On the Asymptotic Distribution of the Correlation Receiver Output for Time-Hopped UWB Signals

J. Fiorina and W. Hachem*

Abstract

In Ultra-Wide Band (UWB) communications based on Time Hopping (TH) Impulse Radio, one of the most frequently studied receivers is the correlation receiver. The Multi-User Interference (MUI) at the output of this receiver is sometimes modeled as a Gaussian random variable. In order to justify this assumption, the conditions of validity of the Central Limit Theorem (CLT) have to be studied in an asymptotic regime where the number of interferers and the processing gain grow toward infinity at the same rate, the channel degree being kept constant. An asymptotic study is made in this paper based on the so-called Lindeberg's condition for the CLT for martingales. We consider non synchronized users sending their signals over independent multi-path channels. These users may also have different powers. It is shown that when the frame length grows and the repetition factor is kept constant, then the MUI does not converge in distribution toward a Gaussian random variable. On the other hand, this convergence can be established if the repetition factor grows at the rate of the frame length. In this last situation, closed form expressions for the Signal to Interference plus Noise Ratio are given for TH Pulse Amplitude Modulation (PAM) and Pulse Position Modulation (PPM) UWB transmissions.

Index Terms

Ultra-Wide Band communications, Time Hopping, Impulse Radio, Pulse Position Modulation, Pulse Amplitude Modulation, Multi-User Interference, Gaussian Approximation, Lindeberg's Condition.

EDICS Category: 3-COMM 3-TRAN 3-PERF

Supélec (Ecole Supérieure d'Electricité), Telecommunications Department, Plateau de Moulon, 91192, Gif-Sur-Yvette Cedex, France. Phone : +33 1 69 85 14 52, Fax : +33 1 69 85 14 69, e-mails: `Jocelyn.Fiorina@supelec.fr`, `Walid.Hachem@supelec.fr`.

(*) Corresponding author.

I. INTRODUCTION

Ultra-Wide Band (UWB) systems [1] are spread spectrum multiple-access communication systems characterized by the fact that the transmitted signals have a very large bandwidth, at least one fourth the value of the center frequency [2], and a low spectral density. Thanks to their low spectral density, these systems could coexist with existing narrow band transmission systems. They can also use profitably the channel multipath diversity. In addition, the high time resolution they are able to provide makes possible the implementation of applications that require an accurate localization.

One class of UWB modulation techniques, termed the Impulse Radio (IR) techniques, consists in transmitting pulses with a duration at the scale of a nanosecond at moments that are subject to a Time Hopping (TH) pattern. In such systems, one symbol interval is divided into N_s frames of N_h time slots each, one pulse being usually transmitted per frame. The position of the time slot that a pulse occupies within a frame follows a TH pseudo-random code specific to the user. Because one information symbol is carried by N_s successive pulses, N_s is called the repetition factor. Symbol encoding can be done either through Pulse Position Modulation (PPM) or through Pulse Amplitude Modulation (PAM). In binary PPM systems [3], the positions of the N_s pulses that code a symbol undergo an additional time delay within their time slots according to whether a one or a zero is transmitted. In PAM, (see for instance [4]), these pulses occupy fixed positions in their slots. Their amplitudes are modulated by the symbol to be transmitted.

A large number of contributions studied the performance of the correlation receiver, a receiver known for its low complexity and for its ability to collect multi-path diversity (see [3], [5], [6], [7], [8], [4] to name these). In a multi-access transmission, the term that affects the most the receiver performance is usually the residual Multi-User Interference (MUI) at its output. In some situations, it is valid to model this MUI term as a Gaussian random variable. References [5] and [6] consider PPM transmissions and resort to the Gaussian approximation of the MUI to provide analytical performance expressions. TH-PAM is one of the modulations considered in [4], where the Gaussian approximation of the distribution of the MUI plus the Inter Symbol Interference (ISI) is discussed. References [9] [7] [8] [10] criticize the MUI Gaussian approximation in the context of PPM transmissions over frequency non selective channels (which usually represent free space communications).

The Gaussian property simplifies the performance calculation, and furthermore, Gaussian noises are well handled by a large number of forward error coding and decoding techniques. From a probabilistic point of view, the Gaussian approximation can be justified naturally by a Central Limit Theorem (CLT) argument. This will be the approach adopted in this paper. To obtain a CLT, one has to define an asymptotic regime where the number of interfering users grows to infinity while the contribution of every interferer to the total MUI becomes infinitesimal. Let us characterize our asymptotic regime beginning with frequency occupation considerations. We denote by T_w the effective pulse width and by T_c the duration of a time slot, *i.e.*, the so-called chip time interval. The frequency band of an UWB signal is of the order of $1/T_w$ and the data symbol rate is equal to $1/(N_s N_h T_c)$. As a spread spectrum system, the UWB system will then have a processing gain of $N_s N_h T_c/T_w$. By a language abuse, we shall ignore in this paper the factor T_c/T_w and call "processing gain" the integer $N = N_s N_h$. Denoting by K the number of users supported by the system, the total number of symbols per second carried by the UWB signal is equal to $K/(N_s N_h T_c)$. In these conditions, it is reasonable to consider the ratio of the number of transmitted symbols per second to the system bandwidth $KT_w/(NT_c)$ as a system load. Let us drop again the factor T_w/T_c and define our load by the factor K/N. Our asymptotic regime is then characterized by the fact that $N \to \infty$ and the number of users $K \to \infty$ in such a way that K/N converges toward a constant $\alpha > 0$, in other words, the number of contributors grow, thus permitting to consider CLT results, but the number of symbols per second per Hertz transmitted by the whole system is constant. This general point of view is often adopted in asymptotic studies for DS-CDMA systems (see for instance [11]).

As $N = N_s N_h$, we have to be specific about the behavior of the repetition factor N_s and the frame length N_h as $N \to \infty$. It will be shown that the MUI term at the receiver output does not converge in distribution toward a Gaussian law if N_h is the only factor of N that grows to infinity while N_s is kept constant. Alternatively, the asymptotic normality of the MUI will be ensured if N_s and N_h grow in such a way that N_s/N_h converges toward a constant $\rho > 0$. These results will be established through the study of the so-called Lindeberg's condition of the CLT for martingales [12]. In our setting, this condition is necessary and sufficient for ensuring the convergence of the MUI distribution toward the Gaussian distribution.

These results show that at high processing gains, the Gaussian character of the MUI term is obtained through repetition. However, in the second case, we also show that the MUI variance grows with ρ. It can even be shown independently that if $N_s/N_h \to \infty$, in other words if N_s grows much faster than N_h, then the MUI variance grows toward infinity.

In practice, a trade-off appears. Assuming N is large enough, when N_s is too small, the Gaussian approximation might not be valid because it would be appropriate to consider N_s as fixed while $N_h \to \infty$. The problem here is that non Gaussian MUI often induces higher Bit Error Rates (BER) than Gaussian MUI at a given Signal to Interference and Noise Ratio (SINR). Alternatively, if we let N_s grow, then it would be possible to use the model $N \to \infty$ and $N_s/N_h \to \rho > 0$. Here, the MUI can be considered as Gaussian, but if ρ is too large, the MUI variance will be large and this results again in a BER increase.

In section II of this paper, the problem is stated in a PAM setting. Non synchronized users with possibly different powers, which send their data over independent multi-path channels, are considered. The convergence of the MUI term distribution is treated in section III. A closed form expression of the asymptotic Signal to Interference plus Noise Ratio (SINR) at the receiver output is given – see Equation (20) – when this interference is Gaussian. In section IV, results equivalent to those of section III are given in the TH-PPM case. Some simulations are finally presented in section V.

In the sequel, $\mathbb{P}$ will denote the probability measure, $\mathbf{1}_{\mathcal{S}}(x)$ the indicator function of the set $\mathcal{S}$, *i.e.*, $\mathbf{1}_{\mathcal{S}}(x) = 1$ if $x \in \mathcal{S}$ otherwise $\mathbf{1}_{\mathcal{S}}(x) = 0$, and $\delta(k)$ the Kronecker delta function $\delta(0) = 1$ and $\delta(k) = 0$ if $k \neq 0$. The σ–field generated by a sequence $X_1, X_2, \ldots$ of random variables will be denoted $\sigma(X_1, X_2, \ldots)$. The conditional expectation given $\sigma(X)$ will be denoted $E\,[\ .\ |\sigma(X)]$ or $E\,[\ .\ |X]$ equivalently. For two real functions $f_1(t)$ and $f_2(t)$, we denote by $r_{f_1 f_2}(t)$ the function $r_{f_1 f_2}(t) = \int f_1(u) f_2(u-t) du$ and by $\mathcal{R}_{f_1 f_2}(t)$ the auto-correlation function of $r_{f_1 f_2}(t)$, *i.e.*, $\mathcal{R}_{f_1 f_2}(t) = \int r_{f_1 f_2}(u) r_{f_1 f_2}(u-t) du$. For reasons that will become apparent in section III, random variables related with the MUI will be denoted with the superscript $^{(K)}$.

II. Problem Formulation

A. The Signal Model

We begin by considering a Time Hopping - PAM (TH-PAM) UWB system. The information symbol $a_{k,m}^{(K)}$ of user k (where $k \in \{1, \ldots, K\}$) at symbol interval m has its values in the set $\{-1, 1\}$. This symbol is repeated over N_s frames, each with a duration $T_f = N_h T_c$. The time hopping code for this user is represented by the sequence $(c_{k,l}^{(K)})_{l \in \mathbb{Z}}$ which elements are discrete random variables equally distributed on $\{0, \ldots, N_h - 1\}$. The random variables $\{c_{k,l}^{(K)}\}_{\substack{k=1,\ldots,K \\ l \in \mathbb{Z}}}$ are furthermore assumed independent. In the case the receiver is synchronized on user k, the contribution of this user to the received signal will be written

$$y_k^{(K)}(t) = \sqrt{\frac{\mathcal{E}_k^{(K)}}{N_s}} \sum_m a_{k,m}^{(K)} \sum_{r=0}^{N_s-1} g_k^{(K)}(t - mN_sT_f - rT_f - c_{k,mN_s+r}^{(K)} T_c) \ . \tag{1}$$

In this expression, $\mathcal{E}_k^{(K)}$ is a constant specific to user k and $g_k^{(K)}(t)$ is the composite channel associated to this user. It is written

$$g_k^{(K)}(t) = \sum_{l=1}^{D} \gamma_{k,l}^{(K)} w(t - \tau_{k,l}^{(K)}) \tag{2}$$

where $w(t)$ is the unit-energy basic pulse waveform with a time support included in $[0, T_c)$, $\boldsymbol{\gamma}_k^{(K)} = [\gamma_{k,1}^{(K)}, \ldots, \gamma_{k,D}^{(K)}]$ is the vector of random zero mean path amplitudes of the radio channel for the signal of user k, $\boldsymbol{\tau}_k^{(K)} = [\tau_{k,1}^{(K)}, \ldots, \tau_{k,D}^{(K)}]$ is the vector of the corresponding random path delays, and D is a uniform upper bound on the number of paths. We assume that the delays $\tau_{k,1}^{(K)}$ are positive, the channel impulse response being causal, and are uniformly bounded with probability one. As a consequence, $g_k^{(K)}(t)$ is supported by the interval $[0, LT_c)$ where $L \in \mathbb{N}^*$ is a uniform upper bound on the lengths of these time supports in chip intervals. We shall need the following assumption on the path amplitudes and delays : for any measurable real function $f(x, y)$, we have

$$E\left[\gamma_{k,l_1}^{(K)}\gamma_{k,l_2}^{(K)} f\left(\tau_{k,l_1}^{(K)}, \tau_{k,l_2}^{(K)}\right)\right] = \delta(l_1 - l_2) E\left[{\gamma_{k,l_1}^{(K)}}^2 f\left(\tau_{k,l_1}^{(K)}, \tau_{k,l_1}^{(K)}\right)\right] \ . \tag{3}$$

This assumption is not restrictive and is satisfied in particular by the so called modified Saleh-Valenzuela model [13], [14], frequently used for representing the UWB channel[1]. This is due to the fact that in this model, the amplitudes can be written as $\gamma_{k,l}^{(K)} = b_{k,l}^{(K)} \rho_{k,l}^{(K)}$ where the random variables $\left\{b_{k,l}^{(K)}\right\}$ indexed by l are independent and have their values in $\{-1, 1\}$ with probabilities 1/2, and furthermore, they are independent from the random variables $\left\{\rho_{k,l}^{(K)}, \tau_{k,l}^{(K)}\right\}$.

Let the channel of user k be represented by the vector $\mathbf{h}_k^{(K)} = [\boldsymbol{\gamma}_k^{(K)}, \boldsymbol{\tau}_k^{(K)}]$. We assume that the K vectors $\{\mathbf{h}_k^{(K)}\}_{k=1,\ldots,K}$ are independent but not necessarily identically distributed. Furthermore, for a given k, the random variables $\{\gamma_{k,l}^{(K)}\}_{l=1,\ldots,D}$ are assumed to satisfy

$$\sum_{l=1}^{D} E\left[{\gamma_{k,l}^{(K)}}^2\right] = 1 \ . \tag{4}$$

In these conditions, it is easy to see that $\mathcal{E}_k^{(K)}$ is the energy per received symbol for user k. The users powers $\mathcal{E}_k^{(K)}$ will be furthermore assumed uniformly bounded, *i.e.*,

$$\exists \ \mathcal{E}_{\sup} > 0 \ : \ \sup_K \max_{k=1,\ldots,K} (\mathcal{E}_k^{(K)}) < \mathcal{E}_{\sup} \ . \tag{5}$$

Assuming that the receiver is perfectly synchronized on user 1, the received signal is written as

$$y^{(K)}(t) = y_1^{(K)}(t) + \sum_{k=2}^{K} y_k^{(K)}(t - \Delta_k^{(K)}) + v(t) \tag{6}$$

where $v(t)$ is a Gaussian noise having a spectral density of $N_0/2$ in the frequency band of $w(t)$. The delay $\Delta_k^{(K)}$ accounts for the absence of synchronization between user k and user 1. It can be checked that for every k, the process $y_k^{(K)}(t)$ is a periodically correlated process with the period $N_s N_h T_c$. Therefore,

[1]Note that modified Saleh-Valenzuela channels have infinite impulse responses. However, truncating these impulse responses to LT_c with L large enough has no practical incidence on the results.

it is natural to assume that the delays $\{\Delta_k^{(K)}\}_{k=2,\ldots,K}$ are random variables uniformly distributed over the interval $[0, N_s N_h T_c)$. Moreover, these delays are independent. Independence assumptions boil down to the independence of the set $\left\{\{a_{k,m}^{(K)}\}_{\substack{k=1,\ldots,K\\ m\in\mathbb{Z}}}, \{c_{k,l}^{(K)}\}_{\substack{k=1,\ldots,K\\ l\in\mathbb{Z}}}, \{\mathbf{h}_k^{(K)}\}_{k=1,\ldots,K}, \{\Delta_k^{(K)}\}_{k=2,\ldots,K}, v(t)\right\}$. In the sequel, we shall drop for convenience the index 1 and the superscript $^{(K)}$ when denoting the quantities relative to user 1.

B. The Correlation Receiver

Assuming a perfect knowledge of $\sqrt{\mathcal{E}}g(t)$ at the receiver, the output of the correlation receiver for symbol a_0 is

$$x = \sqrt{\frac{\mathcal{E}}{N_s}} \sum_{r=0}^{N_s-1} \int y^{(K)}(t) g(t - rN_hT_c - c_rT_c)dt \ , \tag{7}$$

and the decided symbol is $\hat{a}_0 = \text{sign}(x)$. By using the expression (6) of $y^{(K)}(t)$, we get $x = x_{\text{u}} + x_{\text{ISI}} + x_{\text{MUI}}^{(K)} + x_{\text{AWGN}}$ where

$$x_{\text{u}} = \frac{\mathcal{E}}{N_s} a_0 \sum_{r_1,r_2=0}^{N_s-1} r_{gg}\left((r_1 - r_2)N_hT_c + (c_{r_1} - c_{r_2})T_c\right) \tag{8}$$

is the ”useful signal” term,

$$x_{\text{ISI}} = \frac{\mathcal{E}}{N_s} \sum_{m\neq 0} a_m \sum_{r_1,r_2=0}^{N_s-1} r_{gg}\left((r_1 - r_2)N_hT_c + (c_{r_1} - c_{mN_s+r_2})T_c - mN_sN_hT_c\right) \tag{9}$$

is the ISI term,

$$x_{\text{MUI}}^{(K)} = \sum_{k=2}^{K} x_k^{(K)} \tag{10}$$

is the MUI term,

$$x_k^{(K)} = \frac{\sqrt{\mathcal{E}\mathcal{E}_k^{(K)}}}{N_s} \sum_m a_{k,m}^{(K)} \sum_{r_1,r_2=0}^{N_s-1} r_{g_k^{(K)}g}\left((r_1 - r_2)N_hT_c + (c_{r_1} - c_{k,mN_s+r_2}^{(K)})T_c - mN_sN_hT_c - \Delta_k^{(K)}\right) \tag{11}$$

is the contribution of the signal of user k to the MUI term, and

$$x_{\text{AWGN}} = \sqrt{\frac{\mathcal{E}}{N_s}} \sum_{r=0}^{N_s-1} \int v(t) g(t - rN_hT_c - c_rT_c)dt \ . \tag{12}$$

is the term due to the Additive White Gaussian Noise (AWGN) $v(t)$.

