Wyner-Ziv Lattice Coding for Two-Way Relay Channel

Sinda Smirani, Mohamed Kamoun, Mireille Sarkiss
CEA, LIST, Communicating Systems Laboratory
BC 94, Gif Sur Yvette, F91191 - France
{sinda.smirani, mohamed.kamoun, mireille.sarkiss}@cea.fr

Abdellatif Zaidi
Université Paris-Est Marne La Vallée
Champs-sur-Marne, F77454 - France
abdellatif.zaidi@univ-mlv.fr

Pierre Duhamel
CNRS/LSS, Supelec
Gif Sur Yvette, F91192 - France
pierre.duhamel@lss.supelec.fr

***Abstract*—A Two-Way Relay Channel (TWRC) in which duplex transmission between two users via a relay station is considered. A physical layer network coding strategy based on compress-and-forward relaying scheme for the TWRC is proposed. In the underlying coding strategy, we use nested lattices for Wyner-Ziv coding and decoding. The relay uses the weaker side information available at the receivers from the first transmission phase to broadcast a common quantized version of its received signal. We characterize the achievable rate region of the presented scheme. Then we show that lattice codes can achieve random coding rates.**

I. Introduction

The two way relaying problem where two communicating nodes want to exchange information via a relay station is encountered in various wireless communication scenarios: ad-hoc networks, range extension for cellular and local networks ...

While network level routing is the classical solution to this problem it has been shown that network coding (NC) strategies provide better performance by leveraging the side information that is available in each node. In fact, NC allows to improve the rates by combining raw bits or packets at the network layer. The capacity of the system can be further improved when NC is applied to the physical layer. It takes advantage of the broadcast and multiple-access properties of the radio link that are considered usually as an interference nuisance to the system [1]. In this context, we consider a physical network coding (PNC) strategy where the overall communication takes two phases, namely a multiple Access (MAC) phase and a Broadcast (BC) phase.

Among various strategies that can be used in the TWRC, three relaying protocols can be employed: Amplify and Forward (AF), Decode and Forward (DF), and Compress and Forward (CF). The AF scheme is a linear relaying protocol where the relay only scales the received signal to meet its output power constraint. In the DF scheme, the relay decodes separately both messages then re-encodes them before broadcasting the resulting codeword. Finally, CF scheme, introduced first in [2], has been recently extended to TWRC [3]. In this protocol, the relay station sends a quantized version of the received signal.

Performance bounds of these schemes were investigated in [4], [5], [3]. Independently, achievable rate regions of CF relaying have been investigated in [6], [7]. It has been shown that for specific channel conditions, specially symmetric channels, CF outperforms DF for high SNR regimes and AF at all SNR regimes. Besides, the authors in [6] proved that CF relaying scheme achieves rates within one half bit of the capacity region in the Gaussian case for symmetric noise variances.

In the aforementioned references, the derivation of the achievable rate regions has employed random coding tools. Structured codes, on the other hand, have been found to be more advantageous in a practical settings thanks to their reduced complexity in encoding and decoding [8]. Lattice codes represent an important class of structured codes that can be used in many wireless systems. It has been shown in [9] that for an Additive White Gaussian Noise (AWGN) channel, lattice based codes can achieve the Shannon capacity for Gaussian point-to-point communication. Based on this result, lattice coding and decoding schemes have been suggested for TWRC scenario in [10]. In this scheme, the transmitters employ nested lattices as codebooks, and the relay decodes a modulo-lattice sum of the transmitted codewords from the received signal. In addition, all the nodes should transmit with the same power. In [11], this scheme has been extended for different power constraints, however no channel conditions were considered.

In our work, we investigate a CF strategy at the relay for block fading channels with different transmit powers at the transmitting nodes. In the MAC phase, these nodes send simultaneously their messages and the relay receives a mixture of the transmitted signals. This received signal represents the relay source to be encoded. Then, the relay schemes a message which can be decoded by both receivers using the available side information. When only the decoder has access to side information about the source, this setting is equivalent to Wyner-Ziv (WZ) with lossy source compression [12] or Slepian-Wolf with lossless source compression [13]. Nested lattice-code-based lossy source quantization is introduced in [14] for Gaussian sources in point-to-point communication. In this paper, we extend this scheme to the TWRC where a common source is compressed and broadcasted to the receivers. The proposed strategy provides a practical PNC scheme for TWRC.

The rest of the paper is organized as follows. In section II, we introduce our system model. In section III, we outline the fundamentals of nested lattice codes and the lattice based WZ scheme for the TWRC. In section VI, achievable rate region of the considered model will be provided and then compared with random coding achievable rate region. Finally, section V concludes the paper.

II. System Model

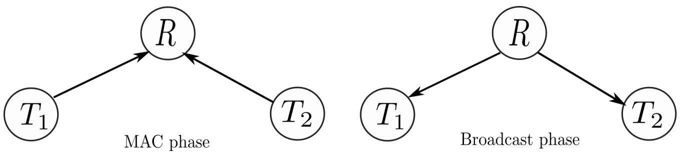

Fig. 1. The two-phase transmission of TWRC

We consider a Gaussian TWRC as presented in Fig.1. The communication takes n channel uses split into two phases MAC and BC. We use the index $i \in \{1,2\}$ to refer to the nodes T_1 and T_2 and r to refer the Relay R. We assume that all nodes are half-duplex so they are not able to transmit and receive at the same time.

