Capacity of Multiple Access Channel with States Known Noncausally at One Encoder and Only Strictly Causally at the Other Encoder

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Abstract— **We consider a two-user state-dependent multiaccess channel in which the states of the channel are known non-causally to one of the encoders and only strictly causally to the other encoder. The two transmitters only send a common message. We study the capacity of this communication model, to which we refer as the *common-message capacity*. We establish the common-message capacity in the discrete memoryless case as well as in the memoryless Gaussian case. The converse proofs show that, for this model, strictly causal knowledge of the state at one of the encoders does not increase capacity if the other is informed non-causally, a result which is somewhat in sharp contrast with very recent results by Lapidoth and Steinberg on a closely connected model with only strictly causal state at both encoders and independent messages.**

I. Introduction

The capacity of a two-user state-dependent multiple access channel with common message and states known non-causally at only one encoder is established in [1], for both discrete memoryless (DM) and memoryless Gaussian cases. Lower and non-trivial upper bounds in the DM case and an alternative proof of the capacity for the Gaussian case are given in [2]. A key element in the converse proofs of [1] and [2] is that one of the encoders, referred to as the *uninformed encoder*, sends inputs which are function of *only* the message to transmit.

In this paper, we generalize the model of [1], [2] by assuming that one encoder knows the states in a non-causal manner as in [1], [2] and, different from [1], [2], the other encoder knows the states in a strictly causal manner. More precisely, let W denote the common message to transmit in, say, n uses of the channel, and $S^n = (S_1, \dots, S_n)$ denote the state affecting the channel during this time. At time i, one of the encoders, say Encoder 1, knows the complete sequence $S^n = (S_1, \dots, S_{i-1}, S_i, \dots, S_n)$ and sends $X_{1i} = \phi_1(W, S^n)$, and Encoder 2 knows *only* $S^{i-1} = (S_1, \dots, S_{i-1})$ and sends $X_{2i} = \phi_{2,i}(W, S^{i-1})$ – the functions ϕ_1 and $\phi_{2,i}$ are some encoding functions.

We show that this model has the *same* capacity as the one in [1], [2]. That is, the knowledge of the states strictly causally at Encoder 2 does *not* increase the capacity; or, equivalently, Encoder 2 does no better than had it known only the message W.

The importance of this result can be seen in regard to very recent results by Lapidoth and Steinberg on closely connected models [3], [4]. In [3], Lapidoth and Steinberg study a state-dependent MAC with states known in a strictly causal manner at both encoders and independent messages, and show that the strict knowledge of the state can be beneficial, in the sense that it increases the capacity for this model. This result is reminiscent of Dueck's proof [5] that feedback can increase capacity in some multiuser channels. In accordance with [5], the main idea of the achievability results in [3] is a block Markov coding scheme in which the two users collaborate to describe the state to the decoder by sending cooperatively a compressed version of it. Although some non-zero rate that otherwise could be used to transmit pure information is spent in describing the state to the decoder, the net effect can be an increase in the capacity.

The converse proof in this paper shows that a coding scheme à-la Lapidoth-Steinberg (or any other) would not increase capacity for our model. The non-utility of a joint description of the state S_{i-1} to the decoder in the spirit of [3] is not due to that the encoders send the same message here.

II. Problem Setup

We consider a stationary memoryless state-dependent MAC $W_{Y|X_1,X_2,S}$ whose output $Y \in \mathcal{Y}$ is controlled by the channel inputs $X_1 \in \mathcal{X}_1$ and $X_2 \in \mathcal{X}_2$ from the encoders and the channel state $S \in \mathcal{S}$ which is drawn according to a memoryless probability law Q_S. We assume that the channel state S^n is known non-causally at Encoder 1, i.e., beforehand, at the beginning of the transmission block. Encoder 2 knows the channel states only strictly-causally; that is, at time i, it knows the states only up to time $i-1$, $S^{i-1} = (S_1, \dots, S_{i-1})$.

We assume that the common message W is a random variable drawn uniformly from the set $\mathcal{M} = \{1, \cdots, M\}$. The sequences X_1^n and X_2^n from the encoders are sent across a state-dependent multiple access channel modeled as a memoryless conditional probability distribution $W_{Y|X_1,X_2,S}$. The joint probability mass function on $\mathcal{W}\times\mathcal{S}^n\times\mathcal{X}_1^n\times\mathcal{X}_2^n\times\mathcal{Y}^n$ is given by

$$P(w, s^n, x_1^n, x_2^n, y^n) = P(w)\prod_{i=1}^{n} Q_S(s_i)P(x_{1,i}|w, s^n)P(x_{2,i}|w, s^{i-1}) \cdot W_{Y|X_1,X_2,S}(y_i|x_{1,i}, x_{2,i}, s_i). \quad (1)$$

The receiver guesses the message sent by the encoders from the channel output Y^n.