III. INTERFERENCE ASYMPTOTIC ANALYSIS

As said in the introduction, we study here the asymptotic regime where the processing gain N and the number of users K grow toward infinity in such a way that $K/N \to \alpha$, a quantity that we designate by the system load. All SINR expressions will naturally depend on the channel vector of user 1, which is assumed to be known at the receiver side. In order to simplify our presentation, we shall treat the channel vector of user 1 as a fixed vector in the sequel.

We begin by studying the asymptotic behavior of the terms x_{u}, x_{ISI}, and x_{AWGN}. The following proposition describes the asymptotic behavior of the useful term x_{u}. The proof is given in appendix A.

Proposition 1: As $N_h \to \infty$, x_{u} converges in probability toward $\mathcal{E} a_0 r_{gg}(0)$.

Let us interpret this result. From (8), the useful term is written as $x_{\text{u}} = \mathcal{E} a_0 r_{gg}(0) + \mathcal{E} a_0 z_g$ where

$$z_g = \frac{1}{N_s} \sum_{\substack{r_1, r_2 = 0 \\ r_1 \neq r_2}}^{N_s - 1} r_{gg}\left((r_1 - r_2) N_h T_c + (c_{r_1} - c_{r_2}) T_c\right) \tag{13}$$

accounts for the Inter Frame Interference (IFI) within the same symbol. Proposition 1 says that this IFI becomes negligible when the frame length is large.

The next proposition is relative the term x_{ISI}. Its proof is rather similar to the proof of Proposition 1, therefore it will be skipped :

Proposition 2: As N grows toward infinity, x_{ISI} converges to zero in probability.

This result can be interpreted intuitively. We recall that the channel lengths measured in chip intervals are uniformly bounded by the constant L. Therefore, as the processing gain grows large, the ISI becomes negligible. Indeed, only the first L chips in a symbol can be corrupted by the interference due to the previous symbol. It is well known that ISI is negligible when the channel length is much smaller than the symbol duration.

Let us consider now the AWGN term x_{AWGN}. The proof of the following proposition is given in appendix B :

Proposition 3: As $N_h \to \infty$, x_{AWGN} converges in distribution toward a Gaussian zero mean random variable with variance

$$\sigma^2_{\text{AWGN}} = \frac{N_0}{2} \mathcal{E} r_{gg}(0) \ .$$

From (12), it can be clearly seen that conditionally to the code vector $\mathbf{c} = [c_0, \ldots, c_{N_s - 1}]$, the distribution of this term is Gaussian. Without conditioning on $\mathbf{c}$, this distribution is not Gaussian in general.

Nevertheless, Proposition 3 asserts that this distribution converges weakly to the Gaussian distribution in the asymptotic regime as $N_h \to \infty$.

We now turn to the main part of the paper, which consists in the asymptotic study of the MUI term. At this point, an assumption on the energies per symbol of the users is needed. Denoting by $\overline{\mathcal{E}}^{(K)}$ the empirical mean $\overline{\mathcal{E}}^{(K)} = \frac{1}{K-1}\sum_{k=2}^{K}\mathcal{E}_k^{(K)}$ of the energies of the interferers, we shall assume that $\overline{\mathcal{E}}^{(K)}$ converges to a limit $\overline{\mathcal{E}}$ as $K \to \infty$.

In appendices C and D, it is shown that the variance of the contribution $x_k^{(K)}$ to the MUI term is given by Eq. (25) and satisfies by consequence

$$E\left[x_k^{(K)2}\right] = \frac{\mathcal{E}\mathcal{E}_k^{(K)}}{T_c}\left(\left(\frac{2}{3N_h^2} - \frac{4}{3N_sN_h^2}\right)\sum_{l=-L}^{L}\mathcal{R}_{wg}(lT_c) + \frac{1}{N_sN_h}\mathcal{R}_{wg}(0)\right) + f_1(N_s, N_h) \qquad (14)$$

where $|f_1(N_s, N_h, \mathcal{E}_k^{(K)})| < C_1\left(\frac{1}{N_s^2N_h^2} + \frac{1}{N_sN_h^3}\right)$ and $C_1 = 34\ \mathcal{E}_{\text{sup}}^2\mathcal{R}_{wg}(0)L^4/T_c$.

Turning to the variance $\sigma_{\text{MUI}}^{(K)\ 2}$ of the MUI term $x_{\text{MUI}}^{(K)}$, we get then

$$\sigma_{\text{MUI}}^{(K)\ 2} = \sum_{k=2}^{K} E\left[x_k^{(K)2}\right] = \frac{\mathcal{E}\overline{\mathcal{E}}^{(K)}}{T_c}\frac{K-1}{N}\left(\left(\frac{2}{3}\frac{N_s}{N_h} - \frac{4}{3}\frac{1}{N_h}\right)\sum_{l=-L}^{L}\mathcal{R}_{wg}(lT_c) + \mathcal{R}_{wg}(0)\right) + f_2(N_s, N_h) \qquad (15)$$

where $|f_2(N_s, N_h)| < C_1\frac{K}{N}\left(\frac{1}{N} + \frac{1}{N_h^2}\right)$.

Let us consider now the asymptotic regime where $N = N_sN_h \to \infty$ while $K/N \to \alpha > 0$. The first case we consider is the case where N grows in such a way that $N_h/N_s \to 0$:

Proposition 4: If $N = N_sN_h \to \infty$ while $K/N \to \alpha > 0$ and $N_h/N_s \to 0$, then $\sigma_{\text{MUI}}^{(K)\ 2} \to \infty$.

This proposition follows directly from Equation (15).

Let us give an intuitive interpretation of this result. Assume for the sake of illustration that $N_h = 1$ and $N_s > 1$. In this situation, time hopping is absent and our system would be a "DS-CDMA" system in which all spreading vectors are equal to $[1, 1, \ldots, 1]$! In this system, if $N_s \to \infty$ and $K \to \infty$, it is clear that all interferers contributions will sum up without any attenuation due to despreading, and therefore, the MUI variance will grow toward infinity. What Proposition 4 asserts is that this will be more generally the case if $N_h/N_s \to 0$: in this situation, time hopping will not be able to separate users contributions reliably in the asymptotic regime due to the small size of the frames.

The two following cases that we shall consider correspond to the situation where N_s is kept constant while $N_h \to \infty$, then to the situation where both N_s and N_h grow toward infinity in such a way that $N_s/N_h \to \rho > 0$. In both situations, the MUI variance will converge to a finite value. Whether this asymptotic MUI will be Gaussian or not will be our main issue.

As is well known, the asymptotic normality is generally established through a Central Limit Theorem. One classical form of this theorem is the following : consider a sequence $z_1, z_2, \ldots$ of centered independent and identically distributed random variables with finite variance σ^2. Then as $K \to \infty$, the random variable $s_K = \frac{1}{\sqrt{K}} \sum_{k=1}^{K} z_k$ converges in distribution toward a Gaussian centered random variable with variance σ^2. In the setting of this paper, if this version of the CLT was to be used, the asymptotic normality of $x_{\mathrm{MUI}}^{(K)} = \sum_{k=2}^{K} x_k^{(K)}$ would have to be established by identifying $\sqrt{K} x_k^{(K)}$ with z_k. However, this cannot be done because in our asymptotic study, as K grows, N_s and/or N_h grow, and this results in a change in the probability distribution of $\sqrt{K} x_k^{(K)}$ through the change of the distributions of of $c_k^{(K)}$ and $\Delta_k^{(K)}$. In our case, the random variables $x_k^{(K)}$ are formally arranged in a so called triangular array $(x_k^{(K)})_{\substack{k=2,\ldots,K \\ K=2,\ldots,\infty}}$, and we have to see whether the sums $x_{\mathrm{MUI}}^{(K)}$ performed along the rows of the array converge in distribution toward the Gaussian law. One additional difference with the classical form of the CLT shown above is that the random variables $(x_k^{(K)})_{k=2,\ldots,K}$ on row K are not independent. Indeed, Eq. (11) shows that all these random variables depend on the code vector $\mathbf{c}$ of user 1. Because of this dependence, we are led to use the CLT for martingales, which generalizes the CLT for independent random variables.

It can be seen from Eq. (11) that $x_k^{(K)}$ is measurable with respect to the σ–field generated by the random variables $\mathbf{c}, (a_{k,m}^{(K)})_m, (c_{k,l}^{(K)})_l$, and $\Delta_k^{(K)}$. Given the sequence of increasing σ–fields $\mathcal{F}_k^{(K)} = \sigma\left(\mathbf{c}, \left\{(a_{n,m}^{(K)})_m, (c_{n,l}^{(K)})_l, \Delta_n^{(K)}\right\}_{n=2,\ldots,k}\right)$, the partial sum $x_{\mathrm{MUI},k}^{(K)} = \sum_{n=2}^{k} x_n^{(K)}$ is therefore measurable with respect to $\mathcal{F}_k^{(K)}$. Furthermore, one can notice that $E\left[\left|x_{\mathrm{MUI},k}^{(K)}\right|\right] < \infty$ and that the conditional expectation $E\left[x_{\mathrm{MUI},k+1}^{(K)} \middle| \mathcal{F}_k^{(K)}\right]$ is equal to $x_{\mathrm{MUI},k}^{(K)}$ with probability one. In these conditions, the sequence $x_{\mathrm{MUI},2}^{(K)}, \ldots, x_{\mathrm{MUI},K}^{(K)}$ is called a martingale relative to the σ–fields $\mathcal{F}_2^{(K)}, \ldots, \mathcal{F}_K^{(K)}$ [12, page 458]. The CLT for martingales takes the following form: let $\sigma_k^{(K)}(\mathbf{c})^2$ be the random variable $\sigma_k^{(K)}(\mathbf{c})^2 = E\left[{x_k^{(K)}}^2 \middle| \mathcal{F}_{k-1}^{(K)}\right] = E\left[{x_k^{(K)}}^2 \middle| \mathbf{c}\right]$, where the last equality can be deduced from Eq. (11), and let $\sigma^{(K)}(\mathbf{c})^2 = \sum_{k=2}^{K} \sigma_k^{(K)}(\mathbf{c})^2$. Assume that $\sigma^{(K)}(\mathbf{c})^2$ converges in probability as $K \to \infty$ to some positive deterministic quantity σ^2. Assume that the so-called Lindeberg condition is satisfied :

$$\forall \varepsilon > 0, \ \lim_{K\to\infty} \sum_{k=2}^{K} E\left[{x_k^{(K)}}^2 \mathbf{1}_{|x_k^{(K)}| \geq \varepsilon}\right] = 0 \ . \tag{16}$$

Then $x_{\mathrm{MUI}}^{(K)}$ converges in distribution toward the centered normal distribution with variance σ^2 [12, Th. 35.12].

In our situation, the asymptotic behavior of $\sigma^{(K)}(\mathbf{c})^2$ is described by the following proposition:

Proposition 5: . Assume that $N \to \infty$, $K/N \to \alpha > 0$, and $N_s/N_h \to \rho \geq 0$. Then

$$\sigma^{(K)^2}(\mathbf{c}) \to \sigma^2_{\text{MUI}} = \frac{\mathcal{E}\overline{\mathcal{E}}}{T_c}\alpha\left(\frac{2\rho}{3}\sum_{l=-L}^{L}\mathcal{R}_{wg}(lT_c) + \mathcal{R}_{wg}(0)\right) \tag{17}$$

in probability.

The proof of this proposition is in Appendix F. Notice that σ^2_{MUI} is also the limit of $\sigma^{(K)}_{\text{MUI}}{}^2$ given by Eq. (15) as can be expected.

We now treat the case where N_s is kept constant while $N_h \to \infty$:

Proposition 6: Assume $N_h \to \infty$ while N_s is kept constant. Then as $K \to \infty$ and $K/N \to \alpha > 0$, the variance $\sigma^{(K)^2}$ converges in probability to $\frac{\mathcal{E}\overline{\mathcal{E}}}{T_c}\alpha\mathcal{R}_{wg}(0)$. Moreover, $x^{(K)}_{\text{MUI}}$ does not converge in distribution toward a Gaussian random variable.

The limiting variance $\frac{\mathcal{E}\overline{\mathcal{E}}}{T_c}\alpha\mathcal{R}_{wg}(0)$ can be deduced directly from Proposition 5. In order to prove the second part of the Proposition, we use the following result, shown in [15] by a refinement of a result of [16]: if $\sigma^{(K)}(\mathbf{c})^2$ converges in probability to a deterministic σ^2, if the conditional distribution functions $F^{(K)}_k(x) = \mathbb{P}\left(x^{(K)}_k \leq x \,\middle|\, \mathcal{F}^{(K)}_{k-1}\right) = \mathbb{P}\left(x^{(K)}_k \leq x \,\middle|\, \mathbf{c}\right)$ are symmetric with probability one, and if $\max_{k=2,\ldots,K} \sigma^{(K)}_k(\mathbf{c})^2 \to 0$ in probability as $K \to \infty$, then the Lindeberg condition (16) is also necessary for convergence of $x^{(K)}_{\text{MUI}}$ toward the normal law $\mathcal{N}(0,\sigma^2)$ ([15, Main theorem and Eq. (6)]). It can be seen from the expression (11) of $x^{(K)}_k$ that $\mathbb{P}\left(x^{(K)}_k \leq x \,\middle|\, \mathbf{c}\right)$ is symmetric. Indeed, $a^{(K)}_{k,m}$ are independent with other random variables and equally distributed over $\{-1,1\}$. Moreover, in Appendix G, it is proven that

$$\max_{k=2,\ldots,K} \sigma^{(K)}_k(\mathbf{c})^2 \to 0 \tag{18}$$

for any value of $\mathbf{c}$ (which is stronger than the convergence in probability required in [15]). We therefore have to show that the random variables $\{x^{(K)}_k\}$ do not satisfy Lindeberg's condition. This is done in appendix H.

Let us give an intuitive interpretation of this fact. The event $x^{(K)}_k \neq 0$ represents a collision between the signal received from user 1 and the signal received from user k. In the model (1) the signal amplitude of a user is multiplied by $1/\sqrt{N_s}$ which is not an infinitesimal value in the setting of Proposition 6, therefore, the values taken by the random variable $x^{(K)}_k$ when $x^{(K)}_k \neq 0$ are not infinitesimal. Yet the variance of this random variable, being of the order $1/N_h$ (see Equation (14)), is infinitesimal. This is because the probability of occurrence of a collision between user 1 and user k is also of the order $1/N_h$. Multiplying $x^{(K)}_k$ by $\mathbf{1}_{|x^{(K)}_k| \geq \varepsilon}$ for ε small enough will not reduce much this variance, and therefore, Lindeberg's condition will not be satisfied.

Alternatively, assume now that the repetition factor N_s also grows in such a way that $N_s/N_h \to \rho > 0$. In this case, many of the pulses of the user of interest carrying one information symbol will undergo collisions, but the effects of these collisions will sum up in such a way that the resulting MUI is asymptotically Gaussian:

Proposition 7: Assume that $\frac{N_s}{N_h} \to \rho > 0$, and that the random variables $\|\boldsymbol{\gamma}_k^{(K)}\|^2$ are uniformly integrable, *i.e.*, that

$$\lim_{a\to\infty} \sup_K \max_{k=1,\ldots,K} E\left[\|\boldsymbol{\gamma}_k^{(K)}\|^2 \mathbf{1}_{\|\boldsymbol{\gamma}_k^{(K)}\|>a}\right] = 0 \ . \tag{19}$$

Then as $K \to \infty$ and $K/N \to \alpha > 0$, $x_{\mathrm{MUI}}^{(K)}$ converges in distribution toward a Gaussian random variable with zero mean and variance σ^2_{MUI} given by (17).

The proof of this proposition is given in appendix I.

