/3}|s|}{2\delta_N}\geqslant\rho$$

(beware that such a condition only depends on s). Then

$$\begin{aligned}2\int_{\rho}^{\infty} e^{\frac{N^{1/3}|s|t}{2\delta_N}-\frac{Ng_N''(\mathfrak{d}_N)}{6}t^3}\mathrm{d}t &\leqslant \frac{2}{\rho}\int_{\rho}^{\infty}\left(3\frac{g_N''(\mathfrak{d}_N)N}{6}t^2-\frac{N^{1/3}|s|}{2\delta_N}\right)e^{\frac{N^{1/3}|s|t}{2\delta_N}-\frac{Ng_N''(\mathfrak{d}_N)}{6}t^3}\mathrm{d}t \\ &\leqslant \frac{2}{\rho}\left[-e^{\frac{N^{1/3}|s|t}{2\delta_N}-\frac{Ng_N''(\mathfrak{d}_N)}{6}t^3}\right]_{\rho}^{\infty} = \frac{2}{\rho}e^{\frac{N^{1/3}|s|\rho}{2\delta_N}-\frac{Ng_N''(\mathfrak{d}_N)}{6}\rho^3}\end{aligned}$$

and we hence obtain the estimate

$$\int_{\Gamma_{res}^{\infty}} \Big|F_{\mathrm{Ai}}(x,z)\Big||\mathrm{d}z| = \int_{\Theta_{res}^{\infty}} \Big|G_{\mathrm{Ai}}(y,w)\Big||\mathrm{d}w| \leqslant Ce^{-C_1N} \tag{4.84}$$

We can now easily handle E_4, E_5 and E_6 and finally obtain

$$\sum_{k=1}^{6} E_k \leqslant Ce^{-C_1N+C_2N^{1/3}\frac{x+y}{\delta_N}}\ . \tag{4.85}$$

Step 3: Conclusions. By combining (4.63), (4.79) and (4.85), we have shown for every $x,y\geqslant s$ and N large enough that

$$\big|\mathrm{K}_N^{(1)}(x,y)-\mathrm{K}_{\mathrm{Ai}}(x,y)\big| \leqslant \frac{C}{N^{1/3}}e^{-\frac{x+y-2s}{2\delta_N}}+C_1e^{-C_2N+C_3N^{1/3}\frac{x+y}{\delta_N}}.$$

As a consequence,

$$\begin{aligned}\Big|\mathrm{Tr}\left(\mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)}(\mathrm{K}_N^{(1)}-\mathrm{K}_{\mathrm{Ai}})\mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)}\right)\Big| &\leqslant \int_s^{\varepsilon N^{2/3}\delta_N}\big|\mathrm{K}_N^{(1)}(x,x)-\mathrm{K}_{\mathrm{Ai}}(x,x)\big|\mathrm{d}x \\ &\leqslant \frac{\delta_N C}{N^{1/3}}+(\varepsilon N^{2/3}\delta_N-s)C_1e^{-N(C_2-2\varepsilon C_3)}\end{aligned}$$

and (4.51) follows provided ε is chosen small enough. Similarly,

$$\begin{aligned}&\big\|\mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)}(\mathrm{K}_N^{(1)}-\mathrm{K}_{\mathrm{Ai}})\mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)}\big\|_2^2 \\ &= \int_s^{\varepsilon N^{2/3}\delta_N}\int_s^{\varepsilon N^{2/3}\delta_N}\big(\mathrm{K}_N^{(1)}(x,y)-\mathrm{K}_{\mathrm{Ai}}(x,y)\big)^2\mathrm{d}x\mathrm{d}y \\ &\leqslant \left(\frac{\delta_N C}{N^{1/3}}\right)^2+(\varepsilon N^{2/3}\delta_N-s)^2C_1'e^{-N(C_2-2\varepsilon C_3)}\end{aligned}$$

where $C_1'>0$ is independent on N. This yields (4.50) as soon as ε is chosen small enough and thus completes the proof of Proposition 4.14. □

We are finally in position to prove Theorem 3 (b).

Proof of Theorem 3 (b). First, we check that the Airy operator K_{Ai} is trace class and Hilbert-Schmidt on $L^2(s,\infty)$ for every $s \in \mathbb{R}$. Indeed, the representation (4.54) provides the factorization $\mathrm{K}_{\mathrm{Ai}} = \mathrm{A}_s^2$ of operators on $L^2(s,\infty)$, where A_s is the integral operator having for kernel $\mathrm{A}_s(x,y) = \mathrm{Ai}(x+y-s)$. The fast decay as $x \to +\infty$ of the Airy function (see [54, p.394])

$$\mathrm{Ai}(x) \leqslant \frac{e^{-\frac{2}{3}x^{3/2}}}{2\pi^{1/2}x^{1/4}}, \qquad x > 0, \tag{4.86}$$

then show that both A_s and K_{Ai} are Hilbert-Schmidt, and moreover that K_{Ai} is trace class being the product of two Hilbert-Schmidt operators.

Next, by using again the upper bound (4.86), it follows that for every $\varepsilon > 0$

$$\lim_{N\to\infty} \left\| \mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)} \mathrm{K}_{\mathrm{Ai}} \mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)} - \mathbf{1}_{(s,\infty)} \mathrm{K}_{\mathrm{Ai}} \mathbf{1}_{(s,\infty)} \right\|_2 = 0,$$
$$\lim_{N\to\infty} \mathrm{Tr}\left(\mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)} \mathrm{K}_{\mathrm{Ai}} \mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)} \right) = \mathrm{Tr}\left(\mathbf{1}_{(s,\infty)} \mathrm{K}_{\mathrm{Ai}} \mathbf{1}_{(s,\infty)} \right).$$

Together with Proposition 4.14, this yields

$$\lim_{N\to\infty} \left\| \mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)} \mathrm{K}_N^{(1)} \mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)} - \mathbf{1}_{(s,\infty)} \mathrm{K}_{\mathrm{Ai}} \mathbf{1}_{(s,\infty)} \right\|_2 = 0,$$
$$\lim_{N\to\infty} \mathrm{Tr}\left(\mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)} \mathrm{K}_N^{(1)} \mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)} \right) = \mathrm{Tr}\left(\mathbf{1}_{(s,\infty)} \mathrm{K}_{\mathrm{Ai}} \mathbf{1}_{(s,\infty)} \right)$$

and, combined moreover with Proposition 4.13 and (4.24), we obtain

$$\lim_{N\to\infty} \left\| \mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)} \widetilde{\mathrm{K}}_N \mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)} - \mathbf{1}_{(s,\infty)} \mathrm{K}_{\mathrm{Ai}} \mathbf{1}_{(s,\infty)} \right\|_2 = 0, \tag{4.87}$$
$$\lim_{N\to\infty} \mathrm{Tr}\left(\mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)} \widetilde{\mathrm{K}}_N \mathbf{1}_{(s,\varepsilon N^{2/3}\delta_N)} \right) = \mathrm{Tr}\left(\mathbf{1}_{(s,\infty)} \mathrm{K}_{\mathrm{Ai}} \mathbf{1}_{(s,\infty)} \right), \tag{4.88}$$

provided we chose ε small enough. Finally, it follows from (4.15)–(4.16), (4.87)–(4.88), and Proposition 4.1 that, for every $s \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{P}\Big(N^{2/3}\delta_N \big(\tilde{x}_{\phi(N)} - \mathfrak{b}_N \big) \leqslant s \Big) = \det\big(I - \mathrm{K}_{\mathrm{Ai}} \big)_{L^2(s,\infty)}.$$

Proof of Theorem 3 (b) is therefore complete.

□

In the next section, we provide a proof for Theorem 3 (a), and thus complete the proof for Theorem 3. We shall see that we can recover the setting of the proof of Theorem 3 (b); the only task left is to prove the existence of appropriate contours for the saddle point analysis which differ from the case of a right edge.

4.6 Asymptotic analysis for the left edges and proof of Theorem 3-(a)

This section is devoted to the end of the proof of Theorem 3. We precisely recall the setting for the analysis of a left regular soft edge $\mathfrak{a}$; we state and prove the counterparts of Proposition 4.6 (i.e. the existence of appropriate contours for the asymptotic analysis), that is Proposition 4.16 for the case where $\mathfrak{c} > 0$ with $\mathfrak{a} = g(\mathfrak{c})$, and Proposition 4.17 for the case where $\mathfrak{c} < 0$.

The remaining of the asymptotic analysis is omitted since we show it is essentially the same than in Section 4.5.

Let $\mathfrak{a}$ be a left regular soft edge; recall the definitions of g, $\mathfrak{c}$, $(\mathfrak{c}_N)$ as provided by Proposition 2.11 and set

$$\mathfrak{a}_N = g_N(\mathfrak{c}_N) \ , \qquad \sigma_N = \left(-\frac{2}{g_N''(\mathfrak{c}_N)}\right)^{1/3} . \tag{4.89}$$

Recall moreover that

$$g_N'(\mathfrak{c}_N) = 0 \ , \quad \lim_{N\to\infty} \mathfrak{c}_N = \mathfrak{c} \ , \quad \lim_{N\to\infty} \mathfrak{a}_N = \mathfrak{a} \ , \quad \lim_{N\to\infty} \sigma_N = \left(-\frac{2}{g''(\mathfrak{c})}\right)^{1/3} . \tag{4.90}$$

In particular, for N large enough, $-g_N''(\mathfrak{c}_N)$ and σ_N are positive numbers and $\mathfrak{c}_N$ and $\mathfrak{c}$ have the same sign.

4.6.1 Reduction to the right edge setting

The definition of the extremal eigenvalue $\tilde{x}_{\varphi(N)}$, see Theorem 2, and Proposition 4.4 yield that for every $\varepsilon > 0$ small enough

$$\mathbb{P}\Big(N^{2/3}\sigma_N\big(\mathfrak{a}_N - \tilde{x}_{\varphi(N)}\big) \leqslant s\Big) = \det\big(I - \mathrm{K}_N\big)_{L^2(\mathfrak{a}_N-\varepsilon,\, \mathfrak{a}_N - s/(N^{2/3}\sigma_N))} + o(1) \tag{4.91}$$

as $N \to \infty$. We then write

$$\det\big(I - \mathrm{K}_N\big)_{L^2(\mathfrak{a}_N-\varepsilon,\, \mathfrak{a}_N - s/(N^{1/3}\sigma_N))} = \det\big(I - \mathbf{1}_{(s,\, N^{2/3}\varepsilon c_N)}\widetilde{\mathrm{K}}_N\, \mathbf{1}_{(s,\, N^{2/3}\varepsilon c_N)}\big)_{L^2(s,\infty)}$$

where the scaled operator $\widetilde{\mathrm{K}}_N$ has for kernel

$$\widetilde{\mathrm{K}}_N(x,y) = -\frac{1}{N^{2/3}\sigma_N}\mathrm{K}_N\left(\mathfrak{a}_N - \frac{x}{N^{2/3}\sigma_N}, \mathfrak{a}_N - \frac{y}{N^{2/3}\sigma_N}\right),$$

and where $\mathrm{K}_N(x,y)$ was introduced in (4.19) (with $\mathfrak{d}_N$ replaced by $\mathfrak{c}_N$). If we introduce the map

$$f_N^*(z) = \mathfrak{a}_N(z - \mathfrak{c}_N) - \log(z) + \frac{1}{N}\sum_{j=1}^{n}\log(1-\lambda_j z), \tag{4.92}$$

which differs from f_N defined in (4.18) by a minus sign and by the fact that $\mathfrak{b}_N$ is replaced by $\mathfrak{a}_N$, then we have

$$\widetilde{\mathrm{K}}_N(x,y) = -\frac{N^{1/3}}{(2i\pi)^2\sigma_N}\oint_\Gamma \mathrm{d}z \oint_\Theta \mathrm{d}w\, \frac{1}{w-z} e^{N^{1/3}x(z-\mathfrak{c}_N)/\sigma_N - N^{1/3}y(w-\mathfrak{c}_N)/\sigma_N - Nf_N^*(z) + Nf_N^*(w)}.$$

Set moreover $\mathrm{K}_N^*(x,y) = \widetilde{\mathrm{K}}_N(y,x)$, then it follows by exchanging z and w in the last integral that

$$\mathrm{K}_N^*(x,y) = \frac{N^{1/3}}{(2i\pi)^2\sigma_N}\oint_\Theta \mathrm{d}z \oint_\Gamma \mathrm{d}w\, \frac{1}{w-z} e^{-N^{1/3}x(z-\mathfrak{c}_N)/\sigma_N + N^{1/3}y(w-\mathfrak{c}_N)/\sigma_N + Nf_N^*(z) - Nf_N^*(w)}. \tag{4.93}$$

Note that, as a consequence of the definition of f_N^* and (4.90), we have

$$(f_N^*)'(\mathfrak{c}_N) = (f_N^*)''(\mathfrak{c}_N) = 0, \qquad (f_N^*)^{(3)}(\mathfrak{c}_N) = -g_N''(\mathfrak{c}_N) > 0. \tag{4.94}$$

Thus, by comparing (4.93) with (4.20) and (4.94) with (4.22), we recover the setting of the proof of Theorem 3-(b), except that we exchanged x and y, the role of Γ and Θ as well, and that we replaced f_N by f_N^*. Since the Airy kernel is symmetric, see (3.1), it is enough to show that

$$\lim_{N\to\infty} \left\| \mathbf{1}_{(s,\varepsilon N^{2/3}\sigma_N)} (\mathrm{K}_N^* - \mathrm{K}_{\mathrm{Ai}}) \mathbf{1}_{(s,\varepsilon N^{2/3}\sigma_N)} \right\|_2 = 0, \tag{4.95}$$

$$\lim_{N\to\infty} \mathrm{Tr}\left(\mathbf{1}_{(s,\varepsilon N^{2/3}\sigma_N)} (\mathrm{K}_N^* - \mathrm{K}_{\mathrm{Ai}}) \mathbf{1}_{(s,\varepsilon N^{2/3}\sigma_N)} \right) = 0, \tag{4.96}$$

in order to prove (3.2), as explained in the proof of Theorem 3-(b).

In the case of left regular soft edges, the analysis substantially changes whether $\mathfrak{c}$ (cf. Prop. 2.11) is positive or not and we consider separately the two cases in the sequel.

4.6.2 The case where $\mathfrak{c}$ is positive

We first consider the case where $\mathfrak{c} > 0$, which is always the case except if $\mathfrak{a}$ is the leftmost edge and $\gamma > 1$, see Proposition 2.4. In particular, $\mathfrak{c}_N > 0$ for all N large enough. We then split Γ into two disjoint contours $\Gamma^{(0)}$ and $\Gamma^{(1)}$ in the following way: $\Gamma^{(0)}$ encloses the λ_j^{-1}'s which are larger that $\mathfrak{c}_N$, while $\Gamma^{(1)}$ encloses the λ_j^{-1}'s which are smaller that $\mathfrak{c}_N$. Proposition 2.4-(e) applied to the measure ν_N shows that the set $\{j\ ,\ 1 \leqslant j \leqslant n\ :\ \lambda_j^{-1} < \mathfrak{c}_N\}$ is not empty and thus the contour $\Gamma^{(1)}$ is always well-defined. If $\mathfrak{c}_N$ is actually larger than all the λ_j^{-1}'s, as it is the case when dealing with the smallest eigenvalue when $\gamma < 1$, then set $\Gamma^{(1)} = \Gamma$, $\mathrm{K}_N^{(1)} = \mathrm{K}_N^*$; any later statement involving $\Gamma^{(0)}$ will be considered as empty. Otherwise, $\Gamma^{(0)}$ is well-defined and we introduce for $\alpha \in \{0,1\}$ the kernels

$$\mathrm{K}_N^{(\alpha)}(x,y) = \frac{N^{1/3}}{(2i\pi)^2\sigma_N} \oint_\Theta \mathrm{d}z \oint_{\Gamma^{(\alpha)}} \mathrm{d}w\, \frac{1}{w-z} e^{-N^{1/3}(z-\mathfrak{c}_N)x/\sigma_N + N^{1/3}(w-\mathfrak{c}_N)y/\sigma_N + Nf_N^*(z) - Nf_N^*(w)}$$

so that $\mathrm{K}_N^*(x,y) = \mathrm{K}_N^{(0)}(x,y) + \mathrm{K}_N^{(1)}(x,y)$. We similarly have for the associated operators that $\mathrm{K}_N^* = \mathrm{K}_N^{(0)} + \mathrm{K}_N^{(1)}$. Observe moreover that we can deform Θ in $\mathrm{K}_N^{(1)}(x,y)$ so that it encloses the origin and $\Gamma^{(1)}$ since the residue we pick at $z = w$ vanishes.