During the MAC phase, each node T_i chooses randomly a message $m_i \in \mathcal{M}_i = \{1, 2, \cdots, M_i\}$ according to a uniform distribution, to be sent to the other terminal via the relay station. The message m_i is then mapped by a function $f_i(.)$ to a codeword $\mathbf{x}_i(m_i) = (x_{i,1}(m_i), x_{i,2}(m_i), \cdots, x_{i,n_1}(m_i))$ of dimension n_1 which is sent to the relay. We assume that the codewords $\mathbf{x}_i$ correspond to the realizations of a random variable $\mathbf{X}_i$ whose probability distribution is given by $p_i(x_i)$. The messages are transmitted through a memoryless channel represented by the Gaussian transition probability $p(y_r|x_1, x_2)$. The relay R receives the signal Y_r that belongs to the output alphabet $\mathcal{Y}_r$.

During the BC phase, an encoding function at the relay generates a codeword $\mathbf{x}_r$ of dimension n_2 from the received signal $\mathbf{y}_r$. The symbols are chosen from a set $\mathcal{X}_r$. The signal X_r is transmitted through a broadcast memoryless channel with Gaussian transition probability $p(y_1, y_2|x_r)$, where Y_i are the received signals at nodes T_i that belong to the output set $\mathcal{Y}_i$. Both nodes use the received signal and their side information taken from their previously sent data to decode their intended messages.

The MAC and BC phases take respectively n_1 and n_2 channel uses such as $n_1 + n_2 = n$. Let $\alpha \in [0,1]$ be the time division coefficient such that $n_1 = \alpha n$ and $n_2 = (1-\alpha)n$. We assume that both phases are statistically independent such that the probability distribution $p(y_r, y_1, y_2|x_1, x_2, x_r) = p(y_r|x_1, x_2)p(y_1, y_2|x_r)$.

All input distributions are real valued: $X_k \sim \mathcal{N}(0, P_k)$, $k \in \{1, 2, r\}$, where $\mathcal{N}(0, P_k)$ denotes a zero mean real Gaussian variable with power P_k. The received signals can be modeled as follows:

$$Y_r = \quad h_1 X_1 + h_2 X_2 + Z_r \tag{1}$$

$$Y_i = \quad h_i X_r + Z_i, \tag{2}$$

where h_i denotes the channel coefficient between T_i and R. We consider channel reciprocity such that the gain between a node and the relay is equal to the gain between the relay and that node i.e. $h_{i\to r} = h_{r \to i} = h_i$. $Z_r \sim \mathcal{N}(0, \sigma_r^2)$ is the additive white Gaussian noise at the relay and $Z_i \sim \mathcal{N}(0, \sigma_i^2)$ the AWGN at node T_i, $i \in \{1, 2\}$. We assume perfect CSI which means that each node is aware of all the channel gains, and the noise components are independent of each other and from the channel inputs.

In the sequel, we investigate the design of an (M_1, M_2, n_1, n_2)-code for TWRC using a Lattice-based Wyner-Ziv strategy. We will provide an achievable rate region for the code given by a set of achievable rate pairs (R_{12}, R_{21}).

Definition A rate pair (R_{12}, R_{21}) is said to be achievable if $\exists$ a sequence of (M_1, M_2, n_1, n_2)-codes with $\frac{\log M_1}{n} \to R_{12}$ and $\frac{\log M_2}{n} \to R_{21}$ such that the decoding error probability approaches zero for n sufficiently large.

III. Wyner-Ziv Lattice coding for TWRC

In this section, we explain our proposed scheme based on WZ encoding for TWRC. In [14], nested lattice codes have been shown to achieve the WZ rate distortion for both binary hamming and the quadratic Gaussian case in point-to-point communication. In our work, the main idea is the following: during BC phase, the relay station sends a compressed version of the signal received during the MAC phase. The relay employs a lossy compression scheme based on nested lattice codes for Wyner-Ziv strategy. This scheme is motivated by the presence of the side information at both receivers. We first outline some preliminaries on lattice coding. More details for real lattices can be found in [9], [14].