Notice that the assumption (19) is needed for mathematical purposes only, because it allows inequality (38) in the proof to be true. It is obvious that for every couple of indices k and K, $E\left[\|\boldsymbol{\gamma}_k^{(K)}\|^2 \mathbf{1}_{\|\boldsymbol{\gamma}_k^{(K)}\|>a}\right]$ converges to 0 as $a \to \infty$. Assumption (19) requires this convergence to be uniform. It is satisfied in all practical cases of interest, and in particular when the vectors $\boldsymbol{\gamma}_k^{(K)}$ are identically distributed.

In the asymptotic conditions of Proposition 7, the SINR at the output of the receiver for TH-PAM signals is

$$\mathrm{SINR}_{\mathrm{PAM}} = \frac{\mathcal{E}^2 r_{gg}(0)^2}{\sigma^2_{\mathrm{AWGN}} + \sigma^2_{\mathrm{MUI}}} = \frac{\mathcal{E} r_{gg}(0)}{\frac{N_0}{2} + \frac{\overline{\mathcal{E}}}{T_c r_{gg}(0)}\alpha\left(\frac{2\rho}{3}\sum_{l=-L}^{L}\mathcal{R}_{wg}(lT_c) + \mathcal{R}_{wg}(0)\right)} \ . \tag{20}$$

where σ^2_{AWGN} and σ^2_{MUI} are given by Propositions 3 and 5 respectively. In these asymptotic conditions, the BER at the output of the receiver is $Q(\sqrt{\mathrm{SINR}_{\mathrm{PAM}}})$ where $Q(.)$ is the Gaussian tail function.

IV. The TH-PPM Case

In the Time Hopping - Pulse Position Modulation (TH-PPM) case, Equation (1) is replaced by

$$y_k^{(K)}(t) = \sqrt{\frac{\mathcal{E}_k^{(K)}}{N_s}} \sum_m \sum_{r=0}^{N_s-1} g_k^{(K)}(t - mN_sT_f - rT_f - c_{k,mN_s+r}^{(K)}T_c - da_{k,m}^{(K)}) \ , \tag{21}$$

where the symbols $\{a_{k,m}^{(K)}\}$ have their values in $\{0,1\}$ and d is the time shift used for position modulation ([3]). The description of the received signal is otherwise unchanged. The output of the correlation receiver for the symbol a_0 is here

$$x = \sqrt{\frac{\mathcal{E}}{N_s}} \sum_{r=0}^{N_s-1} \int y^{(K)}(t) p(t - rN_hT_c - c_rT_c) dt \ .$$

where $p(t) = g(t) - g(t-d)$, and the decision rule is $\hat{a}_0 = 0$ if $x > 0$ and $\hat{a}_0 = 1$ otherwise. Here we have $x = x_{\mathrm{u}} + x_{\mathrm{ISI}} + x_{\mathrm{MUI}}^{(K)} + x_{\mathrm{AWGN}}$ where

$$x_{\mathrm{u}} = \frac{\mathcal{E}}{N_s} \sum_{r_1,r_2=0}^{N_s-1} r_{gp}\left((r_1-r_2)N_hT_c + (c_{r_1} - c_{r_2})T_c - da_0\right) \ ,$$

$$x_{\mathrm{ISI}} = \frac{\mathcal{E}}{N_s} \sum_{m\neq 0} \sum_{r_1,r_2=0}^{N_s-1} r_{gp}\left((r_1-r_2)N_hT_c + (c_{r_1} - c_{mN_s+r_2})T_c - mN_sN_hT_c - da_m\right) \ ,$$

$$x_{\mathrm{MUI}}^{(K)} = \sum_{k=2}^{K} x_k^{(K)} \ ,$$

$$x_k^{(K)} = \frac{\sqrt{\mathcal{E}\mathcal{E}_k^{(K)}}}{N_s} \sum_{m} \sum_{r_1,r_2=0}^{N_s-1} r_{g_k^{(K)}p}\left((r_1-r_2)N_hT_c + (c_{r_1} - c_{k,mN_s+r_2}^{(K)})T_c - mN_sN_hT_c - da_{k,m}^{(K)} - \Delta_k^{(K)}\right) \ ,$$

$$x_{\mathrm{AWGN}} = \sqrt{\frac{\mathcal{E}}{N_s}} \sum_{r=0}^{N_s-1} \int v(t) p(t - rN_hT_c - c_rT_c) dt \ ,$$

and these terms have the same meanings as their equivalents of section II-B.

We shall just give the main results concerning the TH-PPM case, as the proofs and the derivations do not differ much from those of the PAM case. When $N_h \to \infty$, the useful term x_{u} converges in probability toward $\mathcal{E} r_{gp}(-da_0) = \mathcal{E}(1/2 - a_0) r_{pp}(0)$, and the distribution of the AWGN term x_{AWGN} converges toward the Gaussian distribution with the zero mean and the variance $\sigma^2_{\mathrm{PPM,AWGN}} = \frac{N_0}{2}\mathcal{E} r_{pp}(0)$. As for the MUI term $x_{\mathrm{MUI}}^{(K)}$, it does not have a Gaussian limit distribution if N_s is kept constant (*cf.* Proposition 6). On the other hand, if $N_s/N_h \to \rho > 0$, then (*cf.* Proposition 7) it has a Gaussian limit distribution with a zero mean and a variance of

$$\sigma^2_{\mathrm{PPM,MUI}} = \frac{\mathcal{E}\overline{\mathcal{E}}}{T_c}\alpha \left(\frac{2\rho}{3} \sum_{l=-L}^{L} \mathcal{R}_{wp}(lT_c) + \mathcal{R}_{wp}(0) \right) \ .$$

Notice that the only difference between this expression and (17) lies in the fact that $g(t)$ is replaced here by $g(t) - g(t-d)$. The expression of the output SINR is

$$\mathrm{SINR}_{\mathrm{PPM}} = \frac{\mathcal{E}^2 r_{pp}(0)^2}{4(\sigma^2_{\mathrm{PPM,AWGN}} + \sigma^2_{\mathrm{PPM,MUI}})} \qquad (22)$$

V. SIMULATIONS

In order to give an illustration of the results of the previous sections, we carried out some simulations for TH-PAM and TH-PPM transmissions. The basic pulse waveform is the second derivative of a Gaussian pulse with a pulse shape parameter $t_n = 0.4$ns [17]. The time slot number a pulse occupies within a frame is an independent and identically distributed process with the uniform probability distribution on the set $\{0, \ldots, N_h - 1\}$. In TH-PPM, the time shift d satisfies $d/t_n = 0.5422$ as in [1]. The chip period has been set to $T_c = 5\ t_n$ for PAM and $T_c = 6\ t_n$ for PPM. The additional delay of t_n in PPM is due to the presence of the time shift d. The BER resulting from simulations was calculated after having received more than 100 errors. Because in our results, all expectations are conditioned on the channel of user 1, this channel is kept fixed while the channels of all other users change at each simulation trial. Moreover, the transmitted sequences of symbols for all users and the relative delays change at each trial.

In all figures that show Bit Error Rates, the solid line plots indicate the BER versus $2E_{\mathrm{b}}/N_0$ that result from the Gaussian approximation in the asymptotic regime, *i.e.*, $Q(\sqrt{\mathrm{SINR}_{\mathrm{PAM}}})$ or $Q(\sqrt{\mathrm{SINR}_{\mathrm{PPM}}})$ where $\mathrm{SINR}_{\mathrm{PAM}}$ and $\mathrm{SINR}_{\mathrm{PPM}}$ are given by Equations (20) and (22) respectively. The dashed curves are the ones obtained by simulation.

The pertinence of the asymptotic regimes described by Propositions 6 and 7 is first tested for single path channels representing a free space propagation. It is assumed that we have a perfect power control, in other words $\mathcal{E}_1^{(K)} = \cdots = \mathcal{E}_K^{(K)} = \overline{\mathcal{E}}$. Such a scenario has been studied in *e.g.* [3] [18] in a TH-PPM context.

In Figure 1, a TH-PAM transmission is considered, the processing gain is set to $N = 200$ and the number of users is $K = 100$, resulting in a load of $\alpha = 0.5$. One can notice that when $N_h = 200$ and $N_s = 1$, then the transmission conditions can be modeled by the assumptions of Proposition 6, and therefore the Gaussian approximation is not valid as expected. The plain curve representing the Gaussian approximation for $\rho = 1/200$ is plotted for the purpose of comparison : it represents the BER that one would have obtained for the same MUI variance if this MUI was Gaussian. The BER loss due to the non Gaussian character of the MUI is illustrated by the dashed curve obtained for $N_s = 1$ and $N_h = 200$. Figure 1 shows also that when $N_s = 8$ and $N_h = 25$, a situation modeled in Equation (20) by $\rho = 8/25$, then the asymptotic regime of Proposition 7 is practically attained. The behavior described by Propositions 6 and 7 is also illustrated on Fig. 2 where empirical histograms of the random variable $x_{\mathrm{MUI}}^{(K)}$ are shown. The centered Gaussian densities with variances $E\left[x_{\mathrm{MUI}}^2\right]$ are also shown on this figure. From the top to the bottom of this figure, K and N increase in such a way that the load K/N is fixed to $1/2$. In the left column, N_s is fixed to 1. Here, as predicted by Proposition 6, the MUI distribution does not approach the Gaussian distribution as N_h grows. Alternatively, when N_s grows in parallel with N_h (right column), the MUI distribution approaches the Gaussian distribution. Under the same experimental conditions, Figure (3) shows the quantile-quantile plot between the empirical MUI distribution and the Gaussian distribution for different values of N. The couple (N_s, N_h) is chosen equal to $(2, 6)$, $(4, 12)$, and $(8, 25)$ resulting in a ratio N_s/N_h close to $8/25$. The convergence toward the Gaussian distribution with respect to N is clearly seen on this figure.

Simulations were also conducted in more realistic situations where the channels are multi-path channels and the received powers are different. The channel model is the modified Saleh-Valenzuela model described in [13] and [14]. Channels with a RMS delay spread of 5ns are considered. The chosen channel belongs to the set of channels proposed in [13], namely we considered the model characterized by the parameters $\Lambda = 1/22$, $\lambda = 1/0.94$, $\Gamma = 7.6$, $\gamma = 0.94$, and $\sigma = 4.8$ in this reference. The different transmitters are assumed to be uniformly distributed within the ring between the circles with radii 1m and 10m centered on the receiver. The path gains decrease in R^{-2} where R is the distance to the receiver [19]. The power of user 1 is taken equal to the mean power. A processing gain $N = 600$ has been chosen. Because $T_c = 2$ns, the data rate per user is then 833 kbit/s. The system load is $\alpha = 1/2$. Figure 4 which concerns TH-PAM transmissions shows that when $N_h = 600$ and $N_s = 1$, then the Gaussian approximation is not valid. However, when $N_s = 6$ and $N_h = 100$, then the receiver performance can be predicted reliably by the result of Proposition 7. Like for single path channels, the histograms of $x_{\mathrm{MUI}}^{(K)}$ are also shown (Figure 5). The quantile-quantile plot is also shown on Figure 6 for different values of N, the ratio N_s/N_h being set to $3/50$. The convergence to the Gaussian distribution predicted by Proposition 7 can be clearly seen on these figures.

The same results are shown in Figure 7 for the TH-PPM case. In this figure, a curve with $N_s = 3$ and $N_h = 200$ has been added to underline the effect of reducing N_s while keeping N constant.

In Figure 8, we get back to the environment of Figure 4, we fix N_h to 100 and we test the pertinence of the Gaussian asymptotic regime when modifying N or the power distribution. If the users powers are equal, then this regime is attained for $N = 300$. Alternatively, when the powers are unequal as in Figure 4, then at $N = 300$ the Gaussian asymptotic approximation is less accurate. At $N = 600$, we are closer to the Gaussian asymptotic regime. With unequal powers, this asymptotic regime is reached for higher values of N.

In summary, assume that N and K are fixed to large enough values. A too small value of N_s, even though it will provide a small MUI variance, will result in a non-Gaussian MUI distribution which is in general harmful in the sense that the Gaussian approximation predicts a much better BER. The variance reduction does not compensate for the BER degradation. Recall that Proposition 6, that asserts that the Gaussian approximation is not valid in this case, is in agreement with this observation.

We may also notice that to reach the domain of validity of the asymptotic regime, we have to use a large processing gain and a large number of users. We must however note that a complete specification of the domain of validity of this regime is out of the scope of this paper. This study should certainly take into account the statistical model for the channels. In particular, if the Saleh-Valenzuela model is considered, the root mean square of the channels delay spread will play an important role. Other parameters that have an important impact on the convergence are the power distribution and the system load K/N.

It is clear that at a fixed chip rate, a high processing gain leads to a reduced bit rate per user. Therefore, the asymptotic analysis of the Gaussian approximation is valid in the context of networks with relatively low rates per user rather than in the context of high speed WPAN. Impulse Radio UWB is a serious candidate for applications such as sensor networks that use a large number of sensors. In these contexts, the asymptotic analysis can be used.

APPENDIX

A. Proof of Proposition 1

For a given $\varepsilon > 0$, we have

$$\mathbb{P}\left[|x_{\mathrm{u}} - \mathcal{E} a_0 r_{gg}(0)| > \varepsilon\right] \leq \frac{\mathcal{E}}{\varepsilon} E\left[|z_g|\right]$$

by Markov's inequality. We shall prove that $E\left[|z_g|\right] \to 0$ when $N_h \to \infty$. The expectation $E\left[|z_g|\right]$ writes

$$\begin{aligned}
E\left[|z_g|\right] &\leq \frac{1}{N_s} \sum_{\substack{r_1, r_2 = 0 \\ r_1 \neq r_2}}^{N_s - 1} \frac{1}{N_h^2} \sum_{i_1, i_2 = 0}^{N_h - 1} \left| r_{gg}\left((r_2 - r_1) N_h T_c + (i_2 - i_1) T_c\right)\right| \\
&= \frac{1}{N_s} \left(\frac{1}{N_h^2} \left(\sum_{i_1, i_2 = 0}^{N_s N_h - 1} \left| r_{gg}\left((i_1 - i_2) T_c\right)\right| \right) - \frac{N_s}{N_h^2} \left(\sum_{i_1, i_2 = 0}^{N_h - 1} \left| r_{gg}\left((i_1 - i_2) T_c\right)\right| \right) \right) \\
&= \frac{1}{N_s} \frac{1}{N_h^2} \left(\left(\sum_{l = -L + 1}^{L - 1} (N_s N_h - |l|) |r_{gg}(l T_c)| \right) - N_s \left(\sum_{l = -L + 1}^{L - 1} (N_h - |l|) |r_{gg}(l T_c)| \right) \right) \\
&= \frac{N_s - 1}{N_s} \frac{1}{N_h^2} \sum_{l = -L + 1}^{L - 1} |l \; r_{gg}(l T_c)| \;, \qquad (23)
\end{aligned}$$

hence the result.

B. Proof of Proposition 3

Let us denote by $F(x)$ the distribution function (d.f.) of the standard Gaussian law. Conditionally to $\mathbf{c}$, the d.f. of x_{AWGN} is $F(x/\sigma_{\mathbf{c}})$ where $\sigma_{\mathbf{c}} > 0$ and

$$\sigma_{\mathbf{c}}^2 = \frac{N_0 \mathcal{E}}{2N_s} \sum_{r,r'=0}^{N_s-1} r_{gg}\left((r'-r)N_h T_c + (c_{r'} - c_r)T_c\right) = \sigma_{\text{AWGN}}^2 + \sigma^2 \mathcal{E} z_g \tag{24}$$

where z_g is defined in (13). The d.f. of x_{AWGN} is then $E[F(x/\sigma_{\mathbf{c}})]$ where the expectation is taken with respect to $\mathbf{c}$. To prove our proposition, we shall prove that $\chi(x) = E\left[F(x/\sigma_{\mathbf{c}})\right] - F(x/\sigma_{\text{AWGN}}) = E\left[F(x/\sigma_{\mathbf{c}}) - F(x/\sigma_{\text{AWGN}})\right]$ converges to zero as $N_h \to \infty$. For a given $\varepsilon > 0$, we have $|\chi(x)| \leq \chi_1(x,\varepsilon) + \chi_2(x,\varepsilon)$ where $\chi_1(x,\varepsilon) = E\left[|F(x/\sigma_{\mathbf{c}}) - F(x/\sigma_{\text{AWGN}})|\, \mathbf{1}_{|\sigma_{\mathbf{c}}^2 - \sigma_{\text{AWGN}}^2| \leq \varepsilon}\right]$ and $\chi_2(x,\varepsilon) = E\left[|F(x/\sigma_{\mathbf{c}}) - F(x/\sigma_{\text{AWGN}})|\, \mathbf{1}_{|\sigma_{\mathbf{c}}^2 - \sigma_{\text{AWGN}}^2| > \varepsilon}\right]$.