In order to establish (4.95) and (4.96), it is then enough to prove that

$$\lim_{N\to\infty} \left\| \mathbf{1}_{(s,\varepsilon N^{2/3}\sigma_N)} \mathrm{K}_N^{(0)} \mathbf{1}_{(s,\varepsilon N^{2/3}\sigma_N)} \right\|_2 = 0, \tag{4.97}$$

$$\lim_{N\to\infty} \mathrm{Tr}\left(\mathbf{1}_{(s,\varepsilon N^{2/3}\sigma_N)} \mathrm{K}_N^{(0)} \mathbf{1}_{(s,\varepsilon N^{2/3}\sigma_N)} \right) = 0 \tag{4.98}$$

and

$$\lim_{N\to\infty} \left\| \mathbf{1}_{(s,\varepsilon N^{2/3}\sigma_N)} (\mathrm{K}_N^{(1)} - \mathrm{K}_{\mathrm{Ai}}) \mathbf{1}_{(s,\varepsilon N^{2/3}\sigma_N)} \right\|_2 = 0, \tag{4.99}$$

$$\lim_{N\to\infty} \mathrm{Tr}\left(\mathbf{1}_{(s,\varepsilon N^{2/3}\sigma_N)} (\mathrm{K}_N^{(1)} - \mathrm{K}_{\mathrm{Ai}}) \mathbf{1}_{(s,\varepsilon N^{2/3}\sigma_N)} \right) = 0. \tag{4.100}$$

The exact same estimates as in the proof of the Propositions 4.13 and 4.14 show that (4.97)–(4.100) hold true, provided we can show the existence of appropriate contours similarly as in Proposition 4.6. More precisely, it is enough to establish the next proposition in order to prove Theorem 3-(a), in the case where $\mathfrak{c} > 0$.

Proposition 4.16. *For every $\rho > 0$ small enough, there exists a contour $\Upsilon^{(0)}$ independent of N and two contours $\Upsilon^{(1)} = \Upsilon^{(1)}(N)$ and $\widetilde{\Theta} = \widetilde{\Theta}(N)$ which satisfy for every N large enough the following.*

(1) (a) $\Upsilon^{(0)}$ *encircles the λ_j^{-1}'s larger than $\mathfrak{c}_N$,*

(b) $\Upsilon^{(1)}$ *encircles the λ_j^{-1}'s smaller than $\mathfrak{c}_N$,*

(c) $\widetilde{\Theta}$ *encircles the λ_j^{-1}'s smaller than $\mathfrak{c}_N$ and the origin.*

(2) (a) $\Upsilon^{(1)} = \Upsilon_* \cup \Upsilon^{(1)}_{res}$ *where*

$$\Upsilon_* = \{\mathfrak{c}_N - te^{\pm i\pi/3} :\ t \in [0, \rho]\}.$$

(b) $\widetilde{\Theta} = \widetilde{\Theta}_* \cup \widetilde{\Theta}_{res}$ *where*

$$\widetilde{\Theta}_* = \{\mathfrak{c}_N + te^{\pm i\pi/3} :\ t \in [0, \rho]\} \ .$$

(3) *There exists $K > 0$ independent of N such that*

(a) $\mathrm{Re}\left(f_N(w) - f_N(\mathfrak{c}_N)\right) \geqslant K$ *for all* $w \in \Upsilon^{(0)}$,

(b) $\mathrm{Re}\left(f_N(w) - f_N(\mathfrak{c}_N)\right) \geqslant K$ *for all* $w \in \Upsilon^{(1)}_{res}$,

(c) $\mathrm{Re}\left(f_N(z) - f_N(\mathfrak{c}_N)\right) \leqslant -K$ *for all* $z \in \widetilde{\Theta}_{res}$.

(4) *There exists $d > 0$ independent of N such that*

$$\begin{aligned}
&\inf\left\{|z-w| :\ z \in \Upsilon^{(0)},\, w \in \widetilde{\Theta}\right\} \geqslant d \ ,\\
&\inf\left\{|z-w| :\ z \in \Upsilon_*,\, w \in \widetilde{\Theta}_{res}\right\} \geqslant d \ ,\\
&\inf\left\{|z-w| :\ z \in \Upsilon^{(1)}_{res},\, w \in \widetilde{\Theta}_*\right\} \geqslant d \ ,\\
&\inf\left\{|z-w| :\ z \in \Upsilon^{(1)}_{res},\, w \in \widetilde{\Theta}_{res}\right\} \geqslant d \ .
\end{aligned}$$

(5) (a) *The contours $\Upsilon^{(1)}$ and $\widetilde{\Theta}$ lie in a compact subset of $\mathbb{C}$, independent of N.*

(b) *The lengths of $\Upsilon^{(1)}$ and $\widetilde{\Theta}$ are uniformly bounded in N.*

Although the proof uses the same type of arguments than in the proof of Proposition 4.6, the analytical setting is not identical. Thus, although we shall provide less details than in the proof of Proposition 4.6, we shall emphasize on the required changes. Figure 8 may help as a visual support for the argument.

Proof. The regularity assumption yields $\varepsilon > 0$ such that $\lambda_j^{-1} \in (0, +\infty) \setminus B(\mathfrak{c}, \varepsilon)$ for every $1 \leqslant j \leqslant n$ and every N large enough. We then introduce the compact set $\mathcal{K}$ defined by

$$\mathcal{K} = \left(\left[\inf_N \frac{1}{\lambda_n}, \sup_N \frac{1}{\lambda_1}\right] \setminus B(\mathfrak{c}, \varepsilon)\right) \cup \{0\} \tag{4.101}$$

and notice that by construction $\{x \in \mathbb{R} :\ x^{-1} \in \mathrm{Supp}(\nu_N)\} \subset \mathcal{K}$ for every N large enough, and also that $\{x \in \mathbb{R} :\ x^{-1} \in \mathrm{Supp}(\nu)\} \subset \mathcal{K}$. If we introduce the map

$$f^*(z) = \mathfrak{a}(z - \mathfrak{c}) - \log(z) + \gamma \int \log(1 - xz)\nu(\mathrm{d}x) \ ,$$

then, given any simply connected subset of $\mathbb{C} \setminus \mathcal{K}$, we can choose a determination of the logarithm such that both the maps f_N^* and f^* are well-defined and holomorphic there for every N large enough. Notice that the definition of $\operatorname{Re} f^*$ does not depend on the determination of the logarithm. Moreover, the proof of Lemma 4.7-(a) shows that $\operatorname{Re} f_N^*$ converges locally uniformly on $\mathbb{C} \setminus \mathcal{K}$ toward $\operatorname{Re} f^*$, and moreover $\operatorname{Re} f_N^*(\mathfrak{c}_N) \to \operatorname{Re} f^*(\mathfrak{c})$ as $N \to \infty$.

Next, we perform a qualitative analysis for $\operatorname{Re} f^*$ and introduce the sets

$$\Omega_- = \Big\{ z \in \mathbb{C} : \; \operatorname{Re} f^*(z) < \operatorname{Re} f^*(\mathfrak{c}) \Big\}, \quad \Omega_+ = \Big\{ z \in \mathbb{C} : \; \operatorname{Re} f^*(z) > \operatorname{Re} f^*(\mathfrak{c}) \Big\}.$$

Since $\mathfrak{a} > 0$, the asymptotic behavior $\operatorname{Re} f^*(z) = \mathfrak{a} \operatorname{Re}(z - \mathfrak{c}) + O(\log |z|)$ as $z \to \infty$ shows that for every $\alpha \in (0, \pi/2)$ there exisits $R > 0$ large enough such that

$$\Big\{ z \in \mathbb{C} : \quad |z| > R, \quad -\frac{\pi}{2} + \alpha < \arg(z) < \frac{\pi}{2} - \alpha \Big\} \subset \Omega_+ \tag{4.102}$$

and

$$\Big\{ z \in \mathbb{C} : \quad |z| > R, \quad \frac{\pi}{2} + \alpha < \arg(z) < \frac{3\pi}{2} - \alpha \Big\} \subset \Omega_- . \tag{4.103}$$

Notice that the role of Ω_+ and Ω_- has been exchanged compared to the setting of a right edge. Moreover, the arguments of the proof of Lemma 4.8 show that both Ω_+ and Ω_- have a unique unbounded connected component.

As for the behavior of $\operatorname{Re} f^*$ around $\mathfrak{c}$, because $\mathfrak{a} = g(\mathfrak{c})$ it follows from the definition of f^* that $(f^*)'(z) = g(\mathfrak{c}) - g(z)$. Thus, by Proposition 2.11, we have $(f^*)'(\mathfrak{c}) = (f^*)''(\mathfrak{c}) = 0$ and $(f^*)^{(3)}(\mathfrak{c}) = -g''(\mathfrak{c}) > 0$. As a consequence, the same proofs than those of Lemmas 4.9 and 4.7-(b),(c) show there exist $\eta > 0$ and $0 < \theta < \pi/2$ small enough such that

$$\Delta_{2k+1} \subset \Omega_-, \qquad \Delta_{2k} \subset \Omega_+, \qquad k \in \{-1, 0, 1\},$$

where we introduced as in Section 4.4

$$\Delta_k = \Big\{ z \in \mathbb{C} : \quad 0 < |z - \mathfrak{c}| < \eta, \quad \Big| \arg(z - \mathfrak{c}) - k\frac{\pi}{3} \Big| < \theta \Big\}.$$

Notice that the role of Ω_- and Ω_+ is the same than in the right edge setting. We then denote by Ω_{2k+1} the connected component of Ω_- which contains Δ_{2k+1}, and similarly Ω_{2k} the connected component of Ω_+ which contains Δ_{2k}.

The proof of Lemma 4.10 yields that $\operatorname{Re} f^*$ is subharmonic in $\mathbb{C} \setminus \{0\}$ and is superharmonic in $\mathbb{C} \setminus \{x \in \mathbb{R} : \; x^{-1} \in \operatorname{Supp}(\nu)\}$. As a consequence, it follows from the proof of Lemma 4.11 that we obtain a similar statement as in Lemma 4.11 for $\operatorname{Re} f^*$ after having exchanged the role of Ω_+ and Ω_- (to furthermore convince the reader, notice that $\operatorname{Re} f^*(z) - \mathfrak{a} \operatorname{Re}(z - \mathfrak{c}) = -\operatorname{Re} f(z) - \mathfrak{b} \operatorname{Re}(z - \mathfrak{d})$ and that both the maps $z \mapsto \mathfrak{a} \operatorname{Re}(z - \mathfrak{c})$ and $z \mapsto \mathfrak{b} \operatorname{Re}(z - \mathfrak{d})$ are harmonic). Namely,

(1) If Ω_* is a connected component of Ω_-, then Ω_* is open and, if Ω_* is moreover bounded, there exists $x \in \operatorname{Supp}(\nu)$ such that $x^{-1} \in \Omega_*$.

(2) Let Ω_* be a connected component of Ω_+ with non-empty interior.

   (a) If Ω_* is bounded, then $0 \in \Omega_*$.

   (b) If Ω_* is bounded, then its interior is connected.

(c) If $0 \notin \Omega_*$, then the interior of Ω_* is connected.

Equipped with the previous observations we are now in position to provide the counterpart of Lemma 4.12 in the present setting, namely to prove that the following statements hold true.

(A) We have $\Omega_1 = \Omega_{-1}$, the interior of Ω_1 is connected, and for every $0 < \alpha < \pi/2$ there exists $R > 0$ such that

$$\Big\{ z \in \mathbb{C} : \quad |z| > R, \quad \frac{\pi}{2} + \alpha < \arg(z) < \frac{3\pi}{2} - \alpha \Big\} \subset \Omega_1.$$

(B) The interior of Ω_0 is connected and, for every $0 < \alpha < \pi/2$, there exists $R > 0$ such that

$$\Big\{ z \in \mathbb{C} : \quad |z| > R, \quad -\frac{\pi}{2} + \alpha < \arg(z) < \frac{\pi}{2} - \alpha \Big\} \subset \Omega_0.$$

(C) We have $\Omega_2 = \Omega_{-2}$, the interior of Ω_2 is connected, and there exists $\delta > 0$ such that $B(0, \delta) \subset \Omega_2$.

Let us first prove (A). Since by definition Ω_1 is a connected component of Ω_-, its interior is connected by (1). Let us prove by contradiction that Ω_1 is unbounded, from which (A) will follow by using the symmetry $\operatorname{Re} f^*(\overline{z}) = \operatorname{Re} f^*(z)$, the inclusion (4.103) and that Ω_- has a unique unbounded connected component. Assume Ω_1 is bounded. Then (1) yields the existence of $x \in \operatorname{Supp}(\nu)$ such that $x^{-1} \in \Omega_1$. If $x^{-1} < \mathfrak{c}$ (resp. $x^{-1} > \mathfrak{c}$), it then follows from the symmetry $\operatorname{Re} f^*(\overline{z}) = \operatorname{Re} f^*(z)$ that Ω_1 surrounds Ω_2 (resp. Ω_0) so that Ω_2* (resp. Ω_0) is a bounded connected component of Ω_+ which does not contain the origin. Notice that by (4.102), Ω_2 (resp. Ω_0) has non-empty interior. This yields with (2a) a contradiction and our claim follows. Since we just proved that Ω_1 is unbounded, the origin does not belong to Ω_0. As a consequence, (2a) and (2c) yield respectively that Ω_0 is unbounded and has a connected interior. Using moreover the inclusion (4.102) and that Ω_+ has a unique unbounded connected component, (B) follows.

As a byproduct of (A), Ω_2 is bounded. Thus Ω_2 contains the origin by (2a) and has a connected interior by (2b). By using the symmetry $\operatorname{Re} f^*(\overline{z}) = \operatorname{Re} f^*(z)$ and that $\operatorname{Re} f^*(z) \to +\infty$ as $z \to 0$, (C) is proved.