A. Fundamentals on Lattice codes

A real k-dimensional lattice Λ is a subgroup of the Euclidean space $(\mathbb{R}^k, +)$. Thus $\forall \lambda_1, \lambda_2 \in \Lambda$, then $\lambda_1 + \lambda_2 \in \Lambda$. We present below some fundamental properties associated with a lattice:

- The nearest neighbor lattice quantizer of Λ is defined as $Q_\Lambda(\mathbf{x}) = \arg\min_{\lambda \in \Lambda} ||\mathbf{x} - \lambda||$ where $\mathbf{x} \in \mathbb{R}^k$ and $\|.\|$ is the Euclidean norm.
- The basic Voronoi cell of Λ is the set of points in $\mathbb{R}^k$ closest to the zero vector, $\mathcal{V}(\Lambda) = \{\mathbf{x} \mid Q_\Lambda(\mathbf{x}) = \mathbf{0}\}$
- The volume of a lattice $V := \mathrm{Vol}(\mathcal{V}(\Lambda))$
- The mod-Λ operation is defined as $\mathbf{x} \bmod \Lambda = \mathbf{x} - Q_\Lambda(\mathbf{x})$. This law satisfies the distributive law: $(\mathbf{x} \bmod \Lambda + \mathbf{y}) \bmod \Lambda = (\mathbf{x} + \mathbf{y}) \bmod \Lambda$
- **Crypto Lemma**: For a dither vector $\mathbf{T}$ independent of $\mathbf{X}$ and uniformly distributed over $\mathcal{V}(\Lambda)$, then $\mathbf{Y} = (\mathbf{X} + \mathbf{T}) \bmod \Lambda$ is uniformly distributed over $\mathcal{V}(\Lambda)$ and is independent of $\mathbf{X}$ [9]
- The second moment per dimension of Λ is $\sigma^2(\Lambda) := \frac{1}{k} \cdot \frac{1}{V} \int_{\mathcal{V}(\Lambda)} ||\mathbf{x}||^2 d\mathbf{x}$
- The dimensionless normalized second moment is defined as $G(\Lambda) := \frac{\sigma^2(\Lambda)}{V^{2/k}}$

Let G_k, the minimum possible value of $G(\Lambda)$ over all lattices in $\mathbb{R}^k$. The figure of merit of this variable comes from an important result in the quantization theory. It is shown in [15] that for large dimension k there exist lattice quantizers Λ_k^* such that $G(\Lambda_k^*) = G_k \underset{k\to\infty}{\longrightarrow} \frac{1}{2\pi e}$. These lattices are called "good" lattices for quantization. Furthermore, lattices which are good for channel coding as defined in [16] have for large dimension k a volume $V < 2^{k(h(Z)+\epsilon)}$ for any $\epsilon > 0$, where $h(Z) = \frac{1}{2}\log(2\pi e\sigma^2)$ is the differential entropy of Gaussian noise $\mathbf{Z}$ with variance σ^2. In this case, the decoding error probability defined as $P_e = P\{\mathbf{Z} \notin \mathcal{V}(\Lambda)\}$ vanishes when k goes to ∞.

A pair of k-dimensional lattices (Λ_1, Λ_2) is said nested if $\Lambda_2 \subset \Lambda_1$. The fine lattice is Λ_1 with basic Voronoi region $\mathcal{V}_1$ of volume V_1 and the coarse lattice is Λ_2 with basic Voronoi region $\mathcal{V}_2$ of volume V_2. The points of the set $\Lambda_1 \cap \mathcal{V}_2 = \Lambda_1 \mod \Lambda_2$ represent the coset leaders of Λ_2 relative to Λ_1, where for each $\lambda \in \{\Lambda_1 \mod \Lambda_2\}$, the shifted lattice $\Lambda_{2,\lambda} = \Lambda_2 + \lambda$ is called a coset of Λ_2 relative to Λ_1. There are $\frac{V_2}{V_1}$ distinct cosets. It follows that the coding rate when using nested lattices is

$$R = \frac{1}{k}\log_2|\Lambda_1 \cap \mathcal{V}_2| = \frac{1}{k}\log_2\frac{V_2}{V_1} \quad \text{(bits per dimension)} \quad (3)$$

B. Nested Lattice codes for WZ problem

Recalling the described system in II, the global channel model can be summarized as in Fig.2. The messages m_1 and m_2 carry each nR_{12} and nR_{21} bits, these messages are transmitted over n_1 channel uses to the relay. Upon receiving Y_r, the relay generates an index m_r by applying nested lattice codes for WZ source coding. Each terminal T_i regenerates a local version of $\mathbf{Y}_r$ denoted $\hat{Y}_{r,i}$ with a controlled amount of distortion D_i, $i \in \{1,2\}$ such as

$$\frac{1}{n_1}\mathbb{E}\|Y_r - \hat{Y}_{r_i}\|^2 \leq D_i \quad (4)$$

In our system, we consider a single quantization scheme which is adapted to the worst terminal (i.e. the one which has the weakest side information). Without loss of generality we assume that $|h_2|^2P_2 \leq |h_1|^2P_1$ which makes the node T_2 the worst user in terms of side information. Therefore, the source encoding at the relay will be performed with respect to the second decoder T_2 with distortion D_2. In this case, T_1 will undergo the effect of this choice on its decoded signal at the end of transmission.