The function $F(x/\sigma)$ is continuous in the variable σ over the set of the strictly positive real numbers. Therefore, $F(x/\sigma_{\mathbf{c}}) - F(x/\sigma_{\text{AWGN}}) \to 0$ as $\sigma_{\mathbf{c}} \to \sigma_{\text{AWGN}}$. As $|F(x/\sigma_{\mathbf{c}}) - F(x/\sigma_{\text{AWGN}})|\, \mathbf{1}_{|\sigma_{\mathbf{c}}^2 - \sigma_{\text{AWGN}}^2| \leq \varepsilon} \leq 2$, by the dominated convergence theorem, $\chi_1(x,\varepsilon) \to 0$ when $\varepsilon \to 0$.

Considering $\chi_2(x,\varepsilon)$, we have

$$\chi_2(x,\varepsilon) \leq 2E\left[\mathbf{1}_{|\sigma_{\mathbf{c}}^2 - \sigma_{\text{AWGN}}^2| > \varepsilon}\right] = 2\mathbb{P}\left[|\sigma_{\mathbf{c}}^2 - \sigma_{\text{AWGN}}^2| > \varepsilon\right] \leq \frac{2}{\varepsilon} E\left[|\sigma_{\mathbf{c}}^2 - \sigma_{\text{AWGN}}^2|\right]$$

where the last inequality is Markov's inequality. From (24) and (23) we have

$$E\left[|\sigma_{\mathbf{c}}^2 - \sigma_{\text{AWGN}}^2|\right] \leq \frac{N_0}{2} \mathcal{E} \frac{N_s - 1}{N_s} \frac{1}{N_h^2} \sum_{l=-L+1}^{L-1} l |r_{gg}(lT_c)|$$

which converges to zero as $N_h \to \infty$. Therefore, $\chi_2(x,\varepsilon)$ converges to 0 for every ε, thus $\chi(x) \to 0$ as $N_h \to \infty$.

C. Proof of Equation (14).

In appendix D, it is shown that the variance of the interference term $x_k^{(K)}$ writes

$$
\begin{aligned}
E\left[x_k^{(K)2}\right] = \frac{\mathcal{E}\mathcal{E}_k^{(K)}}{N_s^3 N_h T_c}\Big(\quad & \frac{1}{N_h^4}\sum_{i_1,i_2,i_3,i_4=0}^{N_sN_h-1} \mathcal{R}_{wg}\left(\left(i_1-i_2+i_3-i_4\right)T_c\right) \\
- \quad & 2\frac{N_s}{N_h^4}\sum_{i_1,i_2=0}^{N_sN_h-1}\sum_{j_1,j_2=0}^{N_h-1} \mathcal{R}_{wg}\left(\left(i_1-i_2+j_1-j_2\right)T_c\right) \\
+ \quad & \frac{N_s^2}{N_h^4}\sum_{j_1,j_2,j_3,j_4=0}^{N_h-1} \mathcal{R}_{wg}\left(\left(j_1-j_2+j_3-j_4\right)T_c\right) \\
+ \quad & 2\frac{N_s}{N_h^2}\sum_{i_1,i_2=0}^{N_sN_h-1} \mathcal{R}_{wg}\left((i_1-i_2)T_c\right) \\
- \quad & 2\frac{N_s^2}{N_h^2}\sum_{j_1,j_2=0}^{N_h-1} \mathcal{R}_{wg}\left((j_1-j_2)T_c\right) \\
+ \quad & N_s^2\mathcal{R}_{wg}(0)\Big) \ . \qquad (25)
\end{aligned}
$$

This expression can be simplified when $N \to \infty$. The simplification can be done by using the following lemma that will let us enumerate the summands in the right hand side member of (25):

Lemma 1: Let M, Q be two elements of $\mathbb{N}$ and I an element of $\mathbb{Z}$. Then if $|I| \leq Q$,

$$
\sum_{i_1,i_2,i_3,i_4=0}^{Q-1} \delta(i_1-i_2+i_3-i_4-I) = \frac{1}{6}\left(-3|I|+3|I|^3+2Q-6I^2Q+4Q^3\right) \ . \qquad (26)
$$

If $M+|I| \leq Q$, then

$$
\sum_{i_1,i_2=0}^{Q-1}\sum_{j_1,j_2=0}^{M-1} \delta(i_1-i_2+j_1-j_2-I) = \frac{1}{3}\left(-|I|+|I|^3+M-3I^2M-M^3+3M^2Q\right) \ . \qquad (27)
$$

A proof for this lemma is given in appendix E.

Let us study with the help of this lemma the behavior as $N \to \infty$ of the first term in the right hand side member of (25)

$$
\chi = \frac{\mathcal{E}\mathcal{E}_k^{(K)}}{N_s^3 N_h^5 T_c}\sum_{i_1,i_2,i_3,i_4=0}^{N_sN_h-1} \mathcal{R}_{wg}\left(\left(i_1-i_2+i_3-i_4\right)T_c\right) \ .
$$

Noticing that $\mathcal{R}_{wg}(t)$ is equal to zero if $|t| \geq (L+1)T_c$, we have

$$\begin{aligned}\chi &= \frac{\mathcal{E}\mathcal{E}_k^{(K)}}{N_s^3 N_h^5 T_c} \sum_{i_1,i_2,i_3,i_4=0}^{N_s N_h - 1} \sum_{l=-L}^{L} \mathcal{R}_{wg}(lT_c)\delta(i_1 - i_2 + i_3 - i_4 - l) \\ &= \frac{1}{6}\frac{\mathcal{E}\mathcal{E}_k^{(K)}}{N_s^3 N_h^5 T_c} \sum_{l=-L}^{L} \mathcal{R}_{wg}(lT_c)\left(-3|l| + 3|l|^3 + 2N_sN_h - 6l^2 N_s N_h + 4N_s^3 N_h^3\right) \\ &= \frac{2}{3}\frac{\mathcal{E}\mathcal{E}_k^{(K)}}{N_h^2 T_c}\left(\sum_{l=-L}^{L} \mathcal{R}_{wg}(lT_c)\right) + f(N_s, N_h)\end{aligned}$$

where identity (26) of lemma 1 is used and where

$$f(N_s, N_h) = \frac{1}{6}\frac{\mathcal{E}\mathcal{E}_k^{(K)}}{N_s^3 N_h^5 T_c} \sum_{l=-L}^{L} \mathcal{R}_{wg}(lT_c)\left(-3|l| + 3|l|^3 + 2N_sN_h - 6l^2 N_s N_h\right) \ .$$

Using assumption (5), the inequality $|\mathcal{R}_{wg}(lT_c)| \leq \mathcal{R}_{wg}(0)$, and the fact that the absolute value of a sum is less than or equal to the sum of absolute values, we get

$$\begin{aligned}|f(N_s, N_h)| &< \frac{1}{6}\frac{\mathcal{E}_{\sup}^2}{N_s^3 N_h^5 T_c}(2L+1)\mathcal{R}_{wg}(0)\left(3L + 3L^3 + 2N_sN_h + 6L^2 N_s N_h\right) \\ &\leq \frac{1}{2}\frac{\mathcal{E}_{\sup}^2}{N_s^3 N_h^5 T_c}\mathcal{R}_{wg}(0)L\left(3L^3 N_sN_h + 3L^3 N_sN_h + 2L^3 N_sN_h + 6L^3 N_s N_h\right) \\ &= \frac{C}{N_s^2 N_h^4}\end{aligned}$$

where $C = 7\mathcal{E}_{\sup}^2 \mathcal{R}_{wg}(0) L^4 / T_c$.

By performing the same kind of asymptotic derivations on the other terms of the right hand member of (25) (note that for the second term, identity (27) of lemma 1 is required instead of (26)), we obtain Equation (14).

D. Proof of Equation (25).

For clarity, we shall denote by $E_h[.]$, $E_\Delta[.]$ and $E_c[.]$ the expectations with respect to the distribution of the channel $\mathbf{h}_k$, the distribution of the delay $\Delta_k^{(K)}$, and the distribution of the codes $(c_{1,l}^{(K)}, c_{k,l}^{(K)})$ respectively.

Using Equation (11) and the fact that the information symbols $a_{k,m}^{(K)}$ are independent and have their values in $\{-1, 1\}$, we have

$$\begin{aligned}E\left[x_k^{(K)^2}\right] &= \frac{\mathcal{E}\mathcal{E}_k^{(K)}}{N_s^2} E_h\Big[E_\Delta\Big[E_c\Big[\sum_m \sum_{\substack{r_1,r_2=0\\ l_1,l_2=0}}^{N_s-1} \\ &\qquad r_{g_k^{(K)}g}\left((r_1 - r_2)N_hT_c + (c_{r_1} - c_{k,mN_s+r_2}^{(K)})T_c - mN_sN_hT_c - \Delta_k^{(K)}\right) \\ &\qquad r_{g_k^{(K)}g}\left((l_1 - l_2)N_hT_c + (c_{l_1} - c_{k,mN_s+l_2}^{(K)})T_c - mN_sN_hT_c - \Delta_k^{(K)}\right)\Big]\Big]\Big]\end{aligned} \quad (28)$$

We begin by deriving the expectation with respect to the codes. Using the independence of the random variables $\{c_{k,l}^{(K)}\}_{\substack{k=1,\ldots,K\\ l\in\mathbb{Z}}}$ and the fact that they are equally distributed over the set $\{0,\ldots,N_h-1\}$, we obtain

$$\begin{aligned}
E\left[{x_k^{(K)}}^2\right] &= \frac{\mathcal{E}\mathcal{E}_k^{(K)}}{N_s^2}E_h\Big[E_\Delta\Big[\\
&\frac{1}{N_h^4}\sum_{\substack{r_1,l_1=0\\ r_1\neq l_1}}^{N_s-1}\sum_{\substack{r_2,l_2=0\\ r_2\neq l_2}}^{N_s-1}\sum_{i_1,i_2,j_1,j_2=0}^{N_h-1}\sum_m r_{g_k^{(K)}g}\left((r_1-r_2)N_hT_c+(i_1-i_2)T_c-mN_sN_hT_c-\Delta_k^{(K)}\right)\\
&\qquad r_{g_k^{(K)}g}\left((l_1-l_2)N_hT_c+(j_1-j_2)T_c-mN_sN_hT_c-\Delta_k^{(K)}\right)\\
&+\frac{1}{N_h^3}\sum_{\substack{r_1,l_1=0\\ r_1\neq l_1}}^{N_s-1}\sum_{r_2=0}^{N_s-1}\sum_{i_1,i_2,j_1=0}^{N_h-1}\sum_m r_{g_k^{(K)}g}\left((r_1-r_2)N_hT_c+(i_1-i_2)T_c-mN_sN_hT_c-\Delta_k^{(K)}\right)\\
&\qquad r_{g_k^{(K)}g}\left((l_1-r_2)N_hT_c+(j_1-i_2)T_c-mN_sN_hT_c-\Delta_k^{(K)}\right)\\
&+\frac{1}{N_h^3}\sum_{r_1=0}^{N_s-1}\sum_{\substack{r_2,l_2=0\\ r_2\neq l_2}}^{N_s-1}\sum_{i_1,i_2,j_2=0}^{N_h-1}\sum_m r_{g_k^{(K)}g}\left((r_1-r_2)N_hT_c+(i_1-i_2)T_c-mN_sN_hT_c-\Delta_k^{(K)}\right)\\
&\qquad r_{g_k^{(K)}g}\left((r_1-l_2)N_hT_c+(i_1-j_2)T_c-mN_sN_hT_c-\Delta_k^{(K)}\right)\\
&+\frac{1}{N_h^2}\sum_{r_1,r_2=0}^{N_s-1}\sum_{i_1,i_2=0}^{N_h-1}\sum_m\left(r_{g_k^{(K)}g}\left((r_1-r_2)N_hT_c+(i_1-i_2)T_c-mN_sN_hT_c-\Delta_k^{(K)}\right)\right)^2\Big]
\end{aligned} \tag{29}$$

This expression can be simplified by developing the expectation $E_h[E_\Delta[.]]$. Indeed, let us prove that for any couple of integers (n_1,n_2), we have

$$E_h\Big[E_\Delta\Big[\sum_m r_{g_k^{(K)}g}\left(n_1T_c-mN_sN_hT_c-\Delta_k^{(K)}\right)r_{g_k^{(K)}g}\left(n_2T_c-mN_sN_hT_c-\Delta_k^{(K)}\right)\Big]\Big] = \frac{1}{N_sN_hT_c}\mathcal{R}_{wg}\left((n_2-n_1)T_c\right)\ . \tag{30}$$

Because Δ_k is uniformly distributed over the interval $[0,N_sN_hT_c)$, the left hand side member of (30), call it χ, can be written

$$\begin{aligned}
\chi &= \frac{1}{N_sN_hT_c}E_h\left[\int r_{g_k^{(K)}g}\left(n_1T_c-u\right)r_{g_k^{(K)}g}\left(n_2T_c-u\right)du\right]\\
&= \frac{1}{N_sN_hT_c}E_h\left[\int g_k(u_2)g(u_2-n_1T_c+u_1)g_k(u_3)g(u_3-n_2T_c+u_1)du_1du_2du_3\right]
\end{aligned}$$

where the integrals are taken over the whole real line. Let us now replace the function g_k by its expression (2) and compute the expectation. We obtain

$$\begin{aligned}\chi &= \frac{1}{N_s N_h T_c}\sum_{l_1,l_2=1}^{D}\int E\left[\gamma_{k,l_1}^{(K)}\gamma_{k,l_2}^{(K)} w\left(u_2-\tau_{k,l_1}^{(K)}\right) w\left(u_3-\tau_{k,l_2}^{(K)}\right)\right] \\ &\qquad g(u_2-n_1T_c+u_1)g(u_3-n_2T_c+u_1)du_1du_2du_3 \\ &= \frac{1}{N_s N_h T_c}\sum_{l=1}^{D}\int E\left[\gamma_{k,l}^{(K)2} w\left(u_2-\tau_{k,l}^{(K)}\right) w\left(u_3-\tau_{k,l}^{(K)}\right)\right. \\ &\qquad \left. g(u_2-n_1T_c+u_1)g(u_3-n_2T_c+u_1)du_1du_2du_3\right]\end{aligned}$$

where the second equality is due to assumption (3). By doing the change of variables $v_1 = u_1 + \tau_{k,l}^{(K)}$, $v_2 = u_2 - \tau_{k,l}^{(K)}$ and $v_3 = u_3 - \tau_{k,l}^{(K)}$, we notice that the integral in the right hand member of this equality does not depend on $\tau_{k,l}^{(K)}$, therefore

$$\begin{aligned}\chi &= \frac{1}{N_s N_h T_c}\sum_{l=1}^{D} E\left[\gamma_{k,l}^{(K)2}\right]\int w(v_2)\,w(v_3)\,g(v_2-n_1T_c+v_1)g(v_3-n_2T_c+v_1)dv_1dv_2dv_3 \\ &= \frac{1}{N_s N_h T_c}\int w(v_2)\,w(v_3)\,g(v_2-n_1T_c+v_1)g(v_3-n_2T_c+v_1)dv_1dv_2dv_3\end{aligned}$$

by using the normalization (4). Since the integral in this last Equation is equal to $\mathcal{R}_{wg}\left((n_2-n_1)T_c\right)$, Equation (30) is proved.