Finally, as a consequence of (A), (B) and (C), the existence of the contour $\Upsilon^{(0)}$, resp. $\Upsilon^{(1)}$, resp. $\widetilde{\Theta}$, in Proposition 4.16 is proved by choosing $\Upsilon^{(0)}$ in the interior of Ω_0 encircling $\{x \in \mathcal{K} : x > \mathfrak{c}\}$ and intersecting the real axis exactly twice in $\mathbb{R} \setminus \mathcal{K}$ with finite length, resp. by completing $\{\mathfrak{c}_N - te^{\pm i\pi/3} : \; t \in [0, \rho]\}$ for ρ small enough and N large enough so that both the points $\mathfrak{c}_N - \rho e^{i\pi/3}$ and $\mathfrak{c}_N - \rho e^{-i\pi/3}$ lie in Ω_2 into a closed contour with a path lying in the interior of Ω_2 but staying in $\{z \in \mathbb{C} : \; \operatorname{Re}(z) > 0\}$ and intersecting the real line exactly once at the left of $\mathcal{K}$ with finite length, resp. by completing $\{\mathfrak{c}_N + te^{\pm i\pi/3} : \; t \in [0, \rho]\}$ for ρ small enough and N large enough so that both the points $\mathfrak{c}_N + \rho e^{i\pi/3}$ and $\mathfrak{c}_N + \rho e^{-i\pi/3}$ belongs to Ω_1 into a closed contour with a path lying in the interior of Ω_1 and crossing the real axis exactly once at the left of the origin with finite length, and then by using the local uniform convergence of $\operatorname{Re} f_N^* \to \operatorname{Re} f^*$ on $\mathbb{C} \setminus \mathcal{K}$; see the proof of Proposition 4.6 for the details. $\square$

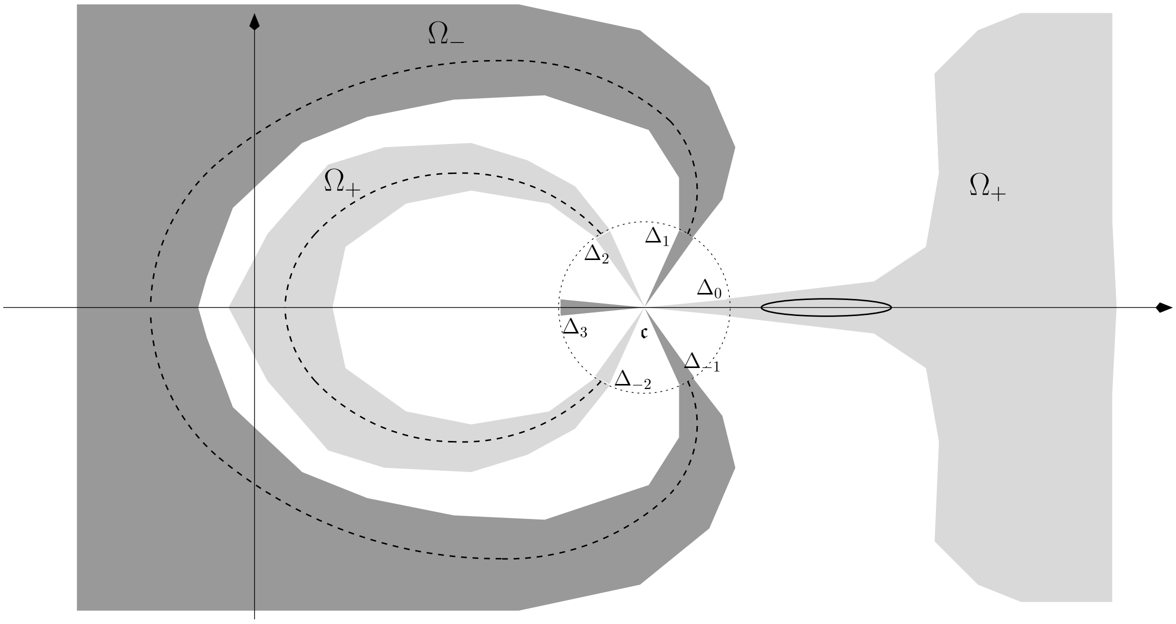

Figure 8: Preparation of the saddle point analysis for a left edge with $\mathfrak{c} > 0$. The path $\Upsilon^{(1)}_{res}$ is close to the inner dotted path at the left of $\mathfrak{c}$. The path $\widetilde{\Theta}_{res}$ is close to the outer dotted path at the left of $\mathfrak{c}$. The contour at the right of $\mathfrak{c}$ is $\Upsilon^{(0)}$.

4.6.3 The case where $\mathfrak{c}$ is negative

Here we consider the case where $\mathfrak{c}$ is negative, which only happens if we are looking at the leftmost edge $\mathfrak{a}$ when $\gamma > 1$, and thus $\mathfrak{c}_N < 0$ for all N large enough. We recall that

$$\mathrm{K}^*_N(x,y) = \frac{N^{1/3}}{(2i\pi)^2\sigma_N}\oint_\Theta \mathrm{d}z \oint_\Gamma \mathrm{d}w \,\frac{1}{w-z} e^{-N^{1/3}x(z-\mathfrak{c}_N)/\sigma_N + N^{1/3}y(w-\mathfrak{c}_N)/\sigma_N + Nf^*_N(z) - Nf^*_N(w)}.$$

Note that the λ_j^{-1}'s are zeros for $e^{f^*_N}$, and that 0 is a zero for $e^{-f^*_N}$. Thus, since the residue picked at $w = z$ vanishes, we can deform Θ and Γ is a way that Γ encircles Θ and all the λ_j^{-1}'s, whereas Θ encircles the origin and possibly some λ_j^{-1}'s.

It is enough to establish the next proposition in order to obtain (4.95) and (4.96) in the case where $\mathfrak{c} < 0$, and thus to complete the proof of Theorem 3-(a), since the same estimates as in the proof of Proposition 4.14 can be used after setting $\mathrm{K}^{(1)}_N = \mathrm{K}^*_N$ and $\Gamma^{(1)} = \Gamma$. The reader may refer to Figure 9 to better visualize the results of the next proposition as well as the proof argument.

Proposition 4.17. *For every $\rho > 0$ small enough, there exist contours $\Upsilon = \Upsilon(N)$ and $\widetilde{\Theta} = \widetilde{\Theta}(N)$ which satisfy for every N large enough the following.*

(1) (a) Υ *encircles $\widetilde{\Theta}$, the origin and all the λ_j^{-1}'s.*

(b) $\widetilde{\Theta}$ *encircles the origin (and possibly some λ_j^{-1}'s).*

(2) (a) $\Upsilon = \Upsilon_* \cup \Upsilon_{res}$ *where*

$$\Upsilon_* = \{\mathfrak{c}_N - te^{\pm i\pi/3} : \; t \in [0,\rho]\}.$$

(b) $\widetilde{\Theta} = \widetilde{\Theta}_* \cup \widetilde{\Theta}_{res}$ *where*

$$\widetilde{\Theta}_* = \{\mathfrak{c}_N + te^{\pm i\pi/3} : \ t \in [0, \rho]\}.$$

(3) *There exists $K > 0$ independent of N such that*

(a) $\mathrm{Re}\left(f_N(w) - f_N(\mathfrak{c}_N)\right) \geqslant K$ *for all* $w \in \Upsilon_{res}$

(b) $\mathrm{Re}\left(f_N(z) - f_N(\mathfrak{c}_N)\right) \leqslant -K$ *for all* $z \in \widetilde{\Theta}_{res}$

(4) *There exists $d > 0$ independent of N such that*

$$\begin{aligned} \inf\left\{|z-w| : \ z \in \Upsilon_*, \, w \in \widetilde{\Theta}_{res}\right\} &\geqslant d \\ \inf\left\{|z-w| : \ z \in \Upsilon_{res}, \, w \in \widetilde{\Theta}_*\right\} &\geqslant d \\ \inf\left\{|z-w| : \ z \in \Upsilon_{res}, \, w \in \widetilde{\Theta}_{res}\right\} &\geqslant d \end{aligned}$$

(5) (a) Υ *and* $\widetilde{\Theta}$ *lie in a bounded subset of* $\mathbb{C}$ *independently of* N

(b) *The lengths of* Υ *and* $\widetilde{\Theta}$ *are uniformly bounded in* N*.*

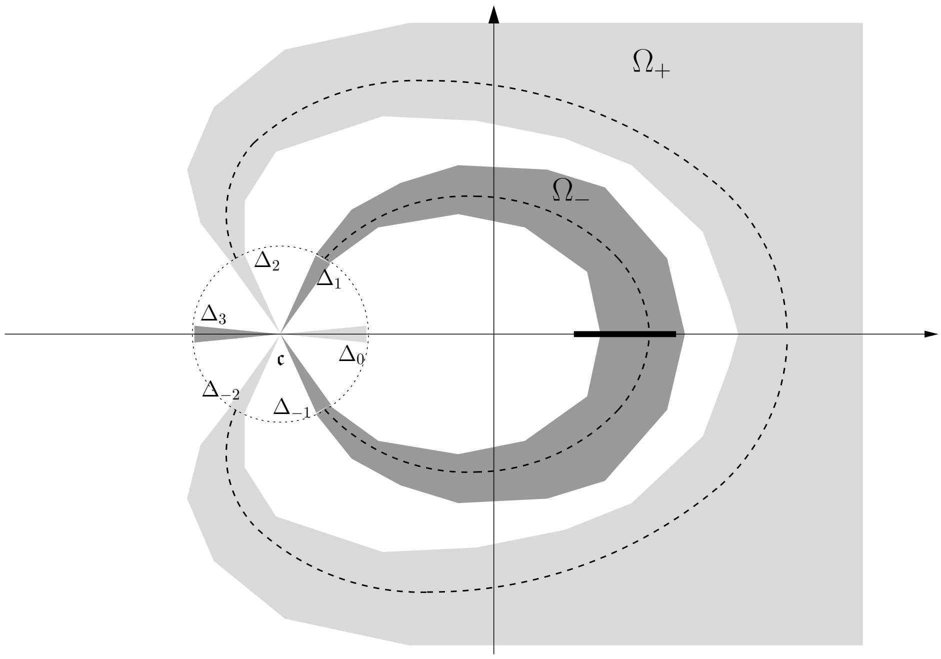

Figure 9: Preparation of the saddle point analysis for a left edge with $\mathfrak{c} < 0$. The path $\widetilde{\Theta}_{res}$ is close to the inner dotted path. The path $\widetilde{\Upsilon}_{res}$ is close to the outer dotted path. The thick segment represents the support of the image of ν by the map $x \mapsto x^{-1}$.

Proof. We use the notations, definitions and properties used in the proof of Proposition 4.16, except for $\mathcal{K}$ that we define by

$$\mathcal{K} = \left[\inf_N \frac{1}{\lambda_n}, \sup_N \frac{1}{\lambda_1}\right] \cup \{0\} \ .$$

Clearly $\{x \in \mathbb{R} : \quad x^{-1} \in \mathrm{Supp}(\nu_N)\} \subset \mathcal{K}$ for every N and moreover $\{x \in \mathbb{R} : \quad x^{-1} \in \mathrm{Supp}(\nu)\} \subset \mathcal{K}$. We now prove that the following facts hold true.

(A) We have $\Omega_1 = \Omega_{-1}$, the interior of Ω_1 is connected, and there exists $x_0 \in \mathrm{Supp}(\nu)$ and $\delta > 0$ such that $B(x_0^{-1}, \delta) \subset \Omega_1$.

(B) We have $\Omega_2 = \Omega_{-2}$, the interior of Ω_2 is connected, and for every $0 < \alpha < \pi/2$ there exists $R > 0$ such that

$$\Big\{ z \in \mathbb{C} : \quad |z| > R, \quad -\frac{\pi}{2} + \alpha < \arg(z) < \frac{\pi}{2} - \alpha \Big\} \subset \Omega_2.$$

The proof will mainly use properties (1) and (2)-(a)/(b)/(c) from the proof of Proposition 4.16. Let us show (A). First, Ω_1 has a connected interior by (1). Let us show by contradiction that Ω_1 is bounded. If Ω_1 is unbounded, then by using the symmetry $\mathrm{Re}\, f^*(\overline{z}) = \mathrm{Re}\, f^*(z)$, the inclusion (4.103) and the uniqueness of the unbounded connected component of Ω_-, it follows that Ω_2 is bounded without containing the origin, which contradicts (2a). Thus Ω_1 is bounded, and has to contain some x_0^{-1} with $x_0 \in \mathrm{Supp}(\nu)$ as a consequence of (1). Moreover, since $\mathrm{Re}\, f^*$ is upper semicontinuous on an open neighborhood of x_0^{-1} (because it is subharmonic on $\mathbb{C} \setminus \{0\}$), there exists $\delta > 0$ such that $B(x_0^{-1}, \delta) \subset \Omega_1$. As a consequence, together with the symmetry $\mathrm{Re}\, f^*(\overline{z}) = \mathrm{Re}\, f^*(z)$, (A) is proved.

Next, since Ω_1 thus surrounds the origin, then Ω_2 has to be unbounded by (2a) and has a connected interior by (2c). Finally, (B) follows from the symmetry $\mathrm{Re}\, f^*(\overline{z}) = \mathrm{Re}\, f^*(z)$, the inclusion (4.102) and the uniqueness of the unbounded connected component of Ω_+.

To construct Υ satisfying the conditions of Proposition 4.17, by (B) we can complete $\{\mathfrak{c}_N - te^{\pm i\pi/3} : \ t \in [0, \rho]\}$, for N large enough and ρ small enough so that both the points $\mathfrak{c}_N - \rho e^{i\pi/3}$ and $\mathfrak{c}_N - \rho e^{-i\pi/3}$ lie in Ω_2, into a closed contour with a path lying in the interior of Ω_2 and intersecting the real line exactly once at the right of $\mathcal{K}$ with finite length, and then use the local uniform convergence of $\mathrm{Re}\, f_N^*$ to $\mathrm{Re}\, f^*$ on $\mathbb{C} \setminus \mathcal{K}$, see the proof of Proposition 4.6 for the details.

To construct $\widetilde{\Theta}$, we need to proceed more carefully since Ω_1 actually crosses $\mathcal{K}$ and $\mathrm{Re}\, f_N^*$ may not converge uniformly to $\mathrm{Re}\, f^*$ there. For N large enough and ρ small enough so that the points $\mathfrak{c}_N + \rho e^{i\pi/3}$ and $\mathfrak{c}_N + \rho e^{-i\pi/3}$ lie in Ω_1, by (A) we can complete $\{\mathfrak{c}_N + te^{\pm i\pi/3} : \ t \in [0, \rho]\}$ into a closed contour with a path Ξ lying in the interior of Ω_1 and crossing the real axis exactly once at x_0^{-1} with finite length. Since $B(x_0^{-1}, \delta) \subset \Omega_1$ we can moreover assume that Ξ crosses the real axis perpendicularly, namely that there exists $\eta_1 > 0$ small enough such that the segment $\{x_0^{-1} + i\eta : \quad |\eta| \leqslant \eta_1\}$ is contained in Ξ. Since $\Omega_1 \subset \Omega_-$, there exists $K > 0$ independent on N such that

$$\mathrm{Re}\, f^*(z) - \mathrm{Re}\, f^*(\mathfrak{c}) \leqslant -4K, \qquad z \in \Xi. \tag{4.104}$$

Notice that the map $z \mapsto \int \log|1 - xz| \nu(\mathrm{d}x)$ is upper semicontinuous on $\mathbb{C}$ since it is subharmonic (see the proof of Lemma 4.10). As a consequence, if $\int \log|1 - xx_0^{-1}| \nu(\mathrm{d}x) = -\infty$, then there exists $\eta_0 \in (0, \eta_1)$ small enough so that

$$\gamma \int \log|1 - x(x_0^{-1} + i\eta_0)| \nu(\mathrm{d}x) \leqslant -2K - \sup_N \Big(\mathfrak{a}_N (x_0^{-1} - \mathfrak{c}_N) - \mathrm{Re}\, f^*(\mathfrak{c}_N) \Big) - \log(x_0). \tag{4.105}$$

If instead $\int \log|1 - x/x_0|\nu(\mathrm{d}x) > -\infty$, then by upper semicontinuity there exists $\eta_0 \in (0, \eta_1)$ small enough to that

$$\gamma \int \log|1 - x(x_0^{-1} + i\eta_0)|\nu(\mathrm{d}x) \leqslant \gamma \int \log|1 - x/x_0|\nu(\mathrm{d}x) + K. \tag{4.106}$$