Definition 1: For positive integers n and M, an (M, n, ϵ) code for the multiple access channel with states known noncausally at one encoder and only strictly causally at the other encoder consists of a mapping

$$\phi_1 : \mathcal{M}\times\mathcal{S}^n \longrightarrow \mathcal{X}_1^n$$

at Encoder 1, a sequence of mappings

$$\phi_{2,i} : \mathcal{M}\times\mathcal{S}^{i-1} \longrightarrow \mathcal{X}_2, \quad i = 1, \ldots, n$$

at Encoder 2, and a decoder map

$$\psi : \mathcal{Y}^n \longrightarrow \mathcal{M}$$

such that the average probability of error is bounded by ϵ,

$$P_e^n = \mathbb{E}_S\Big[\Pr\Big(\psi(Y^n) \neq W|S^n = s^n\Big)\Big] \leq \epsilon.$$

The rate of the code is defined as

$$R = \frac{1}{n}\log M.$$

A rate R is said to be achievable if for every $\epsilon > 0$ there exists an $(2^{nR}, n, \epsilon)$ code for the channel $W_{Y|X_1,X_2,S}$. The common-message capacity of the considered state-dependent MAC is defined as the supremum of all the achievable rates.

III. Main Result and Comments

A. Main Result

The following theorem provides the capacity of the studied DM model.

Theorem 1: The capacity, C, of the multiple access channel with common message and states known noncausally at one encoder and strictly causally at the other encoder is given by

$$C = \max I(U, X_2; Y) - I(U; S|X_2) \quad (2)$$

where the maximization is over joint measures $P_{S,U,X_1,X_2,Y}$ of the form

$$P_{S,U,X_1,X_2,Y} = Q_S P_{X_2} P_{U,X_1|S,X_2}. \quad (3)$$

B. Remarks

Remark 1: The capacity of our model in Theorem 1 is the same as the one of the model with state S^n at Encoder 1 and no state at all at Encoder 2 established in [1]. This shows that the strictly causal knowledge of the state at Encoder 2 does not increase capacity.

In contrast to [1], the converse proof (see below) does not follow directly from the converse part proof of the capacity formula for the standard Gelf'and-Pinsker channel [6] because, at time i, Encoder 2 sends inputs which are function of not only the message to transmit, but also the past state sequence S^{i-1}. For instance, our converse proof includes a redefinition of the involved auxiliary random variable.

Remark 2: Our converse proof proves that, for our model, it is *optimal* to just ignore the known S^{i-1} at Encoder 2 and use the coding scheme of [1] or the alternate scheme of [2]. That is, one can do no better exploitation of the state S^{i-1} at Encoder 2. While one could expect some utility of collaborative transmission of S_{i-1} à-la Lapidoth and Steinberg [3], a direct consequence of our converse proof is that this would be of no help (in the sense that it would not result in a better transmission rate).

In the stated-dependent MAC model with strictly causal side information at the encoders studied in [3], the utility of the strictly causal part of the state known at both encoders is created by utilizing a block Markov coding scheme in which, in block i, the encoders cooperate to send a compressed version of the state S_{i-1} to the decoder in addition to their individual messages. The encoders cannot cooperate in the transmission of the messages but they do in that of the compressed version of S_{i-1}. The decoder estimates S_{i-1} from the output received in block i and then uses it to decode the messages transmitted in block $i-1$. Although some non-zero rate – that otherwise could be used for sending the individual messages, is spent on sending the joint description of the state S_{i-1}, the net effect can be an increase in the capacity because, in block i, each encoder can benefit from the decoder's knowledge of some estimation of the state S_{i-1}.

The effect of the joint transmission of the state S_{i-1} in block i in Lapidoth and Steinberg coding scheme is some reduction of the state effect in block $i-1$ in decoding the individual messages. Our converse proof shows that the net effect is not beneficial in our case. While this can be understood easily in the additive Gaussian case since Encoder 1 knows the state and can cancel its effect completely using a dirty paper scheme [7], i.e., without need to diminishing its effect via the joint transmission of the compressed version of S_{i-1}, the result is less intuitive in the discrete case.