Let us get back to Equation (29). By plugging Equation (30) into (29), we obtain

$$\begin{aligned}E\left[x_k^{(K)2}\right] &= \frac{\mathcal{E}\mathcal{E}_k^{(K)}}{N_s^3 N_h T_c}\Bigg(\\ &\frac{1}{N_h^4}\sum_{\substack{r_1,l_1=0\\ r_1\neq l_1}}^{N_s-1}\sum_{\substack{r_2,l_2=0\\ r_2\neq l_2}}^{N_s-1}\sum_{i_1,i_2,j_1,j_2=0}^{N_h-1}\mathcal{R}_{wg}\left((l_1-r_1-l_2+r_2)N_hT_c+(j_1-i_1-j_2+i_2)T_c\right) \\ &+ 2\frac{N_s}{N_h^2}\sum_{\substack{r_1,l_1=0\\ r_1\neq l_1}}^{N_s-1}\sum_{i_1,j_1=0}^{N_h-1}\mathcal{R}_{wg}\left((l_1-r_1)N_hT_c+(j_1-i_1)T_c\right) \\ &+ N_s^2\mathcal{R}_{wg}(0)\Bigg)\end{aligned} \tag{31}$$

Let us develop the first of the three terms of the right hand member of this Equation. Calling ϕ this term, and using the equality

$$\sum_{\substack{r_1,l_1=0\\ r_1\neq l_1}}^{N_s-1}\sum_{\substack{r_2,l_2=0\\ r_2\neq l_2}}^{N_s-1} = \sum_{r_1,r_2,l_1,l_2=0}^{N_s-1} - \sum_{\substack{r_1,r_2,l_1,l_2=0\\ r_1=l_1}}^{N_s-1} - \sum_{\substack{r_1,r_2,l_1,l_2=0\\ r_2=l_2}}^{N_s-1} + \sum_{\substack{r_1,r_2,l_1,l_2=0\\ r_1=l_1,r_2=l_2}}^{N_s-1}$$

we have

$$\begin{aligned}
\phi \;=\;& \frac{\mathcal{E}\mathcal{E}_k^{(K)}}{N_s^3 N_h^5 T_c}\Big(\sum_{r_1,l_1,r_2,l_2=0}^{N_s-1} \sum_{i_1,i_2,j_1,j_2=0}^{N_h-1} \mathcal{R}_{wg}\left((l_1-r_1-l_2+r_2)N_hT_c + (j_1-i_1-j_2+i_2)T_c\right) \\
& - 2N_s \sum_{r_1,l_1=0}^{N_s-1} \sum_{i_1,i_2,j_1,j_2=0}^{N_h-1} \mathcal{R}_{wg}\left((l_1-r_1)N_hT_c + (j_1-i_1-j_2+i_2)T_c\right) \\
& + N_s^2 \sum_{i_1,i_2,j_1,j_2=0}^{N_h-1} \mathcal{R}_{wg}\left((j_1-i_1-j_2+i_2)T_c\right)\Big) \\
=\;& \frac{\mathcal{E}\mathcal{E}_k^{(K)}}{N_s^3 N_h^5 T_c}\Big(\sum_{i_1,i_2,i_3,i_4=0}^{N_sN_h-1} \mathcal{R}_{wg}\left((i_1-i_2+i_3-i_4)T_c\right) \\
& - 2N_s \sum_{i_1,i_2=0}^{N_sN_h-1} \sum_{j_1,j_2=0}^{N_h-1} \mathcal{R}_{wg}\left((i_1-i_2+j_1-j_2)T_c\right) \\
& + N_s^2 \sum_{j_1,j_2,j_3,j_4=0}^{N_h-1} \mathcal{R}_{wg}\left((j_1-j_2+j_3-j_4)T_c\right)\Big) \; .
\end{aligned}$$

We thus obtain the first three terms of the right hand side member of (25). The following two terms in this Equation are obtained by developing in a similar manner the second term in the right hand member of (31).

E. Proof of lemma 1

We only sketch the proof of (27). The proof of (26) is similar. Assume $I \geq 0$. Then, for any collection of integers $\{i_1, i_2, j_1, j_2\}$ such that $\{i_1, i_2\} \subset \{0, \ldots, Q-1\}$ and and $\{j_1, j_2\} \subset \{0, \ldots, M-1\}$, one can write

$$\delta(i_1 - i_2 + j_1 - j_2 - I) = \sum_{m=I}^{Q+M-2} \delta(i_1 + j_1 - m)\ \delta(i_2 + j_2 + I - m) \; .$$

Similarly, if $I < 0$, then we have

$$\delta(i_1 - i_2 + j_1 - j_2 - I) = \sum_{m=|I|}^{Q+M-2} \delta(i_1 + j_1 + |I| - m)\ \delta(i_2 + j_2 - m) \; .$$

By consequence, the left hand member $\mathcal{I} = \sum_{i_1,i_2=0}^{Q-1} \sum_{j_1,j_2=0}^{M-1} \delta(i_1 - i_2 + j_1 - j_2 - I)$ of Equation (27) can be written $\mathcal{I} = \sum_{m=|I|}^{Q+M-2} \mathcal{I}_1(m)\mathcal{I}_2(m)$ where $\mathcal{I}_1(m) = \sum_{i=0}^{Q-1} \sum_{j=0}^{M-1} \delta(i + j - m)$, and $\mathcal{I}_2(m) = \sum_{i=0}^{Q-1} \sum_{j=0}^{M-1} \delta(i + j + |I| - m)$. We have

$$\mathcal{I}_1(m) = \begin{cases} m+1 & \text{if} \quad |I| \leq m < M \\ M & \text{if} \quad M \leq m < Q \\ Q+M-1-m & \text{if} \quad Q \leq m < Q+M-1 \end{cases}$$

and

$$\mathcal{I}_2(m) = \begin{cases} m - |I| + 1 & \text{if} \quad |I| \le m < M + |I| \\ M & \text{if} \quad M + |I| \le m < Q + |I| \\ M - m + Q - 1 + |I| & \text{if} \quad Q + |I| \le m < Q + M - 1 \end{cases} .$$

By consequence, the sum $\mathcal{I}$ is given by

$$\begin{aligned} \mathcal{I} &= \sum_{m=|I|}^{M-1} (m+1)(m-|I|+1) + \sum_{m=M}^{M+|I|-1} M(m-|I|+1) + \sum_{m=M+|I|}^{Q-1} M^2 \\ &+ \sum_{m=Q}^{Q+|I|-1} M(Q+M-1-m) + \sum_{m=Q+|I|}^{Q+M-1} (Q+M-1-m)(M-m+Q-1+|I|) \ . \end{aligned}$$

The result is obtained by developing this expression and by using the identities $\sum_{k=1}^{n} k = n(n+1)/2$ and $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$.

F. Proof of Proposition 5.

We begin by deriving the expression of $\sigma_k^{(K)}(\mathbf{c})^2 = E\left[{x_k^{(K)}}^2 \middle| \mathbf{c} \right]$. After some derivations similar to those of Appendix D, we have $\sigma_k^{(K)}(\mathbf{c})^2 = \frac{\mathcal{E}\mathcal{E}_k^{(K)}}{T_c} \frac{1}{N} (\phi_1 - \phi_2 + \phi_3)$ where

$$\begin{aligned} \phi_1 &= \frac{1}{N_s^2 N_h^2} \sum_{j_1,j_2,j_3,j_4=0}^{N_s-1} \sum_{i_1,i_2=0}^{N_h-1} \mathcal{R}_{wg}\left((j_1 - j_2 + j_3 - j_4) N_h T_c + (c_{j_1} - c_{j_2} + i_1 - i_2) T_c \right) \\ \phi_2 &= \frac{1}{N_s N_h^2} \sum_{j_1,j_2=0}^{N_s-1} \sum_{i_1,i_2=0}^{N_h-1} \mathcal{R}_{wg}\left((j_1 - j_2) N_h T_c + (c_{j_1} - c_{j_2} + i_1 - i_2) T_c \right) \\ \phi_3 &= \frac{1}{N_s} \sum_{j_1,j_2=0}^{N_s-1} \mathcal{R}_{wg}\left((j_1 - j_2) N_h T_c + (c_{j_1} - c_{j_2}) T_c \right) \ . \end{aligned}$$

It results that ${\sigma^{(K)}}^2 = \frac{\mathcal{E}\overline{\mathcal{E}}^{(K)}}{T_c} \frac{K-1}{N} (\phi_1 - \phi_2 + \phi_3)$. We shall prove that $\phi_1 - \frac{2}{3}\frac{N_s}{N_h} \sum_{l=-L}^{L} \mathcal{R}_{wg}(lT_c)$ and ϕ_2 converge both to zero for every choice of $\mathbf{c}$, and that $\phi_3 - \mathcal{R}_{wg}(0)$ converges to zero in probability. Because $\sigma^{(K)}(\mathbf{c})^2 = \frac{\mathcal{E}\overline{\mathcal{E}}^{(K)}}{T_c} \frac{K-1}{N} (\phi_1 - \phi_2 + \phi_3)$, this will prove our proposition.

Due to the fact that $\mathcal{R}_{wg}(t) = 0$ if $|t| \ge (L+1)T_c$, we have

$$\begin{aligned} \phi_1 - \frac{2}{3}\frac{N_s}{N_h} \sum_{l=-L}^{L} \mathcal{R}_{wg}(lT_c) = \sum_{l=-L}^{L} \mathcal{R}_{wg}(lT_c) \Bigg(\Bigg(\frac{1}{N_s^2 N_h^2} \sum_{j_1,j_2,j_3,j_4=0}^{N_s-1} \sum_{i_1,i_2=0}^{N_h-1} \\ \delta\left((j_1 - j_2 + j_3 - j_4) N_h + c_{j_1} - c_{j_2} + i_1 - i_2 - l \right) \Bigg) - \frac{2}{3}\frac{N_s}{N_h} \Bigg) \end{aligned} \quad (32)$$

We have

$$\delta\left((j_1 - j_2 + j_3 - j_4)\, N_h + c_{j_1} - c_{j_2} + i_1 - i_2 - l\right) = \sum_{k=-2}^{2} \delta\left(j_1 - j_2 + j_3 - j_4 + k\right) \delta\left(c_{j_1} - c_{j_2} + i_1 - i_2 - l - kN_h\right)$$

where all the values of the summand of the right hand member are zero for $|k| > 2$ because $-2N_h - L + 2 \leq c_{j_1} - c_{j_2} + i_1 - i_2 - l \leq 2N_h + L - 2$, and $L \ll N_h$. The term χ between the inner parentheses in Equation (32) now writes

$$\begin{aligned}
\chi &= \frac{1}{N_s^2 N_h^2} \sum_{j_1,j_2,j_3,j_4=0}^{N_s-1} \sum_{k=-2}^{2} \delta\left(j_1 - j_2 + j_3 - j_4 + k\right) \sum_{i_1,i_2=0}^{N_h-1} \delta\left(c_{j_1} - c_{j_2} + i_1 - i_2 - l - kN_h\right) \\
&= \frac{1}{N_s^2 N_h^2} \sum_{j_1,j_2,j_3,j_4=0}^{N_s-1} \sum_{k=-2}^{2} \max\left(0, N_h - |c_{j_1} - c_{j_2} - l - kN_h|\right) \delta\left(j_1 - j_2 + j_3 - j_4 + k\right) \\
&= \frac{1}{N_s^2 N_h^2} \sum_{j_1,j_2,j_3=0}^{N_s-1} \sum_{k=-2}^{2} \sum_{j_4=-k}^{N_s-1-k} \max\left(0, N_h - |c_{j_1} - c_{j_2} - l - kN_h|\right) \delta\left(j_1 - j_2 + j_3 - j_4\right) \\
&\geq \frac{1}{N_s^2 N_h^2} \sum_{j_1,j_2,j_3,j_4=2}^{N_s-3} \delta\left(j_1 - j_2 + j_3 - j_4\right) \sum_{k=-2}^{2} \max\left(0, N_h - |c_{j_1} - c_{j_2} - l - kN_h|\right)
\end{aligned}$$

For any values of c_{j_1} and c_{j_2}, we have $\sum_{k=-2}^{2} \max\left(0, N_h - |c_{j_1} - c_{j_2} - l - kN_h|\right) = N_h$. Therefore, the last term is equal to

$$\frac{1}{N_s^2 N_h} \sum_{j_1,j_2,j_3,j_4=2}^{N_s-3} \delta\left(j_1 - j_2 + j_3 - j_4\right) \; .$$

Using Equation (26) of Lemma 1, we can show along the lines of Appendix C that

$$\frac{1}{N_s^2 N_h} \sum_{j_1,j_2,j_3,j_4=2}^{N_s-3} \delta\left(j_1 - j_2 + j_3 - j_4\right) - \frac{2}{3}\frac{N_s}{N_h} \to 0$$

when $N_s/N_h \to \rho \geq 0$. Getting back to the expression of χ we also have

$$\chi - \frac{2}{3}\frac{N_s}{N_h} \leq \frac{1}{N_s^2 N_h^2} \sum_{j_1,j_2,j_3=0}^{N_s-1} \sum_{j_4=-2}^{N_s+1} \delta\left(j_1 - j_2 + j_3 - j_4\right) \sum_{k=-2}^{2} \max\left(0, N_h - |c_{j_1} - c_{j_2} - l - kN_h|\right) - \frac{2}{3}\frac{N_s}{N_h}$$

By the same argument, the right hand member converges to zero. It results that $\phi_1 - \frac{2}{3}\frac{N_s}{N_h}\sum_{l=-L}^{L}\mathcal{R}_{wg}(lT_c) \to 0$ for every value of $\mathbf{c}$. It can be shown in a similar manner that $\phi_2 \to 0$. Now, we have

$$\begin{aligned} E\left[\left|\phi_3 - \mathcal{R}_{wg}(0)\right|\right] &= \frac{1}{N_s}E\left[\left|\sum_{\substack{j_1,j_2=0\\ j_1\neq j_2}}^{N_s-1}\mathcal{R}_{wg}\left(\left(j_1-j_2\right)N_hT_c + \left(c_{j_1}-c_{j_2}\right)T_c\right)\right|\right] \\ &\leq \frac{1}{N_s}\sum_{\substack{j_1,j_2=0\\ j_1\neq j_2}}^{N_s-1}E\left[\left|\mathcal{R}_{wg}\left(\left(j_1-j_2\right)N_hT_c + \left(c_{j_1}-c_{j_2}\right)T_c\right)\right|\right] \\ &= \frac{1}{N_sN_h^2}\sum_{\substack{j_1,j_2=0\\ j_1\neq j_2}}^{N_s-1}\sum_{i_1,i_2=0}^{N_h-1}\left|\mathcal{R}_{wg}\left(\left(j_1-j_2\right)N_hT_c + \left(i_1-i_2\right)T_c\right)\right| \\ &< \frac{1}{N_sN_h^2}\sum_{i_1,i_2=0}^{N_sN_h-1}\left|\mathcal{R}_{wg}\left(\left(i_1-i_2\right)T_c\right)\right| = \mathcal{O}(1/N_h)\ . \end{aligned}$$

Therefore, $\phi_3 - \mathcal{R}_{wg}(0) \to 0$ in probability by Markov's inequality as in Appendix A.

G. Proof of (18).

We have $\sigma_k^{(K)}(\mathbf{c})^2 = \frac{\mathcal{E}\mathcal{E}_k^{(K)}}{T_c}\frac{1}{N}\left(\phi_1 - \phi_2 + \phi_3\right)$ where ϕ_1, ϕ_2 and ϕ_3 are given in Appendix F and do not depend on the user k. In that Appendix, it can be seen that ϕ_1 is bounded and $\phi_2 \to 0$. Moreover, it is not difficult to prove that ϕ_3 is bounded. Using the boundedness assumption (5) on $\mathcal{E}_k^{(K)}$, we obtain the result.

H. Proof of Proposition 6.

It is shown here that Lindeberg's condition (16) is not satisfied. For this purpose, we shall begin by building a random variable $x_k^{(K)\prime}$ such that $\left|x_k^{(K)\prime}\right| \leq \left|x_k^{(K)}\right|$. Due to this inequality,

$$|x_k^{(K)\prime}|\mathbf{1}_{|x_k^{(K)\prime}|\geq\varepsilon} \leq |x_k^{(K)}|\mathbf{1}_{|x_k^{(K)}|\geq\varepsilon}$$

for every $\varepsilon > 0$, therefore, it will be enough to establish the non validity of Lindeberg's condition over $x_k^{(K)\prime}$ to prove the proposition.