Let η_0 be defined as above and consider a compact tubular neighborhood $\mathcal{T}$ of $\Xi\setminus\{x_0^{-1}+i\eta : |\eta| < \eta_0\}$ small enough so that $\mathcal{T}$ lies in $\mathbb{C} \setminus \mathcal{K}$ and $\operatorname{Re} f^* - \operatorname{Re} f^*(\mathfrak{c}) \leqslant -3K$ there (the latter is possible since $\operatorname{Re} f^*$ is upper semicontinuous on $\mathbb{C} \setminus \{0\}$). Notice that by construction the interior of $\mathcal{T}$ contain both the points $\mathfrak{c}_N + \rho e^{i\pi/3}$ and $\mathfrak{c}_N + \rho e^{-i\pi/3}$ for every N large enough, and the points $x_0^{-1} + i\eta_0$ and $x_0^{-1} - i\eta_0$ as well. Using the local uniform convergence of $\operatorname{Re} f_N^*$ to $\operatorname{Re} f^*$ on $\mathbb{C} \setminus \mathcal{K}$ and the convergence $\operatorname{Re} f_N^*(\mathfrak{c}_N) \to \operatorname{Re} f(\mathfrak{c})$, we can show as in the proof of Proposition 4.6 that for every N large enough we have

$$\operatorname{Re} f_N^*(z) - \operatorname{Re} f_N^*(\mathfrak{c}_N) \leqslant -K$$

for every $z \in \mathcal{T}$. As a consequence, for every N large enough, we can construct the path $\widetilde{\Theta}_{res}$ is the following way: it goes from $\mathfrak{c}_N + \rho e^{-i\pi/3}$ to $x_0^{-1} + i\eta_0$ staying in $\mathcal{T}$, then follows the segment $\{x_0^{-1} + i\eta :\ 0 \leqslant \eta \leqslant \eta_0\}$, and is finally completed by symmetry with respect to the real axis. As for what is happening on $\{x_0^{-1} + i\eta :\ |\eta| < \eta_0\}$, since a priori $\operatorname{Re} f_N^*$ does not converge uniformly toward $\operatorname{Re} f^*$ there, we need an extra argument to complete the proof of Proposition 4.17. Namely, we need to show that for every N large enough, uniformly in $|\eta| < \eta_0$,

$$\operatorname{Re} f_N^*(x_0^{-1} + i\eta) - \operatorname{Re} f_N^*(\mathfrak{c}_N) \leqslant -K\ . \tag{4.107}$$

Let us set for convenience $z_\eta = x_0^{-1} + i\eta$ for any $|\eta| \leqslant \eta_0$. First, since the map $x \mapsto \log|1 - xz_{\eta_0}|$ is bounded and continuous on any compact subset of $\mathbb{R}$, the weak convergence $\nu_N \to \nu$ and the convergence $n/N \to \gamma$ yield that for any N large enough

$$\frac{n}{N} \int \log|1 - xz_{\eta_0}|\nu_N(\mathrm{d}x) \leqslant \gamma \int \log|1 - xz_{\eta_0}|\nu(\mathrm{d}x) + K. \tag{4.108}$$

If we assume $\int \log|1 - x/x_0|\nu(\mathrm{d}x) = -\infty$, then for every N large enough, uniformly in $|\eta| < \eta_0$,

$$\begin{aligned}
&\operatorname{Re} f_N^*(z_\eta) - \operatorname{Re} f_N^*(\mathfrak{c}_N) \\
&\leqslant \quad \sup_N \Big(\mathfrak{a}_N(x_0^{-1} - \mathfrak{c}_N) - \operatorname{Re} f^*(\mathfrak{c}_N)\Big) - \log|z_\eta| + \frac{n}{N}\int \log|1 - xz_\eta|\,\nu_N(\mathrm{d}x) \\
&\leqslant \quad \sup_N \Big(\mathfrak{a}_N(x_0^{-1} - \mathfrak{c}_N) - \operatorname{Re} f^*(\mathfrak{c}_N)\Big) + \log(x_0) + \frac{n}{N}\int \log|1 - xz_{\eta_0}|\,\nu_N(\mathrm{d}x) \\
&\leqslant \quad -K,
\end{aligned}$$

where for the last inequality we used (4.108) and (4.105).

Now, assume instead that $\int \log|1-x/x_0|\nu(\mathrm{d}x) > -\infty$. By using the convergences $\mathfrak{a}_N \to \mathfrak{a}$, $\mathfrak{c}_N \to \mathfrak{c}$ and $\operatorname{Re} f_N^*(\mathfrak{c}_N) \to \operatorname{Re} f^*(\mathfrak{c})$, we obtain for every N large enough (and independently on η)

$$\mathfrak{a}_N \operatorname{Re}(z_\eta - \mathfrak{c}_N) - \operatorname{Re} f_N^*(\mathfrak{c}_N) \leqslant \mathfrak{a} \operatorname{Re}(z_\eta - \mathfrak{c}) - \operatorname{Re} f^*(\mathfrak{c}) + K.$$

Combined with the inequalites (4.104), (4.108) and (4.106), we obtain that for every N large enough and uniformly in $|\eta| < \eta_0$

$$\begin{aligned}
&\operatorname{Re} f_N^*(z_\eta) - \operatorname{Re} f_N^*(\mathfrak{c}_N) \\
&\leqslant \quad K + \operatorname{Re} f^*(z_\eta) - \operatorname{Re} f^*(\mathfrak{c}) + \frac{n}{N}\int \log|1 - xz_\eta|\,\nu_N(\mathrm{d}x) - \gamma\int \log|1 - xz_\eta|\,\nu(\mathrm{d}x) \\
&\leqslant \quad -3K + \frac{n}{N}\int \log|1 - xz_\eta|\,\nu_N(\mathrm{d}x) - \gamma\int \log|1 - xz_\eta|\,\nu(\mathrm{d}x) \\
&\leqslant \quad -3K + \frac{n}{N}\int \log|1 - xz_{\eta_0}|\,\nu_N(\mathrm{d}x) - \gamma\int \log|1 - x/x_0|\,\nu(\mathrm{d}x) \\
&\leqslant \quad -2K + \gamma\int \log|1 - xz_{\eta_0}|\,\nu(\mathrm{d}x) - \gamma\int \log|1 - x/x_0|\,\nu(\mathrm{d}x) \\
&\leqslant \quad -K,
\end{aligned}$$

and this completes the proof of Proposition 4.17.

□

5 Proof of Theorem 4: Asymptotic independence

Our strategy to prove Theorem 4 builds on an approach used by Bornemann [21]. Indeed, the asymptotic independence for the smallest and largest eigenvalues of an $N \times N$ GUE random matrix is established in [21] by showing that the trace class norm of the off-diagonal entries of a two by two operator valued matrix goes to zero as $N \to \infty$. Here we obtain that proving the asymptotic joint independence of several extremal eigenvalues leads to consider a larger operator valued matrix. Moreover, we show that it is actually sufficient to establish that the Hilbert-Schmidt norms of the off-diagonal entries go to zero as $N \to \infty$, instead of the trace class norms. The former can be provided by an asymptotic analysis for double complex integrals as we performed in the previous section.

More generally, our method can be applied to several other determinantal point processes for which a contour integral representation for the kernel and its asymptotic analysis are known, e.g. the eigenvalues of an additive perturbation of a GUE matrix [25].

Conventions: In this section, we fix two finite sets I and J of indices, and real numbers $(s_i)_{i\in I}$ and $(t_j)_{j\in J}$ as well. Assume that $(\mathfrak{a}_i = g(\mathfrak{c}_i))_{i\in I}$ are regular left soft edges and $(\mathfrak{b}_j = g(\mathfrak{d}_j))_{j\in J}$ are regular right edges. We denote by $\mathfrak{c}_{i,N}$ and $\mathfrak{d}_{j,N}$ the sequences associated respectively with $\mathfrak{a}_i$ and $\mathfrak{b}_j$ as specified by Proposition 2.7-(c). We moreover set

$$\mathfrak{a}_{i,N} = g_N(\mathfrak{c}_{i,N}), \qquad \mathfrak{b}_{j,N} = g_N(\mathfrak{d}_{j,N})$$

and

$$\sigma_{i,N} = \left(\frac{2}{g_N''(\mathfrak{c}_{i,N})}\right)^{1/3}, \qquad \delta_{j,N} = \left(\frac{2}{g_N''(\mathfrak{d}_{j,N})}\right)^{1/3},$$

where g_N has been introduced in (2.6). Similarly, $\varphi_i(N)$ (respectively $\phi_j(N)$) denotes the sequence associated with $\mathfrak{a}_{i,N}$ (resp. $\mathfrak{b}_{j,N}$) as in Theorem 2 (see also Propositions 2.11 and 2.12). Finally, we shall consider that the free parameter q introduced in the statement of Proposition 4.2 is zero when dealing with the kernel $\mathrm{K}_N(x,y)$, see Remark 4.3.

Our starting point the following proposition.

Proposition 5.1. *Consider the setting of Theorem 4. Then, for every $\varepsilon > 0$ small enough and for every sequences $(\eta_{i,N})_N$, $(\chi_{j,N})_N$ of positive numbers growing with N to infinity, it holds that*

$$\begin{aligned}\mathbb{P}\Big(\eta_{i,N}\big(\mathfrak{a}_{i,N} - x_{\varphi_i(N)}\big) \leqslant s_i\,,\ \chi_{j,N}\big(x_{\phi_j(N)} - \mathfrak{b}_{j,N}\big) \leqslant t_j\ ,\ i\in I\,,\ j\in J\Big)\\ = \det\big(I - \mathrm{K}_N\big)_{L^2\left((\bigcup_{i\in I}A_i)\cup(\bigcup_{j\in J}B_j)\right)} + o(1)\end{aligned}$$

as $N\to\infty$, where

$$A_i = \left(\mathfrak{a}_{i,N} - \varepsilon,\, \mathfrak{a}_{i,N} - \frac{s_i}{\eta_{i,N}}\right)\ ,\quad B_j = \left(\mathfrak{b}_{j,N} + \frac{t_j}{\chi_{j,N}},\, \mathfrak{b}_{j,N}+\varepsilon\right)\ .$$

The proof is omitted, being very similar to the one of Proposition 4.4. Now, if we specify $\eta_{i,N} = N^{2/3}\sigma_{i,N}$ and $\chi_{j,N} = N^{2/3}\delta_{j,N}$, then Proposition 5.1 reads:

$$\begin{aligned}\mathbb{P}\Big(N^{2/3}\sigma_{i,N}\big(\mathfrak{a}_{i,N} - x_{\varphi_i(N)}\big) \leqslant s_i\,,\ N^{2/3}\delta_{j,N}\big(x_{\phi_j(N)} - \mathfrak{b}_{j,N}\big) \leqslant t_j\,,\ i\in I\,,\ j\in J\Big)\\ = \det\big(I - \mathrm{K}_N\big)_{L^2\left((\bigcup_{i\in I}A_i)\cup(\bigcup_{j\in J}B_j)\right)} + o(1)\end{aligned}\tag{5.1}$$

where

$$A_i = \left(\mathfrak{a}_{i,N} - \varepsilon,\, \mathfrak{a}_{i,N} - \frac{s_i}{N^{2/3}\sigma_{i,N}}\right)\ ,\quad B_j = \left(\mathfrak{b}_{j,N} + \frac{t_j}{N^{2/3}\delta_{j,N}},\, \mathfrak{b}_{j,N}+\varepsilon\right)\ .$$

For every $i\in I$ and $j\in J$, we introduce the maps

$$f^*_{i,N}(z) = \mathfrak{a}_{i,N}(z - \mathfrak{c}_{i,N}) - \log(z) + \frac{1}{N}\sum_{k=1}^{n}\log(1-\lambda_k z) \tag{5.2}$$

$$f_{j,N}(z) = -\mathfrak{b}_{j,N}(z - \mathfrak{d}_{j,N}) + \log(z) - \frac{1}{N}\sum_{k=1}^{n}\log(1-\lambda_k z) \tag{5.3}$$

and the multiplication operators E^*_i and E_j acting on $L^2(A_i)$ and $L^2(B_j)$ respectively by

$$\mathrm{E}^*_i h(x) = e^{Nf^*_{i,N}(\mathfrak{c}_{i,N}) + Nx\mathfrak{c}_{i,N}}h(x), \qquad h\in L^2(A_i),$$
$$\mathrm{E}_j h(x) = e^{-Nf_{j,N}(\mathfrak{d}_{j,N}) + Nx\mathfrak{d}_{j,N}}h(x), \qquad h\in L^2(B_j).$$

The next proposition is the key to obtain Theorem 4.

Proposition 5.2. *For every ε small enough, the following holds true.*

(a) *For every $(i,j)\in J\times J$ such that $i\neq j$, we have*

$$\lim_{N\to\infty}\left\|\mathbf{1}_{B_i}\mathrm{E}_i\,\mathrm{K}_N\mathrm{E}_j^{-1}\mathbf{1}_{B_j}\right\|_2 = 0. \tag{5.4}$$

(b) *For every $(i,j)\in I\times I$ such that $i\neq j$, we have*

$$\lim_{N\to\infty}\left\|\mathbf{1}_{A_i}\mathrm{E}^*_i\,\mathrm{K}_N(\mathrm{E}^*_j)^{-1}\mathbf{1}_{A_j}\right\|_2 = 0. \tag{5.5}$$

(c) *For every* $(i,j)\in I\times J$, *we have*

$$\lim_{N\to\infty}\left\|\mathbf{1}_{A_i}\mathrm{E}_i^*\,\mathrm{K}_N\mathrm{E}_j^{-1}\mathbf{1}_{B_j}\right\|_2=0 \tag{5.6}$$

and

$$\lim_{N\to\infty}\left\|\mathbf{1}_{B_j}\mathrm{E}_j\,\mathrm{K}_N(\mathrm{E}_i^*)^{-1}\mathbf{1}_{A_j}\right\|_2=0. \tag{5.7}$$

Before proving Proposition 5.2, let us show how does it lead to the asymptotic joint independence of the extremal eigenvalues:

Proof of Theorem 4. Our purpose is to show that for large N, the determinant at the right hand side of (5.1) converges to a product of Fredholm determinants involving the Airy kernel. Assume that N is large enough so that all the A_i's and B_j's are disjoint sets. Then, as shown in [22] (see also [37, Chap. 6]), the Fredholm determinant $\det\big(I-\mathrm{K}_N\big)_{L^2\left((\bigcup_{i\in I}A_i)\cup(\bigcup_{j\in J}B_j)\right)}$ admits the operator matrix representation

$$\begin{aligned}&\det\big(I-\mathrm{K}_N\big)_{L^2\left((\bigcup_{i\in I}A_i)\cup(\bigcup_{j\in J}B_j)\right)}\\ =\;&\det\left(I-\begin{bmatrix}\left[\mathrm{K}_{II}^{i,j}\right]_{(i,j)\in I\times I} & \left[\mathrm{K}_{IJ}^{i,j}\right]_{(i,j)\in I\times J}\\ \left[\mathrm{K}_{JI}^{i,j}\right]_{(i,j)\in J\times I} & \left[\mathrm{K}_{JJ}^{i,j}\right]_{(i,j)\in J\times J}\end{bmatrix}\right)_{\left(\bigoplus_{i\in I}L^2(A_i)\right)\oplus\left(\bigoplus_{j\in J}L^2(B_j)\right)}\end{aligned} \tag{5.8}$$

where $\mathrm{K}_{II}^{i,j}:L^2(A_j)\to L^2(A_i)$ denotes the integral operator

$$\mathrm{K}_{II}^{i,j}h(x)=\int_{A_j}\mathrm{K}_N(x,y)h(y)\mathrm{d}y,\qquad x\in A_i,$$

and similarly the operators $\mathrm{K}_{IJ}^{i,j}:L^2(B_j)\to L^2(A_i)$, $\mathrm{K}_{JI}^{i,j}:L^2(A_j)\to L^2(B_i)$ and $\mathrm{K}_{JJ}^{i,j}:L^2(B_j)\to L^2(B_i)$ are defined by restricting K_N on appropriate subspaces of $L^2(\mathbb{R})$. Consider now the diagonal operator

$$\mathrm{E}=\Big(\bigoplus_{i\in I}\mathrm{E}_i^*\Big)\oplus\Big(\bigoplus_{j\in J}\mathrm{E}_j\Big)$$

acting on $\big(\bigoplus_{i\in I}L^2(A_i)\big)\oplus\big(\bigoplus_{j\in J}L^2(B_j)\big)$. Since the A_i's and B_j's are compact sets and