The side information available at T_2 is $\mathbf{S}_2 = h_2\mathbf{X}_2$ with variance $\sigma^2_{S_2} = |h_2|^2P_2$ that is the source is given by $\mathbf{Y}_r = \mathbf{S}_2 + h_1\mathbf{X}_1 + \mathbf{Z}_r$. Let $\mathbf{U}_2 = h_1\mathbf{X}_1 + \mathbf{Z}_r$ the unknown part of the source with variance $\sigma^2_{U_2} = |h_1|^2P_1 + \sigma_z^2$. Thus, the source $\mathbf{Y}_r$ can be written as $\mathbf{Y}_r = \mathbf{S}_2 + \mathbf{U}_2$. By construction, it is clear that $\mathbf{U}_2$ and $\mathbf{S}_2$ are independent random variables of dimension n_1. The WZ rate distortion function $R(D_2)$ of the Gaussian source $\mathbf{Y}_r$ with side information $\mathbf{S}_2$ at the decoder is defined as the minimum achievable rate with distortion D_2. It is given by:

$$R(D_2) = \frac{1}{2}\log_2^+\left(\frac{\mathrm{VAR}(Y_r|S_2)}{D_2}\right), 0 \leq D_2 \leq \mathrm{VAR}(Y_r|S_2) \quad (5)$$

$$= \frac{1}{2}\log_2^+\left(\frac{\sigma^2_{U_2}}{D_2}\right), 0 \leq D_2 \leq \sigma^2_{U_2} \quad (6)$$

$\mathrm{VAR}(Y_r|S_2)$ is the conditional variance of Y_r given S_2 which is equal to $\sigma^2_{U_2}$.

The lattice coding is based on the scheme proposed in [14] and it is adapted to TWRC. We use a pair of n_1-dimensional nested lattices (Λ_1, Λ_2) chosen to be good lattices: the fine lattice Λ_1 is good for quantization with second moment per dimension $\sigma^2(\Lambda_1) = D_2$ and the coarse lattice Λ_2 is chosen to be good lattice for channel coding with moment $\sigma^2(\Lambda_1) = \sigma^2_{U_2}$. According to the properties of good lattices, we have $\frac{1}{n_1}\log_2(V_i) \approx \frac{1}{2}\log_2(2\pi e\sigma^2(\Lambda_i))$, $i \in \{1,2\}$.

Let $\mathbf{t}$ a dither vector uniformly distributed over $\mathcal{V}_1$. We assume that it is available at both the encoder and the decoder. β is a scaling factor to be determined later.

Encoding: It consists in performing a WZ lattice coding (WZLC) with two successive operations: first the signal $\beta\mathbf{y}_r + \mathbf{t}$ is quantized to the nearest point in Λ_1 then the outcome of this operation is processed with a modulo-lattice operation in order to generate a vector $\mathbf{v}_r$ of size n_1 as shown in Fig.3.

$$\mathbf{v}_r = Q_1(\beta\mathbf{y}_r + \mathbf{t}) \mod \Lambda_2 \quad (7)$$

The relay sends the index of $\mathbf{v}_r$ that identifies the coset of Λ_2 relative to Λ_1 that contains $Q_1(\beta\mathbf{y}_r + \mathbf{t})$. The coset leader $\mathbf{v}_r$ is represented with $\frac{V_2}{V_1} \underset{n_1\to\infty}{\longrightarrow} \frac{n_1}{2}\log_2\left(\frac{\sigma^2_{U_2}}{D_2}\right)$ bits. Thus, the corresponding coding rate is equal to WZ distortion rate in (6).

Decoding: At both users, $\mathbf{v}_r$ is decoded first. Then the unknown part of the source is reconstructed with a WZ lattice decoder(WZLD) using the side information $\mathbf{S}_i$ as

$$\hat{\mathbf{u}}_i = \beta((\mathbf{v}_r - \mathbf{t} - \beta\mathbf{s}_i) \mod \Lambda_2) \text{ for } i \in \{1,2\} \quad (8)$$

IV. Achievable rate region

Theorem 4.1: For Gaussian TWRC, the convex hull of the following end-to-end rates (R_{12}, R_{21}) is achievable using WZ nested lattice codes at the relay:

$$R_{12} \leq \frac{\alpha}{2}\log_2\left(1 + \frac{|h_1|^2P_1(|h_1|^2P_1 + \sigma_z^2 - D_2)}{\sigma_z^2(|h_1|^2P_1 + \sigma_z^2 - D_2) + D_2}\right) \quad (9)$$

$$R_{21} \leq \frac{\alpha}{2}\log_2\left(1 + \frac{|h_2|^2P_2(|h_1|^2P_1 + \sigma_z^2 - D_2)}{\sigma_z^2(|h_1|^2P_1 + \sigma_z^2 - D_2) + D_2}\right) \quad (10)$$

subject to:

$$\alpha\log_2\left(\frac{|h_1|^2P_1 + \sigma_z^2}{D_2}\right) \leq (1-\alpha)\min\left\{\log_2\left(1 + \frac{|h_2|^2P_r}{\sigma_2^2}\right), \log_2\left(1 + \frac{|h_1|^2P_r}{\sigma_1^2}\right)\right\} \quad (11)$$

for $\alpha \in [0,1]$.