The random variables $x_k^{(K)\prime}$ will be built in such a way that $x_k^{(K)\prime} \neq 0$ on a certain subset of the probability space where the pulse carried by $c_{k,-1}^{(K)}$ in the signal $y_k^{(K)}(t)$ and the pulse carried by c_0 in the matched filter response $\sqrt{\mathcal{E}/N_s}\sum_{r=0}^{N_s-1} g(t - rN_hT_c - c_rT_c)$ are the only pulses which overlap. Moreover, on this subset, $x_k^{(K)'} = x_k^{(K)}$. Specifically, let $\zeta_k^{(K)}$ be the random variable defined as

$$\zeta_k^{(K)} = 1 \text{ if } \begin{cases} \Delta_k^{(K)} \in [0, \lfloor N_h/3 \rfloor T_c), \\ c_{k,-1}^{(K)} \in \{\lceil 2N_h/3 \rceil, \ldots, N_h - 1\}, \\ c_{k,r}^{(K)} \in \{\lceil N_h/3 \rceil, \ldots, \lfloor 2N_h/3 \rfloor - L - 1\} \text{ for } r = 0, \ldots, N_s - 1, \\ c_r \in \{0, \ldots, \lfloor N_h/3 \rfloor - L - 1\} \text{ for } r = 0, \ldots, N_s - 1 \end{cases}$$

and $\zeta_k^{(K)} = 0$ elsewhere on the probability space. The notations $\lfloor x \rfloor$ (respectively $\lceil x \rceil$) stand for x rounded down (respectively rounded up) to the nearest integer. We put $x_k^{(K)'} = x_k^{(K)} \zeta_k^{(K)}$. It can then be checked that

$$x_k^{(K)'} = \frac{\sqrt{\mathcal{E}\mathcal{E}_k^{(K)}}}{N_s} a_{k,-1}^{(K)} r_{g_k^{(K)} g} \left(N_h T_c + (c_0 - c_{k,-1}^{(K)}) T_c - \Delta_k^{(K)} \right)$$

if $c_0, \ldots, c_{N_s-1}, c_{k,-1}^{(K)}, \ldots, c_{k,N_s-1}^{(K)}$ and $\Delta_k^{(K)}$ satisfy the conditions above, otherwise $x_k^{(K)'} = 0$. We have

$$\begin{aligned} E\left[{x_k^{(K)'}}^2 \mathbf{1}_{|x_k^{(K)'}| \geq \varepsilon} \right] &\geq \varepsilon^2 E\left[\mathbf{1}_{|x_k^{(K)'}| \geq \varepsilon} \right] \\ &= \varepsilon^2 \frac{1}{N_s N_h^3 T_c} \eta_1 \eta_2 \sum_{i_1=0}^{\lfloor \frac{N_h}{3} \rfloor - L - 1} \sum_{i_2 = \lceil 2N_h/3 \rceil}^{N_h - 1} \int_0^{\lfloor \frac{N_h}{3} \rfloor T_c} \mathbb{P}\left(\frac{\sqrt{\mathcal{E}\mathcal{E}_k^{(K)}}}{N_s} |r_{g_k^{(K)} g}((N_h + i_1 - i_2) T_c - t)| \geq \varepsilon \right) dt \\ &= \frac{\varepsilon^2}{N_s N_h^3 T_c} \eta_1 \eta_2 \sum_{i_1=0}^{\lfloor \frac{N_h}{3} \rfloor - L - 1} \sum_{i_2 = \lceil 2N_h/3 \rceil}^{N_h - 1} \sum_{i_3=0}^{\lfloor \frac{N_h}{3} \rfloor - 1} \int_0^{T_c} \mathbb{P}\left(\frac{\sqrt{\mathcal{E}\mathcal{E}_k^{(K)}}}{N_s} |r_{g_k^{(K)} g}((N_h + i_1 - i_2 - i_3) T_c - t)| \geq \varepsilon \right) dt \\ &= \frac{\varepsilon^2}{T_c} \eta_1 \eta_2 \sum_{l=-L+1}^{L} p(l) \int_0^{T_c} \mathbb{P}\left(\frac{\sqrt{\mathcal{E}\mathcal{E}_k^{(K)}}}{N_s} |r_{g_k^{(K)} g}(l T_c - t)| \geq \varepsilon \right) dt \end{aligned}$$

where in these Equations, $\eta_1 = \left(\frac{(\lfloor 2N_h/3 \rfloor - L - \lceil N_h/3 \rceil)}{N_h} \right)^{N_s}$, $\eta_2 = \left(\frac{(\lfloor N_h/3 \rfloor - L)}{N_h} \right)^{N_s - 1}$ and

$$p(l) = \frac{1}{N_s N_h^3} \sum_{i_1=0}^{\lfloor \frac{N_h}{3} \rfloor - L - 1} \sum_{i_2 = \lceil 2N_h/3 \rceil}^{N_h - 1} \sum_{i_3=0}^{\lfloor \frac{N_h}{3} \rfloor - 1} \delta\left(N_h + i_1 - i_2 - i_3 - l \right) \ .$$

Here, η_1 is due to the expectation with respect to the random variables $c_{k,r}^{(K)}$, η_2 is due to the expectation on c_r for $r \neq 0$, and the factor $1/(N_s N_h^3 T_c)$ results from the expectation over $\Delta_k^{(K)}$, c_0 and $c_{k,-1}^{(K)}$. For N_h large enough, as L is constant, we have $\eta_1 > (\frac{1}{4}))^{N_s}$ and $\eta_2 > \left(\frac{1}{4}\right)^{N_s - 1}$. Furthermore, one can show by a technique similar to that of the proof of lemma 1 that that $p(l) \geq C_3 / N_h$ where C_3 does not depend on l nor on N_h. By consequence,

$$\begin{aligned}\sum_{k=2}^{K} E\left[{x_k^{(K)}}'^2 \mathbf{1}_{|x_k^{(K)\prime}|\geq\varepsilon} \right] &\geq C_3\varepsilon^2 \left(\frac{1}{4}\right)^{2N_s-1} \frac{1}{N_h}\sum_{k=2}^{K}\sum_{l=-L+1}^{L}\int_0^{T_c} \mathbb{P}\left(\frac{\sqrt{\mathcal{E}\mathcal{E}_k^{(K)}}}{N_s}|r_{g_k^{(K)}g}(lT_c-t)|\geq\varepsilon\right)dt \\ &= C_3\varepsilon^2 \left(\frac{1}{4}\right)^{2N_s-1} \frac{1}{N_h}\sum_{k=2}^{K}\int \mathbb{P}\left(\frac{\sqrt{\mathcal{E}\mathcal{E}_k^{(K)}}}{N_s}|r_{g_k^{(K)}g}(t)|\geq\varepsilon\right)dt\end{aligned}$$

which does not converge to 0 at least for some $\varepsilon > 0$ because the integral in the right hand member is of the order $\mathcal{O}(1)$ and because N_s is constant and $K/N_h \to \alpha.N_s > 0$.

I. *Proof of Proposition 7.*

We begin with the following lemma :

Lemma 2: Let

$$S_k^{(K)} = \frac{1}{N_s}\sum_m \sum_{r_1,r_2=0}^{N_s-1} \mathbf{1}_{[-LT_c,LT_c)}\left((r_1-r_2)N_hT_c + (c_{r_1} - c_{k,mN_s+r_2}^{(K)})T_c - mN_sN_hT_c - \Delta_k^{(K)}\right) . \tag{33}$$

Then in the asymptotic regime as $N_h\to\infty$, $N_s\to\infty$, and $N_s/N_h\to\rho>0$, we have $E\left[{S_k^{(K)}}^2\right]\leq C_4/N$ and $E\left[{S_k^{(K)}}^3\right]\leq C_5/N^{3/2}$ where C_4 and C_5 are independent of k and of K.

Proof: Let us write $\Delta_k^{(K)} = N_hT_cn_k^{(K)} + T_ct_k^{(K)} + q_k^{(K)}$ where $q_k^{(K)}$ has its range in the interval $[0,T_c)$ and $t_k^{(K)}$ is discrete with $0\leq t_k^{(K)} < N_h$. We can write $S_k^{(K)} = \sum_{l=-L}^{L-1} S_{k,l}^{(K)}$ where

$$S_{k,l}^{(K)} = \frac{1}{N_s}\sum_m \sum_{r_1,r_2=0}^{N_s-1} \delta\left((r_1-r_2)N_h + (c_{r_1} - c_{k,mN_s+r_2}^{(K)}) - mN_sN_h - N_hn_k^{(K)} - t_k^{(K)} - l\right) . \tag{34}$$

Thanks to Minkowski's inequality [12, page 82] which writes here $\left(E\left[{S_k^{(K)}}^p\right]\right)^{1/p}\leq\sum_{l=-L}^{L}\left(E\left[{S_{k,l}^{(K)}}^p\right]\right)^{1/p}$ for every integer $p>0$, it is enough to prove the results for the random variables $\{S_{k,l}^{(K)}\}_{l=-L,\ldots,L-1}$. For this, we shall prove that in the asymptotic regime,

$$E\left[{S_{k,l}^{(K)}}^2 \mid n_k=n, t_k=t\right]\leq C_4'/N \text{ and } E\left[{S_{k,l}^{(K)}}^3 \mid n_k=n, t_k=t\right]\leq C_5'/N^{3/2}$$

where the constant C_4' and C_5' do not depend on n, t, k and K. The random variable ${S_{k,l}^{(K)}}'$ obtained after replacing $t_k^{(K)}$ by t and $n_k^{(K)}$ by n in (34) can be written after some simple manipulations

$$\begin{aligned}{S_{k,l}^{(K)}}' &= \frac{1}{N_s}\sum_{r=0}^{N_s-1}\sum_{i=-\infty}^{\infty}\delta\left(iN_h + c_r - c_{k,r-i-n}^{(K)} - (t+l)\right) \\ &= \frac{1}{N_s}\sum_{r=0}^{N_s-1} Z_{k,l,r}^{(K)}\end{aligned} \tag{35}$$

where $Z_{k,l,r}^{(K)} = \sum_{i=-\infty}^{\infty} \delta\left(iN_h + c_r - c_{k,r-i-n}^{(K)} - (t+l)\right)$. By assuming w.l.o.g. that $t + l < N_h$, and by noting that the range of any of the random variables $\{c_{k,r}^{(K)}\}$ is $\{0, \ldots, N_h - 1\}$, it can be seen that $Z_{k,l,r}^{(K)}$ writes as $Z_{k,l,r}^{(K)} = \delta\left(c_r - c_{k,r-n}^{(K)} - (t+l)\right) + \delta\left(N_h + c_r - c_{k,r-1-n}^{(K)} - (t+l)\right)$. This is a Bernoulli random variable (having its values in $\{0,1\}$) with $E\left[Z_{k,l,r}^{(K)}\right] = 1/N_h$. Further, it can be shown after some computations that $E\left[Z_{k,l,r_1}^{(K)} Z_{k,l,r_2}^{(K)}\right] \leq 1/N_h^2$ for $r_1 \neq r_2$ and that $E\left[Z_{k,l,r_1}^{(K)} Z_{k,l,r_2}^{(K)} Z_{k,l,r_3}^{(K)}\right] \leq 1/N_h^3$ for $r_1 \neq r_2, r_2 \neq r_3$, and $r_1 \neq r_3$. By (35) and these observations, it can be established that $E\left[{S_{k,l}^{(K)}}'^2\right] < \frac{1}{N_s^2}\left(\frac{N_s^2}{N_h^2} + \frac{N_s}{N_h}\right)$ and $E\left[{S_{k,l}^{(K)}}'^3\right] < \frac{1}{N_s^3}\left(\frac{N_s^3}{N_h^3} + 3\frac{N_s^2}{N_h^2} + \frac{N_s}{N_h}\right)$, hence the results. ■

Proof of Proposition 7 :

We can now check the validity of the Lindeberg's condition (16). Let $R_k^{(K)}$ be the random variable $R_k^{(K)} = \max_t \left(|r_{g_k^{(K)}g}(t)|\right)$. We have

$$\begin{aligned}
|x_k^{(K)}| &\leq \frac{\mathcal{E}_{\text{sup}}}{N_s} \sum_m \sum_{r_1,r_2=0}^{N_s-1} \left| r_{g_k^{(K)}g}\left((r_1 - r_2)N_h T_c + (c_{r_1} - c_{k,mN_s+r_2}^{(K)})T_c - mN_s N_h T_c - \Delta_k^{(K)}\right)\right| \\
&\leq \mathcal{E}_{\text{sup}} R_k^{(K)} S_k^{(K)}
\end{aligned}$$

where $S_k^{(K)}$ is defined in (33) and represents the number of summands in the sum. As a consequence, $\mathbf{1}_{|x_k^{(K)}| \geq \varepsilon} \leq \mathbf{1}_{\mathcal{E}_{\text{sup}} R_k^{(K)} S_k^{(K)} \geq \varepsilon}$, and then

$$E\left[{x_k^{(K)}}^2 \mathbf{1}_{|x_k^{(K)}| \geq \varepsilon}\right] \leq \mathcal{E}_{\text{sup}}^2 E\left[{R_k^{(K)}}^2 {S_k^{(K)}}^2 \mathbf{1}_{\mathcal{E}_{\text{sup}} R_k^{(K)} S_k^{(K)} \geq \varepsilon}\right] = \chi_{k,A}^{(K,1)} + \chi_{k,A}^{(K,2)}$$

where

$$\begin{aligned}
\chi_{k,A}^{(K,1)} &= \mathcal{E}_{\text{sup}}^2 E\left[{R_k^{(K)}}^2 {S_k^{(K)}}^2 \mathbf{1}_{\mathcal{E}_{\text{sup}} R_k^{(K)} S_k^{(K)} \geq \varepsilon} \mathbf{1}_{R_k^{(K)} > A}\right] , \\
\chi_{k,A}^{(2)} &= \mathcal{E}_{\text{sup}}^2 E\left[{R_k^{(K)}}^2 {S_k^{(K)}}^2 \mathbf{1}_{\mathcal{E}_{\text{sup}} R_k^{(K)} S_k^{(K)} \geq \varepsilon} \mathbf{1}_{R_k^{(K)} \leq A}\right] ,
\end{aligned}$$

and A is a given constant.

We shall begin by showing that the term $\sum_{k=2}^{K} \chi_{k,A}^{(K,1)}$ can be made as small as possible by increasing A. We have

$$\begin{aligned}
\chi_{k,A}^{(K,1)} &\leq \mathcal{E}_{\text{sup}}^2 E\left[{R_k^{(K)}}^2 {S_k^{(K)}}^2 \mathbf{1}_{R_k^{(K)} > A}\right] \\
&= \mathcal{E}_{\text{sup}}^2 E\left[{R_k^{(K)}}^2 \mathbf{1}_{R_k^{(K)} > A}\right] E\left[{S_k^{(K)}}^2\right] \quad (36) \\
&\leq \mathcal{E}_{\text{sup}}^2 \frac{C_4}{N} E\left[{R_k^{(K)}}^2 \mathbf{1}_{R_k^{(K)} > A}\right] \quad (37)
\end{aligned}$$

where the equality (36) is due to the obvious fact that $R_k^{(K)}$ and $S_k^{(K)}$ are independent, and inequality (37) is deduced from lemma 2. Now, for every real number t, we have $r_{g_k^{(K)}g}(t) = \sum_{l=1}^{D} \gamma_{k,l}^{(K)} r_{wg}(t - \tau_{k,l}^{(K)})$. By

Cauchy-Schwarz inequality, $r_{g_k^{(K)}g}(t)^2 \leq \|\boldsymbol{\gamma}_k^{(K)}\|^2 \sum_{l=1}^{D} r_{wg}(t-\tau_{k,l}^{(K)})^2$. Let $M_{\mathbf{g}} = \int w(t)^2 dt \int g(t)^2 dt$. Also by Cauchy-Schwarz inequality, $r_{wg}(t)^2 \leq M_{\mathbf{g}}$. Therefore, for every value of t, we have $r_{g_k^{(K)}g}(t)^2 \leq D\|\boldsymbol{\gamma}_k^{(K)}\|^2 M_{\mathbf{g}}$. By consequence, the uniform integrability of ${R_k^{(K)}}^2$, *i.e.*,

$$\lim_{A\to\infty} \sup_K \max_{k=1,\ldots,K} E\left[{R_k^{(K)}}^2 \mathbf{1}_{R_k^{(K)}\geq A}\right] = 0$$

results from the assumption (19). Therefore, if we fix $\varepsilon' > 0$, there is a value A_0 for which

$$\sum_{k=2}^{K} \chi_{k,A}^{(K,1)} < \varepsilon' \tag{38}$$

for $A > A_0$, which we assume in the sequel.