K_N is locally trace class, the identity (4.1) then yields

$$\det\left(I-\begin{bmatrix}\left[\mathrm{K}_{II}^{i,j}\right]_{(i,j)\in I\times I} & \left[\mathrm{K}_{IJ}^{i,j}\right]_{(i,j)\in I\times J}\\ \left[\mathrm{K}_{JI}^{i,j}\right]_{(i,j)\in J\times I} & \left[\mathrm{K}_{JJ}^{i,j}\right]_{(i,j)\in J\times J}\end{bmatrix}\right)_{\left(\bigoplus_{i\in I}L^2(A_i)\right)\oplus\left(\bigoplus_{j\in J}L^2(B_j)\right)}$$
$$=\det\left(I-\mathrm{E}\begin{bmatrix}\left[\mathrm{K}_{II}^{i,j}\right]_{(i,j)\in I\times I} & \left[\mathrm{K}_{IJ}^{i,j}\right]_{(i,j)\in I\times J}\\ \left[\mathrm{K}_{JI}^{i,j}\right]_{(i,j)\in J\times I} & \left[\mathrm{K}_{JJ}^{i,j}\right]_{(i,j)\in J\times J}\end{bmatrix}\mathrm{E}^{-1}\right)_{\left(\bigoplus_{i\in I}L^2(A_i)\right)\oplus\left(\bigoplus_{j\in J}L^2(B_j)\right)}$$
$$=\det\left(I-\begin{bmatrix}\left[\mathrm{E}_i^*\mathrm{K}_{II}^{i,j}(\mathrm{E}_j^*)^{-1}\right]_{(i,j)\in I\times I} & \left[\mathrm{E}_i^*\mathrm{K}_{IJ}^{i,j}\mathrm{E}_j^{-1}\right]_{(i,j)\in I\times J}\\ \left[\mathrm{E}_i\mathrm{K}_{JI}^{i,j}(\mathrm{E}_j^*)^{-1}\right]_{(i,j)\in J\times I} & \left[\mathrm{E}_i\mathrm{K}_{JJ}^{i,j}\mathrm{E}_j^{-1}\right]_{(i,j)\in J\times J}\end{bmatrix}\right)_{\left(\bigoplus_{i\in I}L^2(A_i)\right)\oplus\left(\bigoplus_{j\in J}L^2(B_j)\right)}$$
$$=\det\left(I-\begin{bmatrix}\left[\mathbf{1}_{A_i}\mathrm{E}_i^*\,\mathrm{K}_N(\mathrm{E}_j^*)^{-1}\mathbf{1}_{A_j}\right]_{(i,j)\in I\times I} & \left[\mathbf{1}_{A_i}\mathrm{E}_i^*\,\mathrm{K}_N\mathrm{E}_j^{-1}\mathbf{1}_{B_j}\right]_{(i,j)\in I\times J}\\ \left[\mathbf{1}_{B_i}\mathrm{E}_i\,\mathrm{K}_N(\mathrm{E}_j^*)^{-1}\mathbf{1}_{A_j}\right]_{(i,j)\in J\times I} & \left[\mathbf{1}_{B_i}\mathrm{E}_i\,\mathrm{K}_N\mathrm{E}_i^{-1}\mathbf{1}_{B_j}\right]_{(i,j)\in J\times J}\end{bmatrix}\right)_{L^2(\mathbb{R})^{\oplus(|I|+|J|)}} \tag{5.9}$$

where $|I|$ and $|J|$ stand for the cardinalities of I and J respectively. By using the definition (4.2) of $\det_2$, it follows from (5.8) and (5.9) that

$$\begin{aligned}&\det\left(I-\mathrm{K}_N\right)_{L^2\left((\bigcup_{i\in I}A_i)\cup(\bigcup_{j\in J}B_j)\right)}\\ &=\prod_{i\in I}e^{\mathrm{Tr}(\mathbf{1}_{A_i}\mathrm{E}_i^*\,\mathrm{K}_N(\mathrm{E}_i^*)^{-1}\mathbf{1}_{A_i})}\prod_{j\in J}e^{\mathrm{Tr}(\mathbf{1}_{B_j}\mathrm{E}_j\,\mathrm{K}_N\mathrm{E}_j^{-1}\mathbf{1}_{B_j})}\\ &\times\det\nolimits_2\left(I-\begin{bmatrix}\left[\mathbf{1}_{A_i}\mathrm{E}_i^*\,\mathrm{K}_N(\mathrm{E}_j^*)^{-1}\mathbf{1}_{A_j}\right]_{(i,j)\in I\times I} & \left[\mathbf{1}_{A_i}\mathrm{E}_i^*\,\mathrm{K}_N\mathrm{E}_j^{-1}\mathbf{1}_{B_j}\right]_{(i,j)\in I\times J}\\ \left[\mathbf{1}_{B_i}\mathrm{E}_i\,\mathrm{K}_N(\mathrm{E}_j^*)^{-1}\mathbf{1}_{A_j}\right]_{(i,j)\in J\times I} & \left[\mathbf{1}_{B_i}\mathrm{E}_i\,\mathrm{K}_N\mathrm{E}_i^{-1}\mathbf{1}_{B_j}\right]_{(i,j)\in J\times J}\end{bmatrix}\right)_{L^2(\mathbb{R})^{\oplus(|I|+|J|)}}.\end{aligned} \tag{5.10}$$

Let us inspect the diagonal elements of the matrix valued operator in the Fredholm determinant at the right hand side of the previous identity. In Section 4, we have precisely shown that for every $i\in I$ and $j\in J$,

$$\lim_{N\to\infty}\left\|\mathbf{1}_{A_i}\mathrm{E}_i^*\,\mathrm{K}_N(\mathrm{E}_i^*)^{-1}\mathbf{1}_{A_i}-\mathbf{1}_{(s_i,\infty)}\mathrm{K}_{\mathrm{Ai}}\mathbf{1}_{(s_i,\infty)}\right\|_2=0,$$
$$\lim_{N\to\infty}\left\|\mathbf{1}_{B_j}\mathrm{E}_j\,\mathrm{K}_N\mathrm{E}_j^{-1}\mathbf{1}_{B_j}-\mathbf{1}_{(t_j,\infty)}\mathrm{K}_{\mathrm{Ai}}\mathbf{1}_{(t_j,\infty)}\right\|_2=0$$

and

$$\lim_{N\to\infty}\mathrm{Tr}(\mathbf{1}_{A_i}\mathrm{E}_i^*\,\mathrm{K}_N(\mathrm{E}_i^*)^{-1}\mathbf{1}_{A_i})=\mathrm{Tr}(\mathbf{1}_{(s_i,\infty)}\mathrm{K}_{\mathrm{Ai}}\mathbf{1}_{(s_i,\infty)}),$$
$$\lim_{N\to\infty}\mathrm{Tr}(\mathbf{1}_{B_j}\mathrm{E}_j\,\mathrm{K}_N\mathrm{E}_j^{-1}\mathbf{1}_{B_j})=\mathrm{Tr}(\mathbf{1}_{(t_j,\infty)}\mathrm{K}_{\mathrm{Ai}}\mathbf{1}_{(t_j,\infty)}).$$

Proposition 5.2 then yields that the Hilbert-Schmidt norms of the off diagonal entries of the matrix valued operator in the Fredholm determinant at the right hand side of (5.10) converge to zero. Recalling that $\det_2$ is continuous with respect to the Hilbert-Schmidt norm, we obtain from (5.10) that

$$\begin{aligned}
&\lim_{N\to\infty} \det\big(I-\mathrm{K}_N\big)_{L^2\left((\bigcup_{i\in I} A_i)\cup(\bigcup_{j\in J} B_j)\right)} \\
&= \prod_{i\in I} e^{\mathrm{Tr}(\mathbf{1}_{(s_i,\infty)}\mathrm{K}_{\mathrm{Ai}}\mathbf{1}_{(s_i,\infty)})}\det{}_2\big(I-\mathbf{1}_{(s_i,\infty)}\mathrm{K}_{\mathrm{Ai}}\mathbf{1}_{(s_i,\infty)}\big)_{L^2(\mathbb{R})} \\
&\qquad \times \prod_{j\in J} e^{\mathrm{Tr}(\mathbf{1}_{(t_j,\infty)}\mathrm{K}_{\mathrm{Ai}}\mathbf{1}_{(t_j,\infty)})}\det{}_2\big(I-\mathbf{1}_{(t_j,\infty)}\mathrm{K}_{\mathrm{Ai}}\mathbf{1}_{(t_j,\infty)}\big)_{L^2(\mathbb{R})} \\
&= \prod_{i\in I}\det\big(I-\mathrm{K}_{\mathrm{Ai}}\big)_{L^2(s_i,\infty)}\prod_{j\in J}\det\big(I-\mathrm{K}_{\mathrm{Ai}}\big)_{L^2(t_j,\infty)},
\end{aligned}$$

and Theorem 4 is proved. □

Now we turn to the proof of Proposition 5.2.

To do so, we shall deform the contours Γ and Θ in the integral representation of K_N to appropriate contours for the asymptotic analysis, as provided by the propositions 4.6, 4.16 and 4.17. The problem is that since Θ and Γ will be associated to different critical points $\mathfrak{c}_N$'s or $\mathfrak{d}_N$'s, the possibility that they intersect holds true. This raises a problem related to the presence of the factor $(w-z)^{-1}$ in the integral representation of K_N. This problem can be avoided by using the following alternative expression of the kernel K_N, that was established in [17]; since the proof is short, we provide it for the sake of completeness.

Lemma 5.3. *For every $x\neq y$ we have*

$$\mathrm{K}_N(x,y)=\frac{N}{(2i\pi)^2(x-y)}\oint_\Gamma \mathrm{d}z\oint_\Theta \mathrm{d}w\, e^{-Nxz+Nyw}C_N(z,w)\left(\frac{z}{w}\right)^N\prod_{i=1}^n\left(\frac{1-\lambda_i w}{1-\lambda_i z}\right), \tag{5.11}$$

where

$$C_N(z,w)=\frac{1}{zw}-\frac{1}{N}\sum_{j=1}^n\frac{\lambda_j^2}{(1-\lambda_j z)(1-\lambda_j w)}. \tag{5.12}$$

Proof. Starting from (4.5) with $q=0$ and following [17, Section 3.3], we obtain by integrations by parts

$$\begin{aligned}
x\mathrm{K}_N(x,y)=\frac{1}{(2i\pi)^2}\oint_\Gamma \mathrm{d}z\oint_\Theta \mathrm{d}w\, e^{-Nxz+Nyw}\frac{1}{w-z}\left(\frac{z}{w}\right)^N\prod_{i=1}^n\left(\frac{1-\lambda_i w}{1-\lambda_i z}\right)& \\
\times\left(\frac{1}{w-z}+\frac{N}{z}-\sum_{j=1}^n\frac{\lambda_j}{1-\lambda_j z}\right)&
\end{aligned}$$

and

$$\begin{aligned}
y\mathrm{K}_N(x,y)=\frac{1}{(2i\pi)^2}\oint_\Gamma \mathrm{d}z\oint_\Theta \mathrm{d}w\, e^{-Nxz+Nyw}\frac{1}{w-z}\left(\frac{z}{w}\right)^N\prod_{i=1}^n\left(\frac{1-\lambda_i w}{1-\lambda_i z}\right)& \\
\times\left(\frac{1}{w-z}+\frac{N}{w}-\sum_{j=1}^n\frac{\lambda_j}{1-\lambda_j w}\right).&
\end{aligned}$$

This provides

$$(x-y)\mathrm{K}_N(x,y) = \frac{N}{(2i\pi)^2}\oint_\Gamma \mathrm{d}z \oint_\Theta \mathrm{d}w\, e^{-Nxz+Nyw}\left(\frac{z}{w}\right)^N \prod_{i=1}^n\left(\frac{1-\lambda_i w}{1-\lambda_i z}\right) \\ \times\left(\frac{1}{zw}-\frac{1}{N}\sum_{j=1}^n\frac{\lambda_j^2}{(1-\lambda_j z)(1-\lambda_j w)}\right)$$

and Lemma 5.3 follows.

□

Equipped with Lemma 5.3, we are now in position to prove Proposition 5.2.

Proof of Proposition 5.2. Since the sets of indices I and J are finite by assumption, the regularity condition provides $\varepsilon>0$ such that $\lambda_j^{-1}\in(0,+\infty)\setminus\mathcal{B}$ for every $1\leqslant j\leqslant n$ and every N large enough, where

$$\mathcal{B}=\bigcup_{i\in I,\, j\in J}\Big(B(\mathfrak{c}_i,\varepsilon)\cup B(\mathfrak{d}_j,\varepsilon)\Big).$$

We then set

$$\mathcal{K}=\left(\left[\inf_N\frac{1}{\lambda_n},\sup_N\frac{1}{\lambda_1}\right]\setminus\mathcal{B}\right)\cup\{0\},$$

so that $\{x\in\mathbb{R}:\ x^{-1}\in\mathrm{Supp}(\nu_N)\}\subset\mathcal{K}$ for every N large enough and moreover $\{x\in\mathbb{R}:\ x^{-1}\in\mathrm{Supp}(\nu)\}\subset\mathcal{K}$.

We start by proving (a). To do so, we essentially use the estimates from the Section 4.5.2. For any $(i,j)\in J\times J$ such that $i\neq j$, we have

$$\left\|\mathbf{1}_{B_i}\mathrm{E}_i\,\mathrm{K}_N\mathrm{E}_j^{-1}\mathbf{1}_{B_j}\right\|_2^2 \\ =\int_{B_i}\int_{B_j}\left(e^{-Nf_{i,N}(\mathfrak{d}_{i,N})+Nx\mathfrak{d}_{i,N}}\mathrm{K}_N(x,y)e^{Nf_{j,N}(\mathfrak{d}_{j,N})-Ny\mathfrak{d}_{j,N}}\right)^2\mathrm{d}x\mathrm{d}y. \quad (5.13)$$

By using Lemma 5.3 and performing the changes of variables $x\mapsto N^{2/3}\delta_{i,N}(x-\mathfrak{b}_{i,N})$ and $y\mapsto N^{2/3}\delta_{j,N}(y-\mathfrak{b}_{j,N})$, we obtain

$$\int_{B_i}\int_{B_j}\left(e^{-Nf_{i,N}(\mathfrak{d}_{i,N})+Nx\mathfrak{d}_{i,N}}\mathrm{K}_N(x,y)e^{Nf_{j,N}(\mathfrak{d}_{j,N})-Ny\mathfrak{d}_{j,N}}\right)^2\mathrm{d}x\mathrm{d}y \\ =\frac{1}{\delta_{i,N}\delta_{j,N}}\int_{t_i}^{N^{2/3}\delta_{i,N}\varepsilon}\int_{t_j}^{N^{2/3}\delta_{j,N}\varepsilon}\widetilde{\mathrm{K}}_N^{(\mathfrak{b}_i,\mathfrak{b}_j)}(x,y)^2\,\mathrm{d}x\mathrm{d}y, \quad (5.14)$$

where

$$\widetilde{\mathrm{K}}_N^{(\mathfrak{b}_i,\mathfrak{b}_j)}(x,y) \\ =\frac{N^{1/3}}{(2i\pi)^2(\mathfrak{b}_{i,N}-\mathfrak{b}_{j,N}+x/(N^{2/3}\delta_{i,N})-y/(N^{2/3}\delta_{j,N}))}\oint_\Gamma\mathrm{d}z\oint_\Theta\mathrm{d}w\,C_N(z,w) \\ \times e^{-N^{1/3}x\frac{(z-\mathfrak{d}_{i,N})}{\delta_{i,N}}+N(f_{i,N}(z)-f_{i,N}(\mathfrak{d}_{i,N}))}e^{N^{1/3}y\frac{(w-\mathfrak{d}_{j,N})}{\delta_{j,N}}-N(f_{j,N}(w)-f_{j,N}(\mathfrak{d}_{j,N}))}. \quad (5.15)$$

The main point here is that, since $i \neq j$, there exists $C > 0$ independent of N, x and y such that

$$\left| \frac{N^{1/3}}{(2i\pi)^2(\mathfrak{b}_{i,N} - \mathfrak{b}_{j,N} + x/(N^{2/3}\delta_{i,N}) - y/(N^{2/3}\delta_{j,N}))} \right| \leqslant CN^{1/3}. \tag{5.16}$$