Proof: At the relay, the message m_r corresponding to the index of $\mathbf{v}_r$ is mapped to a codeword $\mathbf{x}_r$ of size n_2. Let R and R_r be the common source rate and broadcast rate, respectively. From the Shannon's source-channel separation theorem [17], we have that:

$$n_1R \leq n_2R_r \quad (12)$$

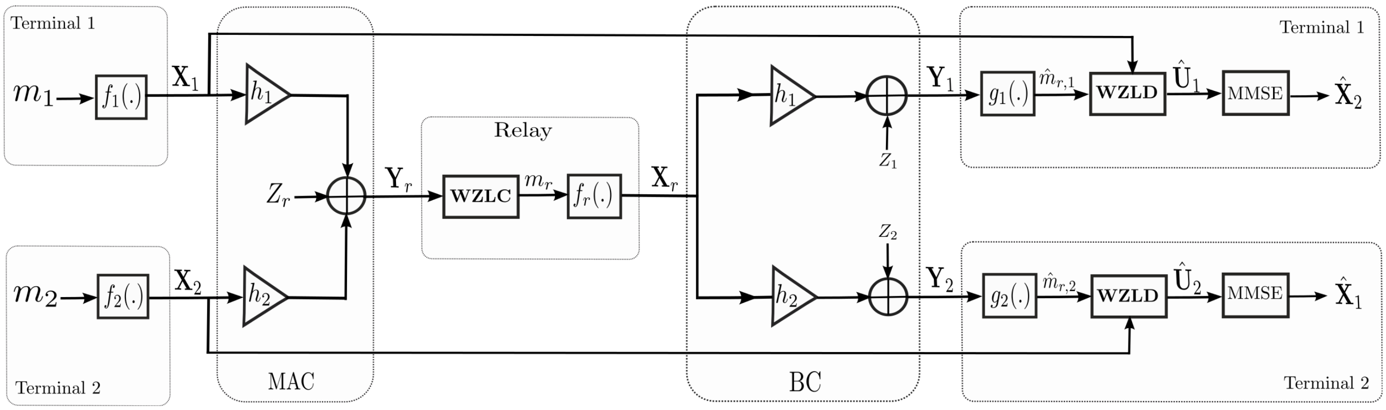

Fig. 2. Chanel Model for TWRC

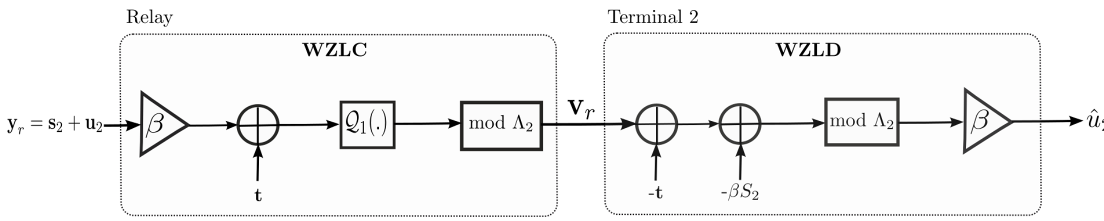

Fig. 3. WZ Lattice Coding and Decoding at T_2

with large dimension lattices we have:

$$R = R(D_2) \tag{13}$$

On the other hand, the nodes receive Y_1 and Y_2 as given by (2). The error probability of decoding X_r vanishes for n_2 sufficiently large if :

$$R_r \leq \min(I(X_r; Y_1), I(X_r; Y_2)) \tag{14}$$

For the real Gaussian case:

$$I(X_r; Y_1) = \frac{1}{2}\log_2\left(1 + \frac{|h_1|^2 P_r}{\sigma_1^2}\right)$$

and

$$I(X_r; Y_2) = \frac{1}{2}\log_2\left(1 + \frac{|h_2|^2 P_r}{\sigma_2^2}\right)$$

Finally (11) is obtained by putting equations (12) and (13) into equation (14) and by replacing n_1 by αn and n_2 by $(1-\alpha)n$. This constraint ensures that the index m_r is transmitted reliably to both terminals and $\mathbf{v}_r$ is available at the input of WZLD at both receivers. In the sequel, (9) and (10) will be demonstrated therefore we present first the details of the source decoders.

Source decoder at receiver T_2

Given the sequence $\mathbf{s}_2$, $\hat{\mathbf{u}}_2$ is reconstructed such as:

$$\begin{aligned}
\hat{\mathbf{u}}_2 &= \hat{\mathbf{y}}_{r,2} - \mathbf{s}_2 && (15)\\
&= \beta((\mathbf{v}_r - \mathbf{t} - \beta\mathbf{s}_2) \bmod \Lambda_2) && (16)\\
&= \beta((Q_1(\beta\mathbf{y}_r + \mathbf{t}) \bmod \Lambda_2 - \mathbf{t} - \beta\mathbf{s}_2) \bmod \Lambda_2) && (17)\\
&= \beta((\beta\mathbf{u}_2 + \mathbf{e}_q) \bmod \Lambda_2) && (18)\\
&\equiv \beta(\beta\mathbf{u}_2 + \mathbf{e}_q) && (19)
\end{aligned}$$

where $\mathbf{e}_q = Q_1(\beta\mathbf{y}_r + \mathbf{t}) - (\beta\mathbf{y}_r + \mathbf{t}) = -(\beta\mathbf{y}_r + \mathbf{t}) \mod \Lambda_1$, is the quantization error. By the Crypto Lemma in Section III-A, $\mathbf{E}_q$ is independent from $\mathbf{Y}_r$ (and therefore from $\mathbf{U}_2$) and it is uniformly distributed over $\mathcal{V}_1$ i.e $\mathrm{VAR}(E_q) = \sigma^2(\Lambda_1) = D_2$. Equation (17) to Equation (18) follows from the lattice distributive law.
The equivalence from (18) to (19) is valid for $n_1 \to \infty$. According to [14], with good channel coding lattices the decoding error probability vanishes asymptotically. i.e.