We turn now to the study of $\chi_{k,A}^{(K,2)}$. From $\mathbf{1}_{|\mathcal{E}_{\sup} R_k^{(K)} S_k^{(K)}|\geq\varepsilon} \mathbf{1}_{R_k^{(K)}\leq A} \;\leq\; \mathbf{1}_{|S_k^{(K)}|\geq \frac{\varepsilon}{\mathcal{E}_{\sup} A}} \mathbf{1}_{R_k^{(K)}\leq A}$, we have

$$\begin{aligned} \chi_{k,A}^{(K,2)} &\leq \mathcal{E}_{\sup}^2 E\left[{R_k^{(K)}}^2 {S_k^{(K)}}^2 \mathbf{1}_{|S_k^{(K)}|\geq \frac{\varepsilon}{\mathcal{E}_{\sup} A}} \mathbf{1}_{R_k^{(K)}\leq A}\right] \\ &\leq \mathcal{E}_{\sup}^2 E\left[{R_k^{(K)}}^2\right] E\left[{S_k^{(K)}}^2 \mathbf{1}_{|S_k^{(K)}|\geq \frac{\varepsilon}{\mathcal{E}_{\sup} A}}\right] . \end{aligned}$$

From the inequality $r_{g_k^{(K)}g}(t)^2 \leq D\|\boldsymbol{\gamma}_k^{(K)}\|^2 M_{\mathbf{g}}$ proved above and (4), it results that $E\left[{R_k^{(K)}}^2\right] \leq DM_{\mathbf{g}}$. We therefore have to establish the fact that $\lim_{K\to\infty} \sum_{k=2}^{K} E\left[{S_k^{(K)}}^2 \mathbf{1}_{|S_k^{(K)}|\geq \frac{\varepsilon}{\mathcal{E}_{\sup} A}}\right] = 0$, which is Lindeberg's condition on $\{S_k^{(K)}\}$. For this, it will be enough to establish Lyapounov's condition

$$\lim_{K\to\infty} \sum_{k=2}^{K} E\left[|S_k^{(K)}|^{2+\eta}\right] = 0 \tag{39}$$

for some $\eta > 0$ ([12, theorem 27.3]). Choosing $\eta = 1$, we have $E\left[{S_k^{(K)}}^3\right] \leq C_5/N^{3/2}$ by lemma 2, hence (39).

In short, for every $\varepsilon' > 0$, $\sum_{k=2}^{K} E\left[{x_k^{(K)}}^2 \mathbf{1}_{|x_k^{(K)}|\geq\varepsilon}\right]$ is bounded above by the sum of a term less than ε', see (38), and a term that converges to 0, therefore, it converges to zero. The variance of the Gaussian limit distribution is the variance σ_{MUI}^2 given by Proposition 5.

REFERENCES

[1] R.A. Scholtz, "Multiple Access with Time-Hopping Impulse Modulation," in *Military Communications Conference*, Bedford, MA, USA, Oct. 1993.

[2] Federal Communication Commision, "Revision of Part 15 of the Commision's Rules Regarding Ultra-Wideband Transmission Systems, First Report and Order," Tech. Rep., FCC, Feb. 2002.

[3] M.Z. Win and R.A. Scholtz, "Ultra-Wide Bandwidth Time-Hopping Spread-Spectrum Impulse Radio for Wireless Multiple-Access Communications," *IEEE Trans. on Communications*, vol. 48, no. 4, pp. 679–691, Oct. 2000.

[4] G. Durisi, J. Romme, and S. Benedetto, “Performance of TH and DS UWB Multiaccess Systems in Presence of Multipath Channel and Narrowband interference,” in *Proc. of the IWUWBS*, Oulu, Finland, June 2003.

[5] F. Ramírez-Mireles, “On the Performance of Ultra-Wide-Band Signals in Gaussian Noise and Dense Multipath,” *IEEE Trans. on Vehicular Technology*, vol. 50, no. 1, pp. 244–249, Jan. 2001.

[6] J.D. Choi and W.E. Stark, “Performance of Ultra-Wideband Communications With Suboptimal Receivers in Multipath Channels,” *IEEE Journal on Selected Areas in Communications*, vol. 20, no. 9, pp. 1754–1766, Dec. 2002.

[7] A. R. Forouzan, M. Nasiri-Kenari, and J. A. Salehi, “Performance Analysis of Time-Hopping Spread-Spectrum Multiple-Access Systems: Uncoded and Coded Schemes,” *IEEE Trans. on Wireless Communications*, vol. 1, no. 4, pp. 671–681, Oct. 2002.

[8] G. Durisi and S. Benedetto, “Performance Evaluation of TH-PPM UWB System in the Presence of Multiuser Interference,” *IEEE Communication Letters*, vol. 7, no. 5, pp. 224–226, May 2003.

[9] G. Durisi and G. Romano, “On the Validity of Gaussian Approximation to Characterize the Multiuser Capacity of UWB TH-PPM,” in *Proc. IEEE Conference on Ultra Wideband Systems and Technologies*, Baltimore, USA, Mar. 2002, pp. 157–162.

[10] B. Hu and N.C. Beaulieu, “Exact Bit Error Rate Analysis of TH-PPM UWB Systems in the Presence of Multiple-Access Interference,” *IEEE Communication Letters*, vol. 7, no. 12, pp. 572–574, Dec. 2003.

[11] J. Zhang, E.K.P. Chong, and D.N.C. Tse, “Output MAI Distributions of Linear MMSE Multiuser Receivers in DS-CDMA Systems,” *IEEE Trans. on Information Theory*, vol. 47, no. 3, pp. 1128–1144, Mar. 2001.

[12] P. Billingsley, *Probability and Measure*, John Wiley, 3rd edition, 1995.

[13] J. Foerster, “Channel Modeling Sub-Committee Report Final,” Tech. Rep., IEEE P802.15-02/490r1-SG3a, Feb. 2002.

[14] J.R. Foerster, M. Pendergrass, and A.F. Molisch, “A Channel Model for Ultrawideband Indoor Communication,” Tech. Rep., MERL, Nov. 2003, `http://www.merl.com/papers/TR2003-73`.

[15] V.M. Kruglov, “On the necessity of the conditional Lindeberg condition for normal convergence of martingales,” *Journal of Mathematical Sciences*, vol. 112, no. 2, pp. 4141–4144, 2002.

[16] A.G. Sholomitskii, “On the necessary conditions of normal convergence for martingales,” *Teor. Veroyatn. Primen. [Theory Probab. Appl.]*, vol. 43, no. 3, pp. 490–508, 1998.

[17] X. Chen and S. Kiaei, “Monocycle Shapes for Ultra Wideband Systems,” in *Proceedings of the Int. Symp. on Circ. And Systems*, May 2002.

[18] K.A. Hamdi and X. Gu, “Bit Error Rate Analysis for TH-CDMA/PPM Impulse Radio Networks,” in *IEEE Conference on Wireless Communication and Networking*, New Orleans, USA, Mar. 2003.

[19] G. Durisi, A. Tarable, J. Romme, and S. Benedetto, “A General Method for Error Probability Computation of UWB Systems for Indoor Multiuser Communications,” *Journal of Communications and Networks*, vol. 5, no. 4, pp. 354–364, Dec. 2003.

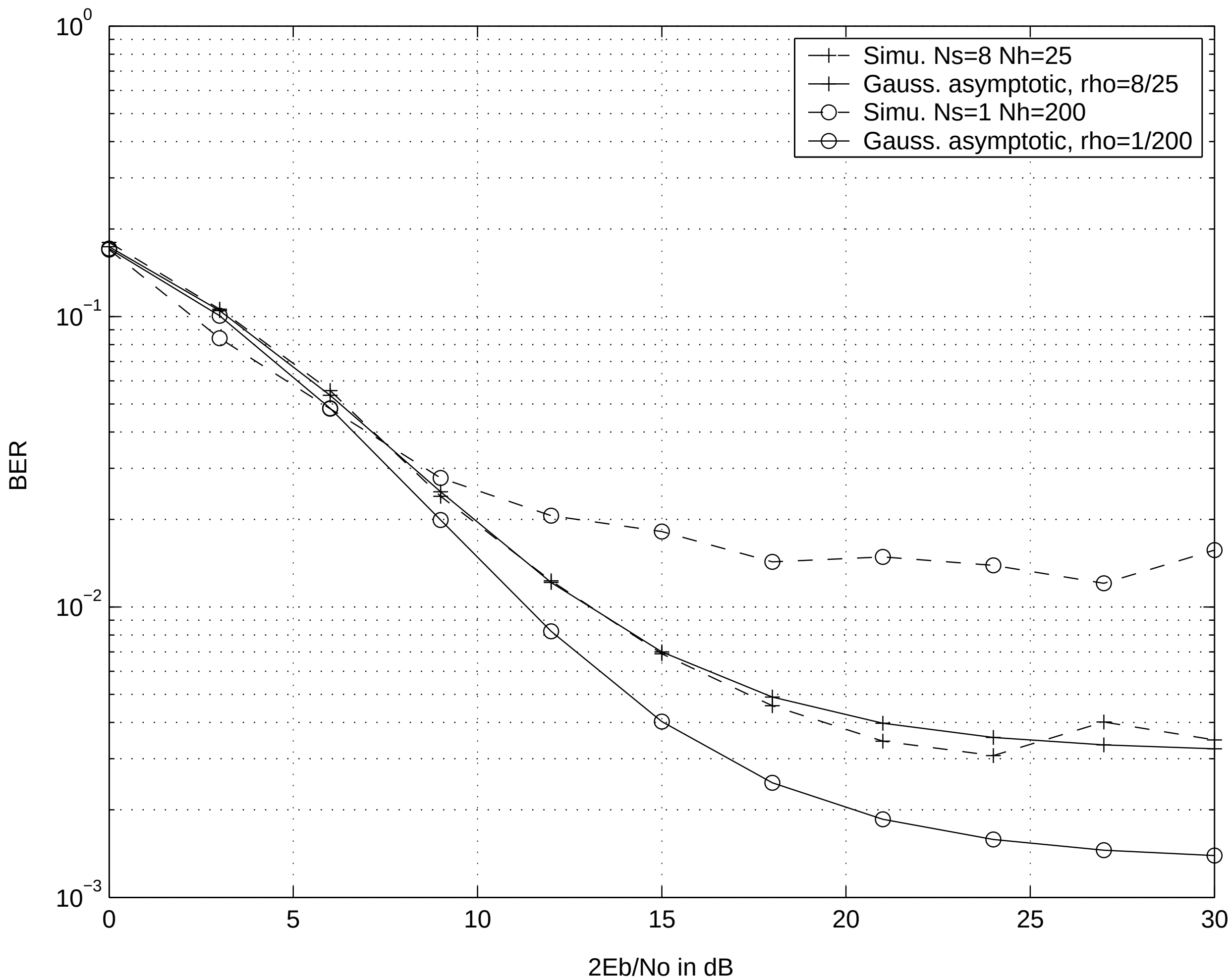
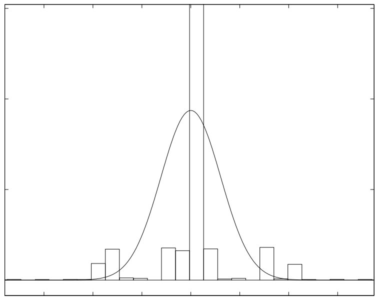
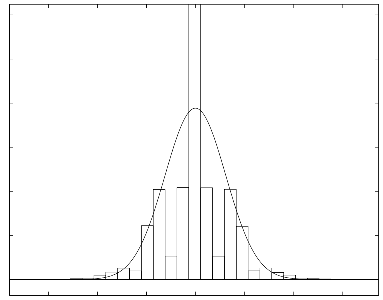


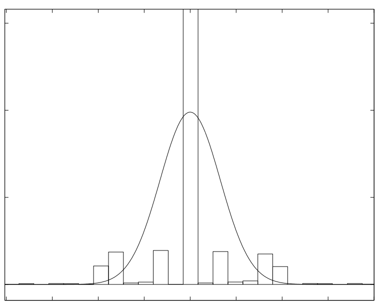
Fig. 1. BER for different values of N_s, N_h. TH-PAM, single path channels, $\alpha = 1/2$.



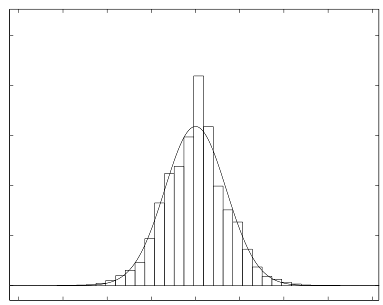
(a) $N_s = 1,\ N_h = 12,\ K = 6$



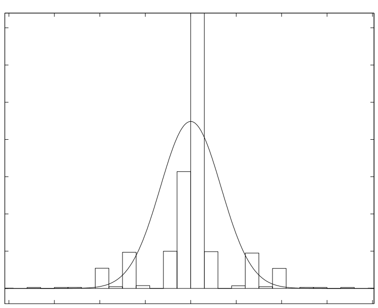
(b) $N_s = 2,\ N_h = 6,\ K = 6$



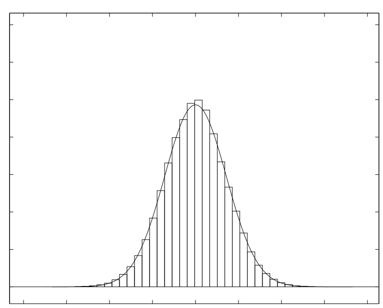
(c) $N_s = 1,\ N_h = 48,\ K = 24$



(d) $N_s = 4,\ N_h = 12,\ K = 24$



(e) $N_s = 1,\ N_h = 200,\ K = 100$



(f) $N_s = 8,\ N_h = 25,\ K = 100$

Fig. 2. Measured MUI histograms (bars) and reference Gaussian distributions with same variances (plain curves). Single path channels.

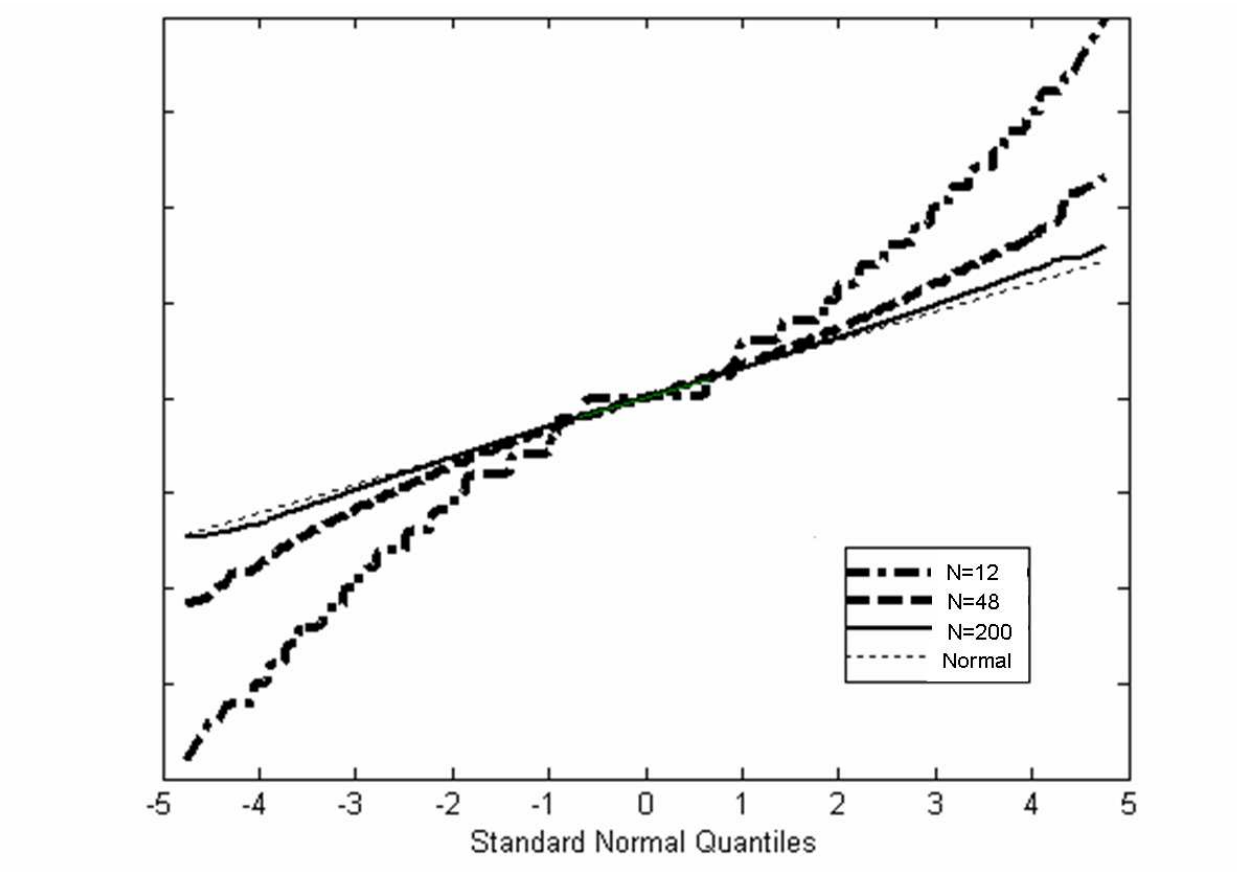


Fig. 3. Q-Q plot, empirical MUI distribution vs Gaussian. Single path channels. $N_s/N_h = 1/3$ or $8/25$, $\alpha = 1/2$.