Then, as we did in Section 4.3 and 4.4, we replace the contour Γ by $\Upsilon^{(0)} \cup \Upsilon^{(1)}$ where the contours $\Upsilon^{(0)}$ and $\Upsilon^{(1)}$ are specified by Proposition 4.6 with $\mathfrak{d}_N = \mathfrak{d}_{i,N}$ (if $\Upsilon^{(0)}$ does not exist, we just deform Γ to $\Upsilon^{(1)}$). Similarly, we deform the contour Θ and replace it with the contour $\widetilde{\Theta}$ specified by Proposition 4.6 with $\mathfrak{d}_N = \mathfrak{d}_{j,N}$. We then deform the contours $\Upsilon^{(1)}$ and $\widetilde{\Theta}$ around the saddle points similarly as it was done in Section 4.5.2. More precisely,

$$\Upsilon^{(1)} = \Upsilon_* \cup \Upsilon^{(1)}_{res} \quad \text{and} \quad \widetilde{\Theta} = \widetilde{\Theta}_* \cup \widetilde{\Theta}_{res}$$

where we introduced

$$\Upsilon_* = \{\mathfrak{d}_{i,N} + N^{-1/3}e^{i\pi\theta} : \ \theta \in [-\pi/3, \pi/3]\} \cup \{\mathfrak{d}_{i,N} + te^{\pm i\pi/3} : \ t \in [N^{-1/3}, \rho]\}$$
$$\widetilde{\Theta}_* = \{\mathfrak{d}_{j,N} + N^{-1/3}e^{i\pi\theta} : \ \theta \in [2\pi/3, 4\pi/3]\} \cup \{\mathfrak{d}_{j,N} - te^{\pm i\pi/3} : \ t \in [N^{-1/3}, \rho]\},$$

with ρ chosen small enough so that Lemma 4.7-(b) applies for both $f_{i,N}$ and $f_{j,N}$. In addition, Proposition 4.6 provides $K > 0$ independent of N such that

$$\mathrm{Re}(f_{i,N}(z) - f_{i,N}(\mathfrak{d}_{i,N})) \leqslant -K, \quad z \in \Upsilon^{(0)} \tag{5.17}$$
$$\mathrm{Re}(f_{i,N}(z) - f_{i,N}(\mathfrak{d}_{i,N})) \leqslant -K, \quad z \in \Upsilon^{(1)}_{res} \tag{5.18}$$
$$\mathrm{Re}(f_{j,N}(w) - f_{j,N}(\mathfrak{d}_{j,N})) \geqslant K, \quad w \in \widetilde{\Theta}_{res}. \tag{5.19}$$

Note that the contours $\Upsilon^{(0)}$ and $\widetilde{\Theta}$ may now intersect, and the contours $\Upsilon^{(1)}$ and $\widetilde{\Theta}$ as well, since the contours are associated with different edges. This raises no problem since $C_N(z,w)$ is analytic on $\mathbb{C} \setminus \mathcal{K}$. More precisely, since by construction the contours $\Upsilon^{(0)}$, $\Upsilon^{(1)}$ and $\widetilde{\Theta}$ lie inside a compact subset of $\mathbb{C} \setminus \mathcal{K}$ which does not dependent on N, there exists $C' > 0$ independent of N such that

$$|C_N(z,w)| \leqslant C', \qquad z \in \Upsilon^{(0)} \cup \Upsilon^{(1)}, \quad w \in \widetilde{\Theta}. \tag{5.20}$$

Next, Lemma 4.7-(b) yields

$$\mathrm{Re}(f_{i,N}(z) - f_{i,N}(\mathfrak{d}_{i,N})) \leqslant g''_N(\mathfrak{d}_{i,N})\,\mathrm{Re}(z - \mathfrak{d}_{i,N})^3/6 + \Delta|z - \mathfrak{d}_{i,N}|^4, \qquad z \in \Upsilon_*$$
$$\mathrm{Re}(f_{j,N}(w) - f_{j,N}(\mathfrak{d}_{j,N})) \geqslant g''_N(\mathfrak{d}_{j,N})\,\mathrm{Re}(w - \mathfrak{d}_{j,N})^3/6 - \Delta|w - \mathfrak{d}_{j,N}|^4, \qquad w \in \widetilde{\Theta}_*,$$

where $\Delta > 0$ is independent of N. We moreover assume we chose ρ small enough so that

$$g''_N(\mathfrak{d}_{i,N}) - \rho\Delta > 0, \qquad g''_N(\mathfrak{d}_{j,N}) - \rho\Delta > 0, \tag{5.21}$$

for all N large enough. Then, by using the same estimates as in Sections 4.5.1 and 4.5.2, we obtain for every $x, y \geqslant s$ and N large enough

$$\int_{\Upsilon^{(0)}} e^{-N^{1/3}\frac{x\,\mathrm{Re}(z-\mathfrak{d}_{i,N})}{\delta_{i,N}} + N\,\mathrm{Re}(f_{i,N}(z) - f_{i,N}(\mathfrak{d}_{i,N}))}|\mathrm{d}z| \leqslant C_1 e^{-C_2N + C_3N^{1/3}\frac{x}{\delta_{i,N}}},$$

$$\int_{\Upsilon^{(1)}} e^{-N^{1/3}\frac{x\,\mathrm{Re}(z-\mathfrak{d}_{i,N})}{\delta_{i,N}} + N\,\mathrm{Re}(f_{i,N}(z) - f_{i,N}(\mathfrak{d}_{i,N}))}|\mathrm{d}z| \leqslant \frac{C}{N^{1/3}} e^{-\frac{x-s}{2\delta_{i,N}}} + C_1 e^{-C_2N + C_3N^{1/3}\frac{x}{\delta_{i,N}}},$$

$$\int_{\widetilde{\Theta}} e^{N^{1/3}\frac{y\,\mathrm{Re}(w-\mathfrak{d}_{j,N})}{\delta_{j,N}} - N\,\mathrm{Re}(f_{j,N}(w) - f_{j,N}(\mathfrak{d}_{j,N}))}|\mathrm{d}w| \leqslant \frac{C}{N^{1/3}} e^{-\frac{y-s}{2\delta_{j,N}}} + C_1 e^{-C_2N + C_3N^{1/3}\frac{y}{\delta_{j,N}}},$$

for some $C, C_1, C_2, C_3 > 0$ independent on N and x, y. Combined with (5.16) and (5.20), it follows from (5.15) that

$$\left|\widetilde{\mathrm{K}}_N^{(\mathfrak{b}_i,\mathfrak{b}_j)}(x,y)\right| \leqslant \frac{C'}{N^{1/3}} e^{-(x-s)/(2\delta_{i,N})-(y-s)/(2\delta_{j,N})} + C_1' e^{-C_2' N + C_3' N^{1/3}(\frac{x}{\delta_{i,N}}+\frac{y}{\delta_{j,N}})}, \tag{5.22}$$

where $C', C_1', C_2', C_3' > 0$ are independent on N and x, y. Finally, by mimicking the Step 3 of the proof of Proposition 4.14, we obtain

$$\lim_{N\to\infty} \left\| \mathbf{1}_{B_i} \mathrm{E}_i \, \mathrm{K}_N \mathrm{E}_j^{-1} \mathbf{1}_{B_j} \right\|_2^2 = 0,$$

as soon as ε is small enough. We thus have proved (a).

Concerning the points (b) and (c), we proceed similarly as for the point (a) and use Lemma 5.3 and the changes of variables $x \mapsto N^{2/3}\sigma_{i,N}(\mathfrak{a}_{i,N} - x)$ and $y \mapsto N^{2/3}\delta_{j,N}(y - \mathfrak{b}_{j,N})$ in order to obtain

$$\left\| \mathbf{1}_{A_i} \mathrm{E}_i^* \, \mathrm{K}_N \mathrm{E}_j^{-1} \mathbf{1}_{B_j} \right\|_2^2 = \frac{1}{\sigma_{i,N}\delta_{j,N}} \int_{s_i}^{N^{2/3}\sigma_{i,N}\varepsilon} \int_{t_j}^{N^{2/3}\delta_{j,N}\varepsilon} \widetilde{\mathrm{K}}_N^{(\mathfrak{a}_i,\mathfrak{b}_j)}(x,y)^2 \, \mathrm{d}x\mathrm{d}y,$$

where

$$\begin{aligned} &\widetilde{\mathrm{K}}_N^{(\mathfrak{a}_i,\mathfrak{b}_j)}(x,y) \\ &\quad = \frac{N^{1/3}}{(2i\pi)^2(\mathfrak{a}_{i,N} - \mathfrak{b}_{j,N} - x/(N^{2/3}\sigma_{i,N}) - y/(N^{2/3}\delta_{j,N}))} \oint_\Gamma \mathrm{d}z \oint_\Theta \mathrm{d}w \, C_N(z,w) \\ &\quad \times e^{N^{1/3}x(z-\mathfrak{c}_{i,N})/\sigma_{i,N} - N(f^*_{i,N}(z) - f^*_{i,N}(\mathfrak{c}_{i,N}))} e^{N^{1/3}y(w-\mathfrak{d}_{j,N})/\delta_{j,N} - N(f_{j,N}(w) - f_{j,N}(\mathfrak{d}_{j,N}))}. \end{aligned} \tag{5.23}$$

If $\mathfrak{c}_i > 0$, then we replace the contour Γ by the contour $\Upsilon^{(0)} \cup \Upsilon^{(1)}$ (if $\Upsilon^{(0)}$ does not exist, we just deform Γ into $\Upsilon^{(1)}$) specified by Proposition 4.16 with $\mathfrak{c}_N = \mathfrak{c}_{i,N}$, and otherwise deform Γ into Υ as in Proposition 4.17. We moreover deform the contour Θ to obtain the contour $\widetilde{\Theta}$ specified by Proposition 4.6 with $\mathfrak{d}_N = \mathfrak{d}_{j,N}$. The same arguments than in the proof of (a) show that

$$\lim_{N\to\infty} \left\| \mathbf{1}_{A_i} \mathrm{E}_i^* \, \mathrm{K}_N \mathrm{E}_j^{-1} \mathbf{1}_{B_j} \right\|_2^2 = 0.$$

Similarly, we have

$$\left\| \mathbf{1}_{B_i} \mathrm{E}_i \, \mathrm{K}_N (\mathrm{E}_j^*)^{-1} \mathbf{1}_{A_j} \right\|_2^2 = \frac{1}{\delta_{i,N}\sigma_{i,N}} \int_{t_i}^{N^{2/3}\delta_{i,N}\varepsilon} \int_{s_j}^{N^{2/3}\sigma_{j,N}\varepsilon} \widetilde{\mathrm{K}}_N^{(\mathfrak{b}_i,\mathfrak{a}_j)}(x,y)^2 \, \mathrm{d}x\mathrm{d}y,$$

and

$$\left\| \mathbf{1}_{A_i} \mathrm{E}_i^* \, \mathrm{K}_N (\mathrm{E}_j^*)^{-1} \mathbf{1}_{A_j} \right\|_2^2 = \frac{1}{\sigma_{i,N}\sigma_{j,N}} \int_{s_i}^{N^{2/3}\sigma_{i,N}\varepsilon} \int_{s_j}^{N^{2/3}\sigma_{j,N}\varepsilon} \widetilde{\mathrm{K}}_N^{(\mathfrak{a}_i,\mathfrak{a}_j)}(x,y)^2 \, \mathrm{d}x\mathrm{d}y,$$

where

$$\begin{aligned} &\widetilde{\mathrm{K}}_N^{(\mathfrak{b}_i,\mathfrak{a}_j)}(x,y) \\ &\quad = \frac{N^{1/3}}{(2i\pi)^2(\mathfrak{b}_{i,N} - \mathfrak{a}_{j,N} + x/(N^{2/3}\delta_{i,N}) + y/(N^{2/3}\sigma_{j,N}))} \oint_\Gamma \mathrm{d}z \oint_\Theta \mathrm{d}w \, C_N(z,w) \\ &\quad \times e^{-N^{1/3}x(z-\mathfrak{d}_{i,N})/\delta_{i,N} + N(f_{i,N}(z) - f_{i,N}(\mathfrak{d}_{i,N}))} e^{-N^{1/3}y(w-\mathfrak{c}_{j,N})/\sigma_{j,N} + N(f^*_{j,N}(w) - f^*_{j,N}(\mathfrak{c}_{j,N}))}, \end{aligned} \tag{5.24}$$

and

$$
\begin{aligned}
&\widetilde{\mathrm{K}}_N^{(\mathfrak{a}_i,\mathfrak{a}_j)}(x,y)\\
&\quad = \frac{N^{1/3}}{(2i\pi)^2(\mathfrak{a}_{i,N}-\mathfrak{a}_{j,N}-x/(N^{2/3}\sigma_{i,N})+y/(N^{2/3}\sigma_{j,N}))}\oint_\Gamma \mathrm{d}z \oint_\Theta \mathrm{d}w\, C_N(z,w)\\
&\quad\times e^{N^{1/3}x(z-\mathfrak{c}_{i,N})/\sigma_{i,N}-N(f^*_{i,N}(z)-f^*_{i,N}(\mathfrak{c}_{i,N}))}e^{-N^{1/3}y(w-\mathfrak{c}_{j,N})/\sigma_{j,N}+N(f^*_{j,N}(w)-f^*_{j,N}(\mathfrak{c}_{j,N}))}. \qquad (5.25)
\end{aligned}
$$

For the kernel (5.24), we split the contour Γ into $\Upsilon^{(0)}$ and $\Upsilon^{(1)}$ where these contours are specified by Proposition 4.6 for $\mathfrak{d}_N = \mathfrak{d}_{i,N}$ (again, if $\Upsilon^{(0)}$ does not exist, we just deform Γ into $\Upsilon^{(1)}$). We also deform Θ to obtain the contour $\widetilde{\Theta}$ as in Proposition 4.16 or Proposition 4.17 with $\mathfrak{c}_N = \mathfrak{c}_{j,N}$, depending on whether $\mathfrak{c}_j > 0$ or not. For the kernel (5.25), we similarly split the contour Γ into $\Upsilon^{(0)}$ and $\Upsilon^{(1)}$ and take these contours as in Proposition 4.16 for $\mathfrak{c}_N = \mathfrak{c}_{i,N}$ if $\mathfrak{c}_i > 0$, and deform Γ into Υ as in Proposition 4.17 otherwise. Moreover, Θ is replaced by $\widetilde{\Theta}$ as specified in Proposition 4.16 or Proposition 4.17 with $\mathfrak{c}_N = \mathfrak{c}_{j,N}$ depending on whether $\mathfrak{c}_j > 0$ or not.

The same line of arguments than in the proof of (a) then shows that (b) and (c) hold true, except when $\mathfrak{c}_{j,N} < 0$. Indeed, in the latter case the contour $\widetilde{\Theta}$ coming with Proposition 4.17 does cross by construction the set $\mathcal{K}$ at a point x_0^{-1} where $x_0 \in \mathrm{Supp}(\nu)$. Thus we cannot use the bound (5.20) anymore.

To overcome this technical point, having in mind the definition (5.3) of $C_N(z,w)$, observe that since by construction $\Upsilon^{(0)} \cup \Upsilon^{(1)}$ or Υ lie in a compact subset of $\mathbb{C} \setminus \mathcal{K}$, the map $z \mapsto (1-z\lambda_\ell)^{-1}$ is bounded there uniformly in $1 \leqslant \ell \leqslant n$ and N large enough. Since moreover by construction $\widetilde{\Theta}$ lies in $\mathbb{C}\setminus\{0\}$, the map $(z,w) \mapsto (zw)^{-1}$ is bounded on the contours uniformly in N large enough. Observe furthermore that for every $1 \leqslant \ell \leqslant n$, we have

$$
\frac{e^{Nf^*_{j,N}(w)}}{1-\lambda_\ell w} = e^{Nf^{*[\ell]}_{j,N}(w)}
$$

where

$$
f^{*[\ell]}_{j,N}(w) = \mathfrak{a}_{j,N}(w-\mathfrak{c}_{j,N}) - \log(w) + \frac{1}{N}\sum_{\substack{k=1\\k\neq\ell}}^{n}\log(1-\lambda_k w). \qquad (5.26)
$$

Namely, the pole at $w=\lambda_\ell$ introduced by $C_N(z,w)$ is actually cancelled by $e^{Nf^*_{j,N}(w)}$. Thus, the items (b) and (c) of the proposition follow provided that the previous estimates continue to hold, uniformly in $1 \leqslant \ell \leqslant n$, after the replacement of $e^{Nf^*_{j,N}}$ by $e^{Nf^{*[\ell]}_{j,N}}$. But this is not hard to obtain because, as a consequence of the definitions (5.2) and (5.26), for every $k \in \mathbb{N}$ and compact subset $B \subset \mathbb{C} \setminus \mathcal{K}$ there exists $C_{B,k} > 0$ independent of N such that

$$
\sup_{w\in B}\max_{1\leqslant \ell\leqslant n}\left|\big(f^{*[\ell]}_{j,N}\big)^{(k)}(w) - \big(f^*_{j,N}\big)^{(k)}(w)\right| \leqslant \frac{C_{B,k}}{N}.
$$

The proof of Proposition 5.2 is therefore complete. $\square$

6 Proof of Theorem 5: Fluctuations at the hard edge

In this section, we provide a proof for Theorem 5.