$$\Pr(\beta\mathbf{U}_2 + \mathbf{E}_q) \notin \mathcal{V}_2 \underset{n_1\to\infty}{\longrightarrow} 0 \tag{20}$$

Note that

$$\begin{aligned}
\frac{1}{n_1}\mathbb{E}||\mathbf{E}_q + \beta\mathbf{U}_2||^2 &= \frac{1}{n_1}\mathbb{E}||\mathbf{E}_q||^2 + \frac{1}{n_1}\mathbb{E}||\beta\mathbf{U}_2||^2\\
&= D_2 + \beta^2\sigma_{U_2}^2
\end{aligned}$$

β should be chosen carefully to guarantee that

$$\frac{1}{n_1}\mathbb{E}\|\mathbf{E}_q + \beta\mathbf{U}_2\|^2 \leq \sigma^2(\Lambda_2) \tag{21}$$

Furthermore, we have:

$$\begin{aligned}
\mathbf{Y}_r - \hat{\mathbf{Y}}_{r,2} &= \mathbf{U}_2 + \mathbf{S}_2 - \hat{\mathbf{U}}_2 + \mathbf{S}_2\\
&= \mathbf{U}_2 - \hat{\mathbf{U}}_2\\
&= (1-\beta^2)\mathbf{U}_2 - \beta\mathbf{E}_q
\end{aligned} \tag{22}$$

So the reconstruction distortion can be expressed as:

$$\begin{aligned}
\frac{1}{n_1}\mathbb{E}\|\mathbf{Y}_r - \hat{\mathbf{Y}}_{r,2}\|^2 &= \frac{1}{n_1}\mathbb{E}\|(1-\beta^2)\mathbf{U}_2 - \beta\mathbf{E}_q\|^2\\
&= \frac{1}{n_1}\mathbb{E}||(1-\beta^2)\mathbf{U}_2||^2 + \frac{1}{n_1}\mathbb{E}||\beta\mathbf{E}_q||^2\\
&= (1-\beta^2)^2\sigma_{U_2}^2 + \beta^2 D_2
\end{aligned}$$

β should guarantee also

$$\frac{1}{n_1}\mathbb{E}\|\mathbf{Y}_r - \hat{\mathbf{Y}}_{r,2}\|^2 \leq D_2 \tag{23}$$

Thus the optimal estimation factor β that verifies (21) and (23) is $\beta = \sqrt{1 - \frac{D_2}{\sigma^2_{U_2}}}$.

Finally from (19), $\hat{\mathbf{U}}_2 = \beta(\beta\mathbf{U}_2 + \mathbf{E}_q)$ which represents an equivalent forward test channel given in Fig.4.

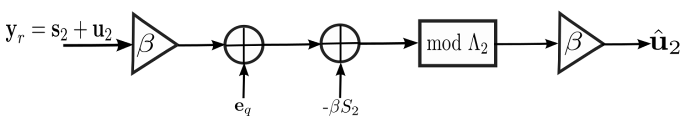

Fig. 4. Equivalent Channel at T_2

By replacing $\mathbf{U}_2$ by its value we conclude that:

$$\hat{\mathbf{U}}_2 = \beta^2 h_1 \mathbf{X}_1 + \beta^2 \mathbf{Z}_r + \beta\mathbf{E}_q \tag{24}$$

Let $\mathbf{Z}_{eq} = \beta^2\mathbf{Z}_r + \beta\mathbf{E}_q$ be the effective additive noise. The communication between T_1 and T_2 is equivalent to a virtual Gaussian channel where the noise is given by $\mathbf{Z}_{eq}$. We approximate $\mathbf{E}_q$ by a Gaussian variable with same variance. The equivalence is valid for asymptotic regime as $n_1 \to \infty$ [18]. The achievable rate of this link satisfies:

$$nR_{12} \leq \frac{n_1}{2}\log_2\left(1 + \frac{\beta^2|h_1|^2 P_1}{\beta^2\sigma_z^2 + D_2}\right)$$

by replacing $\frac{n_1}{n} = \alpha$ and β by its value, (9) is verified.