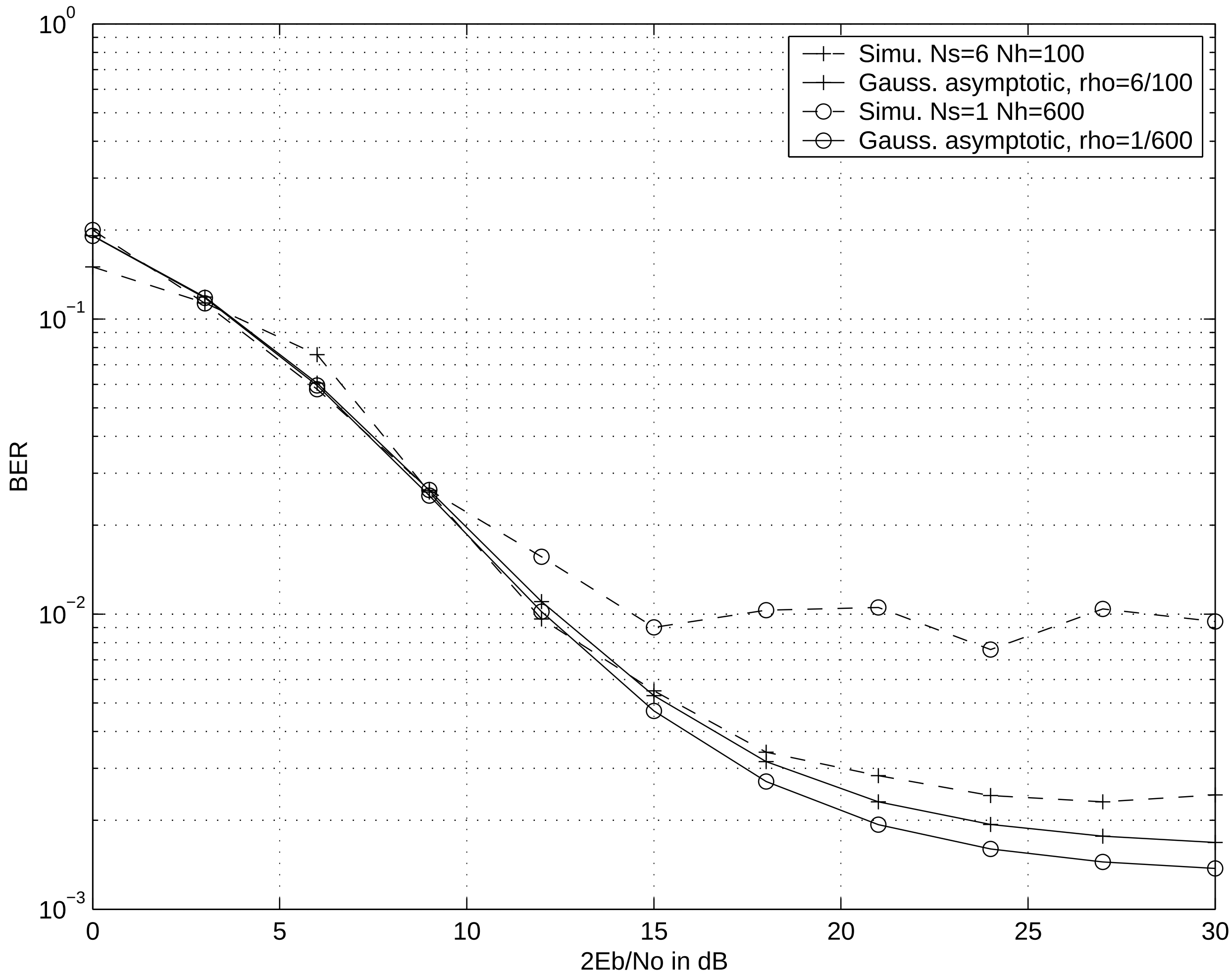
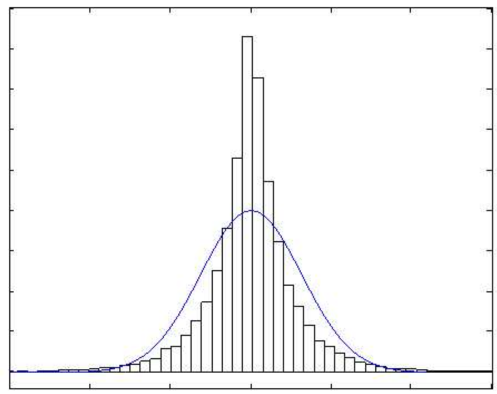
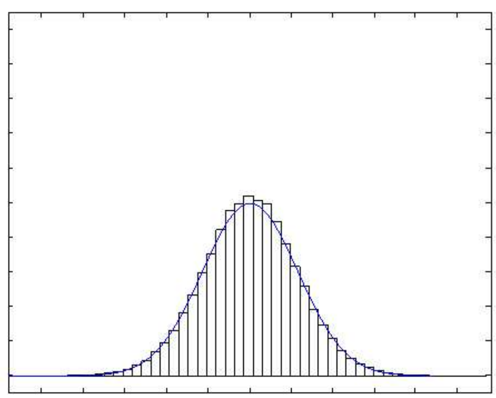


Fig. 4. BER for different values of N_s, N_h. TH-PAM, multi-path channels, $\alpha = 1/2$.



(a) $N_s = 1,\ N_h = 600$



(b) $N_s = 6,\ N_h = 100$

Fig. 5. Measured MUI histograms (bars) and reference Gaussian distributions with same variances (plain curves). Multi-path channels.

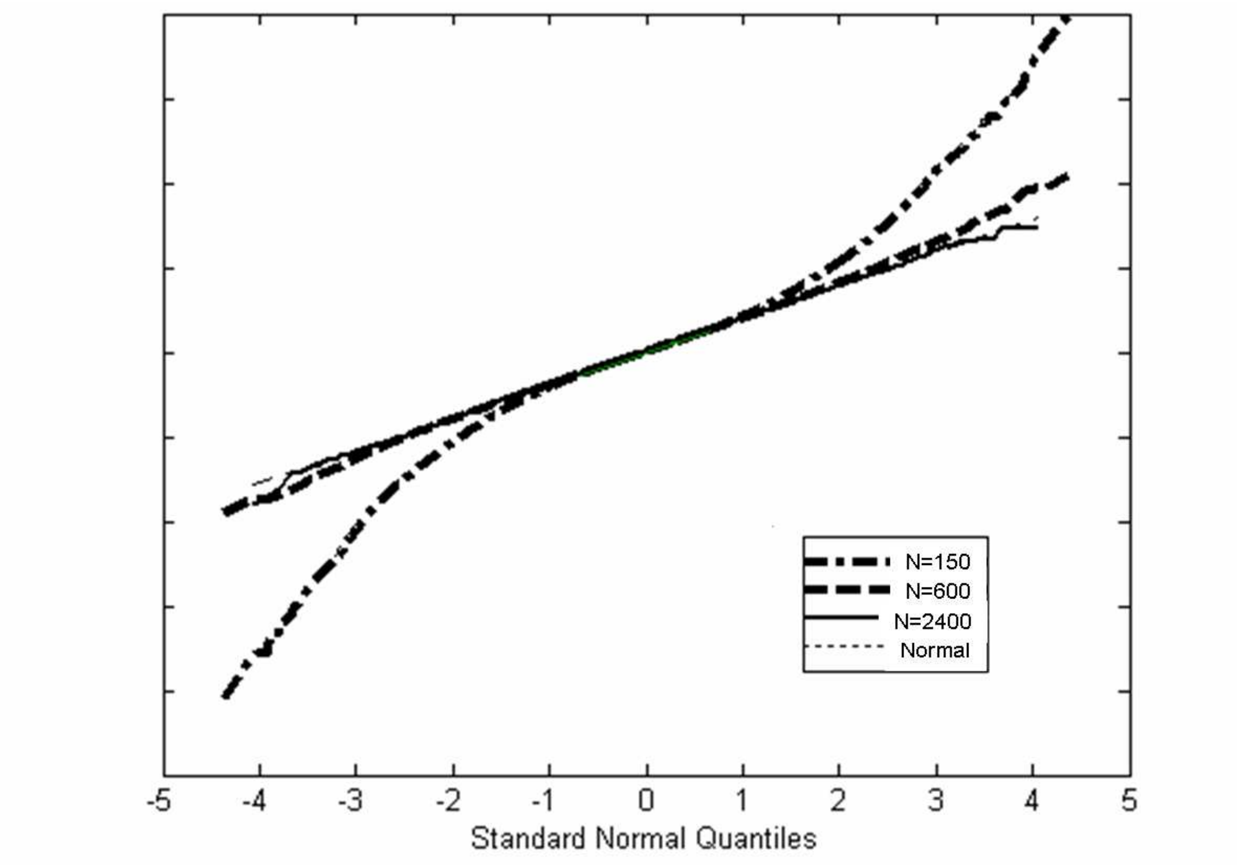


Fig. 6. Q-Q plot, empirical MUI distribution vs Gaussian. Multi-path channels. $N_s/N_h = 3/50$, $\alpha = 1/2$.

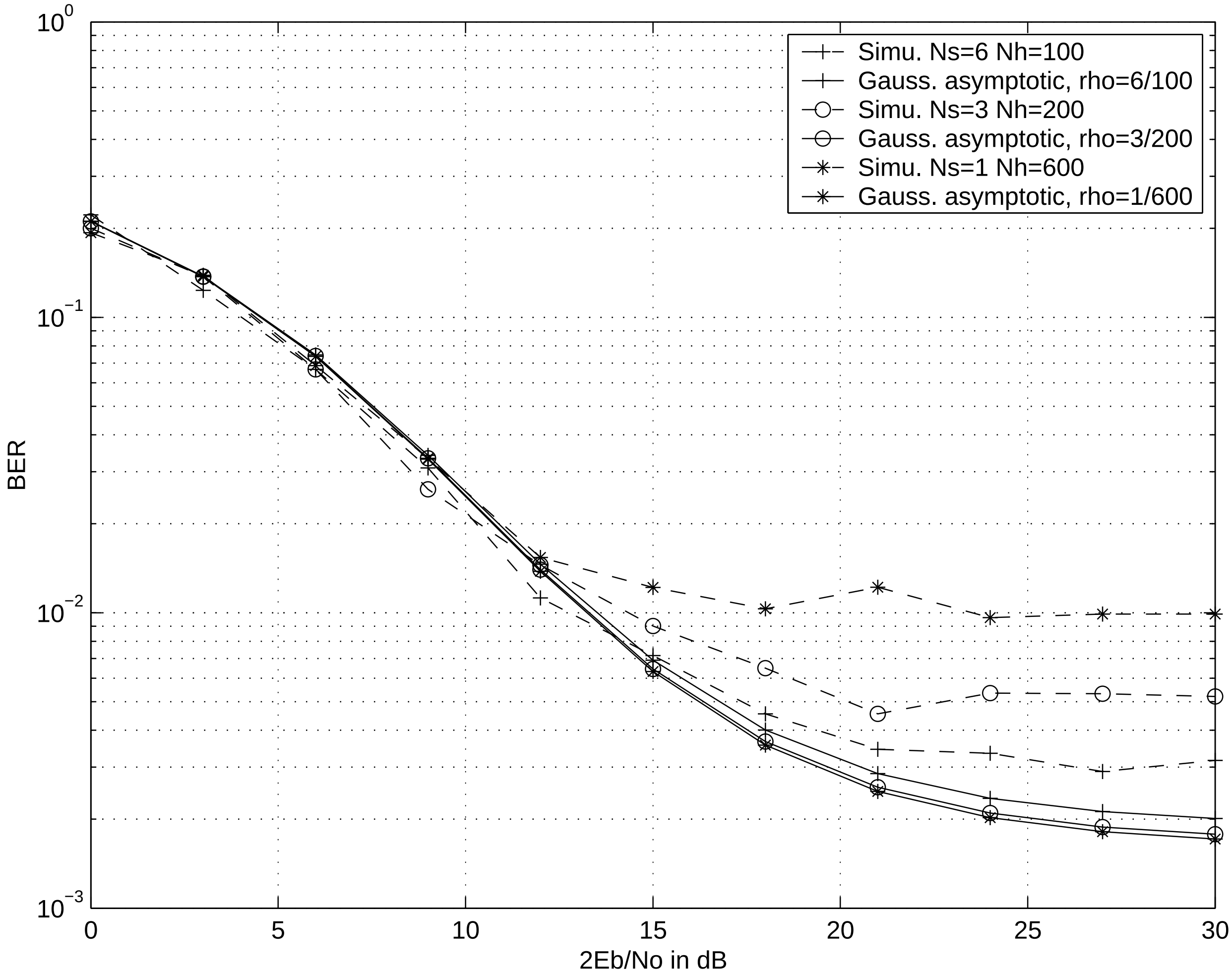


Fig. 7. BER for different values of N_s, N_h. TH-PPM, multi-path channels, $\alpha = 1/2$.

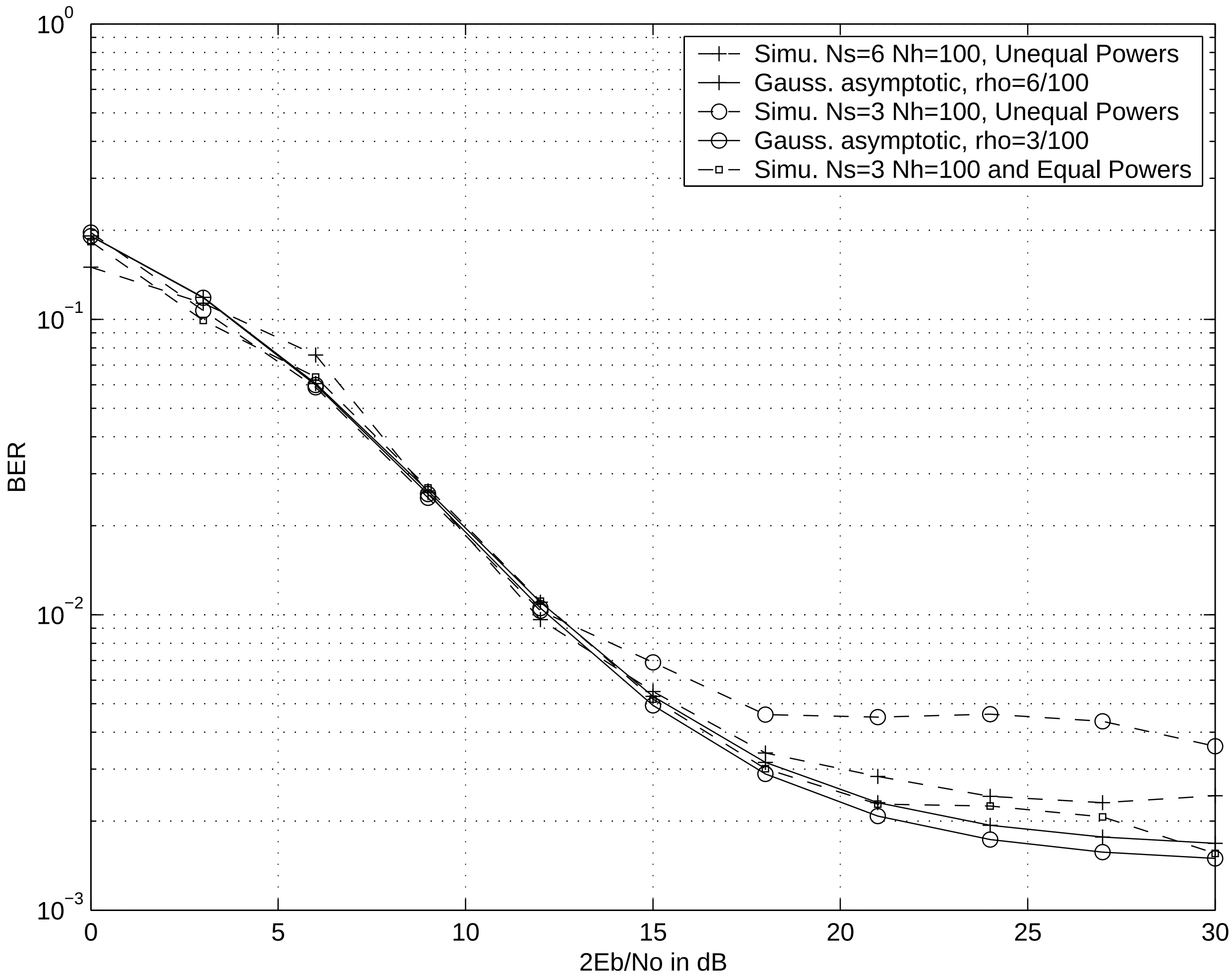


Fig. 8. Asymptotic Approximation vs. N_s and the Power Distribution. TH-PAM, $N_h = 100$, $\alpha = 1/2$.