Let us fix $s > 0$ and $\alpha \in \mathbb{Z}$. We set $n = N+\alpha$ and define σ_N as in (3.7). The representation for the gap probabilities of determinantal point processes as Fredholm determinants yields

$$\mathbb{P}\Big(N^2\sigma_N\, x_{\min} \geqslant s\Big) = \det\big(I - \mathrm{K}_N\big)_{L^2(0,s/(N^2\sigma_N))},$$

where

$$x_{\min} = \begin{cases} x_1 = \tilde{x}_{\alpha+1} & \text{if } \alpha \geqslant 0, \\ x_{1-\alpha} = \tilde{x}_1 & \text{if } \alpha < 0. \end{cases}$$

If we introduce the integral operator $\widetilde{\mathrm{K}}_N$ acting on $L^2(0,s)$ with kernel

$$\widetilde{\mathrm{K}}_N(x,y) = \frac{1}{N^2\sigma_N}\mathrm{K}_N\left(\frac{x}{N^2\sigma_N}, \frac{y}{N^2\sigma_N}\right), \tag{6.1}$$

then it follows from a change of variables that

$$\mathbb{P}\Big(N^2\sigma_N\, x_{\min} \geqslant s\Big) = \det\big(I - \widetilde{\mathrm{K}}_N\big)_{L^2(0,s)}. \tag{6.2}$$

We recall that $\mathrm{K}_{\mathrm{Be},\alpha}(x,y)$ has been introduced in (3.6) and also define the operator E and E^{-1} acting on $L^2(0,s)$ by $\mathrm{E}h(x) = x^{\alpha/2}h(x)$ and $\mathrm{E}^{-1}h(x) = x^{-\alpha/2}h(x)$. Notice that when $\alpha \geqslant 0$ (resp. $\alpha < 0$) the operator E (resp. E^{-1}) is well-defined on $L^2(0,s)$, but E^{-1} (resp. E) is not defined on the whole space. Nevertheless, in the following these operators will always arise premultiplied or post-multiplied by an appropriate operator so that the product is well-defined on $L^2(0,s)$, see below.

The aim of this section is to prove the following.

Proposition 6.1.

$$\lim_{N\to\infty} \sup_{(x,y)\in(0,s]\times(0,s]} \Big|\widetilde{\mathrm{K}}_N(x,y) - \mathrm{E}\mathrm{K}_{\mathrm{Be},\alpha}\mathrm{E}^{-1}(x,y)\Big| = 0.$$

Let us first show how Theorem 5 follows from this proposition.

Proof of Theorem 5. The relation $xJ_\alpha'(x) = \alpha J_\alpha(x) - xJ_{\alpha+1}(x)$, see [32, 7.2.8 (54)], provides

$$\mathrm{K}_{\mathrm{Be},\alpha}(x,y) = \frac{\sqrt{x}J_{\alpha+1}(\sqrt{x}\,)J_\alpha(\sqrt{y}\,) - \sqrt{y}J_{\alpha+1}(\sqrt{y}\,)J_\alpha(\sqrt{x}\,)}{2(x-y)}. \tag{6.3}$$

It then follows from [32, 7.14.1 (9)] that

$$\mathrm{K}_{\mathrm{Be},\alpha}(x,y) = \frac{1}{4}\int_0^1 J_\alpha(\sqrt{xu}\,)J_\alpha(\sqrt{yu}\,)\mathrm{d}u,$$

and, after the change of variables $u \mapsto u/s$, this yields the factorization $\mathrm{K}_{\mathrm{Be},\alpha} = \mathrm{B}_s^2$ as operators of $L^2(0,s)$ where B_s has for kernel $\mathrm{B}_s(x,y) = J_\alpha(\sqrt{xy/s})/(2\sqrt{s})$. The asymptotic behavior as $x \to 0$

$$J_\alpha(\sqrt{x}\,) = \frac{1}{\alpha!}\left(\frac{\sqrt{x}}{2}\right)^\alpha\big(1 + O(x^2)\big), \qquad \text{if } \alpha \geqslant 0,$$

$$J_\alpha(\sqrt{x}\,) = \frac{(-1)^\alpha}{|\alpha|!}\left(\frac{\sqrt{x}}{2}\right)^{|\alpha|}\big(1 + O(x^2)\big), \qquad \text{if } \alpha < 0,$$

which is provided by the series representation (3.5) of J_α, then shows that B_s, $\mathrm{B}_s\mathrm{E}^{-1}$ and $\mathrm{K}_{\mathrm{Be},\alpha}\mathrm{E}^{-1}$ when $\alpha \geqslant 0$, EB_s and $\mathrm{EK}_{\mathrm{Be},\alpha}$ when $\alpha < 0$, and $\mathrm{EK}_{\mathrm{Be},\alpha}\mathrm{E}^{-1}$ are well-defined and Hilbert-Schmidt operators. Moreover, E and $\mathrm{K}_{\mathrm{Be},\alpha}\mathrm{E}^{-1}$ when $\alpha \geqslant 0$, E^{-1} and $\mathrm{EK}_{\mathrm{Be},\alpha}$ when $\alpha < 0$, and $\mathrm{EK}_{\mathrm{Be},\alpha}\mathrm{E}^{-1}$ are trace class being products of two Hilbert-Schmidt operators.

Since $[0,s]$ is compact, it follows from Proposition 6.1 that

$$\lim_{N\to\infty} \big\|\mathbf{1}_{(0,s)}\big(\widetilde{\mathrm{K}}_N - \mathrm{EK}_{\mathrm{Be},\alpha}\mathrm{E}^{-1}\big)\mathbf{1}_{(0,s)}\big\|_2 = 0$$

and

$$\lim_{N\to\infty} \mathrm{Tr}\,\big(\mathbf{1}_{(0,s)}\widetilde{\mathrm{K}}_N\mathbf{1}_{(0,s)}\big) = \mathrm{Tr}\,\big(\mathbf{1}_{(0,s)}\mathrm{EK}_{\mathrm{Be},\alpha}\mathrm{E}^{-1}\mathbf{1}_{(0,s)}\big).$$

We then obtain from Proposition 4.1 that

$$\lim_{N\to\infty} \det\big(I - \widetilde{\mathrm{K}}_N\big)_{L^2(0,s)} = \det\big(I - \mathrm{EK}_{\mathrm{Be},\alpha}\mathrm{E}^{-1}\big)_{L^2(0,s)},$$

which shows together with (6.2) and (4.1) that

$$\lim_{N\to\infty} \mathbb{P}\Big(N^2\sigma_N\, x_{\min} \geqslant s\Big) = \det\big(I - \mathrm{K}_{\mathrm{Be},\alpha}\big)_{L^2(0,s)}.$$

Finally, that $\det\big(I - \mathrm{K}_{\mathrm{Be},0}\big)_{L^2(0,s)} = e^{-s}$ has been observed in [34], and the proof of Theorem 5 is complete.

□

We now focus on the proof of Proposition 6.1.

6.1 The Bessel kernel

We first provide a double complex integral formula for the Bessel kernel.

Lemma 6.2. *With* $\mathrm{K}_{\mathrm{Be},\alpha}(x,y)$ *defined in* (3.6), *for every* $0 < r < R$ *and* $x, y > 0$ *we have*

$$\mathrm{K}_{\mathrm{Be},\alpha}(x,y) = \frac{1}{(2i\pi)^2}\left(\frac{y}{x}\right)^{\alpha/2} \oint_{|z|=r} \frac{\mathrm{d}z}{z} \oint_{|w|=R} \frac{\mathrm{d}w}{w}\,\frac{1}{z-w}\left(\frac{z}{w}\right)^{\alpha} e^{-\frac{x}{z}+\frac{z}{4}+\frac{y}{w}-\frac{w}{4}}. \tag{6.4}$$

We recall that, by convention, all contours of integrations are oriented counterclockwise, and thus the notation $\oint_{|z|=r}$ is unambiguous.

Proof. The Laurent series generating function for the Bessel functions with integer parameters reads, see [32, 7.2.4 (25)],

$$e^{\frac{x}{2}(z-\frac{1}{z})} = \sum_{\alpha\in\mathbb{Z}} J_\alpha(x) z^\alpha, \qquad z \in \mathbb{C}\setminus\{0\}.$$

This yields for every $x, r > 0$ and $\alpha \in \mathbb{Z}$,

$$J_\alpha(\sqrt{x}\,) = \frac{1}{2i\pi}\oint_{|z|=r} z^{-\alpha} e^{\frac{\sqrt{x}}{2}(z-\frac{1}{z})}\frac{\mathrm{d}z}{z}.$$

After the changes of variables $z \mapsto 2\sqrt{x}z$ and $w \mapsto 1/(2\sqrt{y}w)$, this provides for every $x, y > 0$, $0 < r < R$ and $\alpha \in \mathbb{Z}$

$$J_\alpha(\sqrt{x}\,) = \frac{1}{2i\pi(2\sqrt{x}\,)^\alpha}\oint_{|z|=1/r} z^{-\alpha} e^{xz-\frac{1}{4z}}\frac{\mathrm{d}z}{z}\,, \tag{6.5}$$

$$J_\alpha(\sqrt{y}\,) = \frac{(2\sqrt{y}\,)^\alpha}{2i\pi}\oint_{|w|=1/R} w^{\alpha} e^{-yw+\frac{1}{4w}}\frac{\mathrm{d}w}{w}\,. \tag{6.6}$$

By plugging (6.5) and (6.6) into (6.3), we obtain

$$\begin{aligned}&(x-y)\mathrm{K}_{\mathrm{Be},\alpha}(x,y)\\ &\quad= \frac{1}{(2i\pi)^2}\left(\frac{y}{x}\right)^{\alpha/2}\oint_{|z|=1/r}\mathrm{d}z\oint_{|w|=1/R}\mathrm{d}w\,\frac{w^\alpha}{z^{\alpha+1}}e^{xz-1/(4z)-yw+1/(4w)}\left(\frac{1}{4zw}-y\right).\end{aligned} \tag{6.7}$$

We continue the computation by mean of integrations by parts, as explained to us by Manuela Girotti while we discussed a similar formula appearing in her work [36]. Indeed, since $-ye^{-yw} = \frac{\partial}{\partial w}e^{-yw}$, a first integration by parts provides

$$\begin{aligned}&\oint_{|z|=1/r}\mathrm{d}z\oint_{|w|=1/R}\mathrm{d}w\,\frac{w^\alpha}{z^{\alpha+1}}e^{xz-\frac{1}{4z}-yw+\frac{1}{4w}}\left(\frac{1}{4zw}-y\right)\\ &\quad= \oint_{|z|=1/r}\mathrm{d}z\oint_{|w|=1/R}\mathrm{d}w\,\frac{w^\alpha}{z^{\alpha+1}}e^{xz-\frac{1}{4z}-yw+\frac{1}{4w}}\left(\frac{1}{4zw}+\frac{1}{4w^2}-\frac{\alpha}{w}\right)\\ &\quad= \oint_{|z|=1/r}\mathrm{d}z\oint_{|w|=1/R}\mathrm{d}w\,\frac{1}{z-w}\left(\frac{w}{z}\right)^\alpha e^{xz-\frac{1}{4z}-yw+\frac{1}{4w}}\left(\frac{1}{4w^2}-\frac{1}{4z^2}+\frac{\alpha}{z}-\frac{\alpha}{w}\right).\end{aligned} \tag{6.8}$$

Next, by observing that

$$\left(\frac{1}{4w^2}-\frac{1}{4z^2}\right)e^{-\frac{1}{4z}+\frac{1}{4w}} = -\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial w}\right)e^{-\frac{1}{4z}+\frac{1}{4w}},$$

an other integration by parts yields

$$\begin{aligned}&\oint_{|z|=1/r}\mathrm{d}z\oint_{|w|=1/R}\mathrm{d}w\,\frac{1}{z-w}\left(\frac{w}{z}\right)^\alpha e^{xz-\frac{1}{4z}-yw+\frac{1}{4w}}\left(\frac{1}{4w^2}-\frac{1}{4z^2}+\frac{\alpha}{z}-\frac{\alpha}{w}\right)\\ &\quad= (x-y)\oint_{|z|=1/r}\mathrm{d}z\oint_{|w|=1/R}\mathrm{d}w\,\frac{1}{z-w}\left(\frac{w}{z}\right)^\alpha e^{xz-\frac{1}{4z}-yw+\frac{1}{4w}}.\end{aligned} \tag{6.9}$$

By combining (6.7)–(6.9), we obtain

$$\mathrm{K}_{\mathrm{Be},\alpha}(x,y) = \frac{1}{(2i\pi)^2}\left(\frac{y}{x}\right)^{\alpha/2}\oint_{|z|=1/r}\mathrm{d}z\oint_{|w|=1/R}\mathrm{d}w\,\frac{1}{z-w}\left(\frac{w}{z}\right)^\alpha e^{xz-\frac{1}{4z}-yw+\frac{1}{4w}}, \tag{6.10}$$

and the lemma follows after the change of variables $z \mapsto -1/z$ and $w \mapsto -1/w$. □

Corollary 6.3. *For every $0 < r < R$ and $x, y > 0$, we have*

$$\mathrm{E}\mathrm{K}_{\mathrm{Be},\alpha}\mathrm{E}^{-1}(x,y) = \frac{1}{(2i\pi)^2}\oint_{|z|=r}\frac{\mathrm{d}z}{z}\oint_{|w|=R}\frac{\mathrm{d}w}{w}\,\frac{1}{z-w}\left(\frac{z}{w}\right)^\alpha e^{-\frac{x}{z}+\frac{z}{4}+\frac{y}{w}-\frac{w}{4}}.$$

Equipped with Corollary 6.3, we are now in position to establish Proposition 6.1.

6.2 Asymptotic analysis

We now perform an asymptotic analysis for the kernel $\widetilde{\mathrm{K}}_N(x,y)$ as in Section 4. The main idea is that when the leftmost edge is a hard edge, the associated critical point $\mathfrak{c}$ should be at infinity. This leads us to study the integrand of the double integral representation of $\widetilde{\mathrm{K}}_N(x,y)$ in a neighborhood of $z=0$ and $w=0$ after the changes of variables $z\mapsto 1/z$ and $w\mapsto 1/w$.

Proof of Proposition 6.1. By choosing $q=0$ in (4.5), which is possible according to Remark 4.3, we obtain with (6.1)

$$\widetilde{\mathrm{K}}_N(x,y)=\frac{1}{(2i\pi)^2N\sigma_N}\oint_\Gamma \mathrm{d}z\oint_\Theta \mathrm{d}w\,\frac{1}{w-z}\left(\frac{z}{w}\right)^N e^{-\frac{zx}{N\sigma_N}+\frac{wy}{N\sigma_N}}\prod_{j=1}^n\frac{w-\lambda_j^{-1}}{z-\lambda_j^{-1}}, \tag{6.11}$$

where we recall that the contour Γ encloses the λ_j^{-1}'s whereas the contour Θ encloses the origin and is disjoint from Γ. We deform Γ so that it encloses Θ, which is possible since the integrand is analytic at the origin as a function of z and the residue picked at $z=w$ vanishes. Moreover, since the λ_j^{-1}'s are zeros of the integrand as a function of w, we can deform Θ such that it encloses all the λ_j^{-1}'s. More precisely, we specify the contours to be $\Gamma=\{z\in\mathbb{C}: |z|=N\sigma_N/r\}$ and $\Theta=\{z\in\mathbb{C}:\ |z|=N\sigma_N/R\}$ with $0<r<R<\liminf_N\lambda_1/2$. Notice that for N large enough, $\Gamma=\Gamma(N)$ and $\Theta=\Theta(N)$ enclose the λ_j's.