Source decoder at receiver T_1

For terminal T_1, the decoder is adapted in order to fit to the side information $\mathbf{S}_1$. Thus, at the decoder we subtract $\beta\mathbf{s}_1$ and $\hat{\mathbf{u}}_1$ is reconstructed such as:

$$\begin{aligned}
\hat{\mathbf{u}}_1 &= \hat{\mathbf{y}}_{r,1} - \mathbf{s}_1 && (25)\\
&= \beta((\mathbf{v}_r - \mathbf{t} - \beta\mathbf{s}_1) \mod \Lambda_2) && (26)\\
&= \beta((Q_1(\beta\mathbf{y}_r + \mathbf{t})\text{mod}\Lambda_2 - \mathbf{t} - \beta\mathbf{s}_1)\text{mod}\Lambda_2) && (27)\\
&= \beta((\beta\mathbf{u}_1 + \mathbf{e}_q)\text{mod}\Lambda_2) && (28)\\
&\equiv \beta(\beta\mathbf{u}_1 + \mathbf{e}_q) && (29)
\end{aligned}$$

The equality in (29) is conditioned on correct decoding at decoder T_1. It follows from the choice of good shaping lattice Λ_2 that $Pr\{\beta\mathbf{u}_1 + \mathbf{e}_q \notin \mathcal{V}_2\} \underset{n_1\to\infty}{\longrightarrow} 0$. In fact, we have:

$$\begin{aligned}
\frac{1}{n_1}\mathbb{E}\|\mathbf{E}_q + \beta\mathbf{U}_1\|^2 &= D_2 + \beta^2\sigma^2_{U_1}\\
&= \sigma^2_{U_2} + \beta^2(\sigma^2_{U_1} - \sigma^2_{U_2})
\end{aligned}$$

and $\sigma^2_{U_1} \leq \sigma^2_{U_2}$ since $\sigma^2_{S_1} \geq \sigma^2_{S_2}$, thus $\frac{1}{n_1}\mathbb{E}\|\mathbf{E}_q + \beta\mathbf{U}_1\|^2 \leq \sigma^2_{U_2}$. This means that asymptotically $\mathbf{E}_q + \beta\mathbf{U}_1 \in \mathcal{V}_2$. Besides,

$$\begin{aligned}
\mathbf{Y}_r - \hat{\mathbf{Y}}_{r,1} &= \mathbf{U}_1 - \hat{\mathbf{U}}_1\\
&= (1-\beta^2)\mathbf{U}_1 - \beta\mathbf{E}_q
\end{aligned} \tag{30}$$

Thus, the reconstruction distortion at T_1 can be calculated as follows

$$\begin{aligned}
D_1 = \frac{1}{n_1}\mathbb{E}\|\mathbf{Y}_r - \hat{\mathbf{Y}}_{r,1}\|^2 &= \frac{1}{n_1}\mathbb{E}\|(1-\beta^2)\mathbf{U}_1 - \beta\mathbf{E}_q\|^2\\
&= (1-\beta^2)^2\sigma^2_{U_1} + \beta^2 D_2\\
&= D_2 - \frac{D_2^2}{\sigma^4_{U_2}}(\sigma^2_{U_2} - \sigma^2_{U_1})\\
&\leq D_2
\end{aligned}$$

Similarly from (29), $\hat{\mathbf{U}}_1 = \beta^2\mathbf{U}_1 + \beta\mathbf{E}_q$,

$$\hat{\mathbf{U}}_1 = \beta^2 h_2 \mathbf{X}_2 + \mathbf{Z}_{eq} \tag{31}$$

Again the communication between T_1 and T_2 is equivalent to a virtual Gaussian channel with an additive noise $\mathbf{Z}_{eq}$. The rate of this link satisfies:

$$nR_{21} \leq \frac{n_1}{2}\log_2\left(1 + \frac{\beta^2|h_2|^2 P_2}{\beta^2\sigma_z^2 + D_2}\right)$$

which verifies (10). This concludes the proof. ■

In practice, the messages $\hat{\mathbf{X}}_1$ and $\hat{\mathbf{X}}_2$ can be decoded by applying an MMSE estimator after the reconstruction of $\hat{\mathbf{U}}_1$ and $\hat{\mathbf{U}}_2$ and finally both nodes apply a de-mapping function to obtain the messages $\hat{m}_1$ and $\hat{m}_2$. On the other hand, the achievable rates can be optimized by maximizing over all $\alpha \in [0, 1]$.

Comparison with random coding

As detailed earlier, our proposed scheme is a CF relaying strategy based on structured codes at the relay. However, CF scheme have been widely investigated with random coding for TWRC in [4], [6] and [7].

The achievable rate region of random coding CF for TWRC with time division optimization is given by the convex hull of all rates (R_{12}, R_{21}) ([4], [6], [7]):

$$\begin{aligned}
R_{12} &\leq \alpha I(X_1; \hat{Y}_r | X_2, Q) && (32)\\
R_{21} &\leq \alpha I(X_2; \hat{Y}_r | X_1, Q) && (33)
\end{aligned}$$

subject to:

$$\begin{aligned}
\alpha I(\hat{Y}_r; Y_r | X_1, Q) &\leq (1-\alpha) I(X_r; Y_1) && (34)\\
\alpha I(\hat{Y}_r; Y_r | X_2, Q) &\leq (1-\alpha) I(X_r; Y_2) && (35)
\end{aligned}$$

over some joint probability distributions

$$p(q)p(x_1|q)p(x_2|q)p(y_r|x_1, x_2)p(\hat{y}_r|y_r)p(x_r|q)$$

were $\hat{Y}_r$ is the estimate of Y_r at the relay.