IV. Gaussian Model

In this section, we consider a two-user state-dependent Gaussian MAC in which the channel states S^n and the noise are additive and Gaussian. As in Section II, we assume that Encoder 1 knows the channel states non-causally and Encoder 2 knows the channel states strictly causally. The two encoders send only some common message.

At time instant i, the channel output Y_i is related to channel inputs $X_{1,i}$ and $X_{2,i}$ from the two encoders, the channel state S_i and the noise Z_i by

$$Y_i = X_{1,i} + X_{2,i} + S_i + Z_i, \tag{4}$$

where S_i and Z_i are zero-mean Gaussian random variables with variance Q and N, respectively. The random variables S_i and Z_i at time instant $i \in \{1, \cdots, n\}$ are mutually independent, and independent from (S_j, Z_j) for $j \neq i$. Also, at time i, the input $X_{2,i}$ is independent from the state S_i.

We consider the individual power constraints on the transmitted power

$$\sum_{i=1}^{n} X_{1,i}^2 \leq nP_1, \ \sum_{i=1}^{n} X_{2,i}^2 \leq nP_2. \tag{5}$$

The definition of a code for this channel is the same as given in Section II, with the additional power constraint (5).

The following corollary provides the capacity of the studied Gaussian model.

***Corollary** 1:* The capacity, C_{G}, of the Gaussian model (4) is given by

$$C = \max \ \frac{1}{2}\log\Big(1 + \frac{(\sqrt{P_2} + \rho_{12}\sqrt{P_1})^2}{P_1(1-\rho_{12}^2-\rho_{1s}^2) + (\sqrt{Q} + \rho_{1s}\sqrt{P_1})^2 + N}\Big) + \frac{1}{2}\log\Big(1 + \frac{P_1(1-\rho_{12}^2-\rho_{1s}^2)}{N}\Big), \tag{6}$$

where the maximization is over $\rho_{12} \in [0, 1]$, $\rho_{1s} \in [-1, 0]$ such that

$$\rho_{12}^2 + \rho_{1s}^2 \leq 1. \tag{7}$$

Outline Proof: The above results for the DM MAC can be readily extended to memoryless channels with discrete time and continuous alphabets using standard techniques [8]. The capacity of the model (4) with state S^n known at Encoder 1 and S^{i-1} known at Encoder 2 is the same as that of the same model but with no state at all Encoder 2. The capacity of this model can be obtained by specializing the results in [1] and [2], to the case in which the encoders transmit a common message and no individual messages.

V. Proof of Theorem 1

A. Outline Proof of Achievability

The achievability follows by just ignoring the state at Encoder 2 and using the scheme of the common message capacity of [1, Corollary 1] or the alternate scheme of [2, Theorem 1].

B. Proof of Converse

We prove that for any (M, n, ϵ)−code consisting of a mapping $\phi_1 : \mathcal{M}\times\mathcal{S}^n \longrightarrow \mathcal{X}_1^n$ at Encoder 1, a sequence of mappings $\phi_{2,i} : \mathcal{M}\times\mathcal{S}^{i-1} \longrightarrow \mathcal{X}_2$, $i = 1, \ldots, n$, at Encoder 2, and a mapping $\psi : \mathcal{Y}^n \longrightarrow \mathcal{M}$ at the decoder with average error probability $P_e^n \to 0$ as $n \to 0$ and rate $R = n^{-1}\log_2 M$, there exists a quadruple of random variables $(U, S, X_1, X_2) \in \mathcal{U}\times\mathcal{S}\times\mathcal{X}_1\times\mathcal{X}_2$ with joint distribution P_{U,S,X_1,X_2} of the form

$$P_{U,S,X_1,X_2} = Q_S P_{X_2} P_{U,X_1|S,X_2} \tag{8}$$

and such that the marginal distribution of S is $Q_S(s)$, i.e.,

$$\sum_{u,x_1,x_2} P_{U,S,X_1,X_2}(u, s, x_1, x_2) = Q_S(s) \tag{9}$$

and

$$R \leq I(U, X_2; Y) - I(U; S) \tag{10}$$

Fix n and consider a given code of block length n. The joint probability mass function on $\mathcal{W}\times\mathcal{S}^n\times\mathcal{X}_1^n\times\mathcal{X}_2^n\times\mathcal{Y}^n$ is given by

$$p(w, s^n, x_1^n, x_2^n, y^n) = p(w)\prod_{i=1}^{n} p(s_i)p(x_{1i}|w, s^n)p(x_{2i}|w, s^{i-1}) p(y_i|x_{1i}, x_{2i}, s_i), \tag{11}$$

where, $p(x_{1i}|w, s^n)$ is equal 1 if $x_{1i} = f(w, s^n)$ and 0 otherwise.