Next, we perform the changes of variables $z\mapsto N\sigma_N/z$ and $w\mapsto N\sigma_N/w$ in (6.11) in order to get

$$\begin{aligned}
&\widetilde{\mathrm{K}}_N(x,y)\\
&=\frac{1}{(2i\pi)^2}\oint_{|z|=r}\frac{\mathrm{d}z}{z}\oint_{|w|=R}\frac{\mathrm{d}w}{w}\,\frac{1}{z-w}\left(\frac{z}{w}\right)^\alpha e^{-\frac{x}{z}+\frac{y}{w}}\prod_{j=1}^n\frac{\frac{w}{N\sigma_N}-\lambda_j}{\frac{z}{N\sigma_N}-\lambda_j}\\
&=\frac{1}{(2i\pi)^2}\oint_{|z|=r}\frac{\mathrm{d}z}{z}\oint_{|w|=R}\frac{\mathrm{d}w}{w}\,\frac{1}{z-w}\left(\frac{z}{w}\right)^\alpha e^{-\frac{x}{z}+\frac{y}{w}-N(F_N(z)-F_N(0))+N(F_N(w)-F_N(0))},
\end{aligned}$$

where used the fact that $n=N+\alpha$ and we introduced the map

$$F_N(z)=\frac{1}{N}\sum_{j=1}^n\log\left(\frac{z}{N\sigma_N}-\lambda_j\right).$$

Note that for every N large enough and $z\in B(0,R+1)$ we have $|z|/N\sigma_N\leqslant\liminf_N\lambda_1/2-\delta$ for some $\delta>0$. Thus we can choose a branch of the logarithm such that F_N is well-defined and holomorphic on $B(0,R+1)$ for all N sufficiently large. Moreover, recalling that

$$\sigma_N=\frac{4}{N}\sum_{j=1}^n\frac{1}{\lambda_j}$$

and observing the identity $F_N'(0)=-1/(4N)$, a Taylor expansion of F_N around zero yields for every $z\in B(0,R+1)$ and for all N large enough

$$\begin{aligned}
\left|F_N(z)-F_N(0)+\frac{z}{4N}\right| &\leqslant \frac{1}{2}\frac{|z|^2}{N^2\sigma_N^2}\sup_{w\in B(0,R+1)}\left|\frac{1}{N}\sum_{j=1}^n\frac{1}{(\frac{w}{N\sigma_N}-\lambda_j)^2}\right|\\
&\leqslant \frac{n}{2N^3\sigma_N^2\delta^2}(R+1)^2\leqslant\frac{\Delta}{N^2},
\end{aligned}$$

for some $\Delta > 0$ independent of N.

Finally, by using Corollary 6.3 and the inequality (4.71) with

$$u = -N\big(F_N(z) - F_N(0)\big) + N\big(F_N(w) - F_N(0)\big) \qquad \text{and} \qquad v = \frac{z-w}{4} \ ,$$

we obtain for every $0 < x, y \leqslant s$

$$\begin{aligned}
&\Big|\widetilde{\mathrm{K}}_N(x,y) - \mathrm{E}\mathrm{K}_{\mathrm{Be},\alpha}\mathrm{E}^{-1}(x,y)\Big| \\
&\leqslant \frac{\Delta r^{\alpha-1}}{2\pi^2 R^{\alpha+1}(R-r)N} \oint_{|z|=r} e^{-x\,\mathrm{Re}(1/z)+\mathrm{Re}(z)/4+\Delta/N}|\mathrm{d}z| \oint_{|w|=R} e^{y\,\mathrm{Re}(1/w)-\mathrm{Re}(w)/4+\Delta/N}|\mathrm{d}w| \\
&\leqslant \frac{C(s)}{N}
\end{aligned}$$

for some $C(s) > 0$ independent of N and $0 < x, y \leqslant s$, and Proposition 6.1 follows. □

The proof of Theorem 5 is therefore complete.

A Proof of Proposition 2.4

The proof of Proposition 2.4 makes use of [63, Th. 4.3 and 4.4]. In a word, [63, Th. 4.3] says that on any connected component of D, there is at most one interval on which the function g is decreasing while [63, Th. 4.4] says that on any two disjoint open intervals of D where g is decreasing, the images of the closures of these intervals by g are disjoint.

Proof of Proposition 2.4. Let us prove (a). Assume $\gamma > 1$. Since $m(z)$ is the Cauchy-Stieltjes transform of a probability measure supported by $[0, +\infty)$, the function $m(x)$ decreases from zero as x increases from $-\infty$ to the origin. Hence its inverse $g(x)$ decreases to $-\infty$ as x increases to zero. Since

$$xg(x) = 1 + \gamma \int \frac{x\lambda}{1 - x\lambda}\nu(\mathrm{d}\lambda),$$

the dominated convergence theorem implies that $xg(x) \to 1 - \gamma < 0$ as $x \to -\infty$. It results that $g(x) \to 0^+$ as $x \to -\infty$, and $g(x)$ reaches a positive maximum on $(-\infty, 0)$. By [63, Th. 4.3 and 4.4], we obtain that the function $g(x)$ exhibits the behavior described in the statement, and its maximum coincides with $\mathfrak{a}$.

To prove (b), recalling the expression of $xg(x)$ and observing that

$$x^2 g'(x) = -1 + \gamma \int \Big(\frac{x\lambda}{1 - x\lambda}\Big)^2 \nu(\mathrm{d}\lambda),$$

we deduce that when $\gamma \leqslant 1$, the function g is negative and decreasing on $(-\infty, 0)$.

We now show (c). For $x > 2/\eta$ and $\lambda \in \mathrm{Supp}(\nu)$, we have $|1 - x\lambda| \geqslant x\eta - 1 > 1$. Therefore, $g(x) \to 0$ and $x^2 g'(x) \to \gamma - 1 < 0$ as $x \to +\infty$ by the dominated convergence theorem. This shows that $g(x)$ has a positive supremum on $(1/\eta, \infty)$ and it decreases to zero as $x \to +\infty$. By [63, Th. 4.3 and 4.4], we obtain that the function $g(x)$ exhibits the behavior described in the statement, and its supremum coincides with $\mathfrak{a}$.

Turning to (d), assume that $[\mathfrak{d}, \infty) \subset D$. Then by Proposition 2.3–(a) there exists $\varepsilon > 0$ such that $g'(x) < 0$ on $(\mathfrak{d} - \varepsilon, \mathfrak{d})$ and $g'(\mathfrak{d}) = 0$. It is furthermore clear that $g(x) \to 0$ as

$x \to \infty$. Since $\mathfrak{b} = g(\mathfrak{d}) > 0$, we get that there exists an interval in $(\mathfrak{c}, \infty)$ over which g is decreasing. But this contradicts [63, Th. 4.3].

To show (e) we observe that $m(x)$, being the Cauchy-Stieltjes transform of a probability measure, decreases from $\mathfrak{d} = \lim_{x \downarrow \mathfrak{b}} m(x)$ to 0 as x increases over the interval $(\mathfrak{b}, \infty)$. Proposition 2.1 shows then that g decreases from $+\infty$ to $\mathfrak{b}$ as x increases from zero to $\mathfrak{d}$, and that $(0, \mathfrak{d}) \subset (0, 1/\xi)$. Theorem 4.3 of [63] shows that g decreases nowhere on $(\mathfrak{d}, 1/\xi)$. □

B Deformed Tracy-Widom fluctuations

In this section, we consider a particular case of a **non regular positive edge** where our previous analysis still applies. In this case, the fluctuations of the associated extremal eigenvalue will be described by the deformed Tracy-Widom law as introduced in Baik et al [8, Eq. (17)]. Consider the integral operator $\mathrm{K}_{\mathrm{Ai}}^{(k)}$ with kernel

$$\mathrm{K}_{\mathrm{Ai}}^{(k)}(x,y) = \frac{1}{(2i\pi)^2} \oint_{\Xi} \mathrm{d}z \oint_{\Xi'} \mathrm{d}w \, \frac{1}{w-z} \left(\frac{w}{z}\right)^k e^{-xz + \frac{z^3}{3} + yw - \frac{w^3}{3}}, \tag{B.1}$$

where the contours Ξ and Ξ' are the same as in the proof of Lemma 4.15, and the associated distribution[5]

$$F_k(s) = \det\left(I - \mathrm{K}_{\mathrm{Ai}}^{(k)}\right)_{L^2(s,\infty)} .$$

If $k = 0$, we recover the usual Airy kernel (4.56).

Given a right edge $\mathfrak{b}$ associated to the limiting spectral distribution $\mu(\gamma, \nu)$, we assume the following structure for ν_N, which readily implies that $\mathfrak{b}$ is a non regular edge for $k \geqslant 1$:

Assumption 3. (population eigenvalues at criticality $\mathfrak{d}$) Let k be a fixed integer such that there exist eigenvalues $\zeta_1, \dots, \zeta_k \in \{\lambda_1, \dots, \lambda_n\}$ satisfying ${\zeta_j}^{-1} \to \mathfrak{d}$ as $N \to \infty$ for every $1 \leqslant j \leqslant k$.

The following statement may deserve a more formal status, but since we only sketch its proof and do not provide the full details, we simply call it a statement.

Statement. *Let Assumptions 1 and 2 hold true, let $\mathfrak{b}$ be a right edge and $\mathfrak{b} = g(\mathfrak{d})$ with $\mathfrak{d} \in D$ and assume moreover that Assumption 3 holds true[6]. Denote by*

$$\check{\nu}_N = \frac{n}{n-k}\Big(\nu_N - \frac{1}{n}\sum_{j=1}^{k} \delta_{\zeta_j}\Big), \quad \gamma_N = \frac{n-k}{N}, \quad g_N(z) = \frac{1}{z} + \gamma_N \int \frac{\lambda}{1 - z\lambda} \check{\nu}_N(\mathrm{d}\lambda)$$

and let $\mathfrak{d}_N$ and $\tilde{x}_{\phi(N)}$ be the sequences associated to $\check{\nu}_N$ and g_N, as provided in Proposition 2.11. Assume moreover that

$$\lim_{N\to\infty} N^{1/3} \max_{j=1}^{k} |{\zeta_j}^{-1} - \mathfrak{d}_N| = 0 . \tag{B.2}$$

and that the following weak regularity condition holds true:

$$\liminf_{N\to\infty} \min_{j=1,\dots,n,\ \lambda_j \neq \zeta_1,\dots,\zeta_k} \left|\mathfrak{d} - \lambda_j^{-1}\right| > 0 . \tag{B.3}$$

[5] Notice that definition (B.1) is consistent with the one given in [8], as the product of the operators associated with [8, Eq. (120) and Eq. (122)] has kernel $\mathrm{K}_{\mathrm{Ai}}^{(k)}(x,y)$.

[6] In case of a positive left edge, one will consider instead the straightforward counterpart of Assumption 3.

Then

$$\lim_{N\to\infty} \mathbb{P}\Big(N^{2/3}\delta_N\big(\tilde{x}_{\phi(N)} - \mathfrak{b}_N\big) \leqslant s\Big) = \det\big(I - \mathrm{K}_{\mathrm{Ai}}^{(k)}\big)_{L^2(s,\infty)}, \qquad s \in \mathbb{R}. \tag{B.4}$$

where $\mathfrak{b}_N = g_N(\mathfrak{d}_N)$ *and* $\delta_N = (2/g_N''(\mathfrak{d}_N))^{1/3}$.

Outline of proof for the statement. We start by introducing the map

$$f_N(z) = -\mathfrak{b}_N(z - \mathfrak{d}_N) + \log(z) - \frac{n-k}{N}\int \log(1 - xz)\check{\nu}_N(\mathrm{d}x),$$

which is the counterpart of f_N from Section 4. From Proposition 4.4 and a change of variables, we have as $N \to \infty$

$$\mathbb{P}\Big(N^{2/3}\delta_N\big(\tilde{x}_{\phi(N)} - \mathfrak{b}_N\big) \leqslant s\Big) = \det\big(I - \mathbf{1}_{(s,\,\varepsilon N^{2/3}\delta_N)}\widetilde{\mathrm{K}}_N\mathbf{1}_{(s,\,\varepsilon N^{2/3}\delta_N)}\big)_{L^2(s,\infty)} + o(1),$$

where the integral operator $\widetilde{\mathrm{K}}_N$ is associated with the kernel

$$\widetilde{\mathrm{K}}_N(x,y) = \frac{N^{1/3}}{(2i\pi)^2\delta_N}\oint_\Gamma \mathrm{d}z \oint_\Theta \mathrm{d}w\, \frac{1}{w-z}\prod_{j=1}^k\left(\frac{w - \zeta_j{}^{-1}}{z - \zeta_j{}^{-1}}\right) \times e^{-N^{1/3}x\frac{(z-\mathfrak{d}_N)}{\delta_N} + N^{1/3}y\frac{(w-\mathfrak{d}_N)}{\delta_N} + Nf_N(z) - Nf_N(w)}. \tag{B.5}$$

By following the proof of Lemma 4.7, we can see that $\operatorname{Re} f_N$ similarly converges locally uniformly toward (4.27) on an appropriate subset of the complex plane containing $\mathfrak{d}$, and this yields the existence of appropriate contours as in Proposition 4.6 by using the same exact proof. Since by assumption the $\zeta_j{}^{-1}$'s stay in an arbitrary small neighborhood of $\mathfrak{d}$ for every N large enough, the product over the ζ_j's in the integrand $\widetilde{\mathrm{K}}_N(x,y)$ is bounded away from that neighborhood. As a consequence, we can show as in Section 4.5 and in Step 2 of the proof of Proposition 4.14 that, with Υ_* and $\widetilde{\Theta}_*$ respectively defined in (4.60) and (4.62),

$$\widetilde{\mathrm{K}}_N(x,y) = \frac{N^{1/3}}{(2i\pi)^2\delta_N}\oint_{\Upsilon_*} \mathrm{d}z \oint_{\widetilde{\Theta}_*} \mathrm{d}w\, \frac{1}{w-z}\prod_{j=1}^k\left(\frac{w - \zeta_j{}^{-1}}{z - \zeta_j{}^{-1}}\right) \times e^{-N^{1/3}x\frac{(z-\mathfrak{d}_N)}{\delta_N} + N^{1/3}y\frac{(w-\mathfrak{d}_N)}{\delta_N} + Nf_N(z) - Nf_N(w)} \tag{B.6}$$

up to negligible terms, in the sense that the remaining terms do not contribute in the large N limit. Moreover, by proceeding similarly as in Lemma 4.15 and Step 2 of the proof of Proposition 4.14, we have that

$$\mathrm{K}_{\mathrm{Ai}}^{(k)}(x,y) = \frac{N^{1/3}}{(2i\pi)^2\delta_N}\oint_{\Upsilon_*} \mathrm{d}z \oint_{\widetilde{\Theta}_*} \mathrm{d}w\, \frac{1}{w-z}\left(\frac{w-\mathfrak{d}_N}{z-\mathfrak{d}_N}\right)^k \times e^{-N^{1/3}x\frac{(z-\mathfrak{d}_N)}{\delta_N} + Ng_N''(\mathfrak{d}_N)\frac{(z-\mathfrak{d}_N)^3}{6} + N^{1/3}y\frac{(w-\mathfrak{d}_N)}{\delta_N} - Ng_N''(\mathfrak{d}_N)\frac{(w-\mathfrak{d}_N)^3}{6}} \tag{B.7}$$

up to negligible terms. Finally, to conclude we need to estimate the difference between the right hand sides of (B.6) and (B.7), which is the counterpart of Step 1 in the proof of Proposition 4.14; we claim that similar estimates can be performed with minor modifications provided that (B.2) holds true.

□

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