To prove this achievable rate, the source coding strategy is based on random coding WZ encoding.

In the classical Wyner-Ziv coding [12], [19], the reconstruction of Y_r at decoder T_i, $\hat{Y}_{r,i}$, is a function of the compressed source $\hat{Y}_r$ and the side information i.e. $\hat{Y}_{r,i} = f(\hat{Y}_r, X_i)$ and for Gaussian sources $I(Y_r; \hat{Y}_r | X_1) = I(Y_r; \hat{Y}_{r,i} | X_i)$. Broadcasting a common source and considering only the receiver T_2 with weaker side information [20], the optimal forward test channel for a Gaussian source and side information as given in [19] is

$$\hat{Y}_r = a(Y_r + \Psi) \tag{36}$$

where $a = \frac{\text{VAR}(Y_r|X_2) - D_2}{\text{VAR}(Y_r|X_2)} = \frac{\sigma_{U_2}^2 - D_2}{\sigma_{U_2}^2} = \beta^2$ and $\Psi \sim \mathcal{N}(0, \frac{\sigma_{U_2}^2 D_2}{\sigma_{U_2}^2 - D_2})$.

We note that $\text{VAR}(\Psi) = \frac{D_2}{\beta^2}$ and $\hat{Y}_r = \beta^2 Y_r + \beta^2 \Psi$. Thus, we can rewrite the rates according to this Gaussian model as:

$$\begin{aligned}
R_{12} &\leq \alpha I(X_1; \hat{Y}_r|X_2) \\
&= \alpha I(X_1; \beta^2 Y_r + \beta^2 \Psi|X_2) \\
&= \alpha I(X_1; \beta^2(h_2 X_2 + h_1 X_1 + Z_r + \Psi)|X_2) \\
&= \alpha I(X_1; \beta^2(h_1 X_1 + Z_r + \Psi)) \\
&= \alpha h(\beta^2(h_1 X_1 + Z_r + \Psi)) - h(\beta^2(Z_r + \Psi)) \\
&= \frac{\alpha}{2} \log_2 \left(\frac{|h_1|^2 P_1 + \sigma_z^2 + \frac{D_2}{\beta^2}}{\sigma_z^2 + \frac{D_2}{\beta^2}} \right) \\
&= \frac{\alpha}{2} \log_2 \left(1 + \frac{\beta^2 |h_1|^2 P_1}{\beta^2 \sigma_z^2 + D_2} \right)
\end{aligned}$$

R_{21} can be computed similarly. By replacing β with its value we found the same rates as in (9) and (10).
Moreover, limited by the bad receiver T_2,

$$\max(I(\hat{Y}_r; Y_r|X_1), I(\hat{Y}_r; Y_r|X_2)) = I(\hat{Y}_r; Y_r|X_2)$$

Thus the constraints (34) and (35) reduce to

$$\alpha I(\hat{Y}_r; Y_r|X_2) \leq (1-\alpha) \min\{I(X_r; Y_1), I(X_r; Y_2)\} \quad (37)$$

The left-hand side of the inequality can be computed as:

$$\begin{aligned}
I(\hat{Y}_r; Y_r|X_2) &= I(\beta^2 Y_r + \beta^2 \Psi; Y_r|X_2) \\
&= I(\beta^2(h_1 X_1 + Z_r + \Psi); h_1 X_1 + Z_r) \\
&= h(\beta^2(h_1 X_1 + Z_r + \Psi)) - h(\beta^2 \Psi) \\
&= \frac{1}{2} \log_2 \left(\frac{|h_1|^2 P_1 + \sigma_z^2 + \frac{D_2}{\beta^2}}{\frac{D_2}{\beta^2}} \right) \\
&= \frac{1}{2} \log_2 \left(1 + \beta^2 \frac{|h_1|^2 P_1 + \sigma_z^2}{D_2} \right) \\
&= \frac{1}{2} \log_2 \left(\frac{|h_1|^2 P_1 + \sigma_z^2}{D_2} \right) \\
&= R(D_2)
\end{aligned}$$

which corresponds to the WZ rate-distortion function. Thus, the equation (37) corresponds to WZLC scheme equation (11). Therefore, the derived achievable rates for WZLC is equivalent to the ones using random coding for TWRC with CF scheme [4], [6], [7].

V. Conclusion

In this paper, we have proposed a lattice-based WZ coding at the relay using good nested lattices for TWRC. We have derived the achievable rate region of the scheme by considering optimal time division between both transmission phases. Moreover, we have shown that the compress-and-forward rates are achievable for the TWRC using lattices at the relay.
In general, broadcasting a single quantized source is not optimal for both users since they have different channel and side information qualities. However, the problem addressed in this paper gives us an insight on the application of structured codes, namely lattice codes. These codes can achieve rate regions equal to those achievable by random coding schemes while offering practical coding/decoding strategies.

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