The decoding rule ψ recovers W from Y^n with the average error probability $P_e := \sum_{i=1}^{M} \Pr(\hat{W} \neq W)$. Fano's inequality gives

$$\begin{aligned} H(W|Y^n) &\leq H(P_e) + P_e \log(|\mathcal{M}| - 1) && (12)\\ &\leq H(P_e) + P_e \log(|\mathcal{M}|) && (13)\\ &= nP_eR + H(P_e) && (14)\\ &:= n\epsilon_n. && (15) \end{aligned}$$

Then, we have

$$\begin{aligned} nR - n\epsilon_n &\leq I(W; Y^n) && (16)\\ &\overset{(a)}{\leq} I(W; Y^n) \\ &\overset{(b)}{=} I(W; Y^n) - I(W; S^n) && (17)\\ &= \sum_{i=1}^{n} I(W; Y_i|Y_{i+1}^n) - I(W; S_i|S^{i-1}) && (18)\\ &= \sum_{i=1}^{n} I(W, S^{i-1}; Y_i|Y_{i+1}^n) - I(S^{i-1}; Y_i|W, Y_{i+1}^n) - I(W; S_i|S^{i-1}) && (19) \end{aligned}$$

$$= \sum_{i=1}^{n} I(W, S^{i-1}; Y_i|Y_{i+1}^n) - I(W; S_i|S^{i-1}) - \sum_{i=1}^{n} I(S^{i-1}; Y_i|W, Y_{i+1}^n) \tag{20}$$

$$\overset{(c)}{=} \sum_{i=1}^{n} I(W, S^{i-1}; Y_i|Y_{i+1}^n) - I(W; S_i|S^{i-1}) - \sum_{i=1}^{n} I(Y_{i+1}^n; S_i|W, S^{i-1}) \tag{21}$$

$$= \sum_{i=1}^{n} I(W, S^{i-1}; Y_i|Y_{i+1}^n) - H(S_i|S^{i-1}) + H(S_i|W, S^{i-1}, Y_{i+1}^n) \tag{22}$$

$$\overset{(d)}{=} \sum_{i=1}^{n} I(W, S^{i-1}; Y_i|Y_{i+1}^n) - H(S_i) + H(S_i|W, S^{i-1}, Y_{i+1}^n) \tag{23}$$

$$\overset{(e)}{\leq} \sum_{i=1}^{n} I(W, S^{i-1}, Y_{i+1}^n; Y_i) - I(W, S^{i-1}, Y_{i+1}^n; S_i) \tag{24}$$

$$\overset{(f)}{=} \sum_{i=1}^{n} I(U_i; Y_i) - I(U_i; S_i) \tag{25}$$

$$\overset{(g)}{=} \sum_{i=1}^{n} I(U_i, X_{2i}; Y_i) - I(U_i, X_{2i}; S_i) \tag{26}$$

$$= \sum_{i=1}^{n} I(U_i, X_{2i}; Y_i) - H(S_i) + H(S_i|U_i, X_{2i}) \tag{27}$$

$$\overset{(h)}{=} \sum_{i=1}^{n} I(U_i, X_{2i}; Y_i) - H(S_i|X_{2i}) + H(S_i|U_i, X_{2i}) \tag{28}$$

$$= \sum_{i=1}^{n} I(U_i, X_{2i}; Y_i) - I(U_i; S_i|X_{2i}) \tag{29}$$

$$\tag{30}$$

where (*a*) follows by Fano's inequality; (*b*) follows from the fact that W is independent of the state sequence S^n; (*c*) follows from Csiszar and Korner's "summation by parts"-lemma [9]

$$\sum_{i=1}^{n} I(Y_{i+1}^n; S_i|W, S^{i-1}) = \sum_{i=1}^{n} I(S^{i-1}; Y_i|W, Y_{i+1}^n) \tag{31}$$

(*d*) follows from the fact that state S^n is i.i.d.; (*e*) follows from the non-negativeness of $I(Y_{i+1}^n; Y_i)$; (*f*) follows from the substitution $U_i := (W, S^{i-1}, Y_{i+1}^n)$; (*g*) follows from the fact that X_{2i} is a deterministic function of (W, S^{i-1}), and (*h*) follows from the fact that X_{2i} is independent from S_i.

The rest of the proof follows by standard single-letterization.

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